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HARNACK'S INEQUALITY FOR GENERAL SOLUTIONS WITH NONSTANDARD GROWTH

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Abstract. We prove Harnack's inequality for general solutions of elliptic equations

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u)=\mathcal{B}(x,u,\nabla u),$$

where $\mathcal{A}$ and $\mathcal{B}$ satisfy natural structural conditions with respect to a variable growth exponent $p(x)$. The proof is based on a modification of the Caccioppoli inequality, which enables us to use existing versions of the Moser iteration.

1. Introduction

The purpose of this note is to give a proof for Harnack's inequality

$$\operatorname*{ess\,sup}_{B(x,R)} u \le C\big(\operatorname*{ess\,inf}_{B(x,R)} u + R\big) \tag{1.1}$$

for solutions of elliptic equation

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u)=\mathcal{B}(x,u,\nabla u), \tag{1.2}$$

where $\mathcal{A}$ and $\mathcal{B}$ satisfy natural simple structural conditions with respect to a variable growth exponent $p(x)$; see Theorem 3.5 below. The novelty in our argumentation lies in the choice of test functions. We are able to prove under modified assumptions on the test functions exactly the same Caccioppoli estimate as in the case of $p(x)$-Laplacian

$$-\operatorname{div}\big(p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\big)=0.$$

The point is that the Moser iteration technique used in [8] remains valid under our consideration.

The study of Harnack's inequality and Moser iteration for solutions of equations of the type (1.2) (for the constant exponent) goes back to the famous paper [15] of Serrin, in which he extended the ideas of Moser in [12] and [13]. In the case of the variable exponent p-Laplacian, Harnack's inequality was first proved in the paper [1] by Alkhutov. In his paper the constant corresponding to C in (1.1) depends also on the $L^\infty(B(x,4R))$-norm of the function. Later, Harjulehto, Kinnunen and Lukkari [8] were able to improve the argumentation so that their constant C depends on the $L^q(B(x,4R))$-norm of u, $0<q<\infty$, instead of the $L^\infty(B(x,4R))$-norm. Here the exponent q can be made arbitrarily small by choosing R small enough in (1.1).

In the case of general equations of the type (1.2) Harnack's inequality has been studied only in certain special cases, see [16]. Recently, the weak Harnack estimate for supremum was proved in [7] under our assumptions. However, the infimum estimate,

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which is more delicate both in Moser's and De Giorgi's methods, seems to be missing in the literature. The contribution of this note is to verify the infimum Harnack estimate for our solutions by using the trick of modified test functions.

The local boundedness and the local Hölder continuity of solutions were studied in [4]. This is crucial for us since the combination of our approach and results of [4] give Harnack's inequality for the class of solutions studied in [11], see Remark 3.6 (b). Our result is optimal in the sense that even all known proofs for Harnack's inequality of the $p(x)$-Laplace equation end up with the additional term R in (1.1), see e.g. [8], [9] and Chapter 13 of [3]. Notice also that the additional R-term appears for the non-homogenous equations even in the case of constant exponent p, see [15] and [6], Chapter 7.5. Hence our argument is reasonable even for the constant exponent case.

2. Caccioppoli inequality

Throughout, let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $p\colon \Omega \to]1,\infty[$ be a logarithmically Hölder continuous function, i.e. p satisfies

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}$$

with some constant $C > 0$ for all $x, y \in \Omega$ such that $|x-y| \le 1/2$. Logarithmic Hölder continuity is the standard assumption for the regularity methods of $p(x)$-Laplacian type equations; see [2] and [14]. In fact, this continuity assumption is not necessary to prove the Caccioppoli inequality, but we do need it to complete the Moser iteration.

Additionally, we denote $p_E^+ = \sup_E p$ and $p_E^- = \inf_E p$ for any measurable set $E \subset \Omega$, and assume that

$$1 < p_\Omega^- \le p_\Omega^+ < \infty.$$

For a complete account of the variable exponent Sobolev spaces, see [3]. Many of the basic properties were originally proved in [5] and [10].

Let $\mathcal{A}\colon \mathbf{R}\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ and $\mathcal{B}\colon \mathbf{R}\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ be Caratheodory functions. This means that the functions $x \to \mathcal{A}(x,u,\xi)$ and $x \to \mathcal{B}(x,u,\xi)$ are measurable for all $(u,\xi) \in \mathbf{R}\times\mathbf{R}^n$, and the functions $(u,\xi) \to \mathcal{A}(x,u,\xi)$ and $(u,\xi) \to \mathcal{B}(x,u,\xi)$ are continuous for almost all $x \in \Omega$. We assume that there are positive constants a_i, b_i, c_i for $i = 1, 2$ so that

$$\begin{aligned} |\mathcal{A}(x,u,\xi)| &\le a_1|\xi|^{p(x)-1} + a_2, \\ |\mathcal{B}(x,u,\xi)| &\le b_1|\xi|^{p(x)-1} + b_2, \\ \mathcal{A}(x,u,\xi)\cdot\xi &\ge c_1|\xi|^{p(x)} - c_2, \end{aligned} \tag{2.1}$$

and call a Sobolev function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ a *solution* of

