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# BASINS OF ATTRACTION IN LOEWNER EQUATIONS

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**Abstract.** Let $q \geq 2$. We prove that any Loewner PDE on the unit ball $\mathbf{B}^q$ whose driving term $h(z,t)$ vanishes at the origin and satisfies the bunching condition $\ell m(Dh(0,t)) \geq k(Dh(0,t))$ for some $\ell \in \mathbf{R}^+$, admits a solution given by univalent mappings $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)_{t\geq 0}$. This is done by discretizing time and considering the abstract basin of attraction. If $\ell < 2$, then the range $\cup_{t\geq 0} f_t(\mathbf{B}^q)$ of any such solution is biholomorphic to $\mathbf{C}^q$.

## 1. Introduction

Let $\mathbf{B}^q \subset \mathbf{C}^q$ denote the unit ball. The Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = Df_t(z)h(z,t) \quad \text{a.e. } t \geq 0,\ z \in \mathbf{B}^q \tag{1.1}$$

was introduced by Loewner [21] and developed by Kufarev [20] and Pommerenke [26] in the case of the unit disc $\mathbf{D} \doteq \mathbf{B}^1$. The study of this equation culminated with the proof of the Bieberbach conjecture by de Branges [9] and the introduction of the stochastic Loewner evolution by Schramm [27].

The several variables case has been widely studied for its application in geometric function theory by Graham, Hamada, G. Kohr, M. Kohr, Pfaltzgraff and others (see e.g. [16, 25]).

Throughout this paper we will assume that $q \geq 2$. In [5], generalizing the results obtained in the unit disc $\mathbf{D}$ in [8], we explore the connections between this topic and the theory recently developed by Bracci, Contreras and Díaz-Madrigal [6, 7] (see also [4]) of Herglotz non-autonomous vector fields on complete hyperbolic manifolds. An *Herglotz vector field of order* $\infty$ on $\mathbf{B}^q$ is a non-autonomous holomorphic vector field $-h(z,t)\colon \mathbf{B}^q \times \mathbf{R}^+ \to \mathbf{C}^q$ such that

- $-h(z,t)$ is measurable in $t \geq 0$ and for a.e. $\tilde{t} \geq 0$, the holomorphic vector field $-h(z,\tilde{t})$ is an *infinitesimal generator*, that is the "frozen" Cauchy problem

  $$\begin{cases} \dot{z}(s) = -h(z(s),\tilde{t}), \\ z(0) = z_0, \end{cases}$$

  has a solution $z\colon [0,+\infty) \to \mathbf{B}^q$ for all $z_0 \in \mathbf{B}^q$,
- for any compact set $K \subset \mathbf{B}^q$ and any $T > 0$ there exists $c_{K,T} > 0$ satisfying

  $$|h(z,t)| \leq c_{K,T}, \quad z \in K,\ 0 \leq t \leq T.$$

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The solution flow of the Loewner ODE

$$(1.2)\qquad \begin{cases} \frac{\partial}{\partial t}\varphi_{s,t}(z) = -h(\varphi_{s,t}(z),t), \quad z\in \mathbf{B}^q,\ \text{a.e. } t\in[s,\infty),\\ \varphi_{s,s}(z)=z, \quad z\in\mathbf{B}^q,\ s\geq 0,\end{cases}$$

is an *evolution family of order* $\infty$, that is a family of holomorphic mappings $(\varphi_{s,t}\colon \mathbf{B}^q \to \mathbf{B}^q)_{0\leq s\leq t}$ satisfying

- $\varphi_{s,t} = \varphi_{u,t}\circ\varphi_{s,u}$ for all $0\leq s\leq u\leq t$ and $\varphi_{s,s}(z)=z$ for all $s\geq 0$,
- for any compact set $K\subset \mathbf{B}^q$ and for any $T>0$ there exists a $C_{K,T}>0$ satisfying

$$(1.3)\qquad |\varphi_{s,t}(z)-\varphi_{s,u}(z)|\leq C_{K,T}(t-u), \quad z\in K,\ 0\leq s\leq u\leq t<T.$$

