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BAD BOUNDARY BEHAVIOR IN STAR INVARIANT SUBSPACES II

Andreas Hartmann and William T. Ross

Université de Bordeaux, Institut de Mathématiques de Bordeaux
351 cours de la Libération, 33405 Talence, France; hartmann@math.u-bordeaux.fr

University of Richmond, Department of Mathematics and Computer Science
212 Jepson Hall, 28 Westhampton Way, VA 23173, U.S.A.; wross@richmond.edu

Abstract. We continue our study begun in [HR11] concerning the radial growth of functions in the model spaces $(IH^2)^\perp$.

1. Introduction

Suppose $I = BS_\mu$ is an inner function with Blaschke factor B, with zeros $\{\lambda_n\}_{n\geq 1}$ in the open unit disk $\mathbf{D}$ repeated according to multiplicity, and singular inner factor S_μ with associated positive singular measure μ on the unit circle $\mathbf{T}$. The following result was shown by Frostman in 1942 for Blaschke products (see [Fro42] or [CL66]) and by Ahern–Clark for general inner functions [AC71, Lemma 3].

Theorem 1.1. (Frostman, 1942; Ahern–Clark, 1971) *Let $\zeta \in \mathbf{T}$ and I be inner with $\mu(\{\zeta\}) = 0$. Then the following assertions are equivalent.*

(1) *Every divisor of I has a radial limit of modulus one at ζ.*
(2) *Every divisor of I has a radial limit at ζ.*
(3) *The following condition holds*

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{|\zeta-\lambda_n|} + \int_{\mathbf{T}} \frac{1}{|\zeta-e^{it}|}\, d\mu(e^{it}) < \infty. \tag{1.2}$$

Based on a stronger condition than the above, Ahern and Clark [AC70] were able to characterize "good" non-tangential boundary behavior of functions in the model spaces $(IH^2)^\perp$ of the classical Hardy space H^2 (see [Nik86] for a very complete treatment of model spaces).

Theorem 1.3. [AC70] *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\geq 1}$ and associated singular measure μ. For $\zeta \in \mathbf{T}$, the following are equivalent:*

(1) *Every $f \in (IH^2)^\perp$ has a radial limit at ζ.*
(2) *The following condition holds*

$$\sum_{n\geq 1} \frac{1-|\lambda_n|}{|\zeta-\lambda_n|^2} + \int_{\mathbf{T}} \frac{1}{|\zeta-e^{it}|^2}\, d\mu(e^{it}) < \infty. \tag{1.4}$$

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In this paper, we will study what happens when we are somewhere in between the Frostman condition (1.2) and the Ahern–Clark condition (1.4). In order to do so we will introduce an auxiliary function. Let $\varphi\colon (0,2] \to \mathbf{R}^+$ be a positive increasing function such that

(1) $x \to \dfrac{\varphi(x)}{x}$ is bounded,
(2) $x \longmapsto \dfrac{\varphi(x)}{x^2}$ is decreasing,
(3) $\varphi(x) \asymp \varphi(x+o(x))$, $x \downarrow 0$.

Such a function φ will be called *admissible.* One can check that functions such as $\varphi(x) = x^p, 1 \le p \le 2$, and $\varphi(x) = x^p \log(1/x)$, $1 < p < 2$, are admissible. Our main result is the following.

Theorem 1.5. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\ge 1}$ and associated singular measure μ, φ an admissible function, and $\zeta \in \mathbf{T}$. If*

$$\sum_{n\ge 1} \frac{1-|\lambda_n|}{\varphi(|\zeta-\lambda_n|)} + \int_{\mathbf{T}} \frac{1}{\varphi(|\zeta-e^{it}|)}\, d\mu(e^{it}) < \infty, \tag{1.6}$$

then every $f \in (IH^2)^\perp$ satisfies

$$|f(r\zeta)| \lesssim \frac{\sqrt{\varphi(1-r)}}{1-r}. \tag{1.7}$$

When $\varphi(x) = x$ then we are in the Frostman situation (1.2) and no restriction is given for the growth of f since generic functions in H^2 satisfy the growth condition

$$|f(r\zeta)| = o\Big(\frac{1}{\sqrt{1-r}}\Big).$$

On the other hand, when $\varphi(x) = x^2$ we reach the Ahern–Clark situation (1.4) . For other φ such as $\varphi(x) = x^{3/2}$ or perhaps $\varphi(x) = x^2/\log(e/x)$ we get that even though functions in $(IH^2)^\perp$ can be poorly behaved (as in the title of this paper), the growth is controlled.

