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# THE GROWTH AND SINGULAR DIRECTION OF ALGEBROID FUNCTIONS

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**Abstract.** In this paper, we obtain a basic inequation, investigate the relation between the growth as well as the singular direction of algebroid functions and those of their coefficients, and give some applications of the results.

## 1. Introduction and main results

Let $A_\nu(z)\,(\not\equiv 0), A_{\nu-1}(z), \cdots, A_0(z)$ $(z \in \mathbf{C})$ be entire functions without any common zero, where $\nu\,(\geq 1) \in \mathbf{N}$. Then the equation

$$\psi(z,w) \equiv A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0 \tag{1}$$

defines a $\nu$-valued algebroid function $w = w(z)$ in the complex plane (see [2, 4]). When $\nu = 1$, $w(z)$ is a meromorphic function. If $\psi(z,w)$ is irreducible in the polynomial ring $M[w]$ of meromorphic functions (see [4, 8]), then $w(z)$ is called a $\nu$-valued irreducible algebroid function. In this paper, we do not require that the polynomial $\psi(z,w)$ be irreducible. A general $\nu$-valued algebroid function $w(z)$ might be decomposed into $n$ $(1 \leq n \leq \nu)$ $\nu_i$-valued irreducible algebroid functions (including the case of $w = c$ being constant), and $\sum_{i=1}^n \nu_i = \nu$ (see [7, 8]).

Let $S_z$ denote the set of the critical points of $w(z)$(see [2, 4, 8]). Then for any $z_0 \in \mathbf{C} \setminus S_z$, there exists $\nu$ single-valued branches $w_1(z), w_2(z), \cdots, w_\nu(z)$ of $w(z)$ satisfying the equation (1) in some neighborhood of $z_0$, i.e.

$$\psi(z,w) \equiv A_\nu(z)(w - w_1(z))(w - w_2(z)) \cdots (w - w_\nu(z)) = 0, \quad z \in U(z_0).$$

We sometimes use $w(z) := \{w_i(z)\}_{i=1}^\nu$ to denote a $\nu$-valued algebroid functions (see [7, 8]).

In addition, the coefficient $A_j(z)$ of $w^j$ in $\psi(z,w)$ is called the coefficient of the algebroid function $w(z)$, where the coefficient $A_\nu(z)$ of $w^\nu$ is called the leading coefficient of $w(z)$, and the coefficient $A_0(z)$ of $w^0 = 1$ is called the constant coefficient of $w(z)$. If not particularly explained, we generally consider that there is at least one transcendental entire function among $\{A_j(z)\}_{j=0}^\nu$.

**Definition 1.** Let $w(z) = \{w_i(z)\}_{i=1}^\nu$ be an algebroid function defined by (1). Then

$$T(r,w) = m(r,w) + N(r,w) = \frac{1}{\nu}\sum_{i=1}^{\nu} m(r,w_i) + \frac{1}{\nu} N\left(r, \frac{1}{A_\nu}\right)$$

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is said the characteristic function of $w(z)$ or its Nevanlinna characteristic, where $m(r,w)$ is the proximity function, and $N(r,w)$ is the counting function of poles of $w(z)$ (see [2, 4]). The Nevanlinna characteristic of the coefficient $A_j(z)$ is

$$T(r,A_j) = m(r,A_j) + N(r,A_j) = m(r,A_j).$$

$\rho(w)$ denotes the order of $w(z)$, and $\rho(A_j)$ denotes the order of $A_j(z)$. They are respectively

$$\rho(w) = \limsup_{r\to\infty} \frac{\log^+ T(r,w)}{\log r}, \quad \rho(A_j) = \limsup_{r\to\infty} \frac{\log^+ T(r,A_j)}{\log r}.$$

**Definition 2.** Let $A_\nu(z), \cdots, A_0(z)$ be the coefficients in (1). Set

$$A(z) = \max\{|A_j(z)|;\ j = 0,1,\cdots,\nu\} \quad (z \in \mathbf{C}),$$

and define

$$\mu(r,A) = \frac{1}{2\pi\nu}\int_0^{2\pi} \log A(re^{i\theta})\,d\theta, \quad \rho(A) = \limsup_{r\to\infty} \frac{\log^+ \mu(r,A)}{\log r},$$

where $\mu(r,A)$ is said the Valiron characteristic of $w(z)$ (see [11, 2]), and $\rho(A)$ is the order of $\mu(r,A)$.