In [5] we prove that a family $(f_t\colon \mathbf{B}^q\to\mathbf{C}^q)_{t\geq 0}$ of univalent mappings is locally Lipschitz (in the variable $t$) and solves the Loewner PDE (1.1) if and only if it solves the functional equation

$$(1.4)\qquad f_s = f_t\circ\varphi_{s,t}, \quad 0\leq s\leq t.$$

If such a solution $(f_t\colon \mathbf{B}^q\to\mathbf{C}^q)$ exists, then the subset $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)\subset\mathbf{C}^q$ is open and connected and is called the *range* of $(f_t)$. Any other solution $(g_t\colon \mathbf{B}^q\to\mathbf{C}^q)$ is of the form $(\Lambda\circ f_t)$, where $\Lambda\colon \bigcup_{t\geq 0} f_t(\mathbf{B}^q)\to\mathbf{C}^q$ is holomorphic. Thus the ranges of two univalent solutions of (1.1) are biholomorphic.

We are interested in Herglotz vector fields on $\mathbf{B}^q$ whose flow $(\varphi_{s,t})$ is attracting at the origin. A first example is provided by Herglotz vector fields whose linear part does not depend on $t\geq 0$. This has been studied in [10, 14].

**Theorem 1.1.** *Let $-h(z,t)$ be a Herglotz vector field of order $\infty$ on $\mathbf{B}^q$ such that $h(z,t)=Az+O(|z|^2)$ with*

$$(1.5)\qquad 2\min\{\operatorname{Re}\langle Az,z\rangle\colon |z|=1\} > \max\{\operatorname{Re}\lambda\colon \lambda\in\operatorname{sp}(A)\},$$

*where $\langle\cdot,\cdot\rangle$ is the hermitian product on $\mathbf{C}^q$. Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution $(f_t\colon\mathbf{B}^q\to\mathbf{C}^q)$. The range $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ of any such solution is biholomorphic to $\mathbf{C}^q$.*

This result was generalized in [3] (see also [2]), with an approach based on a discretization of time.

**Theorem 1.2.** *Let $-h(z,t)$ be a Herglotz vector field of order $\infty$ on $\mathbf{B}^q$ such that $h(z,t)=Az+O(|z|^2)$, where the eigenvalues of $A$ have strictly positive real part. Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution $(f_t\colon\mathbf{B}^q\to\mathbf{C}^q)$. The range $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ of any such solution is biholomorphic to $\mathbf{C}^q$.*

The same result was obtained independently with different methods by Voda [28], assuming $\min\{\operatorname{Re}\langle Az,z\rangle\colon |z|=1\}>0$. See also [18] for related results.

The next natural step is admitting time-dependent linear parts. Set

$$m(A)\doteq\min\{\operatorname{Re}\langle Az,z\rangle\colon |z|=1\},\quad k(A)=\max\{\operatorname{Re}\langle Az,z\rangle\colon |z|=1\}.$$

The following result is proved in [15, 13]:

**Theorem 1.3.** *Let $-h(z,t)$ be a Herglotz vector field on $\mathbf{B}^q$ of order $\infty$ such that $h(z,t)=A(t)z+O(|z|^2)$, and assume that the family of linear mappings $(A(t))_{t\geq 0}$ satisfies:*

i) *$m(A(t))>0$ for all $t\geq 0$ and $\int_0^\infty m(A(t))\,dt=\infty$,*

ii) $t \mapsto \|A(t)\|$ *is uniformly bounded on* $\mathbf{R}^+$,

iii) *there exists* $\delta > 0$ *such that*

$$2m(A(t)) \geq k(A(t)) + \delta, \quad t \geq 0,$$

iv)

$$\int_s^t A(\tau)\,d\tau \circ \int_r^s A(\tau)\,d\tau = \int_r^s A(\tau)\,d\tau \circ \int_s^t A(\tau)\,d\tau, \quad t \geq s \geq r \geq 0.$$

*Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution* $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)$. *The range* $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ *of any such solution is biholomorphic to* $\mathbf{C}^q$.

In this paper we generalize Theorem 1.3, using the approach of [2][3]. The following is our result.

**Theorem 1.4.** *Let* $-h(z,t)$ *be a Herglotz vector field on* $\mathbf{B}^q$ *of order* $\infty$ *such that* $h(z,t) = A(t)z + O(|z|^2)$, *and assume that the family of linear mappings* $(A(t))_{t\geq 0}$ *satisfies:*

a) $m(A(t)) > 0$ *for all* $t \geq 0$ *and* $\int_0^\infty m(A(t))\,dt = \infty$,

b) $t \mapsto \|A(t)\|$ *is locally bounded on* $\mathbf{R}^+$,

c) *there exists* $\ell \in \mathbf{R}^+$ *such that*

$$\ell m(A(t)) \geq k(A(t)), \quad t \geq 0.$$

*Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution* $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)$. *If* $\ell < 2$ *then the range* $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ *of any such solution is biholomorphic to* $\mathbf{C}^q$.