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) \tag{2.2}$$

in Ω if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega \mathcal{B}(x,u,\nabla u)\varphi\,dx$$

for all test functions $\varphi \in W^{1,p(\cdot)}(\Omega)$ with a compact support in Ω. Similarly, we call $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ a *supersolution* of (2.2) in Ω if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,dx \ge \int_\Omega \mathcal{B}(x,u,\nabla u)\varphi\,dx \tag{2.3}$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega)$ with a compact support in Ω, and a *subsolution* of (2.2) in Ω if the reverse inequality for (2.3) applies.

The following Caccioppoli estimate is the key result of this paper; it corresponds to Lemma 3.2 of [8].

2.4. Lemma. *Let $u > 0$ be a supersolution of* (2.2) *in Ω, $\gamma_0 < 0$, and let η be a compactly supported Lipschitz-function η in Ω with the properties $0 \le \eta \le M$ and $\eta \le M|\nabla\eta|$ for some constant $M \ge 1$. Then for every $\gamma < \gamma_0$ and measurable $E \subset \Omega$, we have*

$$\int_E |\nabla u|^{p_E^-} \eta^{p^+_{\operatorname{spt}\eta}} u^{\gamma-1}\,dx \le C \int_\Omega \eta^{p^+_{\operatorname{spt}\eta}} u^{\gamma-1} + |u|^{\gamma+p(x)-1} |\nabla\eta|^{p(x)}\,dx.$$

Here C depends on γ_0, p, M, and all six structure constants in (2.1).

Proof. For brevity, we write p^+ and p^- for $p^+_{\operatorname{spt}\eta}$ and $p^-_{\operatorname{spt}\eta}$, respectively. Let us take $\varphi = \eta^{p^+} u^\gamma$ as the test function. We plug

$$\nabla\varphi = \gamma\eta^{p^+} u^{\gamma-1}\nabla u + p^+ u^\gamma \eta^{p^+-1}\nabla\eta$$

into the inequality

$$0 \le \int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,dx - \int_\Omega \mathcal{B}(x,u,\nabla u)\varphi\,dx$$

and use the structure assumptions (2.1). Since $\gamma < 0$, we obtain

$$\begin{aligned}
&\int_\Omega (c_1|\nabla u|^{p(x)} - c_2)|\gamma|\eta^{p^+}u^{\gamma-1}\,dx \le \int_\Omega |\gamma|\eta^{p^+}u^{\gamma-1}\mathcal{A}(x,u,\nabla u)\cdot\nabla u\,dx \\
&\le \int_\Omega p^+ u^\gamma \eta^{p^+-1}\mathcal{A}(x,u,\nabla u)\cdot\nabla\eta\,dx + \int_\Omega |\mathcal{B}(x,u,\nabla u)\varphi|\,dx.
\end{aligned}$$

Moving the c_2-term to the right-hand side and using the structure conditions again, we see that

$$\begin{aligned}
&\int_\Omega c_1|\gamma||\nabla u|^{p(x)}\eta^{p^+}u^{\gamma-1}\,dx \\
&\le \int_\Omega c_2|\gamma|\eta^{p^+}u^{\gamma-1}\,dx + \int_\Omega p^+(a_1|\nabla u|^{p(x)-1} + a_2)u^\gamma\eta^{p^+-1}|\nabla\eta|\,dx \\
&\qquad + \int_\Omega \big(b_1|\nabla u|^{p(x)-1} + b_2\big)\,\eta^{p^+}u^\gamma\,dx \\
&\le C|\gamma|\int_\Omega \eta^{p^+}u^{\gamma-1}\,dx + C\int_\Omega (|\nabla u|^{p(x)-1}+1)u^\gamma\eta^{p^+-1}|\nabla\eta|\,dx \\
&\qquad + C\int_\Omega \big(|\nabla u|^{p(x)-1}+1\big)\,\eta^{p^+}u^\gamma\,dx
\end{aligned}$$

for a constant C, which depends only on p^+, a_i, b_i and c_i for $i = 1, 2$. Next we use Young's inequality for the last two integrals on the right-hand side. To estimate the latter integral, we denote $z = \eta + |\nabla\eta|$, and write the trivial estimate

$$\int_\Omega \big(|\nabla u|^{p(x)-1}+1\big)\,\eta^{p^+}u^\gamma\,dx \le C\int_\Omega z u^\gamma\eta^{p^+-1} + z|\nabla u|^{p(x)-1}u^\gamma\eta^{p^+-1}\,dx.$$

