Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 38, 2013, 149–180

MODULUS METHOD AND RADIAL STRETCH MAP IN THE HEISENBERG GROUP

Zoltán M. Balogh, Katrin Fässler and Ioannis D. Platis

Universität Bern, Mathematisches Institut
Sidlerstrasse 5, 3012 Bern, Switzerland; zoltan.balogh@math.unibe.ch

Universität Bern, Mathematisches Institut
Sidlerstrasse 5, 3012 Bern, Switzerland, and
University of Helsinki, Department of Mathematics and Statistics
P.B. 68, FI-00014 University of Helsinki, Finland; katrin.fassler@helsinki.fi

University of Crete, Department of Mathematics
Knossos Avenue, 71409 Heraklion, Crete, Greece; jplatis@math.uoc.gr

Dedicated to Hans Martin Reimann on the occasion of his 70th birthday.

Abstract. We propose a method by modulus of curve families to identify extremal quasiconformal mappings in the Heisenberg group. This approach allows to study minimizers not only for the maximal distortion but also for a mean distortion functional, where the candidate for the extremal map is not required to have constant distortion. As a counterpart of a classical Euclidean extremal problem, we consider the class of quasiconformal mappings between two spherical annuli in the Heisenberg group. Using logarithmic-type coordinates we can define an analog of the classical Euclidean radial stretch map and discuss its extremal properties both with respect to the maximal and the mean distortion. We prove that our stretch map is a minimizer for a mean distortion functional and it minimizes the maximal distortion within the smaller subclass of sphere-preserving mappings.

1. Introduction and statement of results

Modulus of curve families is a classical tool to study properties of quasiconformal mappings in the complex plane and in more general metric spaces. Various versions of the modulus method have been used to identify *extremal quasiconformal* mappings minimizing the maximal distortion within a certain class of mappings in the complex plane or on Riemann surfaces [Grö32, Tei40, Tei40, Str86]. It turns out that this approach can be also applied to study *minimizers for a mean distortion functional* [BFP11] in the class of mappings between annuli in the complex plane. Minimization problems concerning the mean distortion functional have been studied also with other methods in the Euclidean setting for instance in [GM01, AIMO05, Mar09, AIM09]. On the other hand, the method of modulus of curve families seems to be robust enough to work in more general non-Euclidean settings. In the first part of the paper

doi:10.5186/aasfm.2013.3811
2010 Mathematics Subject Classification: Primary 30L10, 30C75.
Key words: Heisenberg group, extremal quasiconformal mappings, modulus of curve families, logarithmic coordinates.
Z.M.B. and K.F. were supported by the Swiss National Science Foundation, the European Science Foundation Project HCAA and the FP7 EU Commission Project CG-DICE. I.D.P. was partly supported by Special Fund for Innovation and Research, (ELKE).

we propose a general version of the modulus method in the sub-Riemannian setting of the first Heisenberg group that can be applied to study extremal quasiconformal mappings minimizing the maximal or the mean distortion functional. The idea is to find a mapping which has the "minimal stretching property" (MSP) for a given curve family which is related to the domain under consideration. A map is said to have the minimal stretching property for a curve family if the largest shrinking of the map is achieved in the direction of a vector field tangential to the curve family. In the second part of the paper we show that this device can be successfully applied to study extremal quasiconformal mappings between spherical rings of the Heisenberg group. As a result we obtain an extremal stretch map minimizing the mean distortion in the class of all quasiconformal mappings between two given spherical rings.

To state our results we recall some notations and preliminary facts about quasiconformal mappings in the Heisenberg group. For details we refer to [KR85, KR95] and Pansu [Pan89b]. In our model the *first Heisenberg group* $\mathbf{H}^1$ is $\mathbf{C}\times\mathbf{R}$ with the group law

$$(z,t)*(z',t')=(z+z',t+t'+2\,\mathrm{Im}(z\bar{z}')),$$

and it is endowed with a natural left-invariant metric (called *Heisenberg distance*)

$$d_H(p,q):=\|p^{-1}*q\|_H,\quad p,q\in\mathbf{H}^1, \tag{1}$$

for $p=(z,t)$ and $q=(z',t')$, where $\|(z,t)\|_H:=(|z|^4+t^2)^{\frac14}$ is the Heisenberg norm. A purely *metric definition* of quasiconformality (in the sense of Mostow [Mos73]) on the Heisenberg group can be given in terms of the Heisenberg distance as follows. Given two domains Ω,Ω' in $\mathbf{H}^1$, a homeomorphism $f\colon\Omega\to\Omega'$ is called *quasiconformal* if

$$H(p,f):=\limsup_{r\downarrow 0}H(p,r,f):=\limsup_{r\downarrow 0}\frac{\max_{d_H(p,q)=r}d_H(f(p),f(q))}{\min_{d_H(p,q)=r}d_H(f(p),f(q))},\quad p\in\Omega, \tag{2}$$

is uniformly bounded from above.

Analogously to the complex plane, there are equivalent analytic definitions for quasiconformal mappings in $\mathbf{H}^1$, besides the metric one, as proved in [Pan89b, KR95, Vod96], see Section 5.4.2 in the Appendix for more details. In addition to Sobolev regularity properties, quasiconformal mappings satisfy the so-called contact condition as they preserve the contact 1-form $\tau=\mathrm{d}t+2\,\mathrm{Im}\,\bar{z}\,\mathrm{d}z$, i.e., $f^*\tau=\lambda\tau$ almost everywhere for some non-vanishing function λ. This implies restrictions imposed on quasiconformal mappings in the Heisenberg group, but it has the advantage that their properties are determined by the behavior of the mapping on its projection to the complex plane. To be more precise, let $f=(f_1+\mathrm{i}f_2,f_3)$ be a quasiconformal mapping between domains in the Heisenberg group, and let $f_I=f_1+\mathrm{i}f_2$. It turns out that the horizontal derivatives

$$Zf_I\quad\text{and}\quad\overline{Z}f_I$$

exist both in the sense of distributions and almost everywhere. Here, f_I is differentiated with respect to the so-called complex horizontal vector fields

$$Z=\tfrac{\partial}{\partial z}+\mathrm{i}\bar{z}\tfrac{\partial}{\partial t}\quad\text{and}\quad\overline{Z}=\tfrac{\partial}{\partial\bar{z}}-\mathrm{i}z\tfrac{\partial}{\partial t}.$$

In analogy to the complex case, the above metric definition implies for an orientation preserving quasiconformal map f, that there exists a measurable complex-valued function μ with $\|\mu\|_\infty<1$ such that

$$\overline{Z}f_I=\mu Zf_I\quad\text{and}\quad\overline{Z}f_{II}=\mu Zf_{II},\ \text{almost everywhere},$$

with $f_{II} = f_3 + \mathrm{i}|f_I|^2$, see [KR85, KR95]. We can define the Beltrami coefficient and the distortion quotient as

$$\mu_f(z,t) := \frac{\overline{Z} f_I(z,t)}{Z f_I(z,t)} \quad \text{and} \quad K((z,t),f) := \frac{|Z f_I(z,t)| + |\overline{Z} f_I(z,t)|}{|Z f_I(z,t)| - |\overline{Z} f_I(z,t)|}$$

for points (z,t) where these expressions exist. Moreover, we set

$$\|\mu_f\|_\infty := \operatorname{ess\,sup}_{(z,t)} |\mu_f(z,t)| \quad \text{and} \quad K_f := \operatorname{ess\,sup}_{(z,t)} K((z,t),f).$$

It is easy to see that μ_f and K_f are related in the following way

$$K((z,t),f) = \frac{1 + |\mu_f(z,t)|}{1 - |\mu_f(z,t)|}, \quad K_f = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

We work mostly with the distortion

$$K((z,t),f)^2 := \frac{(|Z f_I(z,t)| + |\overline{Z} f_I(z,t)|)^2}{(|Z f_I(z,t)| - |\overline{Z} f_I(z,t)|)^2},$$

which can be used also for orientation reversing mappings. The magnitude of K_f^2 is understood to be the deviation of f from conformality. For a quasiconformal map f we have $1 \le K_f^2 < \infty$. Any smooth contact transformation with $1 \le K_f^2 < \infty$ is quasiconformal.

Given two domains Ω and Ω' our main question is to seek within a class $\mathcal{F}$ of quasiconformal maps (possibly subject to certain boundary conditions), mappings which are "as conformal as possible". We say that a mapping f_0 is *extremal with respect to the maximal distortion* in a given class $\mathcal{F}$ of quasiconformal mappings if $f_0 \in \mathcal{F}$ and

$$K_{f_0}^2 = \min_{f \in \mathcal{F}} K_f^2.$$

Similarly, we say that f_0 *is extremal for a mean distortion functional* if

$$\fint_\Omega K(p, f_0)^2 \rho_0(p)^4 \, \mathrm{d}\mathcal{L}^3(p) = \min_{f \in \mathcal{F}} \fint_\Omega K(p, f)^2 \rho_0(p)^4 \, \mathrm{d}\mathcal{L}^3(p), \tag{3}$$

for a certain density ρ_0 which corresponds to the geometry of the domain Ω. Here $\mathcal{L}^3$ is the Lebesgue measure on $\mathbf{R}^3$ which yields a bi-invariant Haar measure on $\mathbf{H}^1$.

By standard normal family arguments, using [KR95, Theorem F], it follows under appropriate conditions on $\mathcal{F}$ that a minimizer exists for the maximal distortion but there is no general method to determine such a minimizer in concrete situations. The issue for the second problem (3) is more difficult, as there are examples of non-existence of the minimizer even in the Euclidean case, see [Mar09]. The purpose of the paper is to give existence results for the minimizers of maximal and mean distortion as well as criteria on how to prove that a certain candidate map is a minimizer by using the modulus of curve families.

In order to state our main results we shall briefly introduce some terminology that is going to be developed in more detail in later parts of the paper.

The first step is to choose an appropriate family Γ of rectifiable curves foliating the domain Ω. Here rectifiability is understood in terms of the Heisenberg metric defined above. It implies that the tangents of the curves are in the space spanned by $\operatorname{Re} Z$ and $\operatorname{Im} Z$ almost everywhere. Usually, we are working with a parameterization of the curve which is absolutely continuous. Such a curve is called *horizontal*. For $\gamma(s) = (\gamma_I(s), \gamma_3(s)) \in \mathbf{C} \times \mathbf{R}$, $s \in [a,b]$, this condition can be written explicitly as

$$\dot{\gamma}_3(s) = \mathrm{i}(\overline{\gamma}_I(s)\dot{\gamma}_I(s) - \gamma_I(s)\dot{\overline{\gamma}}_I(s)) = -2\,\mathrm{Im}(\overline{\gamma}_I(s)\dot{\gamma}_I(s)), \text{ for a.e. } s \in [a,b]. \tag{4}$$

The Heisenberg length of a horizontal curve γ coincides with the usual Euclidean length of its projection γ_I on $\mathbf{C}$. We denote by $\mathrm{adm}(\Gamma)$ the set of non-negative Borel functions $\rho\colon \mathbf{H}^1 \to [0,\infty]$ such that $\int_\gamma \rho\,\mathrm{d}\ell \geq 1$ for all rectifiable $\gamma \in \Gamma$. For a horizontal curve $\gamma\colon [a,b] \to \mathbf{H}^1$, the curve integral with respect to arc-length is given by $\int_\gamma \rho\,\mathrm{d}\ell = \int_a^b \rho(\gamma(s))|\dot{\gamma}_I(s)|\,\mathrm{d}s$.

The conformally invariant 4-modulus of a family Γ of curves in $\mathbf{H}^1$ is defined by

$$M_4(\Gamma) = \inf_{\rho\in\mathrm{adm}(\Gamma)} \int_{\mathbf{H}^1} \rho(p)^4\,\mathrm{d}\mathcal{L}^3(p). \tag{5}$$

Modulus and mean distortion are connected by the following statement. Let $f\colon \Omega \to \Omega'$ be a quasiconformal map between domains Ω, Ω' in $\mathbf{H}^1$. For any family Γ of horizontal curves in Ω we have

$$M_4(f(\Gamma)) \leq \int_\Omega K(p,f)^2 \rho^4(p)\,\mathrm{d}\mathcal{L}^3(p) \quad \text{for all } \rho \in \mathrm{adm}(\Gamma). \tag{6}$$

Here and in the following, we denote by $f(\Gamma) := \{f \circ \gamma \colon \gamma \in \Gamma\}$ the f-image of a given family of curves Γ. In what follows we fix an extremal density ρ_0 for which the infimum in the definition of $M_4(\Gamma)$ is attained. In order to identify a minimizer f_0 for the mean value functional of the distortion we will require that f_0 has the minimal stretching property for a subfamily of curves Γ_0. A $\mathcal{C}^1$ orientation preserving quasiconformal map $f_0\colon \Omega \to \mathbf{H}^1$ has the *minimal stretching property (MSP)* for a family Γ_0 of $\mathcal{C}^1$ horizontal curves in a domain Ω if for all $\gamma \in \Gamma_0$, $\gamma\colon [a,b] \to \Omega$, one has

$$\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma}}_I(s)}{\dot{\gamma}_I(s)} < 0 \quad \text{for all } s \in [a,b].$$

Note that in this definition we require in particular that the expression $\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma}}_I(s)}{\dot{\gamma}_I(s)}$ is real-valued. Suppose that Λ is a domain in $\mathbf{R}^2$. Let $0 < A < B$ and let

$$\gamma\colon (A,B) \times \Lambda \to \Omega$$

be a diffeomorphism which foliates a bounded domain Ω in the Heisenberg group with the property that

$$\gamma(\cdot,\lambda)\colon (A,B) \to \Omega$$

is a horizontal curve with $|\dot{\gamma}_I(s,\lambda)| \neq 0$ for every $\lambda \in \Lambda$ and

$$\mathrm{d}\mathcal{L}^3(\gamma(s,\lambda)) = |\dot{\gamma}_I(s,\lambda)|^4\,\mathrm{d}s\,\mathrm{d}\mu(\lambda)$$

for a measure μ on Λ. Then, it turns out that

$$\rho_0(p) = \begin{cases} \left((B-A)\cdot|\dot{\gamma}_I(\gamma^{-1}(p))|\right)^{-1}, & p = \gamma(s,\lambda) \in \Omega, \\ 0, & p \notin \Omega, \end{cases}$$

is extremal for the curve family

$$\Gamma_0 = \{\gamma(\cdot,\lambda)\colon \lambda \in \Lambda\}$$

with

$$M_4(\Gamma_0) = \frac{1}{(B-A)^3} \int_\Lambda \mathrm{d}\mu(\lambda).$$

Let $f_0\colon \Omega \to \Omega'$ be an orientation preserving quasiconformal diffeomorphism between domains in the Heisenberg group. Let γ be a foliation of Ω as described above. Assume further that f_0 has the MSP for

$$\Gamma_0 = \{\gamma(\cdot, \lambda)\colon \lambda \in \Lambda\}$$

and that the distortion of f_0 is constant along every curve γ, i.e.,

$$K(\gamma(s,\lambda), f_0) \equiv K_{f_0}(\lambda)$$

for all $(s,\lambda) \in (A,B) \times \Lambda$. The main result from the first part of the paper is the following condition for extremality of the mean distortion integral.