Notice that the assumptions ii) and iii) of Theorem 1.3 imply that there exists $\ell < 2$ such that

$$\ell m(A(t)) \geq k(A(t)), \quad t \geq 0.$$

We want to stress the strong analogy between Loewner theory and the theory of discrete non-autonomous complex dynamical systems which has developed around Bedford's conjecture (see [1, 11, 19, 23, 30]). This is reflected in the proof of Theorem 1.4, which is based on a discretization of time, and relies on the study of the abstract basin of attraction performed by Fornaess and Stensønes in [11]:

**Theorem 1.5.** *Let* $(\varphi_{n,n+1})_{n\in\mathbf{N}}$ *be a family of univalent self-mappings of* $r\mathbf{B}^q$. *Assume that there exist* $0 < \nu \leq \mu < 1$ *such that*

$$\nu|z| \leq |\varphi_{n,n+1}(z)| \leq \mu|z|, \quad z \in r\mathbf{B}^q,\ n \in \mathbf{N}. \tag{1.6}$$

*Then, if* $\Omega$ *is the abstract basin of attraction of* $(\varphi_{n,n+1})$, *then there exists an univalent mapping* $\Psi\colon \Omega \to \mathbf{C}^q$.

The abstract basin of attraction comes naturally with a family of univalent mappings $(\omega_n\colon r\mathbf{B}^q \to \Omega)$. Composing this family with the biholomorphism $\Psi$ given by Theorem 1.5 we obtain a family of univalent mappings from $r\mathbf{B}$ to $\mathbf{C}^q$ which we extend to a family $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)$ satisfying the functional equation (1.4).

The range $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ is by construction biholomorphic to $\Omega$ and thus by [11, Theorem 3.1] it is a Stein, Runge domain in $\mathbf{C}^q$ whose Kobayashi pseudometric vanishes identically and which is diffeomorphic to $\mathbf{C}^q$. It is an open question whether $\bigcup_{t\geq 0} f_t(\mathbf{B}^q)$ is biholomorphic to $\mathbf{C}^q$ when $\ell \geq 2$. A positive answer would follow from a proof of Bedford's conjecture (see e.g. [22]):

**Conjecture 1.6.** *Let $(\Phi_{n,n+1})_{n\in\mathbf{N}}$ be a family of automorphisms of $\mathbf{C}^q$. Assume that there exist $0<\nu\le\mu<1$ and $r>0$ such that*

$$\nu|z|\le|\Phi_{n,n+1}(z)|\le\mu|z|,\quad z\in r\mathbf{B}^q,\ n\in\mathbf{N}. \tag{1.7}$$

*Then the basin of attraction*

$$\{z\in\mathbf{C}^q\colon \lim_{n\to\infty}\Phi_{n-1,n}\circ\cdots\circ\Phi_{0,1}=0\}$$

*is biholomorphic to $\mathbf{C}^q$.*

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## 2. Main result

Let $\mathcal{N}$ denote the family of holomorphic mappings $h\colon\mathbf{B}^q\to\mathbf{C}^q$ such that $h(0)=0$ and $\operatorname{Re}\langle h(z),z\rangle>0$, for all $z\neq0$.

**Theorem 2.1.** *Let $h(z,t)\colon\mathbf{B}^q\times\mathbf{R}^+\to\mathbf{C}^q$ be a mapping such that $z\mapsto h(z,t)\in\mathcal{N}$ for all $t\in\mathbf{R}^+$ and $t\mapsto h(z,t)$ is measurable on $\mathbf{R}^+$ for all $z\in\mathbf{B}^q$. Assume that $h(z,t)=A(t)z+O(|z|^2)$ and that the family of linear mappings $(A(t))_{t\ge0}$ satisfies:*

a) *$m(A(t))>0$ for all $t\ge0$ and $\int_0^\infty m(A(t))\,dt=\infty$,*
b) *$t\mapsto\|A(t)\|$ is locally bounded on $\mathbf{R}^+$,*
c) *there exists $\ell\in\mathbf{R}^+$ such that*

$$\ell m(A(t))\ge k(A(t)),\quad t\ge0.$$

*Then the Loewner PDE*

$$\frac{\partial f_t(z)}{\partial t}=Df_t(z)h(z,t),\quad z\in\mathbf{B}^q,\ \text{a.e. } t\ge0$$