The first integrand on the right-hand side is estimated as

$$\begin{aligned} zu^{\gamma}\eta^{p^+-1} &= \big(zu^{(\gamma+p(x)-1)/p(x)}\eta^{p^+-\frac{p^+}{p'(x)}-1}\big)\big(u^{(\gamma-1)/p'(x)}\eta^{p^+/p'(x)}\big) \\ &\le Cz^{p(x)}u^{\gamma+p(x)-1}\eta^{p^+-p(x)}+\frac{1}{2}\eta^{p^+}u^{\gamma-1} \end{aligned}$$

by Young's inequality. Similarly, we obtain

$$z|\nabla u|^{p(x)-1}u^{\gamma}\eta^{p^+-1}\le Cz^{p(x)}u^{\gamma+p(x)-1}\eta^{p^+-p(x)}+\frac{1}{2}|\nabla u|^{p(x)}\eta^{p^+}u^{\gamma-1}.$$

The integral $\int_\Omega \big(|\nabla u|^{p(x)-1}+1\big)\,u^{\gamma}\eta^{p^+-1}|\nabla\eta|\,dx$ is estimated in a similar fashion. Consequently, by combining similar terms, we end up with the inequality

$$\begin{aligned} &|\gamma|\int_\Omega |\nabla u|^{p(x)}\eta^{p^+}u^{\gamma-1}\,dx \\ &\le C|\gamma|\int_\Omega \eta^{p^+}u^{\gamma-1}\,dx + C\int_\Omega (\eta+|\nabla\eta|)^{p(x)}|u|^{\gamma+p(x)-1}\eta^{p^+-p(x)}\,dx. \end{aligned}$$

By the assumptions $\eta+|\nabla\eta|\le (M+1)|\nabla\eta|$ and $|\gamma|\ge|\gamma_0|>0$ we arrive at the estimate

$$\int_\Omega |\nabla u|^{p(x)}\eta^{p^+}u^{\gamma-1}\,dx \le C\int_\Omega \eta^{p^+}u^{\gamma-1}\,dx+\frac{C}{|\gamma_0|}\int_\Omega |u|^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\,dx. \tag{2.5}$$

By [8, Lemma 3.1], we have

$$\int_\Omega |\nabla u|^{p^-}\eta^{p^+}u^{\gamma-1}\,dx-\int_\Omega \eta^{p^+}u^{\gamma-1}\,dx\le\int_\Omega |\nabla u|^{p(x)}\eta^{p^+}u^{\gamma-1}\,dx, \tag{2.6}$$

and the result follows by combining (2.5) with (2.6). □

3. Weak Harnack inequalities

Our Caccioppoli estimate can be adapted to the Moser iteration the same way as in [8]. We state here the required steps and point out the modifications in the proofs.

Throughout this section, let $0<R\le 1$ be so small that $B_{4R}=B(x_0,4R)$ is contained in Ω. For brevity, we write

$$\Phi(f,q,B_r)=\left(\fint_{B_r} f^q\,dx\right)^{1/q}$$

for a positive function f, an exponent $q\in\mathbf{R}$, and an open ball $B_r=B(x_0,r)$.

3.1. Lemma. *Let u be a non-negative supersolution of* (2.2) *in B_{4R}, and let $R\le\rho<r\le 3R$, $s>p^+_{B_{4R}}-p^-_{B_{4R}}$. Then*

$$\Phi(u+R,q\beta,B_r)\le C^{1/|\beta|}(1+|\beta|)^{p^+_{B_{4R}}/|\beta|}\left(\frac{r}{r-\rho}\right)^{p^+_{B_{4R}}/|\beta|}\Phi\left(u+R,\frac{\beta n}{n-1},B_\rho\right)$$

holds for every $\beta<0$, $1<q<n/(n-1)$. Here C depends on n, p, the $L^{q's}(B_{4R})$-norm of u, and all six structure constants of (2.1).

Proof. We construct the test function η as a sum $\eta = \varphi + \psi$, where φ is a Lipschitz function such that $\varphi = 1$ in B_ρ, φ vanishes outside B_r, $|\nabla\varphi| = \frac{1}{r-\rho}$ in $B_r \setminus B_\rho$, and ψ is defined by

$$\psi(x) = \begin{cases} d(x, \partial B_\rho) & \text{for } x \in B_\rho, \\ 0 & \text{for } x \notin B_\rho. \end{cases}$$

Then η vanishes outside B_r and we have $\eta \le 4$ in Ω since $R \le 1$. Moreover, $\frac{r}{R(r-\rho)} \ge \frac{1}{2}$, and therefore

$$|\nabla\eta| \le \frac{2r}{R(r-\rho)}.$$

Since $1 \le \eta \le 4$ in B_ρ, we easily see that $\eta + |\nabla\eta| \le 9|\nabla\eta|$.