**Definition 3.** Suppose that $w(z)$ is a $\nu$-valued algebroid function defined by (1). If for arbitrary $\delta$ $(0 < \delta < \pi/2)$, in the angular region $\triangle(\theta_0,\delta) = \{z \mid |\arg z - \theta_0| < \delta,\ 0 \le \theta_0 < 2\pi\}$,

$$\limsup_{r\to\infty} \frac{\log n(r,\triangle(\theta_0,\delta),a)}{\log r} = \rho \quad (0<\rho<\infty) \tag{2}$$

holds for any $a \in \mathbf{C}\cup\infty$ except at most $2\nu$ values, then the radial $\arg z = \theta_0$ is called a Borel direction of $w(z)$, where $\rho$ is the order of $w(z)$, and $n(r,\triangle(\theta_0,\delta),a)$ denotes the number of $a$-points of $w(z)$ in the sector $\{|z| \le r\} \cap \triangle(\theta_0,\delta)$ counting multiplicities (see [12, 6]).

If (2) is replaced by

$$\limsup_{r\to\infty} \frac{N(r,\triangle(\theta_0,\delta),a)}{T(r,w)} > 0, \tag{3}$$

then the radial $\arg z = \theta_0$ is called a $T$ direction of $w(z)$ (see [13]), where

$$N(r,\triangle(\theta_0,\delta),a) = \frac{1}{\nu}\int_0^r \frac{n(t,\triangle(\theta_0,\delta),a) - n(0,\triangle(\theta_0,\delta),a)}{t}\,dt + \frac{1}{\nu} n(0,\triangle(\theta_0,\delta),a)\log r.$$

Some people have studied the growth of algebroid function from the characteristic function, but an algebroid function is an implicit function, so it is very difficult to calculate its order. However, if we can obtain the order by virtue of its coefficients, then the problem will become simple. This has been studied by Valiron [11], He [2], Katajamäki [4], Sun [9] and so on, where Katajamäki [4] indicated Theorem A by a basic inequation of Selberg [10].

**Theorem A.** *Let $w(z)$ be a $\nu$-valued irreducible algebroid function defined by (1). Then*

$$\rho(w) = \max\left\{\rho\left(\frac{A_i}{A_\nu}\right),\ i = 0, 1, \cdots, \nu - 1\right\}, \tag{4}$$

*where*

$$\rho\left(\frac{A_i}{A_\nu}\right) = \limsup_{r\to\infty} \frac{\log^+ T(r, \frac{A_i}{A_\nu})}{\log r}$$

*is the order of $A_i(z)/A_\nu(z)$.*

Sun and Kong [9] obtained the following result by means of the canonical product theorem (see [16]).

**Theorem B.** *Let $w(z)$ be a $\nu$-valued irreducible algebroid function defined by (1). Then*

$$\rho(w) = \max\left\{\rho\left(\frac{A_i}{A_0}\right),\ i = 1, \cdots, \nu\right\}. \tag{5}$$

For the entire functions $\{A_j(z)\}_{j=0}^{\nu}$ without any common zero, put

$$\Omega = \{w \mid A_{t_\nu}(z)w^\nu + A_{t_{\nu-1}}(z)w^{\nu-1} + \cdots + A_{t_0}(z) = 0\},$$

where $(t_\nu, t_{\nu-1}, \cdots, t_0)$ is one of the permutation of $(0, 1, \cdots, \nu)$. Then $\Omega$ contains $(\nu+1)!$ equations at most, and each equation defines an algebroid function. Hence $\Omega$ contains at most $(\nu+1)!$ algebroid functions. Obviously, $w(z)$ defined by (1) is one element in $\Omega$. When $A_\nu(z), \cdots, A_0(z)$ are all non-vanishing, every element in $\Omega$ is a $\nu$-valued algebroid function; otherwise, there are less-than-$\nu$-valued algebroid functions in $\Omega$.

We have considered the relation between the growth of the algebroid functions in $\Omega$ and that of $\{A_j(z)\}_{j=0}^{\nu}$ (see [14, 15]), and we continue to study the relation. Here Theorem 1 is obtained. It makes Theorems A and B become its special cases.