Theorem 1. *Assume that Γ_0, ρ_0 and f_0 satisfy the above properties. Let $\Gamma \supseteq \Gamma_0$ be a curve family in Ω such that $\rho_0 \in \operatorname{adm}(\Gamma)$ and let $\mathcal{F}$ be the class of all quasiconformal maps f from Ω to Ω' with the property that*

$$M_4(f_0(\Gamma_0)) \le M_4(f(\Gamma)).$$

Then

$$\int_\Omega K(p, f_0)^2 \rho_0^4(p)\, \mathrm{d}\mathcal{L}^3(p) \le \int_\Omega K(p,f)^2 \rho_0^4(p)\, \mathrm{d}\mathcal{L}^3(p)$$

for all $f \in \mathcal{F}$.

The above statement is well suited for applications in order to check extremality of certain quasiconformal mappings. Typically one chooses for Γ_0 an explicitly given family of curves and Γ is a larger family in the same domain with the same modulus. In the second part of the paper we consider the case of spherical rings on the Heisenberg group of the form

$$A(a,b) := \{p \in \mathbf{H}^1\colon a < \|p\|_H < b\} \quad (0 < a < b).$$

This case is important since it is expected that the study of mappings of annuli gives information on the optimal Sobolev and Hölder exponents for quasiconformal mappings, similarly as in the Euclidean setting. In the complex plane, the modulus of annuli has also been used for instance to study conformality at a point [Bra10].

Let Γ be the family of all horizontal curves connecting the two boundary components of $A(a,b)$. It was proved in [KR87] that

$$M_4(\Gamma) = \pi^2 \left(\log\left(\frac{b}{a}\right)\right)^{-3},$$

with the extremal density $\rho_0(z,t) = (\log \frac{b}{a})^{-1} \frac{|z|}{\sqrt{|z|^4+t^2}}$ for $(z,t) \in A(a,b)$.

To exploit the spherical symmetry of $A(a,b)$ it is helpful to use the coordinates introduced in [Pla09]. The idea is to generalize logarithmic coordinates from the complex plane to the Heisenberg setting. One starts with the usual spherical coordinates of the Heisenberg group:

$$(z,t) = (r\cos^{1/2}\theta e^{i\phi}, r^2 \sin\theta), \quad (r,\theta,\phi) \in [0,+\infty) \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times [0, 2\pi),$$

as introduced in [KR87]. Consider now the transformation defined by the equations

$$\xi = 2\log r, \quad \psi = -\theta, \quad \eta = \frac{1}{3}(\pi - \theta - 2\phi).$$

We have $\xi \in \mathbf{R}$, $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi - 3\pi \leq 3\eta < \psi + \pi$. The result is the parameterization of the Heisenberg group by *logarithmic coordinates* (ξ, ψ, η) given by

$$(z,t) = \left(\mathrm{i}\cos^{1/2}\psi \cdot e^{\frac{\xi + i(\psi - 3\eta)}{2}}, -\sin\psi e^{\xi}\right).$$

Using the notations before Theorem 1, set $A = \log a^2$, $B = \log b^2$, $\Lambda = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{2\pi}{3}, \frac{2\pi}{3})$, $\Omega = A(a,b)$ and consider the foliation $\gamma\colon (A,B) \times \Lambda \to \Omega$ of the spherical ring Ω, given in logarithmic coordinates by

$$\gamma(s, (\psi, \eta)) = (s, \psi, \eta - \tfrac{\tan\psi}{3} s).$$

The mapping f_0 will be our extremal map $f_k\colon A(a,b) \to A(a^k, b^k)$, $0 < k < 1$, *the stretch*, which, in logarithmic coordinates, reads as

$$(\xi, \psi, \eta) \mapsto (k\xi, \tan^{-1}(\tfrac{\tan\psi}{k}), \eta).$$

The similarity between the planar and the Heisenberg stretch becomes clear if we write $z \mapsto z|z|^{k-1}$ in planar logarithmic coordinates. The map f_k appeared implicitly already in [Min94, p. 88] in a different context. Notice that rotation in the vertical direction ψ is enforced by the contact property of the map; the obvious "stretch" $(\xi, \psi, \eta) \mapsto (k\xi, \psi, \eta)$ is *not* a contact map. To indicate the complexity of f_k, note that in cartesian coordinates, it is given by

$$f_k(z,t) = \left(k^{\frac{1}{2}} z \left(\frac{|z|^2 + \mathrm{i}t}{k|z|^2 + \mathrm{i}t}\right)^{1/2} ||z|^2 - \mathrm{i}t|^{\frac{k-1}{2}}, t \cdot \frac{||z|^2 - \mathrm{i}t|^k}{|k|z|^2 - \mathrm{i}t|}\right).$$

For $t = 0$, we recover the classical radial stretch, $f_k(z,0) = (z|z|^{k-1}, 0)$. If we formally substitute the exponent k by -1 in the formula for the planar stretch map, we obtain the inversion $z \mapsto z/|z|^2$. A similar phenomenon occurs with the Heisenberg stretch f_k. Starting from $(\xi, \psi, \eta) \mapsto (-\xi, -\psi, \eta)$, we obtain

$$f_{-1}(z,t) = \left(\frac{z}{|z|^2 - \mathrm{i}t}, \frac{-t}{|z|^4 + t^2}\right),$$

which is a conformal inversion in the Korányi unit sphere. We observe that the inversion given in [KR85, p. 315] can be obtained as a composition of the map f_{-1} with a rotation about the t-axis by π.

To formulate the main result of the second part of the paper let us denote by $\mathcal{F}$ the class of all quasiconformal maps $A(a,b) \to A(a^k, b^k)$ which map homeomorphically the inner and outer boundary of $A(a,b)$ to the respective boundary components of $A(a^k, b^k)$.

Theorem 2. *For any $0 < k < 1$, the stretch f_k is an orientation preserving quasiconformal map from $A(a,b)$ to $A(a^k, b^k)$ with maximal distortion $K_{f_k} = \frac{1}{k^2}$. It minimizes the mean distortion within the class $\mathcal{F}$: for all $f \in \mathcal{F}$ we have that*

$$\int_{A(a,b)} K(p, f_k)^2 \rho_0(p)^4 \,\mathrm{d}\mathcal{L}^3(p) \leq \int_{A(a,b)} K(p, f)^2 \rho_0(p)^4 \,\mathrm{d}\mathcal{L}^3(p).$$

An interesting feature of f_k is that it minimizes the maximal distortion in the more restricted class $\mathcal{F}_0$ of sphere preserving quasiconformal mappings that are $\mathcal{C}^1$ smooth and preserve the t-axis. We do not know whether f_k is extremal for the maximal distortion also within the larger class $\mathcal{F}$.

The classical radial stretch map in the complex plane is not only a solution for certain distortion minimization problems, it also proves the sharpness of Astala's result on the optimal Sobolev exponent for K-quasiconformal mappings. We conjecture that the above defined Heisenberg stretch map plays a similar role. A qualitative version of Astala's theorem in the spirit of Gehring's results for Euclidean spaces also holds on the Heisenberg group, i.e., a quasiconformal map f on $\mathbf{H}^1$ lies in $HW^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{H}^1)$ for an exponent $p > 4$. We refer to Section 5.4.2 in the Appendix for the definition of the *horizontal Sobolev space* $HW^{1,p}_{\mathrm{loc}}$. According to the analytic definition stated there, a K-*quasiconformal* map f on $\mathbf{H}^1$ belongs to the horizontal Sobolev space $HW^{1,4}_{\mathrm{loc}}$ and its formal P-differential satisfies $\|D_H f\|^4 \leq KJ(\cdot, f)$ almost everywhere. We denote

$$p(\mathbf{H}^1, K) = \sup\big\{p \geq 1 \colon f \in HW^{1,p}_{\mathrm{loc}}(\mathbf{H}^1, \mathbf{H}^1) \text{ for every } K\text{-quasiconformal} \\ \text{map } f \colon \mathbf{H}^1 \to \mathbf{H}^1\big\},$$

Using the example of the stretch f_k, one can show that $p(\mathbf{H}^1, K) \leq \frac{4K^{\frac14}}{K^{\frac14}-1}$, see Section 5.

Conjecture. We conjecture that the Heisenberg radial stretch map f_k yields the optimal degree of regularity for quasiconformal mappings on the Heisenberg group, that is

$$p(\mathbf{H}^1, K) = \frac{4K^{\frac14}}{K^{\frac14} - 1}.$$

The paper is organized as follows: in Section 2 we discuss the method of curve families and prove Theorem 1. In Section 3 we consider logarithmic coordinates and we develop the analytic quasiconformal machinery in these coordinates. Section 4 is devoted to the stretch map: we shall prove Theorem 2 and the extremality of the stretch map in the class of spheres preserving maps. Section 5 is for final remarks and open questions. Background results concerning Heisenberg geometry and the theory of quasiconformal mappings are collected in an appendix at the end of the paper.

Acknowledgements. Part of this work has been carried out while I.D.P. was visiting the Mathematics Institute of the University of Bern. The hospitality of the institute is gratefully appreciated. We would like to thank Jeremy Tyson for helpful discussions on the subject of this paper and the referee for carefully reading the paper and for his/her comments.

2. Conditions for extremality

As we stressed in the introduction, one of the main goals of this paper is to describe a method by which one can identify mappings which minimize the maximal or a mean distortion functional within a class $\mathcal{F}$ of quasiconformal maps between two domains Ω and Ω' in $\mathbf{H}^1$ (possibly subject to certain boundary conditions). In this section we develop the modulus method to identify such extremal quasiconformal mappings. The modulus inequality (6) yields a lower bound for the distortion. This idea goes back to the early examples by Grötzsch. More recently, it was applied in [Tan96] in order to prove that a certain quasiconformal mapping between CR 3-manifolds is a minimizer for the maximal distortion in its homotopy class.

Definition 3. We say that a $\mathcal{C}^1$ smooth orientation preserving quasiconformal map $f_0\colon \Omega \to \Omega'$ between domains in $\mathbf{H}^1$ has the *minimal stretching property (MSP)* for a family Γ_0 of $\mathcal{C}^1$ horizontal curves in Ω if for all $\gamma \in \Gamma_0$, $\gamma\colon [a,b] \to \mathbf{H}^1$, one has

$$\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma}}_I(s)}{\dot{\gamma}_I(s)} < 0 \quad \text{for all } s \in [a,b] \text{ with } \mu_{f_0}(\gamma(s)) \neq 0. \tag{7}$$

Using the corresponding terminology in the complex plane, we observe that a Teichmüller mapping, i.e., an extremal quasiconformal mapping f^μ with Beltrami coefficient $\mu = k\frac{|\varphi|}{\varphi}$ where φ is a quadratic differential, has the "minimal stretching property" for the vertical trajectories of φ, see [GL00].

If a map f_0 has the MSP for a curve family Γ_0, this means geometrically that Γ_0 consists of curves which are tangential to the direction of the least stretching/largest shrinking of f_0. To make this precise, we observe the following result which follows easily from the chain rule and the definition of a horizontal curve (see Appendix).

Lemma 4. *Let $f\colon \Omega \to \mathbf{H}^1$ be a $\mathcal{C}^1$ map on a domain $\Omega \subset \mathbf{H}^1$ and let also $\gamma\colon [a,b] \to \Omega$ be a horizontal curve. Then*

$$(f_I \dot{\circ} \gamma)(s) = Zf_I(\gamma(s))\dot{\gamma}_I(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma}}_I(s) \quad \text{a.e. } s \in [a,b].$$

Then, for an orientation preserving $\mathcal{C}^1$ quasiconformal map f it follows

$$(|Zf_I(\gamma(s))| - |\overline{Z}f_I(\gamma(s))|)|\dot{\gamma}_I(s)| \le |(f_I \dot{\circ} \gamma)(s)| \le (|Zf_I(\gamma(s))| + |\overline{Z}f_I(\gamma(s))|)|\dot{\gamma}_I(s)|$$

for almost every s. If a map f_0 has the MSP for a family Γ_0, then by (7) we have equality

$$(|Z(f_0)_I(\gamma(s))| - |\overline{Z}(f_0)_I(\gamma(s))|)|\dot{\gamma}_I(s)| = |((f_0)_I \dot{\circ} \gamma)(s)|.$$

2.1. Mappings with constant distortion: minimization of the maximal distortion. To illustrate the method of mappings with MSP and modulus of curve families, we first prove a result for mappings with constant distortion.

Proposition 5. *If an orientation preserving quasiconformal $\mathcal{C}^1$ diffeomorphism $f_0\colon \Omega \to \Omega'$ has the MSP for a family Γ_0 of $\mathcal{C}^1$ horizontal curves, then*

$$M_4(f_0(\Gamma_0)) = \inf_{\rho \in \mathrm{adm}(\Gamma_0)} \int_\Omega \rho^4(z,t) K((z,t),f_0)^2 \, \mathrm{d}\mathcal{L}^3(z,t).$$

If we assume in addition that the distortion $K((z,t),f_0) \equiv K_{f_0}$ is constant, then

$$M_4(f_0(\Gamma_0)) = K_{f_0}^2 M_4(\Gamma_0), \tag{8}$$

Here and in the following, we write

$$\int h(z,t)\, \mathrm{d}\mathcal{L}^3(z,t) := \int h(x+\mathrm{i}y,t)\, \mathrm{d}\mathcal{L}^3(x,y,t).$$

for a function $h\colon \mathbf{C} \times \mathbf{R} \to \mathbf{C}$.

Proof outline. The proof of the modulus inequality

$$M_4(f_0(\Gamma_0)) \le \int_\Omega \rho^4(z,t) K((z,t),f_0)^2 \, \mathrm{d}\mathcal{L}^3(z,t)$$

for all $\rho \in \mathrm{adm}(\Gamma_0)$ goes along the lines of the classical proof in the smooth case, see [Ahl66], using a change of variables formula for quasiconformal mappings on $\mathbf{H}^1$. Equality follows then from the fact that f_0 has the MSP for Γ_0.

To each density $\rho \in \mathrm{adm}(\Gamma_0)$, one assigns a push-forward density

$$\rho'(\zeta,\tau) := \begin{cases} \frac{\rho}{|Z(f_0)_I| - |\overline{Z}(f_0)_I|} \circ f_0^{-1}(\zeta,\tau), & \text{for } (\zeta,\tau) \in \Omega', \\ 0, & \text{elsewhere.} \end{cases}$$

Since f_0 has the MSP for Γ_0 we have

$$\int_{f_0\circ\gamma} \rho' \,\mathrm{d}\ell = \int_a^b \frac{\rho(\gamma(s))}{|Z(f_0)_I(\gamma(s))| - |\overline{Z}(f_0)_I(\gamma((s))|} (|Z(f_0)_I(\gamma(s))| \\ - |\overline{Z}(f_0)_I(\gamma(s))|)|\dot{\gamma}_I(s)| \,\mathrm{d}s = \int_\gamma \rho \,\mathrm{d}\ell$$

for all $\gamma \in \Gamma_0$. This shows that $\{\rho' \colon \rho \in \mathrm{adm}(\Gamma_0)\} = \mathrm{adm}(f_0(\Gamma_0))$. Using the change of variables formula from Section 5.4.2 in the Appendix with

$$J(\cdot, f_0) = (|Z(f_0)_I|^2 - |\overline{Z}(f_0)_I|^2)^2,$$

we compute

$$\int_{\Omega'} \rho'^4(\zeta,\tau) \,\mathrm{d}\mathcal{L}^3(\zeta,\tau) = \int_\Omega \rho^4(z,t) K((z,t),f_0)^2 \,\mathrm{d}\mathcal{L}^3(z,t).$$

We can now conclude the proof as follows:

$$M_4(f_0(\Gamma)) = \inf_{\widetilde{\rho}\in\mathrm{adm}(f_0(\Gamma))} \int_{\Omega'} \widetilde{\rho}^4(\zeta,\tau) \,\mathrm{d}\mathcal{L}^3(\zeta,\tau) = \inf_{\rho\in\mathrm{adm}(\Gamma)} \int_{\Omega'} \rho'^4(\zeta,\tau) \,\mathrm{d}\mathcal{L}^3(\zeta,\tau) \\ = \inf_{\rho\in\mathrm{adm}(\Gamma)} \int_\Omega \rho^4(z,t) K((z,t),f_0)^2 \,\mathrm{d}\mathcal{L}^3(z,t). \qquad \square$$

This fact, together with the inequality which holds for arbitrary quasiconformal mappings, can be used in certain situations to identify a candidate mapping as a distortion minimizer. An example is given in Section 5.1. The corresponding approach for planar quasiconformal mappings is described in more detail in [BFP11]. However, in the Heisenberg group, already in the case of very basic extremal problems, it is hard to find a candidate mapping with a constant distortion.