There is some history behind these types of problems. When $\varphi(x) = x^{2N+2}$, where $N = 0, 1, 2, \cdots$, Ahern and Clark [AC70] showed that (1.6) is equivalent to the condition that $f^{(j)}$, $0 \le j \le N$, have radial limits at ζ for every $f \in (IH^2)^\perp$. When $\varphi(x) = x^p$, $p \in (1,\infty)$, Cohn [Coh86] showed that (1.6) is equivalent to the condition that every $f \in H^q \cap I\overline{H^q_0}$, where $q = p(p-1)^{-1}$, has a finite radial limit at ζ.

Why did we write this second paper? In [HR11] we discussed controlled growth of functions from $(BH^2)^\perp$, where B is a Blaschke product not satisfying the condition (1.4) of the Ahern–Clark theorem. We have a general result but stated in very different terms, and using very different techniques, than the paper here. In particular, in [HR11] we obtain two-sided estimates for the reproducing kernels which yields more precise results. The results presented here are one-sided estimates but are for general inner functions and not just Blaschke products.

2. Proof of the main result

It is well known that $(IH^2)^\perp$ is a reproducing kernel Hilbert space with kernel function

$$k_\lambda^I(z) := \frac{1 - \overline{I(\lambda)}I(z)}{1 - \overline{\lambda}z}.$$

It suffices to prove Theorem 1.5 for $\zeta = 1$. If $\|\cdot\|$ denotes the norm in H^2, the estimate in (1.7) follows from the following result along with the obvious estimate

$$|f(r)| \le \|f\|\|k_r^I\|, \quad f \in (IH^2)^\perp, \quad r \in (0,1).$$

Theorem 2.1. *Let $I = BS_\mu$ be an inner function with zeros $\{\lambda_n\}_{n\ge1}$ and associated singular measure μ and φ be an admissible function. If*

$$\sum_{n\ge1} \frac{1-|\lambda_n|}{\varphi(|1-\lambda_n|)} + \int_{\mathbf{T}} \frac{1}{\varphi(|1-e^{it}|)}\, d\mu(e^{it}) < \infty, \tag{2.2}$$

then

$$\|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2}. \tag{2.3}$$

Proof. Our first observation is that since $x \longmapsto \varphi(x)/x$ is bounded, (2.2) implies condition (1.2). By Theorem 1.1 this implies that $\lim_{r\to1^-} |B(r)| = \lim_{r\to1^-} |S_\mu(r)| = 1$. Hence

$$\|k_r^I\|^2 = \frac{1-|I(r)|^2}{1-r^2} = \frac{1-\exp(\log(|I(r)|^2))}{1-r^2} = \frac{1-\exp(\log(|B(r)|^2 + \log|S_\mu(r)|^2))}{1-r^2},$$

and since $\log|B(r)| \to 0$ and $\log|S_\mu(r)| \to 0$ when $r \to 1$, we get

$$\begin{aligned}\|k_r^I\|^2 &= \frac{1-\exp(\log|B(r)|^2 + \log|S_\mu(r)|^2)}{1-r^2} \\ &= \frac{1-\Big(1+\Big(\log|B(r)|^2+\log|S_\mu(r)|^2\Big)+o\Big(\log|B(r)|^2+\log|S_\mu(r)|^2\Big)\Big)}{1-r^2} \\ &\sim \frac{\log|B(r)|^{-2}+\log|S_\mu(r)|^{-2}}{1-r^2}.\end{aligned}$$

Thus to prove the estimate in (2.3) we need to prove

$$\frac{\log|B(r)|^{-2}}{1-r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2} \tag{2.4}$$

and

$$\frac{\log|S_\mu(r)|^{-2}}{1-r^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}. \tag{2.5}$$

Case 1: The Blaschke product B. First note that from the Frostman condition (1.2) we get

$$\frac{1-|\lambda_n|}{|1-\lambda_n|} \longrightarrow 0. \tag{2.6}$$

This condition implies that for *every* Stolz angle at 1, $\Gamma_\alpha := \{z \in \mathbf{D}\colon |1-z| \le \alpha(1-|z|)\}$, where $\alpha > 1$, there is an index n_0 such that for $n \ge n_0$ the points λ_n are outside Γ_α, implying that $\{\lambda_n\}_{n\ge1}$ goes tangentially to 1. In particuliar, for $n \ge n_0$,

λ_n will be pseudohyperbolically far from the radius $[0,1)$, i.e., there is a δ such that for every $n \geq n_0$ and $r \in [0,1)$,

$$|b_{\lambda_n}(r)| \geq \delta.$$

Here $b_\lambda = (\lambda - z)/(1-\overline{\lambda}z)$ is the usual Blaschke factor and $\rho(\lambda, z) = |b_\lambda(z)|$ defines the pseudohyperbolic distance between λ and z. This implies

$$\log \frac{1}{|b_{\lambda_n}(r)|^2} \asymp 1 - |b_{\lambda_n}(r)|^2.$$

It is well known that

$$1 - |b_{\lambda_n}(r)|^2 = \frac{(1-r^2)(1-|\lambda_n|^2)}{|1-r\overline{\lambda_n}|^2}.$$