**Theorem 1.** *Let $\{A_j(z)\}_{j=0}^{\nu}$ be $\nu+1$ entire functions without common zeros. Then*

$$\rho(w) = \max\left\{\rho\left(\frac{A_i}{A_l}\right);\ i \in \{0, 1, \cdots, \nu\} \setminus \{l\}\right\} \quad (\forall w(z) \in \Omega)$$

*for any $A_l(z) \not\equiv 0$ $(0 \le l \le \nu)$.*

The following Theorem 2 gives the relation between the singular direction of algebroid functions in $\Omega$ and that of their coefficients. In a sense, it gives a way to determine the existence of Borel direction and $T$ direction of a class of algebroid functions.

**Theorem 2.** *Suppose that $\{A_j(z)\}_{j=0}^{\nu}$ are $\nu+1$ entire functions with no common zeros, where $A_l(z)$ $(0 \le l \le \nu)$ is transcendental and the rest functions are constant.*

1. *If $\rho(A_l) = \rho$ $(0 < \rho < \infty)$, then Borel direction of $A_l(z)$ is also that of $w(z)$ for arbitrary $w(z) \in \Omega$.*
2. *If $A_l(z)$ exists $T$ direction, then $T$ direction of $A_l(z)$ is also that of $w(z)$ for arbitrary $w(z) \in \Omega$.*

## 2. Lemmas

In this section, we give three basic inequations, which are needed in the proofs of the theorems.

**Lemma 1.** [2, 11] *Suppose that $w(z)$ is an algebroid function defined by* (1). *Then*

$$\left|T(r,w)-\mu(r,A)+\frac{1}{\nu}\log|c_w|\right|\leq \log 2,$$

*where $c_w$ is the first non-zero coefficient of the Laurent expansion of the leading coefficient of $w(z)$ at the origin.*

**Remark.** $w(z)$ is an irreducible algebroid function in Lemma 1 in the original literature, but the proof process of the result has nothing to do with the reducibility of $w(z)$ (see [2, 11]). Therefore, the result is true for a general algebroid function, so that it is true for each element in $\Omega$. Moreover, the lemma shows that $\rho(w)=\rho(A)$.

**Lemma 2.** [14] *Let $w(z)\in\Omega$. Then for any coefficient $A_t(z)$ and non-vanishing coefficient $A_u(z), t,u\in\{0,1,\cdots,\nu\}$, we have*

$$\mu(r,A)\geq\frac{1}{\nu}T\left(r,\frac{A_t}{A_u}\right)+O(1),\quad \textit{namely}\ \rho\left(\frac{A_t}{A_u}\right)\leq\rho(w).$$

We obtained the basic inequation in Lemma 2 in 2008. In this paper, we obtain another basic inequation. The result is the following Lemma 3.

**Lemma 3.** *Let $\{A_j(z)\}_{j=0}^{\nu}$ be $\nu+1$ entire functions without common zeros. If $A_u(z)$ $(\not\equiv 0)$, $0\leq u\leq\nu$, then*

$$\mu(r,A)\leq\frac{\nu-1}{\nu}\sum_{j\neq u,\,0\leq j\leq\nu}T\left(r,\frac{A_j}{A_u}\right)+O(1).$$

*In particular, if $A_u(z)$ is a unique transcendental function among $\{A_j(z)\}_{j=0}^{\nu}$ and the rest functions are constants, then*

$$\mu(r,A)\leq\frac{1}{\nu}T(r,A_u)+O(1).$$

*Proof.* Set $f_{ju}(z)=\max\{|A_j(z)|,|A_u(z)|\}=|\frac{A_j(z)}{A_u(z)}|^{+}\cdot|A_u(z)|$, where $|\frac{A_j(z)}{A_u(z)}|^{+}=\max\{1,|\frac{A_j(z)}{A_u(z)}|\}$. Then it can be deduced from Definition 2 and Jensen formula that

$$\begin{aligned}\nu\mu(r,A)&\leq\sum_{j\neq u,\,0\leq j\leq\nu}\frac{1}{2\pi}\int_0^{2\pi}\log f_{ju}\big(re^{i\theta}\big)\,d\theta\\&=\sum_{j\neq u,\,0\leq j\leq\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^{+}\left|\frac{A_j(re^{i\theta})}{A_u(re^{i\theta})}\right|d\theta+\frac{\nu-1}{2\pi}\int_0^{2\pi}\log\left|A_u\big(re^{i\theta}\big)\right|d\theta\\&=\sum_{j\neq u,\,0\leq j\leq\nu}m\left(r,\frac{A_j}{A_u}\right)+(\nu-1)\left(N\left(r,\frac{1}{A_u}\right)+\log|c_u|\right),\end{aligned}$$

