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NONLOCALIZATION OF OPERATORS OF SCHRÖDINGER TYPE

Per Sjölin

KTH Royal Institute of Technology, Department of Mathematics
S-100 44 Stockholm, Sweden; pers@math.kth.se

Abstract. Localization properties are studied for operators of Schrödinger type.

1. Introduction

For f belonging to the Schwartz class $\mathcal{S}(\mathbf{R})$ we define the Fourier transform $\hat{f}$ by setting

$$\hat{f}(\xi) = \int_{\mathbf{R}} e^{-i\xi x} f(x)\, dx, \quad \xi \in \mathbf{R}.$$

For $a > 1$ and $f \in \mathcal{S}(\mathbf{R})$ we also set

$$S_t f(x) = \int_{\mathbf{R}} e^{i\xi x} e^{it|\xi|^a} \hat{f}(x)\, dx, \quad x \in \mathbf{R}, \quad t \geq 0.$$

If we set $u(x,t) = S_t f(x)/2\pi$, then $u(x,0) = f(x)$ and in the case $a = 2$, u satisfies the Schrödinger equation $i\,\partial u/\partial t = \partial^2 u/\partial x^2$. We also set

$$m(\xi) = e^{i|\xi|^a}, \quad \xi \in \mathbf{R},$$

and let K denote the Fourier transform of m so that $K \in \mathcal{S}'(\mathbf{R})$. It is known that $K \in C^\infty(\mathbf{R})$ (see Lemma A below) and in the case $t > 0$ it is clear that

$$e^{it|\xi|^a} = m(t^{1/a}\xi)$$

has the Fourier transform

$$K_t(y) = t^{-1/a} K(t^{-1/a} y).$$

One has $S_t f(x) = K_t * f(x)$ for $t > 0$ and $f \in \mathcal{S}(\mathbf{R})$ and we set $S_t f(x) = K_t * f(x)$ for $f \in L^2(\mathbf{R})$ with compact support. We introduce Sobolev spaces H_s by setting

$$H_s = \left\{ f \in \mathcal{S}';\ \|f\|_{H_s} < \infty \right\}, \quad s \in \mathbf{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbf{R}} \left(1 + \xi^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

It is well-known (see Sjölin [4] and Vega [5] and in the case $a = 2$ Carleson [1] and Dahlberg and Kenig [2]) that

$$\lim_{t \to 0} \frac{1}{2\pi} S_t f(x) = f(x)$$

almost everywhere if $f \in H_{1/4}$ and f has compact support. Also it is known that $H_{1/4}$ cannot be replaced by H_s if $s < 1/4$.

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Now assume that $0 \le s < 1/4$.

Here we shall study the problem if there is localization or localization almost everywhere for the above operators S_t and functions $f \in H_s$ with compact support, that is, do we have

$$\lim_{t \to 0} S_t f(x) = 0$$

everywhere or almost everywhere in $\mathbf{R} \setminus (\operatorname{supp} f)$? We shall prove that there is no localization or localization almost everywhere of this type for $0 \le s < 1/4$. In fact we shall prove that there exist two disjoint compact intervals I and J in $\mathbf{R}$ and a function f which belongs to H_s for all $s < 1/4$, with the properties that $\operatorname{supp} f \subset I$ and for every $x \in J$ one does not have

$$\lim_{t \to 0} S_t f(x) = 0.$$

In the special case $a = 2$ this was proved in 2009 by P. Sjölin and F. Soria. The proof for $a > 1$ in this paper is a generalization of the proof of Sjölin and Soria for $a = 2$. We remark that Sjölin and Soria also obtained the corresponding result for $a = 2$ and dimension $n \ge 2$.

2. Proofs

We shall use a theorem of Miyachi to obtain some properties of the kernel K defined in the introduction.