2.2. Mappings with non-constant distortion: minimization of the mean distortion. Proof of Theorem 1. We shall now relax the above condition by requiring that the distortion of the candidate mapping is constant only along the trajectories of a given foliation of the domain.

Proposition 6. *Suppose that Λ is a domain in $\mathbf{R}^2$. Let $0 < A < B$ and*

$$\gamma \colon (A,B) \times \Lambda \to \Omega$$

be a diffeomorphism which foliates a bounded domain Ω in the Heisenberg group with the property that

$$\gamma(\cdot,\lambda) \colon [A,B] \to \overline{\Omega}$$

is a horizontal curve with $|\dot{\gamma}_I(s,\lambda)| \neq 0$ for every $\lambda \in \Lambda$ and

$$\mathrm{d}\mathcal{L}^3(\gamma(s,\lambda)) = |\dot{\gamma}_I(s,\lambda)|^4 \,\mathrm{d}s \,\mathrm{d}\mu(\lambda)$$

for a measure μ on Λ. Then,

$$\rho_0(p) = \begin{cases} ((B-A)\cdot|\dot{\gamma}_I(\gamma^{-1}(p))|)^{-1}, & p = \gamma(s,\lambda) \in \Omega, \\ 0, & p \notin \Omega, \end{cases}$$

is an extremal density for the curve family

$$\Gamma_0 = \{\gamma(\cdot,\lambda)\colon \lambda \in \Lambda\}$$

with

$$M_4(\Gamma_0) = \frac{1}{(B-A)^3}\int_\Lambda \mathrm{d}\mu(\lambda).$$

Here, $\dot\gamma(s,\lambda)$ is $\frac{\partial}{\partial s}\gamma(s,\lambda)$.

Proof. The density ρ_0 is admissible for Γ_0 since for any $\gamma(\cdot,\lambda) \in \Gamma_0$ (which is a horizontal curve by assumption), we have

$$\int_{\gamma(\cdot,\lambda)} \rho_0 \,\mathrm{d}\ell = \int_A^B \rho_0(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|\,\mathrm{d}s = 1.$$

A direct computation yields

$$\int_\Omega \rho_0^4(p)\,\mathrm{d}\mathcal{L}^3(p) = \int_\Lambda\int_A^B \rho_0^4(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|^4\,\mathrm{d}s\,\mathrm{d}\mu(\lambda) = \frac{1}{(B-A)^3}\int_\Lambda \mathrm{d}\mu(\lambda).$$

Therefore,

$$M_4(\Gamma_0) \le \frac{1}{(B-A)^3}\int_\Lambda \mathrm{d}\mu(\lambda).$$

For the reverse inequality, consider an arbitrary density $\rho \in \mathrm{adm}(\Gamma_0)$. Because it is admissible for the curve family Γ_0, by applying Hölder's inequality, one obtains

$$\begin{aligned} 1 \le \int_{\gamma(\cdot,\lambda)} \rho\,\mathrm{d}\ell &= \int_A^B \rho(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|\,\mathrm{d}s \\ &\le \left(\int_A^B \rho^4(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|^4\,\mathrm{d}s\right)^{\frac14}(B-A)^{\frac34} \end{aligned}$$

for every $\lambda \in \Lambda$. Therefore,

$$\frac{1}{(B-A)^3} \le \int_A^B \rho^4(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|^4\,\mathrm{d}s.$$

Integrating both sides of the inequality with respect to $\mathrm{d}\mu$ over Λ yields

$$\frac{1}{(B-A)^3}\int_\Lambda \mathrm{d}\mu(\lambda) \le \int_\Lambda\int_A^B \rho^4(\gamma(s,\lambda))|\dot\gamma_I(s,\lambda)|^4\,\mathrm{d}s\,\mathrm{d}\mu(\lambda) = \int_\Omega \rho^4(p)\,\mathrm{d}\mathcal{L}^3(p).$$

Since ρ was chosen arbitrarily among the admissible densities, we obtain the result. □

Next, we set conditions both on the foliation and the mapping in order to obtain equality in the modulus inequality for the mean distortion.

Proposition 7. *Let $f_0\colon \Omega \to \Omega'$ be an orientation preserving quasiconformal diffeomorphism between domains in the Heisenberg group. Let γ be a foliation of Ω as described above. Assume further that f_0 has the MSP for*

$$\Gamma_0 = \{\gamma(\cdot,\lambda)\colon \lambda \in \Lambda\}$$

and that

$$K(\gamma(s,\lambda), f_0) \equiv K_{f_0}(\lambda)$$

for all $(s, \lambda) \in (A, B) \times \Lambda$. *Then,*

$$M_4(f_0(\Gamma_0)) = \frac{1}{(B-A)^3} \int_\Lambda K_{f_0}^2(\lambda)\,\mathrm{d}\mu(\lambda).$$

Proof. Let $\rho' \in \operatorname{adm}(f_0(\Gamma_0))$ be an arbitrary admissible density. The $\mathcal{C}^1$ map f_0 maps any curve $\gamma(\cdot, \lambda)$ in Γ_0 (which is by assumption $\mathcal{C}^1$ and horizontal) again on a horizontal curve. We find

$$\begin{aligned} 1 &\le \int_A^B \rho'(f_0 \circ \gamma(s,\lambda)) |(f_0 \dot{\circ} \gamma)_I(s,\lambda)|\,\mathrm{d}s \\ &= \int_A^B \rho'(f_0(\gamma(s,\lambda)))(|Z(f_0)_I(\gamma(s,\lambda))| - |\overline{Z}(f_0)_I(\gamma(s,\lambda))|)|\dot{\gamma}_I(s,\lambda)|\,\mathrm{d}s. \end{aligned}$$

For the last equality we have used the fact that the map f_0 has the MSP for the family Γ_0 of horizontal curves. Then applying Hölder's inequality, we obtain

$$\frac{1}{(B-A)^3} \le \int_A^B \rho'(f_0(\gamma(s,\lambda)))^4 \cdot (|Z(f_0)_I(\gamma(s,\lambda))| - |\overline{Z}(f_0)_I(\gamma(s,\lambda))|)^4 |\dot{\gamma}_I(s,\lambda)|^4\,\mathrm{d}s.$$

We multiply both sides of the inequality with $K_{f_0}(\lambda)$ (here it is used that the distortion is constant along the the curves $\gamma(\cdot,\lambda)$) and integrate over Λ with respect to μ. This yields,

$$\frac{1}{(B-A)^3} \int_\Lambda K_{f_0}^2(\lambda)\,\mathrm{d}\mu(\lambda) \le \int_\Lambda \int_A^B \rho'^4(f_0(\gamma(s,\lambda))) J(\gamma(s,\lambda), f_0) |\dot{\gamma}_I(s,\lambda)|^4\,\mathrm{d}s\mathrm{d}\mu(\lambda).$$

Here, we have used the formula

$$(|Z(f_0)_I| - |\overline{Z}(f_0)_I|)^4 K^2(\cdot, f_0) = (|Z(f_0)_I| + |\overline{Z}(f_0)_I|)^2 (|Z(f_0)_I| - |\overline{Z}(f_0)_I|)^2 = J(\cdot, f_0),$$

see Appendix, (38). Then it follows from the splitting of the measure $\mathcal{L}^3$ and the change of variables that

$$\frac{1}{(B-A)^3} \int_\Lambda K_{f_0}^2(\lambda)\,\mathrm{d}\mu(\lambda) \le \int_\Omega \rho'(f_0(p))^4 J(p, f_0)\,\mathrm{d}\mathcal{L}^3(p) = \int_{\Omega'} \rho'(q)^4\,\mathrm{d}\mathcal{L}^3(q),$$

see the Appendix for details. Since ρ' was chosen arbitrarily among the admissible densities of $f_0(\Gamma_0)$, this shows that

$$M_4(f_0(\Gamma_0)) \ge \frac{1}{(B-A)^3} \int_\Lambda K_{f_0}^2(\lambda)\,\mathrm{d}\mu(\lambda).$$

Now consider the push-forward density given by

$$\rho_0'(q) = \begin{cases} \big((B-A) \cdot |\dot{\gamma}_I(s,\lambda)| \cdot (|Z(f_0)_I(\gamma(s,\lambda))| - |\overline{Z}(f_0)_I(\gamma(s,\lambda)|)\big)^{-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q = f_0(\gamma(s,\lambda)) \in \Omega', \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad q \notin \Omega'. \end{cases}$$

This is admissible for $f_0(\Gamma_0)$; indeed,

$$\int_A^B \rho_0'(f_0 \circ \gamma(s,\lambda)) |(f_0 \dot{\circ} \gamma)_I(s,\lambda)|\,\mathrm{d}s = \int_A^B \frac{1}{B-A}\,\mathrm{d}s = 1.$$

Therefore,

$$
\begin{aligned}
M_4(f_0(\Gamma_0)) &\leq \int_{\Omega'} \rho_0'(q)^4 \,\mathrm{d}\mathcal{L}^3(q) = \int_{\Omega} \rho_0'(f_0(p))^4 J(p,f_0)\,\mathrm{d}\mathcal{L}^3(p)\\
&= \int_{\Lambda}\int_A^B \rho_0'(f_0(\gamma(s,\lambda)))^4 J(\gamma(s,\lambda),f_0)|\dot\gamma_I(s,\lambda)|^4 \,\mathrm{d}s\,\mathrm{d}\mu(\lambda)\\
&= \int_{\Lambda}\int_A^B \frac{1}{(B-A)^4} K_{f_0}(\lambda)^2 \,\mathrm{d}s\,\mathrm{d}\mu(\lambda) = \frac{1}{(B-A)^3}\int_{\Lambda} K_{f_0}(\lambda)^2\,\mathrm{d}\mu(\lambda).
\end{aligned}
$$

This concludes the proof. □

Proposition 7 proves the statement of Theorem 1 in the introduction.

3. Logarithmic coordinates

3.1. Coordinates. Interesting examples of quasiconformal and quasiregular mappings in the Heisenberg group can be obtained using the coordinates which were defined in [Pla09]. Another version of these coordinates has been introduced earlier by Korányi and Reimann in [KR87]. The idea is to generalize logarithmic coordinates of the plane to the Heisenberg setting. For clarity, we will present in detail the construction below. One starts with the spherical coordinates of the Heisenberg group. If $\mathbf{H}^1$ is identified with $\mathbf{C}\times\mathbf{R}$, they are given by

$$\mathcal{S}(r,\theta,\phi) = (r\cos^{1/2}\theta e^{i\phi}, r^2\sin\theta) \text{ for each } (r,\theta,\phi)\in[0,\infty)\times[-\tfrac{\pi}{2},\tfrac{\pi}{2}]\times[0,2\pi).$$

We now change the coordinates by applying the transformation

$$T(r,\theta,\phi) = (\xi,\psi,\eta) = \left(2\log r, -\theta, \frac{1}{3}(\pi-\theta-2\phi)\right).$$

The new coordinates are given explicitly by

$$\Phi(\xi,\psi,\eta) = (z,t) = \left(\mathrm{i}\cos^{1/2}\psi\cdot e^{\frac{\xi+i(\psi-3\eta)}{2}}, -\sin\psi e^{\xi}\right). \tag{9}$$

On the domain

$$\widetilde{\mathbf{H}}_0^1 := \mathbf{R}\times(-\tfrac{\pi}{2},\tfrac{\pi}{2})\times\mathbf{R},$$

the map Φ is smooth and locally injective with Jacobian determinant

$$\det\Phi_*(\xi,\psi,\eta) = -\tfrac{3}{4}e^{2\xi}\neq 0. \tag{10}$$

We notice that

$$\Phi\colon \widetilde{\mathbf{H}}_0^1 \to \mathbf{H}_0^1 := \mathbf{H}^1\setminus\{(z,t)\in\mathbf{H}^1\colon z=0\}$$

is a smooth covering map. It is then a well-known fact that for each curve $\gamma\colon[a,b]\to\mathbf{H}_0^1$ and each point (ξ,ψ,η) in the set $\Phi^{-1}(\{\gamma(a)\})$, there exists a unique lifted curve $\tilde\gamma\colon[a,b]\to\widetilde{\mathbf{H}}_0^1$ such that $\Phi\circ\tilde\gamma=\gamma$ and $\tilde\gamma(a)=(\xi,\psi,\eta)$. If γ is absolutely continuous in the Euclidean sense, or if it is $\mathcal{C}^k$ for a $k\in\mathbf{N}_0$, then $\tilde\gamma$ inherits the same regularity properties.

Not only curves can be lifted, but also continuous mappings from simply connected domains in $\widetilde{\mathbf{H}}_0^1$ can be lifted to continuous mappings, see for instance [Lee09]. Yet, not every mapping $\tilde f\colon\widetilde{\mathbf{H}}_0^1\to\widetilde{\mathbf{H}}_0^1$ yields a well-defined map $f\colon\mathbf{H}_0^1\to\mathbf{H}_0^1$ with the property that $\Phi\circ\tilde f = f\circ\Phi$.

Lemma 8. *Let Q be a simply connected subset of $\widetilde{\mathbf{H}}^1_0$ and let $f\colon \Phi(Q) \to \mathbf{H}^1_0$ be a $\mathcal{C}^k$, $k \in \mathbf{N}_0$, map. Then there exists a $\mathcal{C}^k$ map $\tilde{f}\colon Q \to \widetilde{\mathbf{H}}^1_0$ such that $\Phi \circ \tilde{f} = f \circ \Phi$.*

Conversely, if $\tilde{f}\colon Q \to \widetilde{\mathbf{H}}^1_0$ is a $\mathcal{C}^k$ map on a subset $Q \subseteq \widetilde{\mathbf{H}}^1_0$ with the property that for any (ξ, ψ, η) and (ξ', ψ', η') in Q, we have that

$$\Phi(\xi, \psi, \eta) = \Phi(\xi', \psi', \eta') \quad \textit{implies} \quad \Phi(\tilde{f}(\xi, \psi, \eta)) = \Phi(\tilde{f}(\xi', \psi', \eta')),$$

then one can define a $\mathcal{C}^k$ map $f\colon \Phi(Q) \to \mathbf{H}^1_0$ with the relation $\Phi \circ \tilde{f} = f \circ \Phi$.