Thus

$$\frac{\log|B(r)|^{-2}}{1-r^2} = \frac{1}{1-r^2}\sum_{n\geq 1}\log\frac{1}{|b_{\lambda_n}(r)|^2} \asymp \sum_{n\geq 1}\frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}r|^2}. \tag{2.7}$$

Now let $\lambda_n = r_n e^{i\theta_n}$. We need the following two easy estimates:

$$|1-\rho e^{i\theta}|^2 \asymp (1-\rho)^2 + \theta^2, \quad \rho \approx 1,\ \theta \approx 0, \tag{2.8}$$

$$(|z|^2 + |w|^2)^{1/2} \asymp |z| + |w|, \quad z, w \in \mathbf{C}. \tag{2.9}$$

In particular, $|1-\lambda_n|^2 \asymp (1-r_n)^2 + \theta_n^2$. We now remember condition (2.6) which implies that $1 - r_n = 1 - |\lambda_n| = o(|1-\lambda_n|) = o((1-r_n) + \theta_n)$ so that necessarily $1 - r_n = o(\theta_n)$. Hence

$$|1-\overline{\lambda}_n r|^2 \asymp (1-r_n r)^2 + \theta_n^2 = (1 - r_n + r_n(1-r))^2 + \theta_n^2 \asymp (1-r)^2 + \theta_n^2.$$

The estimate in (2.7) yields

$$\begin{aligned}
\frac{\log|B(r)|^{-2}}{1-r^2} &\asymp \sum_{n\geq 1}\frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}r|^2} \asymp \sum_{n\geq 1}\frac{1-r_n}{(1-r)^2+\theta_n^2}\\
&\asymp \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} + \sum_{\{n:1-r\geq\theta_n\}}\frac{1-r_n}{(1-r)^2}\\
&= \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} + \frac{1}{(1-r)^2}\sum_{\{n:1-r\geq\theta_n\}}(1-r_n). \qquad (2.10)
\end{aligned}$$

Let us discuss each summand in (2.10) individually. For the first, we use the fact that φ is admissible and so $\varphi(\theta) \asymp \varphi(|1-e^{i\theta}|)$ to get

$$\begin{aligned}
\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^2} &= \sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\sqrt{\varphi(\theta_n)}\theta_n^2/\sqrt{\varphi(\theta_n)}}\\
&\leq \underbrace{\left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\varphi(\theta_n)}\right)^{1/2}}_{\text{bounded by assumption}}\left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\theta_n^4/\varphi(\theta_n)}\right)^{1/2}\\
&\lesssim \left(\sum_{\{n:1-r<\theta_n\}}\frac{1-r_n}{\varphi(\theta_n)(\theta_n^2/\varphi(\theta_n))^2}\right)^{1/2}.
\end{aligned}$$

Since φ is admissible, $x \to \varphi(x)/x^2$ is decreasing. Hence we can bound $\theta_n^2/\varphi(\theta_n)$ below in this last sum by $(1-r)^2/\varphi(1-r)$. This together with (2.2) gives us

$$\sum_{\{n:1-r<\theta_n\}} \frac{1-r_n}{\theta_n^2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}\left(\sum_{\{n:1-r<\theta_n\}} \frac{1-r_n}{\varphi(\theta_n)}\right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

For the second sum in (2.10) we have

$$\begin{aligned}\sum_{\{n:1-r\geq\theta_n\}}(1-r_n) &= \sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\frac{\sqrt{\varphi(\theta_n)}}{\sqrt{\varphi(\theta_n)}} \\ &\leq \underbrace{\left(\sum_{\{n:1-r\geq\theta_n\}}\frac{(1-r_n)}{\varphi(\theta_n)}\right)^{1/2}}_{\text{bounded by assumption}}\left(\sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\varphi(\theta_n)\right)^{1/2} \\ &\lesssim \sqrt{\varphi(1-r)}\left(\sum_{\{n:1-r\geq\theta_n\}}(1-r_n)\right)^{1/2},\end{aligned}$$

where we have used the fact that φ is increasing. Dividing through the square root of the sum in this last inequality (and then squaring) implies

$$\sum_{\{n:1-r\geq\theta_n\}}(1-r_n) \lesssim \varphi(1-r).$$

This verifies (2.4).

