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# ON THE MATTILA–SJÖLIN THEOREM FOR DISTANCE SETS

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**Abstract.** We extend a result, due to Mattila and Sjölin, which says that if the Hausdorff dimension of a compact set $E \subset \mathbf{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then the distance set $\Delta(E) = \{|x-y| \colon x, y \in E\}$ contains an interval. We prove this result for distance sets $\Delta_B(E) = \{\|x-y\|_B \colon x, y \in E\}$, where $\|\cdot\|_B$ is the metric induced by the norm defined by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature. We also obtain some detailed estimates pertaining to the Radon–Nikodym derivative of the distance measure.

## 1. Introduction

The classical Falconer distance conjecture, originated in 1985, [2], says that if the Hausdorff dimension of a compact subset of $\mathbf{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the set of distances, $\Delta(E) = \{|x-y| \colon x, y \in E\}$ is positive. In [2], Falconer proved the first result in this direction by showing that $\mathcal{L}^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. See also [3] and [7] for a thorough description of the problem and related ideas. The best currently known results are due to Wolff in two dimensions, and to Erdogan [1] in dimensions three and greater. They prove that $\mathcal{L}^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d}{2} + \frac{1}{3}$.

An important addition to this theory is due to Mattila and Sjölin [8] who proved that if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$, then $\Delta(E)$ not only has positive Lebesgue measure, but also contains an interval. This is accomplished by showing that the natural measure on the distance set has a continuous density. However, this set need not contain an interval with left end-point at the origin as illustrated by an example in ([7], p. 165). It was previously shown by Mattila [6] that if the ambient dimension is two or three, then the density of the distance measure is not in general bounded if the Hausdorff dimension of the underlying set $E$ is smaller than $\frac{d+1}{2}$. In higher dimensions, this question is still open for the Euclidean metric, but has been resolved if the Euclidean metric is replaced by a metric induced by a norm defined by a suitably chosen paraboloid. See [5].

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In this paper we give an alternative proof of the Mattila–Sjölin result and extend it to more general distance sets $\Delta_B(E) = \{\|x-y\|_B \colon x, y \in E\}$, where $\|\cdot\|_B$ is the norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.

Our main result is the following.

**Theorem 1.1.** *Let $E$ be a compact subset of $\mathbf{R}^d$, $d \geq 2$, with Hausdorff dimension, denoted by $s$, greater than $\frac{d+1}{2}$. Let $\mu$ be a Frostman measure on $E$. Let $\sigma$ denote the Lebesgue measure on $\partial B$. Define the distance measure $\nu$ by the relation*

$$\int h(t)\, d\nu(t) = \iint h(\|x-y\|_B)\, d\mu(x)\, d\mu(y),$$

*where $\|\cdot\|_B$ is the norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.*

(i) *Then the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure.*

(ii) *We have*

$$\frac{\nu((t-\epsilon, t+\epsilon))}{2\epsilon} = M(t) + R^{\epsilon}(t),$$

*where*

$$M(t) = \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma}(t\xi) t^{d-1}\, d\xi$$

*is the density of $\nu$ and*

$$\sup_{0<\epsilon<\epsilon_0} |R^{\epsilon}(t)| \lesssim \epsilon_0^{s-\frac{d+1}{2}}.$$

(iii) *Moreover, $M \in C^{\lfloor s-\frac{d+1}{2} \rfloor}(I)$ for any interval $I$ not containing the origin, where $\lfloor u \rfloor$ denotes the smallest integer greater than or equal to $u$. In particular, $M$ is continuous away from the origin if $s > \frac{d+1}{2}$ and therefore $\Delta_B(E)$ contains an interval in view of (i).*

(iv) *Suppose that $s > k + \alpha$, where $k$ is a non-negative integer and $0 < \alpha < 1$. Then the $k$th derivative of the density function of $\nu$ is Hölder continuous of order $\alpha$.*

**Remark 1.2.** Metric properties of $\|\cdot\|_B$ are not used in the proof of Theorem 1.1. Let $\Gamma$ be a star shaped body in the sense that for every $\omega \in S^{d-1}$ there exists $1 < r_0(\omega) < 2$ such that $\{r\omega \colon 0 \leq r \leq r_0(\omega)\} \subset \Gamma$ and $\{r\omega \colon r > r_0(\omega)\} \cap \Gamma = \emptyset$. Define $\|x\|_\Gamma = \inf\{t > 0 \colon x \in t\Gamma\}$ and let $\Delta_\Gamma(E) = \{\|x-y\|_\Gamma \colon x, y \in E\}$. Let $\sigma_\Gamma$ denote the Lebesgue measure on the boundary of $\Gamma$. Then if $|\widehat{\sigma}_\Gamma(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}$, the conclusion of Theorem 1.1 holds with the same exponents.

**1.1. Sharpness of results.** As we note above, Mattila's construction [6] shows that if the Hausdorff dimension of $E$ is smaller than $\frac{d+1}{2}$, $d = 2, 3$, then the density of distance measure is not in general bounded in the case of the Euclidean metric. Moreover, Mattila construction can be easily extended to all metrics generated by a bounded convex body $B$ with a smooth boundary and non-vanishing Gaussian curvature.