where $c_u$ is the first non-zero coefficient of the Laurent expansion of $A_u(z)$ at the origin. Since $A_\nu(z),\cdots,A_1(z),A_0(z)$ are entire functions with no common zeros, we have

$$N\left(r, \frac{1}{A_u}\right) \le \sum_{j\neq u,\, 0\le j\le \nu} N\left(r, \frac{A_j}{A_u}\right).$$

Hence

$$\mu(r, A) \le \frac{\nu-1}{\nu} \sum_{j\neq u,\, 0\le j\le \nu} T\left(r, \frac{A_j}{A_u}\right) + \frac{\nu-1}{\nu}\log|c_u|.$$

In particular, let $A_u(z)$ be the unique transcendental function and $A_j(z) \equiv c_j$ ($\in \mathbf{C}$) ($0 \le j \neq u \le \nu$). Without loss of generality, we may assume that $A_t(z) \equiv c_t$ is a non-zero constant. Then for $A_t(z)$, applying the above similar method, we can conclude

$$\begin{aligned}\nu\mu(r, A) &\le \sum_{j\neq t,\, 0\le j\le \nu} \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{A_j(re^{i\theta})}{c_t}\right| d\theta + (\nu-1)\log|c_t| \\ &= m\left(r, \frac{A_u}{c_t}\right) + \sum_{j\neq t,u;\, 0\le j\le \nu} \log^+\left|\frac{c_j}{c_t}\right| + (\nu-1)\log|c_t| \le T(r, A_u) + O(1).\end{aligned}$$

Thus

$$\mu(r, A) \le \frac{1}{\nu} T(r, A_u) + O(1). \qquad \square$$

## 3. Proof of Theorem 1 and its applications

Since $A_l(z) \not\equiv 0$, we obtain from Lemma 2 and Lemma 3 that

$$\frac{1}{\nu}T\left(r, \frac{A_i}{A_l}\right) + O(1) \le \mu(r, A) \le \frac{\nu-1}{\nu}\sum_i T\left(r, \frac{A_i}{A_l}\right) + O(1), \quad i \in \{0, 1, \cdots, \nu\}\setminus\{l\}.$$

Set

$$\rho := \rho\left(\frac{A_{i_0}}{A_l}\right) = \max_{i\neq l, 0\le i\le \nu}\left\{\rho\left(\frac{A_i}{A_l}\right)\right\}, \quad i_0 \in \{0, 1, \cdots, \nu\}\setminus\{l\}.$$

When $\rho < \infty$, for arbitrary $\varepsilon > 0$, there exists $R > 0$ such that

$$T\left(r, \frac{A_{i_0}}{A_l}\right)\left(\frac{1}{\nu} + \frac{O(1)}{T(r, \frac{A_{i_0}}{A_l})}\right) \le \mu(r, A) \le \frac{(\nu-1)^2}{\nu} r^{\rho+\varepsilon} + O(1)$$

holds for $r > R$. This implies that

$$\rho(A) = \rho.$$

In addition, noting that the Valiron characteristic $\mu(r, A)$ only depends on the coefficients $\{A_j(z)\}_{j=0}^{\nu}$ and combining with Lemma 1, we have $\rho(w) = \rho(A)$ for all $w(z) \in \Omega$. Hence

$$\rho(w) = \max\left\{\rho\left(\frac{A_i}{A_l}\right); i \in \{0, 1, \cdots, \nu\}\setminus\{l\}\right\}. \tag{6}$$

When $\rho = \infty$, for any $w(z) \in \Omega$, we have $\rho(w) = \infty$ by Lemma 2. Hence (6) still holds. The theorem is completed.

According to Theorem 1, we know that the order of an algebroid function can be obtained by virtue of its coefficients, and the order of every element in $\Omega$ is equal.