In the following, we will heavily use these logarithmic coordinates: we will define a quasiconformal map f between domains in the Heisenberg group by specifying a formula for $\tilde{f}$, and conversely, we will work with the lift $\tilde{f}$ of $f \circ \Phi$ when it is convenient.

The reason why we shall make use of logarithmic coordinates and do not simply use the Heisenberg spherical coordinates is two–fold: first, it appears that the contact form and the contact conditions have a simpler and nicer form in this way. Secondly, there is purely geometric interpretation; in the logarithmic coordinates, the coordinate planes $\xi = 0$, $\psi = 0$ and $\eta = 0$ are the Heisenberg unit sphere, the complex plane $\mathbf{C}$ (these are boundaries of bisectors, [Gol99]), and the boundary of the standard flat pack, [Pla09] respectively.

In the following we shall express the contact condition, formulas for volume and curve integrals, for horizontal vector fields, the Beltrami coefficient and the condition for the minimal stretching property in logarithmic coordinates. We use the notation

$$\tilde{f}(\xi, \psi, \eta) = (\Xi(\xi, \psi, \eta), \Psi(\xi, \psi, \eta), H(\xi, \psi, \eta))$$

and we write $\Xi_\xi = \frac{\partial \Xi}{\partial \xi}$ and similarly for the other partial derivatives.

3.2. Contact condition. The contact form and the contact conditions have a simple form when expressed in terms of logarithmic coordinates. Let $\tilde{U}$ be an open set in $\widetilde{\mathbf{H}}^1_0$ such that a local coordinate chart on $\mathbf{H}^1_0$ can be defined using $(\Phi|_{\tilde{U}})^{-1} = (\xi, \psi, \eta)$. Then the contact form τ has the local expression

$$\tau = -e^{\xi}(\sin\psi \, \mathrm{d}\xi + 3\cos\psi \, \mathrm{d}\eta). \tag{11}$$

Proposition 9. *Let Q be an open set in $\widetilde{\mathbf{H}}^1_0$ and assume that there exist $\mathcal{C}^k$, $k \in \mathbf{N}$, maps $\tilde{f}\colon Q \to \widetilde{\mathbf{H}}^1_0$ and $f\colon \Phi(Q) \to \mathbf{H}^1_0$ such that $\Phi \circ \tilde{f} = f \circ \Phi$ on Q. Then the following conditions are equivalent*

(1) *The map f is an injective contact map.*
(2) *The map $\tilde{f} = (\Xi, \Psi, H)$ has the property that*

$$\Phi(\tilde{f}(\xi, \psi, \eta)) = \Phi(\tilde{f}(\xi', \psi', \eta')) \quad \textit{if and only if} \quad \Phi(\xi, \psi, \eta) = \Phi(\xi', \psi', \eta'), \tag{12}$$

for $(\xi, \psi, \eta), (\xi', \psi', \eta') \in Q$, and there exists a nowhere vanishing function $\tilde{\lambda}\colon Q \to \mathbf{R}$ such that

$$\begin{aligned} e^{\Xi}\,(\sin\Psi\,\Xi_\xi + 3\cos\Psi\,H_\xi) &= \tilde{\lambda} e^{\xi} \sin\psi, \\ e^{\Xi}\,(\sin\Psi\,\Xi_\psi + 3\cos\Psi\,H_\psi) &= 0, \\ e^{\Xi}\,(\sin\Psi\,\Xi_\eta + 3\cos\Psi\,H_\eta) &= \tilde{\lambda} e^{\xi} 3\cos\psi. \end{aligned} \tag{13}$$

Moreover, if (13) *holds, then*

$$H_\psi + \tfrac{1}{3}\tan\Psi\cdot\Xi_\psi = 0,$$
$$W_{\xi,\eta}H + \tfrac{1}{3}\tan\Psi\cdot W_{\xi,\eta}\Xi = 0, \tag{14}$$

with $W_{\xi,\eta} = \frac{\partial}{\partial\xi} - \frac{\tan\psi}{3}\frac{\partial}{\partial\eta}$.

Proof. If the two maps f and $\tilde f$ are related by $\Phi\circ\tilde f = f\circ\Phi$, then it is easy to see that f is injective if and only if (12) holds.

Concerning the contact condition, we observe that at each point $p\in\Phi(Q)$ and $f(p)\in f(\Phi(Q))$ we can define local coordinate charts using the map Φ such that the local expression of f with respect to these coordinate charts is exactly $\tilde f$. This is because of the uniqueness of the continuous lift through a given point. The contact form τ is then given by (11). The condition that there exists $\lambda(p)\neq 0$ such that $(f^*\tau)_p = \lambda(p)\tau_p$ is equivalent to (13) with $\tilde\lambda = \lambda\circ\Phi$. Notice that $\psi,\Psi\in(-\frac{\pi}{2},\frac{\pi}{2})$, which allows us to divide the equations by $\cos\psi$ and $\cos\Psi$. From the second equation it follows that

$$\tan\Psi\,\Xi_\psi + 3H_\psi = 0.$$

Moreover, we can solve the third equation for $\tilde\lambda$ and insert the result in the first equation. This yields

$$\tan\Psi(\Xi_\xi - \tfrac{1}{3}\tan\psi\Xi_\eta) + 3(H_\xi - \tfrac{1}{3}\tan\psi H_\eta) = 0.$$

The result follows if we introduce the abbreviating notation with the differential operator $W_{\xi,\eta}$. □

3.3. Volume and curve integral. Let $\Omega\subseteq\mathbf{H}^1$ be a measurable set and let $Q\subseteq\widetilde{\mathbf{H}}^1_0$ be an open set such that $\Phi(Q)$ coincides with Ω up to a null set and such that $\Phi|_Q$ is invertible. Then a function $h\colon\Omega\to\mathbf{R}$ is integrable if and only if $(h\circ\Phi)|\det\Phi_*|$ is integrable on Q and in this case we have

$$\int_\Omega h(p)\,\mathrm{d}\mathcal{L}^3(p) = \frac{3}{4}\int_Q e^{2\xi}h(\Phi(\xi,\psi,\eta))\,\mathrm{d}\mathcal{L}^3(\xi,\psi,\eta).$$

In the particular case of the spherical annulus,

$$A(a,b) = \{p\in\mathbf{H}^1\colon a<\|p\|_H<b\},\quad 0<a<b,$$

for every integrable function $h\colon A(a,b)\to\mathbf{R}$ we have

$$\int_{A(a,b)} h(p)\,\mathrm{d}\mathcal{L}^3(p) = \frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{2\pi}{3}}^{\frac{2\psi}{3}}\int_{\log a^2}^{\log b^2} h(\Phi(\xi,\psi,\eta))e^{2\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\psi.$$

In order to apply the modulus method, the following formula for curve integrals in terms of logarithmic coordinates is useful. The proof of the first part of the proposition is a straight-forward computation. The proof gets a bit harder if we assume that the curve γ may intersect the t-axis; a case that we will encounter in our applications.

Proposition 10. *A curve* $\gamma\colon[a,b]\to\mathbf{H}^1_0$ *is horizontal if and only if there exists an absolutely continuous curve*

$$\tilde\gamma\colon[a,b]\to\widetilde{\mathbf{H}}^1_0,\quad \tilde\gamma(s) = (\xi(s),\psi(s),\eta(s))$$

with $\Phi \circ \tilde{\gamma} = \gamma$ *and*

$$\sin(\psi(s))\dot{\xi}(s) + 3\cos(\psi(s))\dot{\eta}(s) = 0 \quad \textit{for almost every } s \in [a,b]. \tag{15}$$

If a horizontal curve $\gamma\colon [a,b] \to \mathbf{H}^1 \setminus \{0\}$ *satisfies* $\gamma(s) \in \mathbf{H}^1_0$ *for almost every* $s \in [a,b]$*, then there exists*

$$\tilde{\gamma}\colon [a,b] \to \mathbf{R} \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times \mathbf{R}, \quad s \mapsto (\xi(s), \psi(s), \eta(s))$$

with $s \mapsto \xi(s)$ *absolutely continuous, such that for* $s \in [a,b] \cap \gamma^{-1}(\mathbf{H}^1_0)$ *we have* $\Phi(\tilde{\gamma}(s)) = \gamma(s)$ *and* (15) *holds almost everywhere.*

Moreover, for any Borel function $\rho\colon \mathbf{H}^1 \to [0,\infty]$*, we have*

$$\int_\gamma \rho \, \mathrm{d}\ell = \int_a^b \rho(\Phi(\tilde{\gamma}(s)))\tfrac{1}{2} e^{\frac{\xi(s)}{2}} (1 + \tan^2(\psi(s)))^{\frac14} (\dot{\xi}(s)^2 + \dot{\psi}(s)^2)^{\frac12} \, \mathrm{d}s. \tag{16}$$

Proof. It $\tilde{\gamma}\colon [a,b] \to \widetilde{\mathbf{H}}^1_0$ is an absolutely continuous function satisfying (15), then we consider the absolutely continuous curve $\gamma := \Phi \circ \tilde{\gamma}$. Conversely, if $\gamma\colon [a,b] \to \mathbf{H}^1_0$ is horizontal, we take $\tilde{\gamma}\colon [a,b] \to \widetilde{\mathbf{H}}^1_0$ to be a lift of γ with respect to the covering map Φ. The situation is more complicated if $\gamma([a,b])$ is not entirely contained in $\mathbf{H}^1_0$.

For the moment, let us assume that we are given almost everywhere differentiable mappings $\gamma\colon [a,b] \to \mathbf{H}^1 \setminus \{0\}$ and $\tilde{\gamma}\colon [a,b] \to \mathbf{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbf{R}$ such that $\Phi \circ \tilde{\gamma} = \gamma$ for all $s \in [a,b] \cap \gamma^{-1}(\mathbf{H}^1_0)$. Let s be a point of differentiability in $[a,b] \cap \gamma^{-1}(\mathbf{H}^1_0)$. There exists a neighborhood of s where we also have $\Phi \circ \tilde{\gamma} = \gamma$. It follows

$$\dot{\gamma}_I(s) = \mathrm{i}\sqrt{\cos(\psi(s))}\, e^{\frac{\xi(s) + \mathrm{i}(\psi(s) - 3\eta(s))}{2}} \tfrac12 \big((\dot{\xi}(s) - \tan(\psi(s))\dot{\psi}(s)) + \mathrm{i}(\dot{\psi}(s) - 3\dot{\eta}(s))\big) \tag{17}$$

and the condition for a horizontal curve, (4), reads as

$$-e^{\xi(s)}(\cos(\psi(s))\dot{\psi}(s) + \sin(\psi(s))\dot{\xi}(s)) = -\cos(\psi(s))e^{\xi(s)}(\dot{\psi}(s) - 3\dot{\eta}(s)).$$

This proves (15) under the assumption that $\gamma = \Phi \circ \tilde{\gamma}$ almost everywhere. Since $\cos(\psi(s)) \neq 0$, we can further solve for $\dot{\eta}(s) = -\frac13 \tan(\psi(s))\dot{\xi}(s)$. Then (17) yields

$$\begin{aligned}
|\dot{\gamma}_I(s)| &= \sqrt{\cos(\psi(s))}\, e^{\frac{\xi(s)}{2}} \tfrac12 \sqrt{(\dot{\xi}(s) - \tan(\psi(s))\dot{\psi}(s))^2 + (\dot{\psi}(s) + \tan(\psi(s))\dot{\xi}(s))^2} \\
&= \sqrt{\cos(\psi(s))}\, e^{\frac{\xi(s)}{2}} \tfrac12 \sqrt{1 + \tan^2(\psi(s))}\sqrt{\dot{\xi}(s)^2 + \dot{\psi}(s)^2} \\
&= \tfrac12 e^{\frac{\xi(s)}{2}} (1 + \tan^2(\psi(s)))^{\frac14} (\dot{\xi}(s)^2 + \dot{\psi}(s)^2)^{\frac12}.
\end{aligned}$$

For such a horizontal curve $\gamma\colon [a,b] \to \mathbf{H}^1 \setminus \{0\}$, the formula for the curve integral follows then immediately since $\int_\gamma \rho \, \mathrm{d}\ell = \int_a^b \rho(\gamma(s))|\dot{\gamma}_I(s)| \, \mathrm{d}s$.

It remains to prove the existence of $\tilde{\gamma}$ for a horizontal curve γ which crosses the t-axis. If $s \in [a,b]$ is such that $\gamma(s) \in \mathbf{H}^1_0$, then by continuity of γ, we also have $\gamma(t) \in \mathbf{H}^1_0$ for t in a neighborhood of s and the map $\tilde{\gamma}$ can be defined locally. To make this precise, let $J(s)$ be the largest interval such that $\gamma(t) \in \mathbf{H}^1_0$ for all $t \in J(s)$. Then for two points $s, t \in [a,b]$ with $\gamma(s), \gamma(t) \in \mathbf{H}^1_0$ we have either $J(s) = J(t)$ or $J(s) \cap J(t) = \emptyset$. Thus there exists a family J_i, $i \in I$, of disjoint nonempty intervals such that $[a,b] \cap \gamma^{-1}(\mathbf{H}^1_0) = \bigcup_{i \in I} J_i$. Now for each $i \in J$ the curve $\gamma|_{J_i}\colon J_i \to \mathbf{H}^1_0$ has a lift $\tilde{\gamma}$, i.e., there exists an absolutely continuous curve $\tilde{\gamma}|_{J_i}\colon J_i \to \widetilde{\mathbf{H}}^1_0$, such that

$\Phi \circ \tilde{\gamma}|_{J_i} = \gamma|_{J_i}$. We define $\tilde{\gamma}\colon [a,b] \to \mathbf{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbf{R}$ by setting

$$\tilde{\gamma}(s) := (\xi(s), \psi(s), \eta(s)) := \begin{cases} \tilde{\gamma}|_{J_i}(s), & s \in J_i, \\ (\log(\gamma_3(s)), -\frac{\pi}{2}, 0), & s \in [a,b] \setminus \left(\bigcup_{i \in I} J_i\right), \ \gamma_3(s) > 0, \\ (\log(-\gamma_3(s)), \frac{\pi}{2}, 0), & s \in [a,b] \setminus \left(\bigcup_{i \in I} J_i\right), \ \gamma_3(s) < 0. \end{cases}$$

One sees immediately that $\Phi \circ \tilde{\gamma} = \gamma$ for $s \in [a,b] \cap \gamma^{-1}(\mathbf{H}_0^1)$, that $\tilde{\gamma}$ is almost everywhere differentiable in $[a,b]$ and the component $s \mapsto \xi(s)$ is absolutely continuous. □