Case 2: The singular inner factor S_μ. This case is very similar to the first case. Indeed,

$$\frac{\log|S_\mu(r)|^{-2}}{1-r^2} = 2\int_{\mathbf{T}}\frac{1}{|1-re^{i\theta}|^2}\,d\mu(e^{i\theta}) \asymp \int_{\mathbf{T}}\frac{1}{(1-r)^2+\theta^2}\,d\mu(e^{i\theta}),$$

where we have again used (2.8). As in the Blaschke situation we split the integral into two parts depending on which term in the denominator dominates:

$$\begin{aligned}\frac{\log|S_\mu(r)|^{-2}}{1-r^2} &\lesssim \int_{\{\theta:1-r\leq\theta\}}\frac{1}{(1-r)^2+\theta^2}\,d\mu(e^{i\theta}) + \int_{\{\theta:1-r\geq\theta\}}\frac{1}{(1-r)^2+\theta^2}\,d\mu(e^{i\theta}) \\ &\asymp \int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}\,d\mu(e^{i\theta}) + \frac{1}{(1-r)^2}\int_{\{\theta:1-r\geq\theta\}}d\mu(e^{i\theta}). \qquad (2.11)\end{aligned}$$

Let us consider the first integral:

$$\begin{aligned}\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^2}\,d\mu(e^{i\theta}) &= \int_{\{\theta:1-r\leq\theta\}}\frac{1}{\sqrt{\varphi(\theta)}\theta^2/\sqrt{\varphi(\theta)}}\,d\mu(e^{i\theta}) \\ &\leq \left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\varphi(\theta)}\,d\mu(e^{i\theta})\right)^{1/2}\left(\int_{\{\theta:1-r\leq\theta\}}\frac{1}{\theta^4/\varphi(\theta)}\,d\mu(e^{i\theta})\right)^{1/2}.\end{aligned}$$

Again, $|1-e^{i\theta}| \asymp \theta$. Then using the hypothesis of admissibility we have $\varphi(\theta) \asymp \varphi(|1-e^{i\theta}|)$ and so

$$\int\frac{1}{\varphi(\theta)}\,d\mu(e^{i\theta}) \asymp \int\frac{1}{\varphi(|1-e^{i\theta}|)}\,d\mu(e^{i\theta})$$

which is bounded by assumption. Hence,

$$\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\theta^2}\, d\mu(e^{i\theta}) \lesssim \left(\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\theta^4/\varphi(\theta)}\, d\mu(e^{i\theta})\right)^{1/2} = \left(\int_{\{\theta:1-r\leq\theta\}} \frac{\varphi^2(\theta)}{\varphi(\theta)\theta^4}\, d\mu(e^{i\theta})\right)^{1/2}.$$

Now using the fact that $x \longrightarrow \varphi(x)/x^2$ is decreasing we obtain

$$\varphi^2(\theta)/\theta^4 \leq (\varphi(1-r))^2/(1-r)^4,$$

and

$$\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\theta^2}\, d\mu(e^{i\theta}) \lesssim \frac{\varphi(1-r)}{(1-r)^2} \left(\int_{\{\theta:1-r\leq\theta\}} \frac{1}{\varphi(\theta)}\, d\mu(e^{i\theta})\right)^{1/2} \lesssim \frac{\varphi(1-r)}{(1-r)^2}.$$

We turn to the second integral in (2.11) to get

$$\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) = \int_{\{\theta:1-r\geq\theta\}} \frac{\sqrt{\varphi(\theta)}}{\sqrt{\varphi(\theta)}}\, d\mu(e^{i\theta}) \leq \left(\int_{\{\theta:1-r\geq\theta\}} \varphi(\theta)\, d\mu(e^{i\theta})\right)^{1/2} \left(\int_{\{\theta:1-r\geq\theta\}} \frac{1}{\varphi(\theta)}\, d\mu(e^{i\theta})\right)^{1/2}.$$

We have already seen above that the second factor is bounded by assumption. Using the fact that φ is increasing we get

$$\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) \lesssim \left(\int_{\{\theta:1-r\geq\theta\}} \varphi(\theta)\, d\mu(e^{i\theta})\right)^{1/2} \leq \sqrt{\varphi(1-r)} \left(\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta})\right)^{1/2}.$$

Dividing through by the integral (and then squaring), we obtain

$$\int_{\{\theta:1-r\geq\theta\}} d\mu(e^{i\theta}) \lesssim \varphi(1-r),$$

which verifies (2.5). □

3. An example

The Blaschke situation was discussed in [HR11] where we obtained two-sided estimates for the reproducing kernels. It can be shown with concrete examples that the estimates from Theorem 2.1 are in general weaker than those obtained in [HR11] for Blaschke products.