Therefore, the order of two algebroid functions is equal provided that their non-vanishing coefficients are the same even if they are different-valued functions. For example, let

$$C(z)=\sum_{n=2}^{\infty}\left(\frac{1}{n\log n}\right)^{2n}z^n,\quad D(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^n.$$

Then $C(z)$ is an entire function with the order $\rho(C)=1/2$ (see [2,P10]), and $D(z)$ is an entire function with the order $\rho(D)=1/2$ (see [15]). Set

$$\psi(z,w)\equiv C(z)w^4+cw^3+D(z)=0,\quad \phi(z,w)\equiv D(z)w^2+cw+C(z)=0,$$

where $c\in\mathbf{C}$ is a non-zero constant. Choose $c$ as $A_l(z)$ in Theorem 1, then it is easy to see that the 4-valued algebroid function defined by $\psi(z,w)=0$ and the 2-valued algebroid function defined by $\phi(z,w)=0$ have the same order 1/2 (here $\Omega$ can be considered as the set of algebroid functions with the coefficients $\{0,0,c,C(z),D(z)\}$), while the result is not easily obtained by Theorem A or Theorem B.

Next we give other two applications of Theorem 1.

**1.** First, we can study the growth of an algebroid function compounding an entire function.

Let $w(z)$ be a $\nu$-valued algebroid function defined by (1), and $g(z)$ be a non-constant entire function. Then $\omega(z):=w(g(z))$ is a $\nu$-valued algebroid function defined by the following equation (see [18])

$$A_\nu(g(z))\omega^\nu+\cdots+A_1(g(z))\omega+A_0(g(z))=0.$$

Hence by Theorem 1, the order of $w(g(z))$ is

$$\rho(w(g))=\rho(\omega)=\max\left\{\rho\left(\frac{A_i(g)}{A_l(g)}\right);\, i\in\{0,1,\cdots,\nu\}\setminus\{l\},\ \forall A_l(g)\not\equiv 0\right\}. \tag{7}$$

Gross [1] discussed the growth of the composite function of a meromorphic function and an entire function, and obtained the following result.

**Theorem C.** *Let $f(z)$ be a meromorphic function with the order $\rho(f)>0$. Then $\rho(f(g))=\infty$ for arbitrary transcendental entire function $g(z)$.*

For an algebroid function compounding an entire function, Zheng and Yang [18] proved Theorem D by getting an inequation on their characteristic functions.

**Theorem D.** *Suppose that $w(z)$ is an algebroid function of $\rho(w)>0$, and $g(z)$ is an arbitrary transcendental entire function. Then $\rho(w(g))=\infty$.*

In fact, we can also obtain the result of Theorem D easily by Theorem 1 and Theorem C: Since $\rho(w)>0$, there exists $i,l\in\{0,1,\cdots,\nu\}$ such that

$$\rho\left(\frac{A_i}{A_l}\right)=\rho(w)>0.$$

Then by Theorem C,

$$\rho\left(\frac{A_i(g)}{A_l(g)}\right)=\infty.$$

Combining with (7), we get the result.

**2.** Second, we can study the growth of the derivative of an algebroid function.

Let $w(z)$ be a $\nu$-valued algebroid function defined by (1). Then its derivative $w'(z)$ is also a $\nu$-valued algebroid function. Without loss of generality, we may assume that $w'(z)$ is defined by the following equation

$$C_\nu(z)w'^\nu + \cdots + C_1(z)w' + C_0(z) = 0, \tag{8}$$

where $C_\nu(z)\ (\not\equiv 0), \cdots, C_1(z), C_0(z)$ are entire functions without any common zero.

Jacobson [3, 5] indicated that if (1) was written as

$$w^\nu + B_{\nu-1}(z)w^{\nu-1} + \cdots + B_1(z)w + B_0(z) = 0,$$

where $B_i(z) = A_i(z)/A_\nu(z)\ (0 \le i \le \nu-1)$, then (8) could be obtained by calculating the resultant $((\nu-1)+\nu$-order determinant) of the polynomials

$$w^\nu + B_{\nu-1}w^{\nu-1} + \cdots + B_1 w + B_0$$

and

$$(\nu w' + B'_{\nu-1})w^{\nu-1} + \cdots + (2B_2 w' + B'_1)w + B_1 w' + B'_0$$

and being properly multiplied by a factor at both ends. Then we can get $\rho(w')$ by Theorem 1.