3.4. Vector fields. Below we express the vector fields Z, $\overline{Z}$ and T in terms of logarithmic coordinates which are given locally by Φ. Straightforward calculations induce

$$Z = -\mathrm{i}\sqrt{\cos\psi}\, e^{\frac{-\xi + \mathrm{i}3(\eta - \psi)}{2}} \left(\frac{\partial}{\partial \xi} - \frac{\tan\psi}{3}\frac{\partial}{\partial \eta} - \mathrm{i}\frac{\partial}{\partial \psi}\right), \tag{18}$$

$$\bar{Z} = \mathrm{i}\sqrt{\cos\psi}\, e^{\frac{-\xi + \mathrm{i}3(\psi - \eta)}{2}} \left(\frac{\partial}{\partial \xi} - \frac{\tan\psi}{3}\frac{\partial}{\partial \eta} + \mathrm{i}\frac{\partial}{\partial \psi}\right), \tag{19}$$

$$T = -\frac{\sin\psi}{e^{\xi}}\frac{\partial}{\partial \xi} - \frac{\cos\psi}{e^{\xi}}\frac{\partial}{\partial \psi} - \frac{1}{3}\frac{\cos\psi}{e^{\xi}}\frac{\partial}{\partial \eta}. \tag{20}$$

3.5. Beltrami coefficient and distortion. Let f and $\tilde{f} = (\Xi, \Psi, H)$ be $\mathcal{C}^1$ maps as in Proposition 9. Set

$$W := W_{\xi,\eta} - \mathrm{i}\frac{\partial}{\partial \psi} := \left(\frac{\partial}{\partial \xi} - \frac{\tan\psi}{3}\frac{\partial}{\partial \eta}\right) - \mathrm{i}\frac{\partial}{\partial \psi},$$
$$\overline{W} := W_{\xi,\eta} + \mathrm{i}\frac{\partial}{\partial \psi} := \left(\frac{\partial}{\partial \xi} - \frac{\tan\psi}{3}\frac{\partial}{\partial \eta}\right) + \mathrm{i}\frac{\partial}{\partial \psi}.$$

The Beltrami coefficient of f is given by

$$\mu_f(\Phi(\xi, \psi, \eta)) = -e^{\mathrm{i}3(\psi - \eta)} \left.\frac{\overline{W}(\Xi + \mathrm{i}\Psi)}{W(\Xi + \mathrm{i}\Psi)}\right|_{(\xi,\psi,\eta)}. \tag{21}$$

3.6. The minimal stretching property. Let again f and $\tilde{f} = (\Xi, \Psi, H)$ be $\mathcal{C}^1$ maps as in Proposition 9 and assume in addition that f is an orientation preserving quasiconformal map. Let $\tilde{\Gamma}$ be a family of $\mathcal{C}^1$ curves

$$\tilde{\gamma}\colon [a,b] \to \widetilde{\mathbf{H}}_0^1, \quad \tilde{\gamma}(s) = (\xi(s), \psi(s), \eta(s))$$

such that

$$\sin(\psi(s))\dot{\xi}(s) + 3\cos(\psi(s))\dot{\eta}(s) = 0 \quad \text{for all } s \in (a,b)$$

and

$$\frac{\dot{\xi}(s) - \mathrm{i}\dot{\psi}(s)}{\dot{\xi}(s) + \mathrm{i}\dot{\psi}(s)} \left.\frac{\overline{W}(\Xi + \mathrm{i}\Psi)}{W(\Xi + \mathrm{i}\Psi)}\right|_{\tilde{\gamma}(s)} < 0 \tag{22}$$

for $s \in (a,b)$ with $\mu_f(\Phi(\tilde{\gamma}(s)) \neq 0$. Then f has the MSP for the family $\Gamma = \{\Phi \circ \tilde{\gamma}\colon \tilde{\gamma} \in \tilde{\Gamma}\}$.

4. Extremality of the stretch map

In this section we define a stretch mapping on the Heisenberg group and discuss its properties. We prove Theorem 2 stated in the introduction and we show that the stretch is a minimizer for the maximal distortion within a class of sphere-preserving mappings.

4.1. The stretch map. Proof of Theorem 2. Let $A(a,b)$, $0<a<b$, be the spherical annulus and let $0<k<1$. In what follows, we shall give an example of a quasiconformal map $f_k\colon A(a,b)\to A(a^k,b^k)$, *the stretch*, which turns out to be a minimizer of a certain mean distortion. In logarithmic coordinates, the map is given by

$$\tilde{f}_k(\xi,\psi,\eta)=(k\xi,\tan^{-1}(\tfrac{\tan\psi}{k}),\eta). \tag{23}$$

Let us briefly explain the origin of this formula. Motivated by the planar radial stretch map in logarithmic coordinates, we search for a mapping $\tilde{f}_k=(\Xi,\Psi,H)$ with $\Xi(\xi,\psi,\eta)=\Xi(\xi)=k\xi$. From the contact condition (13) it follows then immediately that $H_\psi=0$, thus $H(\xi,\psi,\eta)=H(\xi,\eta)$ with periodicity condition $H(\xi,\eta+\frac{4\pi}{3})=H(\xi,\eta) \pmod{\frac{4\pi}{3}}$. Moreover,

$$\Psi(\xi,\psi,\eta)=\tan^{-1}\left(\frac{H_\eta(\xi,\eta)\tan\psi-3H_\xi(\xi,\eta)}{k}\right)$$

with $H_\eta(\xi,\eta)\neq 0$. The stretch map is obtained by setting $H(\xi,\eta)=\eta$. In cartesian coordinates, the formula for the associated map $f_k|_{\mathbf{H}^1_0}$ reads as

$$f_k(z,t)=\left(k^{\frac{1}{2}}z\left(\frac{|z|^2+\mathrm{i}t}{k|z|^2+\mathrm{i}t}\right)^{1/2}\big||z|^2-\mathrm{i}t\big|^{\frac{k-1}{2}},\,t\cdot\frac{\big||z|^2-\mathrm{i}t\big|^k}{\big|k|z|^2-\mathrm{i}t\big|}\right).$$

It is not hard to see that $\|f_k(z,t)\|_H=\|(z,t)\|_H^k$. If we extend f_k to the t-axis by setting $f_k(0,t)=(0,t|t|^{k-1})$, this yields a homeomorphism $f_k:\overline{A(a,b)}\to\overline{A(a^k,b^k)}$. This map has already appeared implicitly in [Min94]; there, it was generated by the flow method due to Korányi and Reimann.

Proof of Theorem 2. First we prove that f_k is quasiconformal. Since $f_k\colon A(a,b)\to A(a^k,b^k)$ is a homeomorphism, it suffices to prove that $f_k|_{A(a,b)\cap\mathbf{H}^1_0}$ is a quasiconformal mapping. Then, the quasiconformality of f_k follows by the removability result of [BKR07, Theorem 1.2].

The hypotheses of Proposition 9 are satisfied by the smooth map $\tilde{f}_k\colon\widetilde{\mathbf{H}}^1_0\to\widetilde{\mathbf{H}}^1_0$; thus the stretch $f_k|_{A(a,b)\cap\mathbf{H}^1_0}$ is a smooth contact transformation onto its image. Formula (21) yields that

$$\mu_{f_k}(\Phi(\xi,\psi,\eta))=-e^{\mathrm{i}3(\psi-\eta)}\frac{k^2-1}{k^2+1+2\tan^2\psi}$$

for all $(\xi,\psi,\eta)\in\widetilde{\mathbf{H}}^1_0$. A direct computation shows that $\|\mu_{f_k}\|_\infty<1$ and

$$K(\Phi(\xi,\psi,\eta),f_k)=\frac{1+\tan^2\psi}{k^2+\tan^2\psi}.$$

Thus we have proved that f_k is a smooth orientation preserving quasiconformal map on $A(a,b)\cap\mathbf{H}^1_0$. The extension to the whole annulus is quasiconformal with

$$\|\mu_{f_k}\|_\infty=\frac{1-k^2}{1+k^2},$$

and with distortion given by $K_{f_k}=\frac{1}{k^2}$.

We prove next that

$$\int_{A(a,b)}K(p,f_k)^2\rho_0(p)^4\,\mathrm{d}\mathcal{L}^3(p)\le\int_{A(a,b)}K(p,f)^2\rho_0(p)^4\,\mathrm{d}\mathcal{L}^3(p)$$

with $\rho_0(z,t) = (\log \frac{b}{a})^{-1} \frac{|z|}{\sqrt{|z|^4+t^2}}$ for all $f \in \mathcal{F}$. To do so, we shall use Theorem 1.

Let $A = \log a^2$, $B = \log b^2$, and $\Lambda = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{2\pi}{3}, \frac{2\pi}{3})$. We define

$$\tilde{\gamma}\colon (A,B) \times \Lambda \to \widetilde{\mathbf{H}}^1_0, \quad \tilde{\gamma}(s,\psi,\eta) = (\xi(s), \psi(s), \eta(s)) = (s, \psi, \eta - \tfrac{\tan\psi}{3} s). \tag{24}$$

and

$$\gamma\colon (A,B) \times \Lambda \to \Omega, \quad \gamma(s,\psi,\eta) = \Phi(\tilde{\gamma}(s,\psi,\eta)).$$

The set $\Omega := \gamma((A,B) \times \Lambda)$ is a bounded domain inside $A(a,b) \cap \mathbf{H}^1_0$, and γ defines a smooth diffeomorphism between $(A,B) \times \Lambda$ and Ω with nowhere vanishing Jacobian determinant $|\det \gamma_*(s,\psi,\eta)| = \frac{3}{4} e^{2s}$. Moreover, for each fixed $(\psi,\eta) \in \Omega$ the curve

$$\gamma(\cdot,\psi,\eta)\colon (A,B) \to \Omega, \quad s \mapsto \Phi(s,\psi,\eta - \tfrac{\tan\psi}{3} s)$$

is horizontal: just observe that

$$\sin(\psi(s))\dot{\xi(s)} + 3\cos(\psi(s))\dot{\eta}(s) = \sin\psi + 3\cos\psi\left(-\tfrac{\tan\psi}{3}\right) = 0$$

and then use Proposition 10. Moreover,

$$|\dot{\gamma}_I(s,\psi,\eta)| = \tfrac{1}{2} e^{\frac{s}{2}} (\cos\psi)^{-\frac{1}{2}} \neq 0 \quad \text{for all } s \in (A,B).$$

Therefore, the volume element can be written as

$$\mathrm{d}\mathcal{L}^3(\gamma(s,\lambda)) = \tfrac{3}{4} e^{2s}\, \mathrm{d}s\, \mathrm{d}\psi\, \mathrm{d}\eta = |\dot{\gamma}_I(s,\psi,\eta)|^4\, \mathrm{d}s\, \mathrm{d}\mu(\psi,\eta)$$

where

$$\mathrm{d}\mu(\psi,\eta) = 12\cos^2\psi \mathrm{d}\psi\, \mathrm{d}\eta.$$

Our model curve family will be the family

$$\Gamma_0 = \{\gamma(\cdot,\psi,\eta)\colon (\psi,\eta) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times (-\tfrac{2\pi}{3}, \tfrac{2\pi}{3})\}.$$

According to Proposition 6, an extremal density for Γ_0 is ρ_0 where

$$\rho_0(p) = \begin{cases} (\log \frac{b}{a})^{-1} \sqrt{\cos\psi} e^{-\frac{s}{2}}, & p = \gamma(s,\psi,\eta) \in \Omega, \\ 0, & p \notin \Omega. \end{cases}$$

Finally, we use the criterion given in (22) to verify whether the stretch map f_k has the MSP for the family Γ_0. Indeed, we have that

$$\frac{\dot{\xi}(s) - \mathrm{i}\dot{\psi}(s)}{\dot{\xi}(s) + \mathrm{i}\dot{\psi}(s)} \left.\frac{\overline{W}(\Xi + \mathrm{i}\Psi)}{W(\Xi + \mathrm{i}\Psi)}\right|_{\tilde{\gamma}(s,\psi,\eta)} = \frac{k^2-1}{k^2+1+2\tan^2\psi} < 0$$

for all s. This holds true for $0 < k < 1$. Therefore, by Proposition 7, we have

$$\begin{aligned} M_4(f_k(\Gamma_0)) &= \frac{1}{2^3(\log\frac{b}{a})^3} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K^2_{f_k}(\psi,\eta) 12\cos^2\psi\, \mathrm{d}\psi\, \mathrm{d}\eta \\ &= \int_\Omega K(p,f_k)^2 \rho_0^4(p)\, \mathrm{d}\mathcal{L}^3(p). \end{aligned}$$

In order to apply Theorem 1, we have to find a larger curve family $\Gamma \supseteq \Gamma_0$ for which ρ_0 is still admissible and such that $M_4(f_k(\Gamma_0)) \leq M_4(f(\Gamma))$ for all $f \in \mathcal{F}$. A natural choice for Γ is the family of *all* horizontal curves which connect the two boundary components in $A(a,b)$. The boundary conditions for mappings in the class $\mathcal{F}$ guarantee that the image $f(\Gamma)$ will essentially be a family of the same type in $A(a^k,b^k)$. Using the absolute continuity of quasiconformal mappings on almost every curve up to a negligible family of curves with zero modulus, and using the boundary conditions, one can show that

$$M_4(f_k(\Gamma_0)) \leq M_4(f(\Gamma)) \quad \text{for all } f \in \mathcal{F}.$$

It remains to prove that the density ρ_0 can be modified on a zero measure set such that $\rho_0 \in \mathrm{adm}(\Gamma)$. This follows from the computation of the modulus of the spherical ring, [KR87, p. 20]. Below we give a proof in terms of logarithmic coordinates.

We observe that the set $A(a,b) \setminus \Omega$ is of $\mathcal{L}^3$-measure zero set and we may set ρ_0 equal to infinity on a Borel set of zero measure which contains the set $A(a,b) \setminus \Omega$. The modified density is admissible for the larger family Γ and it agrees with the original density almost everywhere, therefore the mean distortion integral is not at all affected.

We have to check that ρ_0 is admissible for the extended family Γ. To do that, let γ be a curve in this family. We may without loss of generality assume that $\gamma(s) \in \mathbf{H}^1_0$ for almost every s. Observe that in logarithmic coordinates,

$$\rho_0(\Phi(\xi,\psi,\eta)) = \left(\log \tfrac{b}{a}\right)^{-1} \sqrt{\cos\psi}\, e^{-\frac{\xi}{2}}.$$

If γ is parameterized by arc-length, it follows from the proof of Proposition 10 that

$$\sqrt{\cos(\psi(s))}\, e^{-\frac{\xi(s)}{2}} = \frac{1}{2}(\dot{\xi}(s)^2 + \dot{\psi}(s)^2)^{\frac{1}{2}}$$

for almost every s with $\gamma(s) \in \mathbf{H}^1_0$. Then,

$$\begin{aligned}
\int_\gamma \rho_0(z,t)\,\mathrm{d}\ell &\geq \frac{1}{\log\frac{b}{a}} \int_0^{\ell(\gamma)} \sqrt{\cos\psi(s)}\, e^{-\frac{\xi(s)}{2}}\,\mathrm{d}s = \frac{1}{\log\frac{b}{a}} \int_0^{\ell(\gamma)} \frac{1}{2}(\dot{\xi}(s)^2 + \dot{\psi}(s)^2)^{\frac{1}{2}}\,\mathrm{d}s \\
&\geq \frac{1}{\log\frac{b}{a}} \int_0^{\ell(\gamma)} \frac{1}{2}\dot{\xi}(s)\,\mathrm{d}s = \frac{1}{2\log\frac{b}{a}} \left[\xi(s)\right]_{s=0}^{s=\ell(\gamma)} \\
&= \frac{1}{2\log\frac{b}{a}}(\log b^2 - \log a^2) = 1.
\end{aligned}$$

Here we have used for the evaluation of the integral the fact that $s \mapsto \xi(s)$ is an absolutely continuous function; see Proposition 10.