Let us discuss the simplest case, in fact close enough to a Blaschke product, that a singular inner function S_μ with a discrete measure μ. Let

$$\mu = \sum_{n\geq 1} \alpha_n \delta_{\zeta_n},$$

where $\delta_{\zeta_n} \in \mathbf{T}$ and α_n are positive numbers with $\sum_n \alpha_n < \infty$ guaranteeing that μ is a finite measure on $\mathbf{T}$. Let us fix

$$\zeta_n = e^{i\theta_n} = e^{i/2^n}, \quad \alpha_n = \left(\frac{1}{2^\varepsilon}\right)^n, \quad n = 1, 2, \dots .$$

Also let $\varphi(t) = t^\gamma$ which defines an admissible function for $1 < \gamma < 2$. In order to have condition (2.2) it is necessary and sufficient to have

$$\sum_n \alpha_n \frac{1}{\varphi(|1-e^{i\theta_n}|)} \simeq \sum_n \frac{1}{2^{n\varepsilon}} \frac{1}{\varphi(1/2^n)} \simeq \sum_n 2^{(\gamma-\varepsilon)n} < \infty$$

which is equivalent to $\gamma < \varepsilon$. We suppose that

$$\gamma < \varepsilon < 2. \tag{3.1}$$

By Theorem 2.1 we deduce that

$$\|k_r^I\|^2 \lesssim \frac{\varphi(1-r)}{(1-r)^2} = \left(\frac{1}{1-r}\right)^{2-\gamma},$$

and hence

$$|f(r)| \lesssim \frac{1}{(1-r)^{1-\gamma/2}}, \quad f \in (S_\mu H^2)^\perp,$$

which is slower growth than the standard estimate

$$|f(r)| \lesssim \frac{1}{(1-r)^{1/2}}, \quad f \in H^2.$$

In this situation, it is actually possible to get a double-sided estimate for the reproducing kernel: since φ is admissible, Theorem 1.1 implies that $I(r) \longrightarrow \eta \in \mathbf{T}$ when $r \to 1^-$. In particular for $r \in (0,1)$, this implies that

$$|I(r)| = \exp\left(-\sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}\right) \sim 1 - \sum_n \alpha_n \frac{1-r^2}{|\zeta_n - r|^2}.$$

Let us consider the reproducing kernel of $(S_\mu H^2)^\perp$ at $r = \rho_N = 1 - 2^{-N}$. Indeed,

$$\|k_{\rho_N}^I\|^2 = \frac{1-|I(\rho_N)|^2}{1-\rho_N^2} \asymp \sum_n \frac{\alpha_n}{|\zeta_n - \rho_N|^2}.$$

Now using (2.8)

$$|\zeta_n - \rho_N|^2 \asymp \frac{1}{2^{2n}} + \frac{1}{2^{2N}},$$

and so

$$\begin{aligned}\|k_{\rho_N}^I\|^2 &\asymp \sum_n \frac{\alpha_n}{1/2^{2n} + 1/2^{2N}} = \sum_{n \le N} \frac{\alpha_n}{1/2^{2n}} + \sum_{n>N} \frac{\alpha_n}{1/2^{2N}} \\ &\asymp \sum_{n\le N} 2^{(2-\varepsilon)n} + 2^{2N} \sum_{n>N} \frac{1}{2^{\varepsilon n}} \asymp 2^{(2-\varepsilon)N} = \left(\frac{1}{1-\rho_N}\right)^{2-\varepsilon}\end{aligned}$$

or, equivalently,

$$\|k_{\rho_N}^I\| \asymp \left(\frac{1}{1-\rho_N}\right)^{1-\varepsilon/2} \tag{3.2}$$

(the estimate extends to the whole radius). As a consequence, the estimate from Theorem 2.1 is not optimal, though it is possible to come closer to it by choosing e.g., $\varphi(t) = t^\varepsilon / \log^{1+\gamma}(1/t)$, $\gamma > 0$.

4. A lower estimate

We finish this paper with a construction of an $f \in (S_\mu H^2)^\perp$, with μ the discrete measure discussed in the previous section, getting close to the growth given by the norm of the reproducing kernels thoughout a whole Stolz angle at 1. As in [HR11] our construction will be based on unconditional sequences. We need to recall some material on generalized interpolation in Hardy spaces for which we refer the reader to [Nik02, Section C3]. Let $I = \prod_n I_n$ be a factorization of an inner function I into inner functions I_n, $n \in \mathbf{N}$. The sequence $\{I_n\}_{n\geq 1}$ satisfies the generalized Carleson condition, sometimes called the Carleson–Vasyunin condition, which we will write $\{I_n\}_{n\geq 1} \in (CV)$, if there is a $\delta > 0$ such that

$$|I(z)| \geq \delta \inf_{n\geq 1} |I_n(z)|, \quad z \in \mathbf{D}. \tag{4.1}$$

In the special case of a Blaschke product $B = B_\Lambda$ with simple zeros $\Lambda = \{\lambda_n\}_{n\geq 1}$ and $I_n = b_{\lambda_n}$, this is equivalent to the well-known Carleson condition $\inf_n |B_{\Lambda\setminus\{\lambda_n\}}(\lambda_n)| \geq \delta > 0$.