*admits a locally Lipschitz solution given by univalent mappings $(f_t\colon\mathbf{B}^q\to\mathbf{C}^q)$. If $l<2$, then the range $\bigcup_{t\ge0}f_t(\mathbf{B}^q)$ of any such solution is biholomorphic to $\mathbf{C}^q$. Any other solution given by holomorphic mappings $(g_t\colon\mathbf{B}^q\to\mathbf{C}^q)$ is of the form $(\Lambda\circ f_t)$, where $\Lambda\colon\bigcup_{t\ge0}f_t(\mathbf{B}^q)\to\mathbf{C}^q$ is holomorphic.*

*Proof.* Notice that for all $A\in\mathcal{L}(\mathbf{C}^q)$,

$$m(A)\le k(A)\le\|A\|,$$

and thus $k(t)$ and $m(t)$ are also locally bounded on $\mathbf{R}^+$. By [14, Lemma 1.2] one has for a.e. $t\ge0$

$$|h(z,t)|\le\frac{4r}{(1-r)^2}\|A(t)\|,\quad |z|\le r<1,$$

hence $-h(z,t)$ is a Herglotz vector field of order $\infty$ on $\mathbf{B}^q$. Let $(\varphi_{s,t})$ be the associated evolution family of order $\infty$, that is the solution of the Loewner ODE

$$\begin{cases}\frac{\partial}{\partial t}\varphi_{s,t}(z)=-h(\varphi_{s,t}(z),t),\quad z\in\mathbf{B}^q,\ \text{a.e. } t\in[s,\infty),\\ \varphi_{s,s}(z)=z,\quad z\in\mathbf{B}^q, s\ge0.\end{cases} \tag{2.1}$$

Recall that $\varphi_{s,t}\colon\mathbf{B}^q\to\mathbf{B}^q$ is an univalent mapping for all $0\le s\le t$ and that $t\mapsto\varphi_{s,t}(z)$ is locally Lipschitz continuous on $[s,\infty)$ uniformly on compact sets with respect to $z\in\mathbf{B}^q$.

Fix $s \geq 0$ and $z \in \mathbf{B}^q \setminus \{0\}$. Then for a.e. $\tau \geq s$,

$$\frac{\partial}{\partial \tau}|\varphi_{s,\tau}(z)|^2 = 2\mathrm{Re}\left\langle \frac{\partial}{\partial \tau}\varphi_{s,\tau}(z), \varphi_{s,\tau}(z)\right\rangle = -2\mathrm{Re}\,\langle h(\varphi_{s,\tau}(z),\tau), \varphi_{s,\tau}(z)\rangle.$$

Set $C(r) \doteq \frac{1+r}{1-r}$ and $c(r) \doteq \frac{1-r}{1+r}$ for all $r \geq 0$. Gurganus proved [17] that for a.e. $t \geq 0$,

$$\mathrm{Re}\,\langle A(t)w, w\rangle c(|w|) \leq \mathrm{Re}\,\langle h(w,t), w\rangle \leq \mathrm{Re}\,\langle A(t)w, w\rangle C(|w|), \quad w \in \mathbf{B}^q \setminus \{0\}.$$

Since $|\varphi_{s,\tau}(z)| \leq |z|$ one has

$$-2k(A(\tau))C(|z|) \leq \frac{\frac{\partial}{\partial \tau}|\varphi_{s,\tau}(z)|^2}{|\varphi_{s,\tau}(z)|^2} \leq -2m(A(\tau))c(|z|), \quad \text{a.e. } \tau \geq 0,$$

$$-2C(|z|)\int_s^t k(A(\tau))\,d\tau \leq \int_s^t \frac{\frac{\partial}{\partial \tau}|\varphi_{s,\tau}(z)|^2}{|\varphi_{s,\tau}(z)|^2}\,d\tau \leq -2c(|z|)\int_s^t m(A(\tau))\,d\tau, \quad 0 \leq s \leq t,$$

$$e^{-C(|z|)\int_s^t k(A(\tau))\,d\tau} \leq \frac{|\varphi_{s,t}(z)|}{|z|} \leq e^{-c(|z|)\int_s^t m(A(\tau))\,d\tau}, \quad 0 \leq s \leq t. \tag{2.2}$$