We conclude that $\rho_0 \in \mathrm{adm}(\Gamma)$, and from Theorem 1 it follows that

$$\int_{A(a,b)} K(p,f_k)^2 \rho_0^4(p)\,\mathrm{d}\mathcal{L}^3(p) \leq \int_{A(a,b)} K(p,f)^2 \rho_0^4(p)\,\mathrm{d}\mathcal{L}^3(p) \quad \text{for all } f \in \mathcal{F}.$$

The proof is complete. □

Theorem 2 has some interesting consequences which are described by the following remarks.

Remark 11. *Stretch for* $k > 1$. By the same arguments as in the first part of the above proof, one can show that the map f_k, $k > 1$, is quasiconformal with $K_{f_k} = k^2$. To prove extremality in the case $k > 1$, a different argument is needed.

Remark 12. *Modulus of the spherical annulus.* The curves in the family Γ_0 are the integral curves of the renormalized horizontal gradient of the Heisenberg norm $\|\cdot\|_H$, see [KR87]. In our notation, they are integral curves of the vector field $W_{\xi,\psi}$.

These are straight lines in $\widetilde{\mathbf{H}}^1_0$. Using the same notation as in the proof of Theorem 2, one obtains according to Proposition 6 that

$$M_4(\Gamma_0) = \frac{1}{(\log b^2 - \log a^2)^3} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos^2\psi \,\mathrm{d}\psi \,\mathrm{d}\eta = \pi^2 \left(\log \tfrac{b}{a}\right)^{-3}.$$

Since the (modified) density ρ_0 is still admissible for the larger family $\Gamma \supseteq \Gamma_0$ of all horizontal curves which connect the two boundary components in $A(a,b)$, this gives the well-known modulus of the spherical annulus, as in [KR87]:

$$M_4(\Gamma) = \pi^2 \left(\log \frac{b}{a}\right)^{-3}.$$

Remark 13. *Maximal distortion of the stretch map.* The Beltrami coefficient $|\mu_{f_k}(z,t)|$ and the pointwise distortion $K((z,t), f_k)$ of the stretch map are not constant and therefore, it is not possible to apply Proposition 5 in order to prove extremality of f_k, $0<k<1$, in $\mathcal{F}$ with respect to the *maximal* distortion. A direct computation shows that the equality (8) does not hold for the stretch map f_k and the family Γ of all horizontal curves connecting the two boundary components of $A(a,b)$. Indeed, the image family $f_k(\Gamma)$ will be a curve family of the same form in $A(a^k, b^k)$ and one has

$$M_4(\Gamma) = \pi^2 \left(\log\left(\frac{b}{a}\right)\right)^{-3} \quad \text{and} \quad M_4(f_k(\Gamma)) = k^{-3}\pi^2 \left(\log\left(\frac{b}{a}\right)\right)^{-3},$$

by Remark 12 above. We observe

$$\frac{M_4(f_k(\Gamma))}{M_4(\Gamma)} = k^{-3} \lneq k^{-4} = K_{f_k}^2.$$

We do not know whether f_k is extremal for the maximal distortion in $\mathcal{F}$.

Remark 13 motivates the discussion in the following section.

4.1.1. Extremality among sphere-preserving maps. The fact that the radial stretch minimizes a certain mean distortion functional does *not* necessarily imply that it is also extremal for the maximal distortion in the same class. There is, however, a positive result in this direction: the radial stretch minimizes the maximal distortion within a class of sphere-preserving maps.

We call a map $f\colon \Omega \to \Omega'$ between domains in the Heisenberg group *sphere-preserving* if for all $r \geq 0$ there exists $r' \geq 0$ such that

$$f(\{p \in \Omega\colon \|p\|_H = r\}) = \{p \in \Omega'\colon \|p\|_H = r'\}.$$

As before, let $0<a<b$ and $0<k<1$. We will consider the class $\mathcal{F}_0$ of orientation and spheres-preserving quasiconformal $\mathcal{C}^1$ diffeomorphisms

$$A(a,b) \to A(a^k, b^k)$$

which send homeomorphically the inner boundary to the inner boundary and the outer boundary to the outer boundary and which also preserve the t-axis.

Theorem 14. *For all $f \in \mathcal{F}_0$, the maximal distortion is at least as big as the maximal distortion of the radial stretch,*

$$K_f \geq K_{f_k} = \frac{1}{k^2}.$$

Proof. Proposition 9 can be applied to any $f \in \mathcal{F}_0$ in order to deduce the existence of a $\mathcal{C}^1$ map

$\tilde{f} = (\Xi, \Psi, H) : Q = [\log a^2, \log b^2] \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R} \to [k \log a^2, k \log b^2] \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R}$

such that

$$H_\psi + \tfrac{1}{3} \tan \Psi \cdot \Xi_\psi = 0 \tag{25}$$

and

$$W_{\xi,\eta} H + \tfrac{1}{3} \tan \Psi \cdot W_{\xi,\eta} \Xi = 0 \tag{26}$$

hold for all $(\xi, \psi, \eta) \in Q$ and the periodicity and injectivity conditions (12) in Proposition 9 are fulfilled.

From the assumption that f preserves spheres, it follows that

$$\Xi(\xi, \psi, \eta) = \Xi(\xi)$$

is a function of ξ only. It is continuous on $[\log a^2, \log b^2]$ and differentiable on $(\log a^2, \log b^2)$ by assumption, with boundary values $\Xi(\log a^2) = k \log a^2$ and $\Xi(\log b^2) = k \log b^2$. By the Mean Value Theorem, there exists $\xi_0 \in (\log a^2, \log b^2)$ such that

$$k \log b^2 - k \log a^2 = \Xi(\log b^2) - \Xi(\log a^2) = \Xi_\xi(\xi_0)(\log b^2 - \log a^2)$$

and thus

$$\Xi_\xi(\xi_0) = k. \tag{27}$$

Moreover, the fact that Ξ does not depend on ψ implies together with (25) that $H_\psi(\xi, \psi, \eta) = 0$ for all $(\xi, \psi, \eta) \in Q$. Hence,

$$H(\xi, \psi, \eta) = H(\xi, \eta) \quad \text{for all } (\xi, \psi, \eta) \in Q.$$

From the second contact equation (26) it follows that

$$\Psi(\xi_0, \psi, \eta) = \tan^{-1}\left(\tfrac{H_\eta(\xi_0,\eta)}{k} \tan \psi - \tfrac{3H_\xi(\xi_0,\eta)}{k}\right) \quad \text{for all } (\psi, \eta) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times \mathbf{R}.$$

Since $\eta \mapsto H(\xi_0, \eta + \frac{4\pi}{3}) - H(\xi_0, \eta)$ is a continuous function on a connected set which, by the periodicity condition (12), can only take discrete values, there must exist $m \in \mathbf{Z}$ such that

$$H(\xi_0, \eta + \tfrac{4\pi}{3}) = H(\xi_0, \eta) + m\tfrac{4\pi}{3} \quad \text{for all } \eta \in \mathbf{R}.$$

We claim that the integer m is non-zero. Assume towards a contradiction that $m = 0$. Then $H(\xi_0, \eta + \frac{4\pi}{3}) = H(\xi_0, \eta)$ for all $\eta \in \mathbf{R}$, in particular $H(\xi_0, -\frac{2\pi}{3}) = H(\xi_0, \frac{2\pi}{3})$. Again by the Mean Value Theorem and the differentiability of $\eta \mapsto H(\xi_0, \eta)$, there exists $\eta' = \eta'(\xi_0) \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$ such that

$$H_\eta(\xi_0, \eta') = 0.$$

We have

$$\tilde{f}(\xi_0, \psi, \eta') = (\Xi(\xi_0), \tan^{-1}\left(\tfrac{-3H_\xi(\xi_0,\eta')}{k}\right), H(\xi_0, \eta')) = \tilde{f}(\xi_0, \psi', \eta'),$$

for arbitrary $\psi, \psi' \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This contradicts the injectivity assumption of f, see (12) in Proposition 9. Therefore we have in fact $m \neq 0$.

We can again apply the Mean Value Theorem in order to prove the existence of a number $\eta_0 \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$ with the property that

$$m\tfrac{4\pi}{3} = H(\xi_0, \tfrac{2\pi}{3}) - H(\xi_0, -\tfrac{2\pi}{3}) = H_\eta(\xi_0, \eta_0)\tfrac{4\pi}{3}.$$

Thus, there exists $\eta_0 \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$ such that

$$H_\eta(\xi_0, \eta_0) = m \neq 0. \tag{28}$$

Let us now consider

$$\psi_0 := \tan^{-1}\left(\frac{3H_\xi(\xi_0, \eta_0)}{m}\right) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{29}$$

This number is chosen such that

$$H_\eta(\xi_0, \eta_0) \tan\psi_0 = 3H_\xi(\xi_0, \eta_0).$$

The formula for the Beltrami coefficient is of a particularly simple form in this situation. We need the partial derivatives of Ξ, Ψ and H in the point (ξ_0, ψ_0, η_0). Since $\Xi_\xi(\xi_0) = k \neq 0$, we have $\Xi_\xi \neq 0$ in a neighborhood of ξ_0 and we can write

$$\Psi(\xi, \psi, \eta) = \tan^{-1}\left(\frac{-3H_\xi(\xi, \eta) + \tan\psi H_\eta(\xi, \eta)}{\Xi_\xi(\xi)}\right).$$

We differentiate with respect to ψ and insert (27), (28) and (29). This yields

$$\Psi_\psi(\xi_0, \psi_0, \eta_0) = \frac{m}{k}\left(1 + \frac{9H_\xi(\xi_0, \eta_0)^2}{m^2}\right).$$

Direct computations yield

$$(W_{\xi,\eta}\Xi)(\xi_0) = k \quad \text{and} \quad \Xi_\psi(\xi_0) = 0.$$

We compute using (21)

$$\left|\frac{\overline{W}(\Xi + \mathrm{i}\Psi)}{W(\Xi + \mathrm{i}\Psi)}(\xi_0, \psi_0, \eta_0)\right| = \left|\frac{\left(k^2 - m\left(1 + \frac{9H_\xi^2(\xi_0,\eta_0)}{m^2}\right)\right) + \mathrm{i}(kW_{\xi,\eta}\Psi(\xi_0, \psi_0, \eta_0))}{\left(k^2 + m\left(1 + \frac{9H_\xi^2(\xi_0,\eta_0)}{m^2}\right)\right) + \mathrm{i}(kW_{\xi,\eta}\Psi(\xi_0, \psi_0, \eta_0))}\right|.$$

Let us observe that

$$\frac{m^2 + 9H_\xi^2(\xi_0, \eta_0)}{m} \neq -k^2,$$

since the absolute value of the left-hand side is bigger or equal than 1, whereas the absolute value of the right-hand side is less than 1.

Since f is assumed to be orientation preserving, we have $|\mu_f| < 1$ almost everywhere. By continuity of $|\mu_f|$ we necessarily have

$$|\mu_f(\Phi(\xi_0, \psi_0, \eta_0))| < 1.$$

This happens exactly if

$$\left|k^2 - m\left(1 + \frac{9H_\xi^2(\xi_0, \psi_0, \eta_0)}{m^2}\right)\right| < \left|k^2 + m\left(1 + \frac{9H_\xi^2(\xi_0, \psi_0, \eta_0)}{m^2}\right)\right|,$$

which is possible only if $m > 0$. We conclude that $m \in \mathbf{N}$ and thus

$$a := m\left(1 + \frac{9H_\xi^2(\xi_0, \eta_0)}{m^2}\right) \geq 1.$$

We are now in a position to show that

$$|\mu_f(\Phi(\xi_0, \psi_0, \eta_0))| \geq \frac{1 - k^2}{1 + k^2} = \|\mu_{f_k}\|_\infty. \tag{30}$$

In this context, we emphasize that by continuity of $|\mu_f|$, the inequality (30) in the point $\Phi(\xi_0, \psi_0, \eta_0)$ is enough to show that $\|\mu_f\|_\infty \geq \|\mu_{f_k}\|_\infty$.

The estimate (30) can be proved as follows. First, we have for any $a \geq 1$ that

$$\frac{a-k^2}{a+k^2} \geq \frac{1-k^2}{1+k^2}. \tag{31}$$

Furthermore, $a \geq 1 > k^2 > 0$ implies that

$$\left|\frac{a-k^2+\mathrm{i}b}{a+k^2+\mathrm{i}b}\right| \geq \left|\frac{a-k^2}{a+k^2}\right| \tag{32}$$

for arbitrary $b \in \mathbf{R}$. Together with (31) this yields the desired result (30). □

5. Remarks and open questions

5.1. Further extremal problems. Proposition 5 can be used to solve a question similar to the classical Grötzsch problem in the complex plane (see [Ahl54]). Let $a, b > 0$ and consider the rectangle $R_{a,b} = \{z \in \mathbf{C} \colon 0 \leq \mathrm{Re}(z) \leq a,\, 0 \leq \mathrm{Im}(z) \leq b\}$; this is foliated by curves $\gamma_{x,I}(s) = x + \mathrm{i}s$, $s \in [0, b]$, $x \in [0, a]$. Such a curve can be lifted to a horizontal curve in the Heisenberg group, uniquely up to vertical translation. In this way we obtain for $c > 0$ the domain

$$\Omega = \{(z, t - \mathrm{Im}(z^2)) \in \mathbf{H}^1 \colon z \in R_{a,b},\, t \in (0, c)\}$$

which shall be mapped to

$$\Omega' = \{(z, t - \mathrm{Im}(z^2)) \in \mathbf{H}^1 \colon z \in R_{a',b'},\, t \in (0, c')\}, \quad a', b', c' > 0.$$

We impose similar boundary conditions as in the classical case. For $y \geq 0$, we denote

$$\partial\Omega_y = \{(z, t - \mathrm{Im}(z^2)) \colon \mathrm{Im}(z) = y\}$$

and we consider the class $\mathcal{F}$ of all quasiconformal mappings $f \colon \Omega \to \Omega'$ which extend homeomorphically to the boundary with

$$f(\partial\Omega_0) = \partial\Omega'_0 \quad \text{and} \quad f(\partial\Omega_b) = \partial\Omega'_{b'}.$$

If we assume in addition that

$$\tfrac{c'}{c} = \tfrac{a'b'}{ab} \quad \text{and} \quad a'b > ab',$$

then the map f_0, given by

$$f_0(z, t) = \left(\tfrac{1}{2}\left(\tfrac{a'}{a} + \tfrac{b'}{b}\right) z + \tfrac{1}{2}\left(\tfrac{a'}{a} - \tfrac{b'}{b}\right) \overline{z}, \tfrac{a'b'}{ab} t\right)$$

is an orientation preserving quasiconformal map which belongs to the class $\mathcal{F}$. By applying Proposition 5 to the family

$$\Gamma_0 = \{\gamma_{x,t} \colon x \in (0, a),\, t \in (0, c)\}$$

with $\gamma_{x,t}(s) = (x + \mathrm{i}s, t - \mathrm{Im}((x + \mathrm{i}s)^2))$, and using standard modulus arguments, it can be shown that f_0 is extremal in $\mathcal{F}$ in the sense that

$$\int_\Omega K(\cdot, f_0)^2 \,\mathrm{d}\mathcal{L}^3 \leq \int_\Omega K(\cdot, f)^2 \,\mathrm{d}\mathcal{L}^3 \quad \text{and} \quad K_{f_0}^2 \leq K_f^2$$

for all $f \in \mathcal{F}$.