The rest of the proof is identical to the proof of Lemma 3.5 in [8]. We use our test function η together with Lemma 2.4. The fact that we have the condition $1 \le \eta \le 4$ instead of $\eta = 1$ in B_ρ is irrelevant. □

3.2. Lemma. *Let u be a non-negative supersolution of* (2.2) *in B_{4R}, and let $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then there exist constants $q_0 > 0$ and C depending on n, p and the $L^s(B_{4R})$-norm of u such that*

$$\Phi(u + R, q_0, B_{3R}) \le C\Phi(u + R, -q_0, B_{3R}).$$

Proof. Let $B_{2r} \subset B_{4R}$. We choose η as in the proof of Lemma 3.1, replacing ρ and r with r and $2r$, respectively. Then we have $\eta \ge 1$ in B_r and $|\nabla\eta| \le C/r$ in B_{2r}, since $r \le 2$. Taking $E = B_r$ and $\gamma = 1 - p^-_{B_r}$ in Lemma 2.4, we have

$$\fint_{B_r} |\nabla \log v|^{p^-_{B_r}}\, dx \le C\left(\fint_{B_{2r}} v^{-p^-_{B_r}} + \fint_{B_{2r}} v^{p(x)-p^-_{B_r}} r^{-p(x)}\, dx\right).$$

The rest of the proof is identical to that of Lemma 3.6 in [8]. □

The following weak Harnack inequality is the main result of this note. The proof follows from lemmas 3.1 and 3.2 as in [8].

3.3. Theorem. *Let u be a non-negative supersolution of* (2.2) *in B_{4R}, $1 < q < n/(n-1)$, and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then*

$$\left(\fint_{B_{2R}} u^{q_0}\, dx\right)^{1/q_0} \le C\left(\operatorname*{ess\,inf}_{x\in B_R} u(x) + R\right),$$

where C depends on n, p, q and the $L^{q's}(B_{4R})$-norm of u, and all six structure constants of (2.1).

For the Harnack supremum estimate we adapt Theorem 1.2 of [7].

3.4. Lemma. *Let u be a non-negative subsolution of* (2.2) *in B_{4R} and let $s > p^+_{B_{4R}} - p^-_{B_{4R}}$. Then*

$$\operatorname*{ess\,sup}_{x\in B_R} u(x) \le C\left(\fint_{B_{2R}} u^t\, dx\right)^{1/t} + R$$

for every $t > 0$, where C depends on n, p, b_1, b_2, t, and the $L^{ns}(B_{4R})$-norm of u.

3.5. Theorem. (Harnack's inequality) *Let u be a non-negative solution of* (2.2) *in Ω, and let $B_{4R} \subset \Omega$, $1 < q < n/(n-1)$, and $s > p^+_{B_{4R}} - p^-_{B_{4R}}$ with $0 < R \le 1$. Then*

$$\operatorname*{ess\,sup}_{x \in B_R} u(x) \le C \left(\operatorname*{ess\,inf}_{x \in B_R} u(x) + R \right),$$

where C depends on n, p, q and the $L^{ns}(B_{4R})$-norm of u, and all six structure constants of (2.1).

3.6. Remark. (a) Since our exponent p is logarithmically Hölder continuous, we may choose R so small that $ns \le p^-_\Omega$. Hence, in the local sense, the dependence of C on u is similar to the variable exponent p-Laplacian case studied in [7].

(b) In [11], Lukkari considers the boundary continuity of solutions under the structure conditions

$$\begin{aligned} |\mathcal{A}(x,u,\xi)| &\le a_1|\xi|^{p(x)-1} + a_2|u|^{p(x)-1} + a_3, \\ |\mathcal{B}(x,u,\xi)| &\le b_1|\xi|^{p(x)-1} + b_2|u|^{p(x)-1} + b_3, \\ \mathcal{A}(x,u,\xi)\cdot\xi &\ge c_1|\xi|^{p(x)} - c_2|u|^{p(x)} - c_3, \end{aligned}$$

where a_i, b_i, c_i for $i = 1,2,3$ are positive constants and p is logarithmically Hölder continuous. These structure conditions are included in the more general approach of [4] and hence by [4], Theorem 4.1, the solutions are locally bounded whenever p is log-Hölder continuous. Consequently our structure conditions (2.1) are fulfilled in any open set $\Omega' \Subset \Omega$ for the solutions of [11] with constants a_2, b_2 and c_2 depending on $\|u\|_{L^\infty(\Omega')}$. Hence the claim of Theorem 3.5 holds in Ω' for solutions of [11] with a constant C depending on $\|u\|_{L^\infty(\Omega')}$ instead of the $L^{q's}(B_{4R})$-norm of u.

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