In dimensions four and higher, all we know at the moment (see the main result in [5]) is that there exists a bounded convex body $B$ with a smooth boundary and non-vanishing curvature, such that the density of the distance measure is not in general bounded if the Hausdorff dimension of the underlying set $E$ is less than $\frac{d+1}{2}$. We do no know what happens when the Hausdorff dimension of $E$ equals $\frac{d+1}{2}$ in any dimension and for any smooth metric.

It would be very interesting if any of these results actually depended on the underlying convex body $B$ in a non-trivial way. This would mean that smoothness and non-vanishing Gaussian curvature of the level set do not tell the whole story. There is some evidence that this may be the case. See, for example, [4], where connections between problems of Falconer type and distribution of lattice points in thin annuli are explored.

If the Hausdorff dimension of $E$ is less than $\frac{d}{2}$, then the density of the distance measure, for any metric induced by a bounded convex body $B$ with a smooth boundary and non-vanishing curvature is not in general bounded by a construction due to Falconer [2].

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## 2. Proof of Theorem 1.1

**2.1. Proof of items (i) and (ii).** The proof of item (i) of Theorem 1.1 is due to Falconer [2] and Mattila [6]. This brings us to item (ii). Recall that every compact set $E$ in $\mathbf{R}^d$, of Hausdorff dimension $s > 0$ possesses a Frostman measure (see e.g. [7], p. 112), which is a probability measure $\mu$ with the property that given any $\delta > 0$, for every ball of radius $r^{-1}$, denoted by $B_{r^{-1}}$, there exists $C_\delta > 0$ such that

$$\mu(B_{r^{-1}}) \leq C_\delta r^{-s+\delta}.$$

Let

$$\nu^\epsilon(t) = \frac{\nu((t-\epsilon, t+\epsilon))}{2\epsilon} = \frac{1}{2\epsilon}\, \mu \times \mu\{(x,y)\colon t-\epsilon \leq \|x-y\|_B \leq t+\epsilon\}.$$

We shall prove that $\lim_{\epsilon\to 0} \nu^\epsilon(t)$ exists and is a $C^{\lfloor s-\frac{d+1}{2}\rfloor}$ function.

Let $\rho$ be a smooth cut-off function, identically equal to 1 in the unit ball and vanishing outside the ball of radius 2, with $\int \rho = 1$. Let $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. Since $\sigma_t * \rho_\epsilon$ is supported on the annulus of radius $t$ and width $\approx \epsilon$, and is $\approx \epsilon^{-1}$ on this annulus, there is no harm in working with the measure

$$\iint \sigma_t * \rho_\epsilon(x-y)\, d\mu(x)\, d\mu(y), \tag{2.1}$$

where $\sigma_t$ is the surface measure on the set $\{x\colon \|x\|_B = t\}$. By abusing notation slightly, we shall refer to this measure as $\nu^\epsilon(t)$.

By the Fourier inversion formula,

$$\begin{aligned}\nu^\epsilon(t) &= \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma}_t(\xi)\widehat{\rho}(\epsilon\xi) d\xi \\ &= \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma}_t(\xi) d\xi - \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma}_t(\xi)(1-\widehat{\rho}(\epsilon\xi)) d\xi = M(t) + R^\epsilon(t).\end{aligned}$$

We shall prove that $M(t)$ is a $C^{\lfloor s-\frac{d+1}{2}\rfloor}$ function and that $\lim_{\epsilon\to 0} R^\epsilon(t) = 0$. We start with the latter. We shall need the following stationary phase estimate. See, for example, [10], [9] or [11].

**Lemma 2.1.** *Let $\sigma$ be the surface measure on a compact piece of a smooth convex surface in $\mathbf{R}^d$, $d \geq 2$, with everywhere non-vanishing Gaussian curvature. Then*

$$|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}, \tag{2.2}$$

*where here, and throughout, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$. Moreover,*

$$|D^\alpha \widehat{\sigma}(\xi)| \leq C_{\alpha,d}|\xi|^{-\frac{d-1}{2}}, \tag{2.3}$$

*where $D^\alpha$ is the differential operator with respect to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$.*

We shall also need the following well-known estimate. See, for example, [3] and [7].

**Lemma 2.2.** *Let $\mu$ be a Frostman measure on a compact set $E$ of Hausdorff dimension $s > 0$. Then*

$$\int_{2^j\leq|\xi|\leq 2^{j+1}} |\widehat{\mu}(\xi)|^2 \, d\xi \lessapprox 2^{j(d-s)},$$

*and, consequently,*

$$\int |\widehat{\mu}(\xi)|^2 |\xi|^{-\gamma} \, d\xi = c \iint |x-y|^{-d+\gamma} \, d\mu(x) \, d\mu(y) \lesssim 1$$

*if $\gamma > d-s$. Here, and throughout, $X \lessapprox Y$, with the controlling parameter $r$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon r^\epsilon Y$.*

To prove the lemma, observe that

$$\int_{2^j\leq|\xi|\leq 2^{j+1}} |\widehat{\mu}(\xi)|^2 \, d\xi \lesssim \int |\widehat{\mu}(\xi)|^2 \psi(2^{-j}\xi) \, d\xi,$$

where $\psi$ is a suitable smooth function supported in $\{x \in \mathbf{R}^d \colon 1/2 \leq |x| \leq 4\}$ and identically equal to 1 in the unit annulus. By definition of the Fourier transform and the Fourier inversion theorem, this expression is equal to

$$2^{dj} \iint \widehat{\psi}(2^j(x-y)) \, d\mu(x) \, d\mu(y) \lessapprox 2^{j(d-s)}$$

since $\widehat{\psi}$ decays rapidly at infinity.