5.2. More examples in logarithmic coordinates. The logarithmic type coordinates from Section 3 are well suited to describe spheres-preserving quasiconformal mappings in the Heisenberg group. Often, they allow us to describe counterparts of planar quasiconformal mappings in an easier way than it would be the case for the usual cartesian coordinates.

For instance, an interesting map is given by

$$\tilde{f}_k(\xi, \psi, \eta) = (k\xi, \psi, k\eta).$$

For $0 < k < 1$, this defines an quasiconformal mapping of $\mathbf{H}^1_0 \setminus \{(z,t)\colon \arg(z) = 0 \pmod{2\pi}\}$ onto its image with constant distortion $K(\cdot, f_k) \equiv k^{-1}$.

Given $k > 0$, a quasiconformal mapping on the Heisenberg group similar to the *logarithmic spiral map* in the complex plane is given by

$$\tilde{f}_k(\xi, \psi, \eta) = (\xi, \tan^{-1}(\tan\psi - 3k), \eta + k\xi).$$

This map has been studied in [Pla09]. Its distortion $K(\Phi(\xi, \psi, \eta), f_k)$ is a function of ψ only.

Another interesting map is the Heisenberg *Fenchel–Nielsen twist* f_k which in logarithmic coordinates is defined by the function

$$\tilde{f}_k(\xi, \psi, \eta) = \left(\xi + k\psi, \psi, \eta + \frac{k}{3}\log(\cos\psi)\right), \quad k \in \mathbf{R} \setminus \{0\}.$$

Obviously f_k preserves the complex plane $\mathbf{C}$ and is defined everywhere sufficiently away from the vertical axis. It can be shown that f_k is quasiconformal with Beltrami coefficient and distortion given by

$$\mu_{f_k}(z,t) = \frac{ik}{2 - ik}\frac{z}{\bar{z}}\frac{(-|z|^2 + \mathrm{i}t)}{(-|z|^2 - \mathrm{i}t)} \quad \text{and} \quad K((z,t), f_k) = \frac{|2 - ik| + |k|}{|2 + ik| - |k|} = K_{f_k}.$$

The study of the extremal properties of these mappings could be a subject for further research.

5.3. Open questions related to the radial stretching. Let $0 < a < b$, $0 < k < 1$ and let $\mathcal{F}$ be the class of all quasiconformal mappings form $A(a,b)$ to $A(a^k, b^k)$ which extend homeomorphically to the boundary, mapping the inner boundary of $A(a,b)$ to the respective boundary component of $A(a^k, b^k)$.

Problem A. Standard normal family arguments, using Theorem F in [KR95], guarantee the existence of a minimizer for the maximal distortion within the class $\mathcal{F}$. Is the radial stretch map (23) such a minimizer?

The classical radial stretch map on the complex plane is an important example of a quasiconformal mapping, as it is extremal for various problems. Astala [Ast94] has proved Gehring's conjecture [Geh73] on the exponent of higher integrability in the two-dimensional case, showing that a planar K-quasiconformal mapping lies in the Sobolev space $W^{1,p}_{\mathrm{loc}}$ with $p < 2K/(K-1)$. The example of the radial stretch map

$$f(z) = z|z|^{(1/K)-1}$$

demonstrates that the given bound is sharp. Moreover, it is known [Ahl54] that a K-quasiconformal map in the plane is locally Hölder continuous with exponent $1/K$ and again the stretch map can be used to show that this exponent cannot be improved.

It seems natural to analyze the Heisenberg stretch (23) with respect to its Hölder and Sobolev exponents to see whether it has similar extremal properties as the Euclidean stretch map. Korányi and Reimann [KR95] established the analogue of Gehring's higher integrability result for quasiconformal mappings on the Heisenberg group, i.e., a quasiconformal map f on $\mathbf{H}^1$ lies in $HW^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{H}^1)$ for an exponent

$p > 4$. An upper bound for p in the spirit of Astala's result is not known. For our radial stretch map (23) with $0 < k < 1$ we find

$$J(\Phi(\xi,\psi,\eta), f_k) = e^{2(k-1)\xi}\frac{k^2}{k^2\cos^2\psi + \sin^2\psi}.$$

Thus, by the analytic definition of quasiconformality,

$$\begin{aligned}\int_{B_H(0,r_0)} \|D_H f_k\|^p \,\mathrm{d}\mathcal{L}^3 &\le (K_{f_k}^2)^{p/4}\int_{B_H(0,r_0)} J(\cdot, f_k)^{p/4}\,\mathrm{d}\mathcal{L}^3\\ &= (K_{f_k}^2)^{p/4}k^{p/2}\pi\int_{-\infty}^{\xi_0} e^{2\xi(k\frac{p}{4}-\frac{p}{4}+1)}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{(k^2\cos^2\psi+\sin^2\psi)^{p/4}}\,\mathrm{d}\psi\right)\mathrm{d}\xi\end{aligned}$$

for $r_0 = e^{\xi_0/2} > 0$. It follows that if $f_k \in HW^{1,p}_{\mathrm{loc}}$, then necessarily

$$p < \frac{4}{1-k} = \frac{4(K_{f_k}^2)^{1/4}}{(K_{f_k}^2)^{1/4}-1}.$$

Recall that a quasiconformal map f is K-quasiconformal according to the analytic definition in Section 5.4.2 with $K = K_f^2$. Denoting

$$p(\mathbf{H}^1, K) = \sup\{p \ge 1\colon f \in HW^{1,p}_{\mathrm{loc}}(\mathbf{H}^1,\mathbf{H}^1) \text{ for every } K\text{-quasiconformal map } f\colon \mathbf{H}^1 \to \mathbf{H}^1\},$$

we obtain that necessarily $p(\mathbf{H}^1, K) \le \frac{4K^{\frac14}}{K^{\frac14}-1}$.

Concerning Hölder continuity, it is known that quasiconformal mappings on the Heisenberg group [KR95, p. 53] and more general Carnot groups [Hei95b] are locally Hölder continuous; a bound for the Hölder exponent in terms of the distortion has been given in [BHT02]. In particular, this result shows that a K-quasiconformal map f between domains in the Heisenberg group is locally $K^{-1/3}$ Hölder continuous.

Problem B. A consideration at the origin indicates that the stretch map f_k, $0 < k < 1$, given by (23), is Hölder continuous with exponent at most $k = K_{f_k}^{-1/2} = K^{-1/4}$. If this is the actual Hölder exponent of f_k, then our stretch map could not be used to prove sharpness of Theorem 6.6 in [BHT02]. Can one find a quasiconformal mapping on the Heisenberg group which proves that the bound for the Hölder exponent given in [BHT02] is optimal?

Appendix

5.4. Background results on quasiconformal mappings on the Heisenberg group.

5.4.1. The Heisenberg group. The *(first) Heisenberg group* is a Lie group with underlying manifold $\mathbf{R}^3$. It is convenient to identify the Heisenberg group with $\mathbf{C}\times\mathbf{R}$, as described in the introduction. Elements $p \in \mathbf{H}^1$ are written in the form (z,t) with $z = x + \mathrm{i}y \in \mathbf{C}$ and $t \in \mathbf{R}$. The complex vector fields Z and $\bar{Z}$ satisfy the non-trivial commutator relation $[Z,\overline{Z}] = -2\mathrm{i}T$. The Lie algebra of left invariant vector fields of the Heisenberg group has a grading $\mathfrak{h}^1 = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ with

$$\mathfrak{v}_1 = \mathrm{span}_{\mathbf{R}}\{\operatorname{Re} Z, \operatorname{Im} Z\} \quad \text{and} \quad \mathfrak{v}_2 = \mathrm{span}_{\mathbf{R}}\{T\}.$$

Elements of the first layer $\mathfrak{v}_1$ are called *horizontal left invariant vector fields*. The *horizontal bundle* $H\mathbf{H}^1$ is the subbundle of the tangent bundle $T\mathbf{H}^1$ whose fibers are the *horizontal subspaces*

$$H_p\mathbf{H}^1 = \mathrm{span}_{\mathbf{R}}\{\mathrm{Re}\, Z_p,\ \mathrm{Im}\, Z_p\}, \quad p \in \mathbf{H}^1.$$

The *contact form* of $\mathbf{H}^1$ is the form

$$\tau = \mathrm{d}t - \mathrm{i}\bar{z}\mathrm{d}z + \mathrm{i}z\mathrm{d}\bar{z}.$$

A *contact transformation* $f\colon \Omega \to \Omega'$ on $\mathbf{H}^1$ is a diffeomorphism between domains Ω and Ω' in $\mathbf{H}^1$ which preserves the contact structure, i.e.,

$$f^*\tau = \lambda\tau \tag{33}$$

for some non-vanishing real valued function λ. We identify $\mathbf{H}^1$ with $\mathbf{C}\times\mathbf{R}$ and write $f = (f_I, f_3)$, $f_I = f_1 + \mathrm{i}f_2$. A contact map f is determined by f_I in the sense that

$$\overline{f}_I Z f_I - f_I Z \overline{f}_I + \mathrm{i}Z f_3 = 0 \tag{34}$$

$$f_I \overline{Z f}_I - \overline{f}_I \overline{Z} f_I - \mathrm{i}\overline{Z} f_3 = 0 \tag{35}$$

$$-\mathrm{i}(\overline{f}_I T f_I - f_I T \overline{f}_I + \mathrm{i}T f_3) = \lambda. \tag{36}$$

5.4.2. Quasiconformal mappings. As shown in [KR85], a smooth quasiconformal map is a contact transformation. Yet typical quasiconformal maps are not smooth but they belong to an appropriate Sobolev class and they are contact almost everywhere. To formulate this precisely, let $1 \le p < \infty$ and let Ω be a domain in $\mathbf{H}^1$. We say that a function $u\colon \Omega \to \mathbf{C}$ belongs to the *horizontal Sobolev space*, $u \in HW^{1,p}(\Omega, \mathbf{C})$, if $u \in L^p(\Omega, \mathbf{C})$ and there exist functions $v, w \in L^p(\Omega, \mathbf{C})$ such that

$$\int_\Omega v\varphi \,\mathrm{d}\mathcal{L}^3 = -\int_\Omega uZ\varphi\,\mathrm{d}\mathcal{L}^3 \quad \text{and} \quad \int_\Omega w\varphi\,\mathrm{d}\mathcal{L}^3 = -\int_\Omega u\overline{Z}\varphi\,\mathrm{d}\mathcal{L}^3$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega, \mathbf{R})$. If this is the case, we denote by Zu and $\overline{Z}u$ the weak horizontal complex derivatives v and w. This definition is an equivalent reformulation of the definitions given in [KR95] and [Dai00]. A map $f = (f_I, f_3)\colon \Omega \to \mathbf{H}^1$ is said to belong to $HW^{1,p}(\Omega, \mathbf{H}^1)$ if f_I, f_3 are in $HW^{1,p}(\Omega, \mathbf{C})$. The local horizontal Sobolev spaces $HW^{1,p}_{\mathrm{loc}}$ are defined in the obvious way. If a continuous function $u\colon \Omega \to \mathbf{C}$ on a domain $\Omega \subseteq \mathbf{H}^1$ belongs to $HW^{1,p}(\Omega)$, then it is absolutely continuous on almost every curve of a fibration determined by a horizontal left invariant vector field, see [KR95]. It follows that the pointwise derivatives $(\mathrm{Re}\, Z)u$ and $(\mathrm{Im}\, Z)u$ exist almost everywhere and coincide with the distributional derivatives almost everywhere. We recall that the Hausdorff dimension of $(\mathbf{H}^1, d_H)$ is 4. It turns out that the correct regularity assumption in the analytic definition of quasiconformal mappings on the Heisenberg group is $HW^{1,p}_{\mathrm{loc}}$ for $p = 4$.

A mapping $f \in HW^{1,4}_{\mathrm{loc}}(\Omega, \mathbf{H}^1)$ is called *weakly contact* if it satisfies (34) and (35) almost everywhere in Ω. For such a mapping, one can define the *formal horizontal differential* $D_H f(p)\colon H_p\mathbf{H}^1 \to H_{f(p)}\mathbf{H}^1$ for almost every $p \in \Omega$, see [Dai00]. The mapping $D_H f(p)$ can be extended to a Lie algebra homomorphism, which is called the *formal P-differential* $D_0 f(p)$ of f at p. Using complex notation, we have

$$\|D_H f(p)\| := \max\{\|D_H(p)V\|\colon \|V\| = 1\} = |Zf_I(p)| + |\overline{Z}f_I(p)| \quad \text{a.e.}$$

and

$$J(p,f) := \det D_0 f(p) = (\det D_H f(p))^2 = (|Zf_I(p)|^2 - |\overline{Z} f_I(p)|^2)^2 \quad \text{a.e.}$$

There are various analytic definitions for quasiconformality on the Heisenberg group which are equivalent to the metric definition stated in the introduction, see [Hei95a] and [Vod96]. We use the following definition which appears in [Dai00].

Definition 15. (Analytic definition) A homeomorphism $f\colon \Omega \to \Omega'$ between domains Ω, Ω' in $\mathbf{H}^1$ is K-*quasiconformal* if $f \in HW^{1,4}_{\mathrm{loc}}(\Omega, \mathbf{H}^1)$ is weakly contact, and there exists a constant $K \geq 1$ such that

$$\|D_H f(p)\|^4 \leq K J(p,f) \quad \text{for almost every } p \in \Omega. \tag{37}$$

A map is *quasiconformal*, if it is K-quasiconformal for some K.

A homeomorphism is quasiconformal according to this analytic definition if and only if it is quasiconformal according to the metric definition.

It can be proved that $J(p,f) \neq 0$ a.e. for a quasiconformal mapping. The above considerations show for a quasiconformal map $f\colon \Omega \to \Omega'$ between two domains in the Heisenberg group that

$$K(p,f)^2 = \frac{\|D_H f(p)\|^4}{J(p,f)} = \left(\frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{|Zf_I(p)| - |\overline{Z} f_I(p)|}\right)^2 \tag{38}$$

a.e. on Ω. We set $K(p,f)^2 = 1$ at the nonregular points. This defines a measurable function on Ω which is finite almost everywhere.

A quasiconformal map $f\colon \Omega \to \Omega'$ between domains in the Heisenberg group is called *orientation preserving* if

$$\det D_H f(p) > 0 \quad \text{for almost every } p \in \Omega.$$

To prove the modulus inequality for quasiconformal mappings on $\mathbf{H}^1$, one uses the absolute continuity of quasiconformal mappings on curves and in measure similarly as in the Euclidean case. See [Dai00] for the change of variables formula in the case of quasiconformal mappings on the Heisenberg group which we state below.