Lemma A. *One has $K \in C^\infty(\mathbf{R})$ and there exists a number $\alpha \ge 0$ such that*

$$|K(x)| \le C\left(1 + |x|^\alpha\right) \quad \text{for } x \in \mathbf{R}. \tag{1}$$

Proof. Let $\psi \in C^\infty(\mathbf{R})$ with

$$\psi(\xi) = 1, \quad |\xi| \ge 2, \quad \text{and} \quad \psi(\xi) = 0, \quad |\xi| \le 1.$$

We have $m = m_1 + m_2$, where

$$m_1(\xi) = \left(1 - \psi(\xi)\right) e^{i|\xi|^a} \quad \text{and} \quad m_2(\xi) = \psi(\xi)\, e^{i|\xi|^a}.$$

Let m_1 and m_2 have Fourier transforms K_1 and K_2 respectively. We have

$$K_1(x) = \int_{|\xi| \le 2} e^{-ix\xi} \left(1 - \psi(\xi)\right) e^{i|\xi|^a} d\xi, \quad x \in \mathbf{R},$$

and it is easy to see that K_1 is bounded and belongs to C^∞.

Also Miyachi [3] has proved that $K_2 \in C^\infty$ and that

$$|K_2(x)| \le C\,|x|^{(1-a/2)/(a-1)}$$

for $|x|$ large. It follows that $K \in C^\infty$ and that (1) holds with $\alpha = 0$ for $a \ge 2$ and $\alpha = (1 - a/2)/(a-1)$ for $1 < a < 2$. Hence Lemma A is proved. □

We shall use the inverse Fourier transform defined by

$$\check{f}(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i\xi x} f(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbf{R}).$$

Now choose $g \in \mathcal{S}(\mathbf{R})$ such that $\operatorname{supp} \check{g} \subset (-1, 1)$, $\check{g}(0) \ne 0$, and set

$$f_v(x) = e^{-ix/v^2} \check{g}(x/v), \quad 0 < v < 1.$$

It follows that $\operatorname{supp} f_v \subset (-v, v)$ and $f_v \in \mathcal{S}(\mathbf{R})$. We shall use the functions f_v to construct the counter-example mentioned in the introduction. We remark that similar functions were used by Dahlberg and Kenig [2]. We need the following lemma, which is essentially contained in [2].

Lemma 1. *One has* $\hat{f}_v(\xi) = v\, g(v\xi + 1/v)$ *for* $0 < v < 1$ *and* $\|f_v\|_{H_s} \le C\, v^{1/2-2s}$ *for* $0 < v < 1$ *and* $0 < s < 1/4$.

In our counter-example we shall use the following estimate.

Lemma 2. *There exist positive numbers* c_0, δ *and* v_0 *such that*

$$\big|S_{xv^{2a-2}/a} f_v(x)\big| \ge c_0$$

for $0 < v < v_0$ *and* $0 < x < \delta$.

Proof. We have $\int g(\xi)\, d\xi \ne 0$ and we choose a large number L such that

$$\int_{|\xi|\ge L} \big|g(\xi)\big|\, d\xi \le \frac{1}{100}\left|\int g(\xi)\, d\xi\right|.$$

Setting $\eta = v\xi + 1/v$ we obtain

$$\begin{aligned} S_t f_v(x) &= \int e^{ix\xi}\, e^{it|\xi|^a}\, v\, g(v\xi + 1/v)\, d\xi \\ &= \int e^{ix(\eta/v - 1/v^2)} e^{it|\eta/v - 1/v^2|^a} g(\eta)\, d\eta = \int e^{iF} g\, d\xi, \end{aligned}$$

where

$$F = F(x,\xi,t,v) = \frac{x}{v}\Big(\xi - \frac{1}{v}\Big) + \frac{t}{v^a}\left|\xi - \frac{1}{v}\right|^a.$$

We now take $v_0 = 1/(2L)$ and v such that $0 < v < v_0$. One has

$$S_t f(x) = \int_{-L}^{L} e^{iF}\, g\, d\xi + \int_{|\xi|\ge L} e^{iF}\, g\, d\xi$$

and

$$\big|S_t f_v(x)\big| \ge \left|\int_{-L}^{L} e^{iF}\, g\, d\xi\right| - \left|\int_{|\xi|\ge L} e^{iF}\, g\, d\xi\right| \ge \left|\int_{-L}^{L} e^{iF}\, g\, d\xi\right| - \frac{1}{100}\left|\int g\, d\xi\right|.$$