For example, if $w(z)$ is a 2-valued algebroid function, then by the resultant equaling zero:

$$\begin{vmatrix} 1 & B_1 & B_0 \\ 2w' + B'_1 & B_1 w' + B'_0 & \\ & 2w' + B'_1 & B_1 w' + B'_0 \end{vmatrix} = 0,$$

we get

$$(4B_0 - B_1^2)w'^2 + B'_1(4B_0 - B_1^2)w' + B_0'^2 - B_1 B'_0 B'_1 + B_0 B_1'^2 = 0. \tag{9}$$

If $4B_0 - B_1^2 \equiv 0$, then $w(z)$ is a reducible algebroid function defined by

$$w^2 + B_1 w + \frac{B_1^2}{4} = \left(w + \frac{B_1}{2}\right)^2 = 0.$$

Its two branches are $w_1(z) = w_2(z) = -B_1(z)/2 = -A_1(z)/(2A_2(z))\ (\forall z \in \mathbf{C})$, then $w(z)$ is equivalent to the meromorphic function $-A_1(z)/(2A_2(z))$. Hence, we generally discuss the case of $4B_0 - B_1^2 \not\equiv 0$. Combining (8) with (9), we deduce

$$\frac{C_1}{C_2} = B'_1, \quad \frac{C_0}{C_2} = \frac{B_0'^2 - B_1 B'_0 B'_1 + B_0 B_1'^2}{4B_0 - B_1^2}.$$

Then $\rho(w')$ can be obtained by Theorem 1. If $w(z)$ is a more-than-2-valued algebroid function, then we can calculate the resultant by Matlab. Therefore, we can also obtain $\rho(w')$.

## 4. Proof of Theorem 2

Let $B(z) := A_l(z)$. Then it follows from Theorem 1 that $\rho(w) = \rho(B)\ (\forall w(z) \in \Omega)$.

**1.** If $\theta_0$ is an arbitrary Borel direction of the entire function $B(z)$, then for arbitrary $\delta\ (0 < \delta < \pi/2)$, in the angular region $\triangle(\theta_0, \delta)$,

$$\limsup_{r\to\infty} \frac{\log n(r, \triangle(\theta_0,\delta), B = a)}{\log r} = \rho$$

holds, where $a \in \mathbf{C}$ with at most one exceptional value. We distinguish three cases below.

**Case 1.** $B(z)$ is the leading coefficient of $w(z)$. Without loss of generality, we assume that $w(z)$ is defined by

$$\psi(z,w) \equiv B(z)w^{\nu} + c_{\nu-1}w^{\nu-1} + \ldots + c_1 w + c_0 = 0,$$

where $c_i \in \mathbf{C}$, $0 \le i \le \nu - 1$.

For any $a \in \mathbf{C} \cup \infty$, we have from [2,P77] that

$$n(r, w = a) = n(r, \psi(z,a) = 0), \quad a \in \mathbf{C}; \tag{10}$$

$$n(r, w) = n(r, B = 0), \quad a = \infty. \tag{11}$$

(i) If $a \in \mathbf{C} \setminus \{0\}$, then

$$\psi(z,a) = 0 \iff B(z) = -\frac{c_{\nu-1}a^{\nu-1} + \cdots + c_1 a + c_0}{a^{\nu}} := \widetilde{a}. \tag{12}$$

Therefore, if $\widetilde{a}$ is not an exceptional value of $B(z)$ in the Borel direction $\theta_0$, then by (10) and (12), we have

$$\limsup_{r\to\infty} \frac{\log n(r, \triangle(\theta_0,\delta), w = a)}{\log r} = \limsup_{r\to\infty} \frac{\log n(r, \triangle(\theta_0,\delta), B = \widetilde{a})}{\log r} = \rho. \tag{13}$$

If $\widetilde{a}$ is an exceptional value of $B(z)$ in the direction $\theta_0$, then there are at most $\nu$ exceptional values of $w(z)$ in $\theta_0$ ($a$ is just one of them). These exceptional values are the roots of the equation

$$\widetilde{a}w^{\nu} + c_{\nu-1}w^{\nu-1} + \cdots + c_1 w + c_0 = 0.$$

(ii) If $a = 0$, then $\psi(z, 0) = c_0$.

When $c_0 = 0$, we have $\psi(z, 0) \equiv 0$. Then by (10), for any $z \in \mathbf{C}$, it is deduced that $w = 0$ (in fact, some branch(es) of $\nu$-valued algebroid function $w(z)$ is(are) identically vanishing). Obviously, 0 is not an exceptional value of $w(z)$.