Set for all $0 \leq s \leq t$,

$$\nu_{s,t} \doteq e^{-C(|z|)\int_s^t k(A(\tau))\,d\tau}, \quad \text{and} \quad \mu_{s,t} \doteq e^{-c(|z|)\int_s^t m(A(\tau))\,d\tau}.$$

One has, thanks to assumption c),

$$\log_{\mu_{s,t}} \nu_{s,t} = \frac{\log \nu_{s,t}}{\log \mu_{s,t}} = C^2(|z|)\frac{\int_s^t k(A(\tau))\,d\tau}{\int_s^t m(A(\tau))\,d\tau} \leq C^2(|z|)\ell, \quad 0 \leq s \leq t. \tag{2.3}$$

Let $n \in \mathbf{N}$ and let $u_n \in \mathbf{R}^+$ be defined by

$$\int_0^{u_n} m(A(\tau))\,d\tau = n.$$

Let now $h \in \mathbf{N}$ be the least integer strictly greater than $\ell$, and let $r > 0$ be such that $C^2(r) < h/\ell$. Set $\mu \doteq e^{-c(r)}$ (notice that $\mu = \mu_{u_n,u_{n+1}}$ for all $n \geq 0$) and $\nu \doteq \min\{\nu_{u_n,u_{n+1}} \colon n \geq 0\}$. By (2.2) and (2.3) one has that

$$\nu|z| \leq |\varphi_{u_n,u_{n+1}}(z)| \leq \mu|z|, \quad z \in r\mathbf{B}^q, n \geq 0.$$

and

$$\mu^h < \nu. \tag{2.4}$$

The *abstract basin of attraction* or *tail space* $\Omega$ of the family $(\varphi_{u_n,u_{n+1}} \colon r\mathbf{B}^q \to r\mathbf{B}^q)$ is defined in [11] (see also [1]) as its topological inductive limit endowed with a natural complex structure. $\Omega$ is the quotient of the set

$$\left\{z \in \prod_{m \geq n} r\mathbf{B}^q \colon n \in \mathbf{N},\ z_{m+1} = \varphi_{u_m,u_{m+1}}(z_m), \quad 0 \leq n \leq m\right\},$$

obtained identifying $z$ and $z'$ if $z_m = z'_m$ for $m$ large enough, and the holomorphic structure is induced by a family of open inclusions $(\omega_n \colon r\mathbf{B}^q \to \Omega)$ defined as

$$\omega_n(z) \doteq (\varphi_{u_n,u_m}(z))_{m \geq n}, \quad n \in \mathbf{N},$$

which are thus by definition biholomorphisms with their image and satisfy

$$\omega_n(z) = \omega_m \circ \varphi_{u_n,u_m}(z), \quad 0 \leq n \leq m,\ z \in r\mathbf{B}^q. \tag{2.5}$$

By [11, Theorem 2.2] there exists an univalent mapping $\Psi\colon \Omega \to \mathbf{C}^q$. We claim that, for all $s \geq 0$, the sequence $(\Psi \circ \omega_m \circ \varphi_{s,u_m})_{m\geq 0}$ converges uniformly on compact sets in $\mathrm{Hol}(\mathbf{B}^q, \mathbf{C}^q)$. Indeed by equation (2.2) and assumption a) one has that for all $s \geq 0$,

$$\lim_{m\to\infty} \varphi_{s,u_m}(z) = 0,$$

uniformly on compact sets. Thus, if $0 < v < 1$, there exist $m(v) \in \mathbf{N}$ such that for all $j \geq m(v)$, one has $\varphi_{s,u_j}(v\mathbf{B}^q) \subset r\mathbf{B}^q$. Let $j, h$ be integers such that $m(v) \leq j \leq h$, then by (2.5),

$$\Psi \circ \omega_h \circ \varphi_{s,u_h}(z) = \Psi \circ \ \omega_j \circ \varphi_{s,u_j}(z), \quad z \in v\mathbf{B}^q.$$

Thus the sequence $(\Psi \circ \omega_m \circ \varphi_{s,u_m})$ is eventually constant in $\mathrm{Hol}(v\mathbf{B}^q, \mathbf{C}^q)$.