For $|\xi| \le L$ we have

$$F = \frac{x}{v}\left(\xi - \frac{1}{v}\right) + \frac{t}{v^a}\left(\frac{1}{v} - \xi\right)^a$$

and using a Taylor expansion one obtains

$$\begin{aligned} \left(\frac{1}{v} - \xi\right)^a &= \frac{1}{v^a}\big(1 - v\xi\big)^a = \frac{1}{v^a}\Big(1 - av\xi + \frac{1}{2}a(a-1)\, v^2\xi^2 + O(v^3|\xi|^3)\Big) \\ &= \frac{1}{v^a} - a\xi v^{1-a} + \frac{1}{2}a(a-1)v^{a-2}\xi^2 + O(v^{3-a}). \end{aligned}$$

Hence

$$F = \frac{x\xi}{v} - \frac{x}{v^2} + \frac{t}{v^{2a}} - a\xi t v^{1-2a} + \frac{1}{2}a(a-1)t v^{2-2a}\xi^2 + O(t v^{3-2a}).$$

Setting $t = xv^{2a-2}/a$ we get

$$
\begin{aligned}
F &= \frac{x\xi}{v} - \frac{x}{v^2} + \frac{x}{av^2} - \xi x v^{2a-2} v^{1-2a} + \frac{1}{2}(a-1) x v^{2a-2} v^{2-2a} \xi^2 + O(xv) \\
&= \frac{x\xi}{v} - \frac{x}{v^2} + \frac{x}{av^2} - \frac{x\xi}{v} + \frac{1}{2}(a-1) x \xi^2 + O(xv)
\end{aligned}
$$

for $x > 0$. It follows that

$$
F = \frac{x}{av^2} - \frac{x}{v^2} + \frac{1}{2}(a-1) x \xi^2 + O(xv)
$$

and hence

$$
\begin{aligned}
\left| \int_{-L}^{L} e^{iF} g \, d\xi \right| &= \left| \int_{-L}^{L} e^{i\frac{1}{2}(a-1)x\xi^2} e^{iO(xv)} g(\xi) \, d\xi \right| \\
&= \left| \int_{-L}^{L} e^{i\frac{1}{2}(a-1)x\xi^2} g(\xi) \, d\xi + \int_{-L}^{L} e^{i\frac{1}{2}(a-1)x\xi^2} \left(e^{iO(xv)} - 1 \right) g(\xi) \, d\xi \right| \\
&\geq \left| \int_{-L}^{L} e^{i\frac{1}{2}(a-1)x\xi^2} g(\xi) \, d\xi \right| - C\, x \geq \frac{1}{2} \left| \int_{-L}^{L} g(\xi) \, d\xi \right|
\end{aligned}
$$

for $0 < x < \delta$ if δ is small.

We conclude that

$$
\begin{aligned}
\left| S_{xv^{2a-2}/a} f_v(x) \right| &\geq \frac{1}{2} \left| \int_{-L}^{L} g \, d\xi \right| - \frac{1}{100} \left| \int g \, d\xi \right| \\
&\geq \frac{1}{2} \left| \int g \, d\xi \right| - \frac{1}{100} \left| \int g \, d\xi \right| - \frac{1}{100} \left| \int g \, d\xi \right| \geq \frac{1}{4} \left| \int g \, d\xi \right|
\end{aligned}
$$

for $0 < v < v_0$ and $0 < x < \delta$. Hence Lemma 2 is proved. □

In the remaining part of this paper δ and v_0 are given by Lemma 2 and we may also assume that $\delta < 1$. We need two more lemmas.

Lemma 3. *For $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, and $\delta/2 < x < \delta$ one has*

$$
\left| S_t f_v(x) \right| \leq C \, \frac{v}{t^\gamma}
$$

where $\gamma = (1+\alpha)/a > 0$.

Proof. Using the estimate in Lemma A we obtain

$$
\left| K_t(y) \right| \leq t^{-1/a} C \left(1 + |t^{-1/a} y|^\alpha \right) \leq C\, t^{-1/a} \left(1 + t^{-\alpha/a} \right) \leq C\, t^{-(1+\alpha)/a}
$$

for $0 < t < 1$ and $|y| \leq 2$.