When $c_0 \ne 0$, then $w \ne 0$ for any $z \in \mathbf{C}$. Hence 0 is an exceptional value of $w(z)$.

(iii) If $a = \infty$, combining with (11), then an equality similar to (13) holds when 0 is not an exceptional value of $B(z)$ in the Borel direction $\theta_0$, else $\infty$ is an exceptional value of $w(z)$ in $\theta_0$ when 0 is a Borel exceptional value of $B(z)$.

Summing up the above discussions: if $B(z)$ does not have exceptional values in the direction $\theta_0$, then $w(z)$ has at most one exceptional value $a = 0$. If $B(z)$ has one finite exceptional value in $\theta_0$, then $w(z)$ has at most $\max\{\nu+1, 2\} = \nu+1$ exceptional values.

Therefore, we obtain that (2) holds for arbitrary $a \in \mathbf{C} \cup \infty$ with at most $\nu + 1$ ($\le 2\nu$) exceptional values. Then $\theta_0$ is a Borel direction of $w(z)$.

**Case 2.** $B(z)$ is the coefficient of $w^j$. Without loss of generality, we assume that $w(z)$ is defined by

$$\psi(z,w) \equiv c_t w^t + \cdots + B(z)w^j + \cdots + c_0 = 0,$$

where $1 \le j < t \le \nu$, $c_i \in \mathbf{C}$, $0 \le i(\ne j) \le t$.

(i) For any $a \in \mathbf{C}\setminus\{0\}$, in the similar way, we have

$$\psi(z,a) = 0 \iff B(z) = -\frac{c_t a^t + \cdots + c_{j+1}a^{j+1} + c_{j-1}a^{j-1} \cdots + c_0}{a^j} := \widetilde{a}.$$

(13) holds when $\widetilde{a}$ is not an exceptional value of $B(z)$ in $\theta_0$. Otherwise, corresponding to the exceptional value $\widetilde{a}$ of $B(z)$, $w(z)$ has at most $t$ exceptional values ($a$ is one of them) in $\theta_0$. They are the roots of the equation

$$c_t w^t + \cdots + \widetilde{a} w^j + \cdots + c_0 = 0.$$

(ii) $a = 0$. We can discuss the case in the similar way to (ii) of Case 1.

(iii) $a = \infty$. Since the leading coefficient of an algebroid function is not identically vanishing, we have $c_t \neq 0$. Then $w(z) \neq \infty$. Hence $\infty$ is an exceptional value of $w(z)$.

Noting $t > 1$, thus $t + 2 \leq 2t$. Combining (i)–(iii) in Case 2, similar to Case 1, we show that (2) holds for any $a \in \mathbf{C} \cup \infty$ except at most $t + 2$ values. Then $\theta_0$ is a Borel direction of $w(z)$.

**Case 3.** $B(z)$ is the constant coefficient. Without loss of generality, we assume that $w(z)$ is defined by

$$\psi(z, w) \equiv c_t w^t + c_{t-1} w^{t-1} + \cdots + c_1 w + B(z) = 0,$$

where $1 \leq t \leq \nu$, $c_i \in \mathbf{C}$, $1 \leq i \leq t$.

(i) For any $a \in \mathbf{C} \backslash \{0\}$, in the similar way, we have

$$\psi(z, a) = 0 \iff B(z) = -(c_t a^t + \cdots + c_1 a) := \widetilde{a}.$$

When $\widetilde{a}$ is not an exceptional value of $B(z)$ in the direction $\theta_0$, (13) holds. When $\widetilde{a}$ is an exceptional value of $B(z)$ in $\theta_0$, $w(z)$ has at most $t$ exceptional values ($a$ is one of them). They are the roots of the equation

$$c_t w^t + c_{t-1} w^{t-1} + \cdots + c_1 w + \widetilde{a} = 0.$$

(ii) If $a = 0$, then $\psi(z, 0) = B(z)$. Therefore, (13) holds for $w = 0$ when 0 is not an exceptional value of $B(z)$ in $\theta_0$, else 0 is also an exceptional value of $w(z)$ in the direction $\theta_0$.

(iii) If $a = \infty$, similar to (iii) of Case 2, we obtain that $\infty$ is an exceptional value of $w(z)$.