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} |R^\epsilon(t)| &\lessapprox \int_{|\xi|>\frac{1}{\epsilon}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} \, d\xi \\ &\leq \int_{|\xi|>\frac{1}{\epsilon}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} \, d\xi = \sum_{j>\log_2(1/\epsilon)} \int_{2^j\leq|\xi|\leq 2^{j+1}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} \, d\xi \\ &\lessapprox \sum_{j>\log_2(1/\epsilon)} 2^{j(d-s)} 2^{-j\frac{d-1}{2}} \lessapprox \epsilon^{s-\frac{d+1}{2}}, \end{aligned} \tag{2.4}$$

and thus

$$\sup_{0<\epsilon\le\epsilon_0} |R_\epsilon(t)| \le \epsilon_0^{s-\frac{d+1}{2}}.$$

In order to handle $|R^\epsilon(t)|$ over the integral when $|\xi| < \frac{1}{\epsilon}$, we notice that $(1-\widehat{\rho}(\epsilon\xi))$ is 0 when $\xi = (0,\dots,0)$ and, by continuity, is small in a neighborhood of the origin. This calculation establishes all the claims in part ii) of Theorem 1.1.

We note that the weaker statement showing that $\lim_{\epsilon\to 0} |R_\epsilon(t)| = 0$ follows in an easier way from the dominated convergence theorem.

**2.2. Proof of item (iii).** Once again, by Lemma 2.1, we have

$$|M(t)| \lesssim \int |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}}\, d\xi$$

and by the calculation identical to the one in the previous paragraph, we see that this quantity is $\lesssim 1$ if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. Continuity follows by the Lebesgue dominated convergence theorem. The convergence of the integral allows us to differentiate inside the integral sign. We obtain

$$M'(t) = \int |\widehat{\mu}(\xi)|^2 \frac{d}{dt}\left\{t^{d-1}\widehat{\sigma}(t\xi)\right\} d\xi.$$

We have

$$\frac{d}{dt}\left\{t^{d-1}\widehat{\sigma}(t\xi)\right\} = (d-1)t^{d-2}\widehat{\sigma}(t\xi) + t^{d-1}\nabla\widehat{\sigma}(t\xi)\cdot\xi.$$

Applying (2.2) and (2.3) of Lemma 2.1 once more, it follows that

$$\left|\frac{d}{dt}\left\{t^{d-1}\widehat{\sigma}(t\xi)\right\}\right| \lesssim |\xi|^{-\frac{d-1}{2}+1}.$$

Repeating the argument in 2.4, we see that $M'(t)$ exists if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}+1$. Proceeding in the same way one establishes that

$$\frac{d^m}{dt^m}\left\{t^{d-1}\widehat{\sigma}(t\xi)\right\} \lesssim |\xi|^{-\frac{d-1}{2}+m}$$

and the conclusion of Theorem 1.1 follows.

**2.3. Proof of item (iv).** We shall deal with the case $k=0$, as the other cases follow from a similar argument. Let

$$\lambda(t) = t^{d-1}\widehat{\sigma}(t\xi).$$

We must show that

$$|M(u)-M(v)| \le C|u-v|^\alpha.$$

We have

$$\begin{aligned} M(u)-M(v) &= \int |\widehat{\mu}(\xi)|^2(\lambda(u)-\lambda(v))\, d\xi \\ &= \int |\widehat{\mu}(\xi)|^2(\lambda(u)-\lambda(v))^\alpha(\lambda(u)-\lambda(v))^{1-\alpha}\, d\xi. \end{aligned}$$

Now,

$$\lambda(u)-\lambda(v) = (u-v)\lambda'(c),$$

where $c\in(u,v)$, by the mean-value theorem. It follows that

$$|\lambda(u)-\lambda(v)|^\alpha \le |u-v|^\alpha|\lambda'(c)|^\alpha.$$

On the other hand,

$$|\lambda(u) - \lambda(v)|^{1-\alpha} \leq |\lambda(u)|^{1-\alpha} + |\lambda(v)|^{1-\alpha}.$$

We have already shown above that

$$|\lambda(u)| \lesssim |\xi|^{-\frac{d-1}{2}} \text{ and } |\lambda'(u)| \lesssim |\xi|^{-\frac{d-1}{2}+1}.$$

It follows that

$$|M(u) - M(v)| \lesssim |u-v|^{\alpha} \int |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}+\alpha} d\xi \lesssim |u-v|^{-\alpha},$$

where the last step follows by Lemma 2.2, and so the item iv) follows.

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