One has

$$
S_t f_v(x) = \int e^{it|\xi|^a} \hat{f}_v(\xi) \, e^{ix\xi} \, d\xi = \int K_t(y) \, f_v(y+x) \, dy.
$$

If $\delta/2 < x < \delta$ and $|y| \geq 2$, we obtain $|y+x| \geq |y| - |x| \geq 2 - 1 = 1$ and $f_v(y+x) = 0$ and hence

$$
S_t f_v(x) = \int_{|y| \leq 2} K_t(y) \, f_v(y+x) \, dy
$$

for $\delta/2 < x < \delta$. It follows that

$$\begin{aligned}\big|S_t f_v(x)\big| &\le \int_{|y|\le 2} \big|K_t(y)\big|\,\big|f_v(y+x)\big|\,dy \le C\,t^{-(1+\alpha)/a}\int \big|f_v(y)\big|\,dy\\ &= C\,t^{-(1+\alpha)/a}\int \big|\check{g}(y/v)\big|\,dy = C\,\frac{v}{t^{\gamma}}\end{aligned}$$

where $\gamma = (1+\alpha)/a$. □

Lemma 4. *For $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, and $\delta/2 < x < \delta$ one has*

$$\big|S_t f_v(x)\big| \le C\,\frac{t}{v^{\beta}}$$

where $\beta = 2a$.

Proof. We have

$$S_t f_v(x) = \int \big(e^{it|\xi|^a} - 1\big)\,e^{ix\xi}\,\hat{f}_v(\xi)\,d\xi + \int e^{ix\xi}\hat{f}_v(\xi)\,d\xi.$$

The second integral on the above right hand side equals $2\pi f_v(x)$ which vanishes since $x > \delta/2$ and $\operatorname{supp} f_v \subset (-v, v) \subset (-\delta/4, \delta/4)$. Setting $\eta = v\xi$ we obtain

$$\begin{aligned}\big|S_t f_v(x)\big| &\le \int t|\xi|^a\big|\hat{f}_v(\xi)\big|\,d\xi = t\int |\xi|^a v\,\big|g(v\xi + 1/v)\big|\,d\xi\\ &= t\int \Big|\frac{\eta}{v}\Big|^a\,\Big|g\big(\eta + \frac{1}{v}\big)\Big|\,d\eta = \frac{t}{v^a}\int |g(\xi)|\,\Big|\xi - \frac{1}{v}\Big|^a d\xi\\ &\le \frac{t}{v^a}\bigg(C\int |g(\xi)|\,|\xi|^a d\xi + C\int |g(\xi)|\,\frac{1}{v^a}\,d\xi\bigg) \le C\,\frac{t}{v^{2a}},\end{aligned}$$

and the proof of Lemma 4 is complete. □

Now take v_1 such that $0 < v_1 < \min(v_0, \delta/4)$ and set $\varepsilon_k = 2^{-k}$ for $k = 1, 2, 3, \ldots$. Also set

$$v_k = \varepsilon_k\,v_{k-1}^{\mu}, \quad k = 2, 3, 4, \ldots,$$

where

$$\mu = \max\big((2a-2)\gamma,\ \beta/(2a-2)\big).$$

Since $\beta = 2a$ it is clear that $\mu > 1$. By induction we prove that $v_k < 1$ for $k = 1, 2, 3, \ldots$. It follows that $0 < v_k \le \varepsilon_k$, $k = 1, 2, 3, \ldots$.

Also we have $v_k \le \varepsilon_k v_{k-1} \le \frac{1}{2}\,v_{k-1}$ for $k = 2, 3, 4, \ldots$. It follows that

$$\sum_{j=k+1}^{\infty} v_j \le 2v_{k+1}, \quad k = 1, 2, 3, \ldots,$$

and

$$\sum_{j=1}^{k-1} \frac{1}{v_j^{\beta}} \le C\frac{1}{v_{k-1}^{\beta}}, \quad k = 2, 3, 4, \ldots.$$

Now set $f = \sum_{k=1}^{\infty} f_{v_k}$. Then $f \in H_s$ for $s < 1/4$, since

$$\|f\|_{H_s} \le \sum_1^{\infty} \|f_{v_k}\|_{H_s} \le C\sum_1^{\infty} v_k^{1/2-2s} \le C\sum_1^{\infty} \varepsilon_k^{1/2-2s} < \infty$$

for $0 < s < 1/4$. It is clear that $\operatorname{supp} f \subset (-\delta/4, \delta/4)$.