If $\{I_n\}_{n\geq 1} \in (CV)$ then $\{(I_n H^2)^\perp\}_{n\geq 1}$ is an unconditional basis for $(IH^2)^\perp$ meaning that every $f \in (IH^2)^\perp$ can be written uniquely as

$$f = \sum_{n\geq 1} f_n, \quad f_n \in (I_n H^2)^\perp,$$

with

$$\|f\|^2 \asymp \sum_{n\geq 1} \|f_n\|^2.$$

In our situation we have $I = S_\mu$ and

$$I_n = e^{\alpha_n \frac{z+\zeta_n}{z-\zeta_n}}.$$

The corresponding spaces $(I_n H^2)^\perp$ are known to be isometrically isomorphic to the Paley–Wiener space of analytic functions of exponential type $\alpha_n/2$ and square integrable on the real axis. In this situation, a sufficient condition for (4.1) is known:

$$\sup_{n\geq 1} \sum_{k\neq n} \frac{\mu(\{\zeta_n\})\mu(\{\zeta_k\})}{|\zeta_n - \zeta_k|^2} < \infty$$

(see [Nik86, Corollary 6, p. 247]). So, since $\varepsilon > 1$ by (3.1), we have

$$\begin{aligned}\sum_{k\neq n} \frac{1/2^{\varepsilon n} 1/2^{\varepsilon k}}{|1/2^n - 1/2^k|^2} &\simeq 1/2^{\varepsilon n} \sum_{k<n} \frac{1/2^{\varepsilon k}}{|1/2^k|^2} + 1/2^{\varepsilon n} \sum_{k>n} \frac{1/2^{\varepsilon k}}{|1/2^n|^2} \\ &= 1/2^{\varepsilon n} \sum_{k<n} 2^{(2-\varepsilon)k} + 2^{(2-\varepsilon)n} \sum_{k>n} 1/2^{\varepsilon k} \asymp 2^{2(1-\varepsilon)n},\end{aligned}$$

which is uniformly bounded in n. Hence $(IH^2)^\perp$ is an ℓ^2-sum of Paley–Wiener spaces (each of which possesses, for instance, the harmonic unconditional basis). In particular, picking

$$\lambda_n := r_n \zeta_n = r_n e^{i/2^n}, \quad r_n = 1 - \frac{1}{2^n},$$

the sequence $\{K_n\}_{n\geq 1}$, where

$$K_n = \frac{k_{\lambda_n}^{I_n}}{\|k_{\lambda_n}^{I_n}\|} \in (I_n H^2)^\perp,$$

is an unconditional sequence in $(IH^2)^\perp$. We can introduce the family of functions

$$f_\beta := \sum_{n\geq n_0} \beta_n K_n,$$

where $\|f_\beta\|^2 \asymp \sum_{n\geq 1} |\beta_n|^2 < \infty$, and n_0 will be determined later. Let us estimate the norms $\|k_{\lambda_n}^{I_n}\|$. First observe that

$$\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n} = \alpha_n \frac{r_n + 1}{r_n - 1} = \frac{1}{2^{\varepsilon n}} \frac{2 - 1/2^n}{-1/2^n} = -\frac{2 - 1/2^n}{2^{(\varepsilon - 1)n}} \longrightarrow 0, \quad n \to \infty.$$

Hence

$$\begin{aligned}\|k_{\lambda_n}^{I_n}\|^2 &= \frac{1 - |I_n(\lambda_n)|^2}{1 - r_n^2} \asymp \frac{1 - |I_n(\lambda_n)|}{1 - r_n} = \frac{1 - \exp\Big(\log|I_n(\lambda_n)|\Big)}{1 - r_n} \\ &= \frac{1 - \exp\Big(\alpha_n \frac{\lambda_n + \zeta_n}{\lambda_n - \zeta_n}\Big)}{1 - r_n} \sim \frac{1 - \Big(1 + \alpha_n \frac{r_n + 1}{r_n - 1}\Big)}{1 - r_n} \sim \frac{2\alpha_n}{(1 - r_n)^2},\end{aligned}$$

so that

$$\|k_{\lambda_n}^{I_n}\| \asymp \sqrt{\frac{\alpha_n}{(1 - r_n)^2}} = \frac{\sqrt{2^{-(\varepsilon n)}}}{1/2^n} = 2^{(1 - \varepsilon/2)n}. \tag{4.2}$$