Therefore, for arbitrary $a \in \mathbf{C} \cup \infty$ with at most $\max\{t + 1, 2\} = t + 1 \,(\leq 2t)$ exceptional values, (2) holds. Then $\theta_0$ is a Borel direction of $w(z)$.

**2.** Suppose that $\theta_0$ is any $T$ direction of $B(z)$. Then for arbitrary $\delta$ $(0 < \delta < \pi/2)$, in the angular region $\triangle(\theta_0, \delta)$,

$$\limsup_{r \to \infty} \frac{N(r, \triangle(\theta_0, \delta), B = a)}{T(r, B)} > 0$$

holds, where $a \in \mathbf{C}$ with at most one exceptional value. Similar to the proof of the Borel direction, from the above three cases, we can show that $\theta_0$ is also a $T$ direction of $t$-valued $(1 \leq t \leq \nu)$ algebroid function $w(z) \,(\in \Omega)$. In the process, just noting

$$N(r, \triangle(\theta_0, \delta), w = a) = \frac{1}{t} N(r, \triangle(\theta_0, \delta), \psi(z, a) = 0) = \frac{1}{t} N(r, \triangle(\theta_0, \delta), B = \widetilde{a}),$$

$a \in \mathbf{C}\backslash\{0\}$, and combining Lemma 1 with Lemma 3, we deduce

$$\begin{aligned}\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),w=a)}{T(r,w)} &\geq \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{t\mu(r,A)+O(1)}\\ &\geq \limsup_{r\to\infty}\left(\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{T(r,B)}\cdot\frac{1}{1+\frac{O(1)}{T(r,B)}}\right)\\ &= \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{T(r,B)}.\end{aligned}$$

In addition, combining Lemma 1, Lemma 2 with the first fundamental theorem, we obtain

$$\begin{aligned}\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),w=a)}{T(r,w)} &\leq \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{t\mu(r,A)+O(1)}\\ &\leq \limsup_{r\to\infty}\left(\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{T(r,B)}\cdot\frac{1}{1+\frac{O(1)}{T(r,B)}}\right)\\ &= \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{T(r,B)}.\end{aligned}$$

Then

$$\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),w=a)}{T(r,w)} = \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_0,\delta),B=\widetilde{a})}{T(r,B)}.$$

Therefore, when $\widetilde{a}$ is not an exceptional value of $B(z)$ in $\theta_0$, $a$ is not an exceptional value of $w(z)$ in $\theta_0$ either. Otherwise, corresponding to the exceptional value $\widetilde{a}$ of $B(z)$, $w(z)$ has at most $t$ exceptional values ($a$ is just one of them) in $\theta_0$.

Moreover, if $a = 0$ or $a = \infty$, we can discuss these cases similar to 1.

Therefore, for any $t$-valued ($1 \leq t \leq \nu$) algebroid function $w(z) \in \Omega$, (3) holds for arbitrary $a \in \mathbf{C} \cup \infty$ except at most $s\,(\leq 2t)$ values, where $s = t + 1$ if $t \geq 1$, $s = t + 2$ if $t > 1$. Then $\theta_0$ is a $T$ direction of $w(z)$. Theorem 2 is completed.

In addition, since any transcendental entire function exists Julia direction (see [17]), we can show similarly that Julia direction of $A_l(z)$ is also that of $w(z)$ ($\forall w(z) \in \Omega$). Next we put forward two questions.

**Question 1.** For $\Omega$ in Theorem 2 and arbitrary $w(z) \in \Omega$, are the above singular directions of $w(z)$ also those of $A_l(z)$? If it is true, then all the singular directions of the algebroid functions in $\Omega$ are the same.

**Question 2.** For a general set $\Omega$ of algebroid functions, let $w(z) \in \Omega$ be defined by (1). If $A_0(z) \not\equiv 0$, then $w^{-1}(z)$ which is the reciprocal element of $w(z)$ is defined by (see [8])

$$A_0(z)w^{\nu} + A_1(z)w^{\nu-1} + \ldots + A_\nu(z) = 0.$$

It is obvious that $w(z)$ and $w^{-1}(z)$ have the same singular directions, such as the same Borel direction, $T$ direction and Julia direction. Therefore, it is natural to ask whether it is true for all the algebroid functions in $\Omega$.

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