Let $f_t\colon \mathbf{B}^q \to \mathbf{C}^q$ the univalent mapping defined as

$$f_t(z) \doteq \lim_{m\to+\infty} \Psi \circ \omega_m \circ \varphi_{t,u_m}(z). \tag{2.6}$$

One easily verifies that

$$f_s(z) = f_t \circ \varphi_{s,t}(z), \quad 0 \leq s \leq t, \ z \in \mathbf{B}^q, \tag{2.7}$$

and that

$$\bigcup_{t\geq 0} f_t(\mathbf{B}^q) = \Psi(\Omega).$$

Notice that the abstract basin of attraction of the family $(\varphi_{u_n,u_{n+1}})$ is thus biholomorphic to the Loewner range of the family $(\varphi_{s,t})$ defined in [5]. This can be checked directly since both objects are defined as direct limits.

By [5, Theorem 4.10] one has that $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)$ is a Loewner chain of order $\infty$, that is

- $f_s(\mathbf{B}^q) \subset f_t(\mathbf{B}^q)$ for all $0 \leq s \leq t$,
- for any compact set $K \subset \mathbf{B}^q$ and for any $T > 0$ there exists a $k_{K,T} > 0$ satisfying

$$|f_t(z) - f_s(z)| \leq k_{K,T}(t-s), \quad z \in K, \ 0 \leq s \leq t < T. \tag{2.8}$$

By [5, Theorem 5.2] one obtains finally

$$\frac{\partial f_t(z)}{\partial t} = Df_t(z)h(z,t), \quad z \in \mathbf{B}^q, \ \text{a.e. } t \geq 0.$$

Thus any univalent mapping $\Psi\colon \Omega \to \mathbf{C}^q$ gives rise to a univalent solution $(f_t\colon \mathbf{B}^q \to \mathbf{C}^q)$ of the Loewner PDE. Following [1, Remark A.4] we recall a way to construct such a univalent mapping $\Psi\colon \Omega \to \mathbf{C}^q$. Given any polynomial map $p\colon \mathbf{C}^q \to \mathbf{C}^q$ of degree at most $k$ with $Dp(0)$ invertible there exists [12][29] an holomorphic automorphism $\Phi$ of $\mathbf{C}^q$ such that

$$\Phi(z) = p(z) + O(|z|^{k+1}).$$

We choose a sequence of automorphisms $(\Phi_{n,n+1}\colon \mathbf{C}^q \to \mathbf{C}^q)$ which is uniformly bounded on a neighborhood of the origin and which satisfies

$$\Phi_{n,n+1}(z) = \varphi_{u_n,u_{n+1}}(z) + O(|z|^h), \quad n \geq 0,$$

where $h \in \mathbf{N}$ is as in (2.4). We denote the basin of attraction of the sequence $(\Phi_{n,n+1})$ by

$$\mathfrak{A}(\Phi_{n,n+1}) \doteq \{z \in \mathbf{C}^q\colon \lim_{n\to\infty} \Phi_{n-1,n} \circ \cdots \circ \Phi_{0,1} = 0\}.$$

It follows from [1, Theorem A.1] that there exists a biholomorphism

$$\Psi\colon \Omega \to \mathfrak{A}(\Phi_{n,n+1}) \subset \mathbf{C}^q.$$

If $\ell < 2$, then $h = 2$ and, by [30, Theorem 4], one has that the basin of attraction $\mathfrak{A}(\Phi_{n,n+1})$ is biholomorphic to $\mathbf{C}^q$.

By [5, Theorem 4.10] any solution $(g_t\colon \mathbf{B}^q \to \mathbf{C}^q)$ of the Loewner PDE has to satisfy $g_s = g_t \circ \varphi_{s,t}$ for all $0 \le s \le t$ and thus [5, Theorem 4.7] yields that the family $(g_t)$ is of the form $(\Lambda \circ f_t)$, where $\Lambda\colon \bigcup_{t\ge 0} f_t(\mathbf{B}^q) \to \mathbf{C}^q$ is holomorphic. □

## References


[1] Abate, M., A. Abbondandolo, and P. Majer: Stable manifolds for holomorphic automorphisms. - Preprint, arXiv:1104.4561v2 [math.DS].

[2] Arosio, L.: Resonances in Loewner equations. - Adv. Math. 227, 2011, 1413–1435.

[3] Arosio, L.: Loewner equations on complete hyperbolic domains. - Preprint, arXiv:1102.5454 [math.CV].