We can now formulate our theorem.

Theorem 1. *Let f be the function we have just constructed. With $t_k = t_k(x) = x\, v_k^{2a-2}/a$ one has*

$$\left|S_{t_k(x)} f(x)\right| \geq c_0/2$$

for $\delta/2 < x < \delta$ and $k \geq k_0$. Here c_0 denotes a positive constant. Hence we do not have $\lim_{t\to 0} S_t f(x) = 0$ in the interval $(\delta/2, \delta)$. Thus we do not have localization or localization almost everywhere for all functions in H_s if $s < 1/4$.

Proof. We have

$$S_{t_k(x)} f(x) = \sum_{j=1}^{\infty} S_{t_k(x)} f_{v_j}(x)$$

and

$$|S_{t_k(x)} f(x)| \geq |S_{t_k(x)} f_{v_k}(x)| - \sum_{j \neq k} |S_{t_k(x)} f_{v_j}(x)|$$

and using Lemma 2 we obtain

$$|S_{t_k(x)} f(x)| \geq c_0 - \sum_{j=1}^{k-1} |S_{t_k(x)} f_{v_j}(x)| - \sum_{j=k+1}^{\infty} |S_{t_k(x)} f_{v_j}(x)|.$$

We shall estimate the two sums on the right hand side for $\delta/2 < x < \delta$. For $j \geq k+1$ we have

$$|S_{t_k(x)} f_{v_j}(x)| \leq C \frac{v_j}{(t_k(x))^{\gamma}}$$

according to Lemma 3. Hence

$$|S_{t_k(x)} f_{v_j}(x)| \leq C \frac{v_j}{(x v_k^{2a-2})^{\gamma}} \leq C \frac{v_j}{v_k^{(2a-2)\gamma}}$$

and

$$\sum_{j=k+1}^{\infty} |S_{t_k(x)} f_{v_j}(x)| \leq C \frac{1}{v_k^{(2a-2)\gamma}} \sum_{j=k+1}^{\infty} v_j \leq C \frac{v_{k+1}}{v_k^{(2a-2)\gamma}}.$$

Since $\mu \geq (2a-2)\gamma$ we have $v_{k+1} \leq \varepsilon_{k+1} v_k^{(2a-2)\gamma}$ and hence

$$\sum_{j=k+1}^{\infty} |S_{t_k(x)} f_{v_j}(x)| \leq C \varepsilon_{k+1}.$$

For $1 \leq j \leq k-1$ we have

$$|S_{t_k(x)} f_{v_j}(x)| \leq C \frac{t_k(x)}{v_j^{\beta}} \leq C \frac{v_k^{2a-2}}{v_j^{\beta}}$$

according to Lemma 4. It follows that

$$\sum_{j=1}^{k-1} |S_{t_k(x)} f_{v_j}(x)| \leq C v_k^{2a-2} \sum_{j=1}^{k-1} \frac{1}{v_j^{\beta}} \leq C v_k^{2a-2} \frac{1}{v_{k-1}^{\beta}}.$$

Since $\mu \geq \beta/(2a-2)$ we obtain

$$v_k \leq \varepsilon_k v_{k-1}^{\beta/(2a-2)}$$

and

$$v_k^{2a-2} \leq \varepsilon_k^{2a-2} v_{k-1}^{\beta}.$$

We conclude that

$$\sum_{j=1}^{k-1} |S_{t_k(x)} f_{v_j}(x)| \le C\varepsilon_k^{2a-2}.$$

Thus for $k \ge k_0$ one obtains

$$|S_{t_k(x)} f(x)| \ge c_0/2$$

for $\delta/2 < x < \delta$ and the proof of the theorem is complete. □

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