Observe now that the λ_n's belong to a Stolz domain with vertex at 1: Γ_α for some $\alpha > 1$. Indeed,

$$1 - |\lambda_n| = 1 - r_n = 1/2^n \simeq |1 - \zeta_n| \asymp |1 - \lambda_n|$$

(this follows from (2.8)). Absorbing the equivalence constants appearing in (4.2) into $\beta = \{\beta_n\}_{n\geq 1} \in \ell^2$ with $\beta_n \geq 0$, and picking a suitable real number φ_0 (also to be determined later), we will be interested in the real part of

$$f_{e^{i\varphi_0}\beta}(z) = \sum_{n\geq n_0} e^{i\varphi_0} \beta_n 2^{(\varepsilon - 1/2)n} \frac{1 - \overline{I_n(\lambda_n)} I_n(z)}{1 - \overline{\lambda_n} z}$$

for $z \in \Gamma_\alpha$. Note that when $z \in \Gamma_\alpha$ there exists a unique closest λ_N to z (in the pseudohyperbolic metric) and $|b_{\lambda_N}(z)| \leq \rho < 1$. We have already seen that $\mathbf{R} \ni I_n(\lambda_n) \longrightarrow 1$, $n \to \infty$, and

$$I_n(\lambda_n) \sim 1 - \alpha_n \frac{1 + r_n}{1 - r_n} \sim 1 - \frac{2}{2^{(\varepsilon - 1)n}}.$$

For $I_n(z)$ we need to consider

$$\alpha_n \frac{z + \zeta_n}{z - \zeta_n}.$$

Note that since $|b_{\lambda_N}(z)| \leq \rho < 1$,

$$\left|\alpha_n \frac{z + \zeta_n}{z - \zeta_n}\right| \asymp \left|\alpha_n \frac{\lambda_N + \zeta_n}{\lambda_N - \zeta_n}\right|.$$

For n and N bigger than some n_0, we have $\mathrm{Re}(\lambda_N+\zeta_n)\asymp|\lambda_N+\zeta_n|\asymp 2$. We thus have to consider the denominator. We observe that by (2.8)

$$|\lambda_N-\zeta_n|=|1-\overline{\zeta_n}\lambda_N|\asymp(1-r_N)+\left|\frac{1}{2^n}-\frac{1}{2^N}\right|=\frac{1}{2^N}+\left|\frac{1}{2^n}-\frac{1}{2^N}\right|$$

$$\asymp\begin{cases}\frac{1}{2^n} & \text{if } n<N,\\ \frac{1}{2^N} & \text{if } n\geq N.\end{cases} \tag{4.3}$$

As a consequence,

$$\left|\alpha_n\frac{z+\zeta_n}{z-\zeta_n}\right|\asymp\left|\alpha_n\frac{\lambda_N+\zeta_n}{\lambda_N-\zeta_n}\right|\longrightarrow 0,\quad n,N\to\infty,$$

and

$$I_n(z)\sim 1+\alpha_n\frac{z+\zeta_n}{z-\zeta_n}.$$

Hence

$$\begin{aligned}1-\overline{I_n(\lambda_n)}I_n(z)&\sim 1-\left(1+\alpha_n\frac{r_n+1}{r_n-1}\right)\left(1+\alpha_n\frac{z+\zeta_n}{z-\zeta_n}\right)\sim\alpha_n\frac{1+r_n}{1-r_n}+\alpha_n\frac{\zeta_n+z}{\zeta_n-z}\\&=\alpha_n\left(\frac{1+r_n}{1-r_n}+\frac{\zeta_n+z}{\zeta_n-z}\right)=\alpha_n\frac{(1+r_n)(\zeta_n-z)+(1-r_n)(\zeta_n+z)}{(1-r_n)(\zeta_n-z)}\\&=2\alpha_n\frac{\zeta_n-r_nz}{(1-r_n)(\zeta_n-z)}=2\alpha_n\zeta_n\frac{1-\overline{\zeta_n}r_nz}{(1-r_n)(\zeta_n-z)}\\&=2\alpha_n\zeta_n\frac{1-\overline{\lambda_n}z}{(1-r_n)(\zeta_n-z)}.\end{aligned}$$