[4] Arosio, L., and F. Bracci: Infinitesimal generators and the Loewner equation on complete hyperbolic manifolds. - Anal. Math. Phys. 1:4, 2011, 337–350.

[5] Arosio, L., F. Bracci, H. Hamada, and G. Kohr: An abstract approach to Loewner chains. - J. Anal. Math. (to appear), arXiv:1002.4262v1 [math.CV].

[6] Bracci, F., M. D. Contreras, and S. Díaz-Madrigal: Evolution families and the Loewner equation I: the unit disc. - J. Reine Angew. Math. (to appear), DOI:10.1515/crelle.2011.167.

[7] Bracci, F., M. D. Contreras, and S. Díaz-Madrigal: Evolution families and the Loewner equation II: complex hyperbolic manifolds. - Math. Ann. 344, 2009, 947–962

[8] Contreras, M. D., S. Díaz-Madrigal, and P. Gumenyuk: Loewner chains in the unit disc. - Rev. Mat. Iberoamericana 26, 2010, 975–1012.

[9] de Branges, L.: A proof of the Bieberbach conjecture. - Acta Math. 154, 1985, 137–152.

[10] Duren, P., I. Graham, H. Hamada, and G. Kohr: Solutions for the generalized Loewner differential equation in several complex variables. - Math. Ann. 347:2, 2010, 411–435.

[11] Fornaess, J. E., and B. Stensønes: Stable manifolds of holomorphic hyperbolic maps. - Internat. J. Math. 15, 2004, 749–758

[12] Forstneric, F.: Interpolation by holomorphic automorphisms and embeddings in $\mathbf{C}^n$. - J. Geom. Anal. 9, 1999, 93–117.

[13] Graham, I., H. Hamada, and G. Kohr: On subordination chains with normalization given by a time-dependent linear operator. - Complex Anal. Oper. Theory 5:3, 2011, 787–797.

[14] Graham, I., H. Hamada, G. Kohr, and M. Kohr: Asymptotically spirallike mappings in several complex variables. - J. Anal. Math. 105, 2008, 267–302.

[15] Graham, I., H. Hamada, G. Kohr, and M. Kohr: Spirallike mappings and univalent subordination chains in $\mathbf{C}^n$. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2008, 717–740.

[16] Graham, I., and G. Kohr: Geometric function theory in one and higher dimensions. - Marcel Dekker Inc., New York, 2003.

[17] Gurganus, K.: Ψ-like holomorphic functions in $\mathbf{C}^n$ and Banach spaces. - Trans. Amer. Math. Soc. 205, 1975, 389–406.

[18] Hamada, H.: Polynomially bounded solutions to the Loewner differential equation in several complex variables. - J. Math. Anal. Appl. 381, 2011, 179–186.

[19] Jonsson, M., and D. Varolin: Stable manifolds of holomorphic diffeomorphisms. - Invent. Math. 149:2, 2002, 409–430.

[20] Kufarev, P. P.: On one-parameter families of analytic functions. - Mat. Sb. 13, 1943, 87–118 (in Russian).

[21] Loewner, C.: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. - Math. Ann. 89, 1923, 103–121.

[22] Peters, H.: Non-autonomous complex dynamical systems. - Ph.D. thesis, Univ. of Michigan, 2005.

[23] Peters, H.: Perturbed basins of attraction. - Math. Ann. 337, 2007, 1–13.

[24] Peters, H., and E. F. Wold: Non-autonomous basins of attraction and their boundaries. - J. Geom. Anal. 15, 2005, 123–136.

[25] Pfaltzgraff, J. A.: Subordination chains and univalence of holomorphic mappings in $\mathbf{C}^n$. - Math. Ann. 210, 1974, 55–68.

[26] Pommerenke, Ch.: Über die Subordination analytischer Funktionen. - J. Reine Angew. Math. 218, 1965, 159–173.

[27] Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. - Israel J. Math. 118, 2000, 221–288.

[28] Voda, M.: Solution of a Loewner chain equation in several variables. - J. Math. Anal. Appl. 375:1, 2011, 58–74.

[29] Weickert, B.: Automorphisms of $\mathbf{C}^n$. - Ph.D. thesis, Univ. of Michigan, 1997.

[30] Wold, E. F.: Fatou–Bieberbach domains. - Internat. J. Math. 16, 2005, 1119–1130.

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