From here we have

$$\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\sim\frac{2\alpha_n\zeta_n}{(1-r_n)(\zeta_n-z)}=\frac{2}{2^{(\varepsilon-1)n}}\frac{\zeta_n}{\zeta_n-z}=\frac{2}{2^{(\varepsilon-1)n}}\frac{1-\zeta_n\overline{z}}{|\zeta_n-z|^2}, \tag{4.4}$$

and

$$e^{i\varphi_0}\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}=e^{i\varphi_0}\frac{2}{2^{(\varepsilon-1)n}}\frac{1-\zeta_n\overline{z}}{|\zeta_n-z|^2}(1+\varepsilon_{n,z}),$$

where $\varepsilon_{n,z}$ is arbitrarily small for n,N sufficiently big (note that $|b_{\lambda_N}(z)|\leq\rho<1$). Since $\zeta_n=e^{i/2^n}$, $n\geq n_0$, and z is in Γ_α, we observe that

$$-\frac{\pi}{2}\leq\arg(1-\zeta_n\overline{z})\leq\eta$$

for some $\eta<\pi/2$ (η is actually given by half of the opening angle of Γ_α). Let now $\varphi_0=(\pi/2-\eta)/2$ so that

$$-\frac{\pi}{2}<-\frac{\pi}{4}-\frac{\eta}{2}\leq\arg\left(e^{i\varphi_0}(1-\zeta_n\overline{z})\right)\leq\frac{\pi}{4}+\frac{\eta}{2}<\frac{\pi}{2}.$$

Choosing n_0 sufficiently big, we can suppose that

$$|\arg(1+\varepsilon_{n,z})|\leq\frac{\varphi_0}{2},$$

which implies that for $n,N\geq n_0$,

$$\left|\arg\left[e^{i\varphi_0}\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\right]\right|\leq\frac{\pi}{2}-\varphi_0,$$

so that taking into account (4.4)

$$\begin{aligned}\operatorname{Re}\left(e^{i\varphi_0}\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\right) &\asymp \left|e^{i\varphi_0}\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\right| = \left|\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\right| \\ &\asymp \frac{2}{2^{(\varepsilon-1)n}}\frac{1}{|\zeta_n - z|}.\end{aligned}$$

For $n_0 \le n < N$ this expression is positive, and for $n \ge N$ (4.3) gives

$$|\zeta_n - z| \asymp |\zeta_n - \lambda_N| = |1-\zeta_n\overline{\lambda}_N| \asymp \frac{1}{2^N}.$$

Hence for $n \ge N$,

$$\operatorname{Re}\left(e^{i\varphi_0}\frac{1-\overline{I_n(\lambda_n)}I_n(z)}{1-\overline{\lambda_n}z}\right) \asymp 2\frac{2^N}{2^{(\varepsilon-1)n}},$$

and

$$\operatorname{Re} f_\beta(z) \gtrsim \sum_{n\ge N}\beta_n \frac{1}{2^{(1-\varepsilon/2)n}}\frac{2^N}{2^{(\varepsilon-1)n}} \gtrsim 2^N \sum_{n\ge N}\frac{\beta_n}{2^{n\varepsilon/2}}.$$

Pick for instance $\beta_n = n^{-(1+\gamma)/2}$, where $\gamma > 0$ is arbitrary, so that obvioulsy $\beta_n \ge 0$ and $\beta \in \ell^2$. Then

$$\begin{aligned}\operatorname{Re} f_\beta(z) &\gtrsim 2^N \sum_{n\ge N}\frac{1}{n^{(1+\gamma)/2}}\frac{1}{2^{n\varepsilon/2}} \ge 2^N \frac{1}{N^{(1+\gamma)/2}}\frac{1}{2^{N\varepsilon/2}} = \frac{2^{(1-\varepsilon/2)N}}{N^{(1+\gamma)/2}} \\ &\gtrsim \left(\frac{1}{1-|\lambda_N|}\right)^{1-\varepsilon/2}\frac{1}{\log^{(1+\gamma)/2}\left(\frac{1}{1-|\lambda_N|}\right)} \\ &\gtrsim \left(\frac{1}{1-|z|}\right)^{1-\varepsilon/2}\frac{1}{\log^{(1+\gamma)/2}\left(\frac{1}{1-|z|}\right)}\end{aligned}$$

so that we loose a logarithmic term with respect to the upper estimate of the reproducing kernel (3.2).

We should mention that using the biorthogonal system to $(K_n)_n$ in the space generated by $(K_n)_n$, we could also have obtained a lower estimate, but only at the points λ_N, whereas in the above construction, as already mentioned in the beginning of the section, the lower estimate holds throughout the whole Stolz angle.

Finally, we point out that when $I(z) \longmapsto 1$ when $z \to 1$ in a fixed z domain, it is, in general, particularly difficult to decide whether or not a sequence of reproducing kernels for $(IH^2)^\perp$, with the parameter in a Stolz domain with vertex at 1, is an unconditional basis or not. Even when $\sup_n |I(\lambda_n)| < 1$, there is a characterization known for unconditional basis which is, in general, difficult to check.

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