Theorem 16. *Let $f\colon \Omega \to \Omega'$ be a quasiconformal mapping between domains $\Omega, \Omega' \subseteq \mathbf{H}^1$. Then the following transformation formula holds: If $u\colon \mathbf{H}^1 \to \mathbf{R}$ is a measurable nonnegative function, then the function $p \mapsto (u \circ f)(p)|J(p,f)|$ is measurable and we have*

$$\int_\Omega (u \circ f)(p)|J(p,f)| \, \mathrm{d}\mathcal{L}^3(p) = \int_{\Omega'} u(q) \, \mathrm{d}\mathcal{L}^3(q).$$

5.5. Modulus of curve families.

5.5.1. Curves in the Heisenberg group. By a curve γ on $\mathbf{H}^1$ we shall always mean a continuous map on an interval. The points on a curve $\gamma\colon [a,b] \to \mathbf{H}^1$ are denoted by

$$\gamma(t) = (\gamma_I(t), \gamma_3(t)) \in \mathbf{C} \times \mathbf{R}.$$

The curve γ is called *rectifiable* if its *length* $\ell(\gamma)$ with respect to the Heisenberg metric d_H is finite. In this case it has a uniquely determined 1-Lipschitz parameterization by arc-length

$$\tilde{\gamma}\colon [0, \ell(\gamma)] \to (\mathbf{H}^1, d_H).$$

We let $\rho\colon \mathbf{H}^1 \to [0,\infty]$ be a non-negative Borel function and write

$$\int_\gamma \rho \,\mathrm{d}\ell := \int_0^{\ell(\gamma)} \tilde{\gamma}(s)\,\mathrm{d}s$$

for the curve integral with respect to arc-length.

An absolutely continuous curve $\gamma\colon [a,b] \to \mathbf{H}^1$ (in the Euclidean sense) is called *horizontal* if

$$\dot{\gamma}(t) \in H_{\gamma(t)}\mathbf{H}^1 \quad \text{for almost every } t \in [a,b].$$

A curve $\gamma\colon [a,b] \to \mathbf{H}^1$ is absolutely continuous with respect to the Heisenberg distance d_H if and only if it is a horizontal curve, see [Pan89b]. Moreover, the length of a smooth rectifiable curve $\gamma = (\gamma_I, \gamma_3)$ with respect to d_H is given by the integral over the norm of the horizontal part of the tangent vector,

$$\ell(\gamma) = \int_a^b |\dot{\gamma}_I(t)|\,\mathrm{d}t,$$

see [Kor85]. In fact, the same formula holds for arbitrary horizontal curves and for integrals $\int_\gamma \rho\,\mathrm{d}\ell$, where not necessarily $\rho = 1$.

Proposition 17. *If $\gamma\colon [a,b] \to \mathbf{H}^1$ is horizontal (equivalently: absolutely continuous with respect to d_H), then, for any non-negative Borel function $\rho\colon \mathbf{H}^1 \to [0,\infty]$ we have*

$$\int_\gamma \rho\,\mathrm{d}\ell = \int_a^b \rho(\gamma(t))|\dot{\gamma}_I(t)|\,\mathrm{d}t.$$

5.5.2. Modulus of a curve family. The definition for the conformally invariant 4-modulus of a family Γ of curves in $\mathbf{H}^1$ has been given in the introduction, see (5). In the case of curves $\gamma\colon (a,b) \to \mathbf{H}^1$ the notion of rectifiability is replaced by local rectifiability. The curve γ is said to be *locally rectifiable* if all its closed subcurves γ' are rectifiable. In this case we set

$$\int_\gamma \rho\,\mathrm{d}\ell = \sup_{\gamma'} \int_{\gamma'} \rho\,\mathrm{d}\ell.$$

Note that a family which consists merely of curves that are not locally rectifiable has modulus zero. Similarly to the Euclidean result in [Fug57], quasiconformal mappings of the Heisenberg group are absolutely continuous on almost every curve, [Sha00, HKST01]. Thus, given a quasiconformal map $f\colon \Omega \to \Omega'$ between domains in the Heisenberg group and given a family Γ of closed rectifiable curves in Ω, one has

$$M_4(\{\gamma \in \Gamma\colon f \circ \tilde{\gamma} \text{ not absolutely continuous}\}) = 0.$$

Notice that if $\gamma\colon [a,b] \to (\mathbf{H}^1, d_H)$ is absolutely continuous and f is absolutely continuous on γ, then also $f \circ \gamma\colon [a,b] \to (\mathbf{H}^1, d_H)$ is absolutely continuous.

5.5.3. Modulus inequality. It is well known that quasiconformal mappings in the complex plane satisfy a modulus inequality [Ahl66], see [LV73] for a proof in the non-smooth case. Tang proved in [Tan96] a modulus inequality for $\mathcal{C}^2$ maps on compact, smooth and strongly pseudoconvex CR 3-manifolds. Pansu established in [Pan89a] the quasi-invariance of the so-called “module grossier”, see also [Pan89b] and [KR95] for the analogous inequality for the capacity of a condenser. More recently, the equivalence of the metric definition of quasiconformality with a geometric condition involving the modulus of curve families has been proved in [HKST01] for a large class of metric measure spaces with bounded geometry.

We use a formulation of the modulus inequality in terms of a mean distortion functional. This type of inequality can essentially be derived from the proof of a result in the more general setting of quasimeromorphic mappings on H-type Carnot groups in [MV06]. For clarity, we will include here the simpler proof in the case of quasiconformal mappings on the Heisenberg group. The second half of the proof can be simplified in our situation since quasiconformal maps have quasiconformal inverses.

Theorem 18. *Suppose that $f\colon \Omega \to \Omega'$ is a quasiconformal map between two domains in $\mathbf{H}^1$ and Γ is a family of curves in Ω. Then*

$$M_4(\Gamma) \le \int_{\Omega'} K(f^{-1}(\zeta,\tau), f)^2 \widetilde{\rho}^4(\zeta,\tau)\, \mathrm{d}\mathcal{L}^3(\zeta,\tau) \quad \text{for all } \widetilde{\rho} \in \mathrm{adm}(f(\Gamma)), \tag{39}$$

$$M_4(f(\Gamma)) \le \int_{\Omega} K((z,t), f)^2 \rho^4(z,t)\, \mathrm{d}\mathcal{L}^3(z,t) \quad \text{for all } \rho \in \mathrm{adm}(\Gamma), \tag{40}$$

and thus

$$\frac{1}{K_f^2} M_4(\Gamma) \le M_4(f(\Gamma)) \le K_f^2 M_4(\Gamma). \tag{41}$$

If the map f is conformal, then it is a smooth map with $\overline{Z} f_I = 0$ and $M_4(f(\Gamma)) = M_4(\Gamma)$.

Proof. We start by proving (39), from where (40) and (41) follow easily.

Without loss of generality we may assume that the curves in Γ are defined on a closed interval $[a,b]$. If a curve $\gamma\colon (a,b) \to \Omega$ is defined on an open interval, we consider its closed subcurves γ' and the proof reduces to the case of closed curves.

Let Γ_0 be the family of all rectifiable curves in Γ on which f is absolutely continuous (the non-rectifiable curves have modulus zero). Since f is quasiconformal, we have $M_4(\Gamma) = M_4(\Gamma_0)$. Moreover, f is *P-differentiable*, that is, differentiable in the sense of Pansu [Pan89b], almost everywhere. For the P-differential we have versions of the chain rule which we will use in the following. Take now an arbitrary density $\tilde{\rho}$ which is admissible for $f(\Gamma)$ and let E_0 be the set of points $(z,t) \in \Omega$ on which f is not P-differentiable. Using the quasiconformality of f, it follows that $\mathcal{L}^3(E_0) = 0$. Since $\mathcal{L}^3$ is a Borel measure, there exists a Borel set $E \supseteq E_0$ with $\mathcal{L}^3(E) = 0$. To $\tilde{\rho}$ we can now assign a pull-back density $\rho_{\tilde{\rho}}$ defined by

$$\rho_{\tilde{\rho}}(z,t) := \begin{cases} \tilde{\rho}(f(z,t))(|Zf_I(z,t)| + |\bar{Z} f_I(z,t)|), & (z,t) \in \Omega \setminus E, \\ \infty, & (z,t) \in E, \\ 0, & (z,t) \in \mathbf{H}^1 \setminus \Omega. \end{cases}$$

Clearly, $\rho_{\tilde{\rho}}$ is Borel since E is a Borel set, $\tilde{\rho}$ a Borel function and the continuity of f implies that the directional derivatives Zf_I, $\overline{Z} f_I$ are Borel functions on the domain where they exist. We will show that $\rho_{\tilde{\rho}}$ is admissible for Γ_0. To this end, let $\gamma\colon [a,b] \to \Omega$ be an arbitrary curve in Γ_0. By definition of Γ_0, it is rectifiable and therefore has a parameterization by arc-length, $\tilde{\gamma}\colon [0, \ell(\gamma)] \to \Omega$. We have to distinguish two cases:

Case 1. If $\mathcal{L}^1(\{s \in [0,\ell(\gamma)] \colon \tilde\gamma(s) \in E\}) > 0$, then

$$\int_\gamma \rho_{\tilde\rho}\,\mathrm{d}\ell = \int_0^{\ell(\gamma)} \rho_{\tilde\rho}(\tilde\gamma(s))\,\mathrm{d}s = \infty.$$

Case 2. If $\mathcal{L}^1(\{s \in [0,\ell(\gamma)] \colon \tilde\gamma(s) \in E\}) = 0$, then the image curves have nice differentiability properties. Indeed, $\tilde\gamma(s) \notin E$ for almost every $s \in [0,\ell(\gamma)]$, which is equivalent to say that f is P-differentiable on $\tilde\gamma(s)$ for almost every $s \in [0,\ell(\gamma)]$. In this context, we recall from [Pan89b] that if $\tilde\gamma \colon [0,\ell(\gamma)] \to (\mathbf{H}^1, d_H)$ is Lipschitz, then $\tilde\gamma$ is P-differentiable almost everywhere with

$$\lim_{c\to 0} \delta_{\frac{1}{c}}(\tilde\gamma(s)^{-1} * \tilde\gamma(s+c)) = (\dot{\tilde\gamma}_I(s), 0) \in \mathbf{C}\times\mathbf{R} = \mathbf{H}^1$$

for almost every $s \in [0,\ell(\gamma)]$. By the chain rule for the P-differential we get $(f_I \dot\circ \tilde\gamma)(s) = Zf_I(\tilde\gamma(s))\dot{\tilde\gamma}_I(s) + \bar Z f_I(\tilde\gamma(s))\dot{\bar{\tilde\gamma}}_I(s)$ for almost every $s \in [0,\ell(\gamma)]$, and therefore,

$$|(f_I \dot\circ \tilde\gamma)(s)| \le (|Zf_I(\tilde\gamma(s))| + |\bar Z f_I(\tilde\gamma(s))|)|\dot{\tilde\gamma}_I(s)| \quad \text{a.e. } s \in [0,\ell(\gamma)].$$

Now $\tilde\gamma$ is absolutely continuous on $(\mathbf{H}^1, d_H)$ (since it is Lipschitz). Moreover, $f\circ\tilde\gamma$ is absolutely continuous with respect to d_H by assumption and Proposition 17 applies. Altogether, this yields

$$\begin{aligned}\int_\gamma \rho_{\tilde\rho}\,\mathrm{d}\ell &= \int_0^{\ell(\gamma)} \rho_{\tilde\rho}(\tilde\gamma(s))|\dot{\tilde\gamma}_I(s)|\,\mathrm{d}s \\ &= \int_0^{\ell(\gamma)} \widetilde\rho(f(\tilde\gamma(s)))(|Zf_I(\tilde\gamma(s))| + |\bar Z f_I(\tilde\gamma(s))|)|\dot{\tilde\gamma}_I(s)|\,\mathrm{d}s \\ &\ge \int_0^{\ell(\gamma)} \widetilde\rho(f(\tilde\gamma(s)))|(f_I \dot\circ \tilde\gamma)(s)|\,\mathrm{d}s = \int_{f\circ\tilde\gamma} \widetilde\rho\,\mathrm{d}\ell = \int_{f\circ\gamma} \widetilde\rho\,\mathrm{d}\ell \ge 1.\end{aligned}$$

Combining the two cases, we deduce that $\rho_{\tilde\rho} \in \mathrm{adm}(\Gamma_0)$. This allows us to conclude as follows:

$$\begin{aligned}&M_4(\Gamma_0) = \inf_{\rho\in\mathrm{adm}(\Gamma_0)} \int_\Omega \rho^4(z,t)\,\mathrm{d}\mathcal{L}^3(z,t) \\ &\le \int_\Omega \rho_{\tilde\rho}^4(z,t)\,\mathrm{d}\mathcal{L}^3(z,t) = \int_\Omega \widetilde\rho^4(f(z,t))(|Zf_I(z,t)| + |\bar Z f_I(z,t)|)^4\,\mathrm{d}\mathcal{L}^3(z,t) \\ &= \int_\Omega \widetilde\rho^4(f(z,t))\left(\frac{|Zf_I(z,t)| + |\bar Z f_I(z,t)|}{|Zf_I(z,t)| - |\bar Z f_I(z,t)|}\right)^2 (|Zf_I(z,t)|^2 - |\bar Z f_I(z,t)|^2)^2\,\mathrm{d}\mathcal{L}^3(z,t) \\ &= \int_\Omega \widetilde\rho^4(f(z,t))K((z,t),f)^2 J((z,t),f)\,\mathrm{d}\mathcal{L}^3(z,t) \\ &= \int_{\Omega'} \widetilde\rho^4(\zeta,\tau)K(f^{-1}(\zeta,\tau),f)^2\,\mathrm{d}\mathcal{L}^3(\zeta,\tau)\end{aligned}$$

for all $\widetilde\rho \in \mathrm{adm}(f(\Gamma))$. Here, we have applied in the last step the transformation formula for quasiconformal mappings, see Theorem 16.

This reasoning yields

$$M_4(\Gamma) = M_4(\Gamma_0) \le K_f^2 M_4(f(\Gamma)),$$

which concludes the proof of the first half of the theorem.

Since the inverse of the quasiconformal map f is again quasiconformal, we may apply (39) to the map f^{-1} and the curve family $f(\Gamma)$. Thus

$$M_4(f(\Gamma)) \le \int_\Omega K(f(z,t), f^{-1})^2 \rho(z,t)^4 \, \mathrm{d}\mathcal{L}^3(z,t) \tag{42}$$

for all $\rho \in \operatorname{adm}(\Gamma)$. We note that

$$K((z,t), f) = K(f(z,t), f^{-1}) \quad \text{almost everywhere.} \tag{43}$$

This follows by the chain rule, see also [KR95, p. 64]. Combining (42) and (43) we obtain the desired result. □

References

[Ahl54] Ahlfors, L. V.: On quasiconformal mappings. - J. Anal. Math. 3, 1954, 1–58; correction, 207–208.

[Ahl66] Ahlfors, L. V.: Lectures on quasiconformal mappings. - Van Nostrand Mathematical Studies 10, D. Van Nostrand Co., 1966.

[Ast94] Astala, K.: Area distortion of quasiconformal mappings. - Acta Math. 173:1, 1994, 37–60.

[AIM09] Astala, K., T. Iwaniec, and G. Martin: Elliptic partial differential equations and quasiconformal mappings in the plane. - Princeton Math. Ser. 48, Princeton Univ. Press, Princeton, NJ, 2009.

[AIMO05] Astala, K., T. Iwaniec, G. J. Martin, and J. Onninen: Extremal mappings of finite distortion. - Proc. London Math. Soc. (3) 91:3, 2005, 655–702.

[BFP11] Balogh, Z., and K. Fässler, and I. Platis: Modulus of curve families and extremality of spiral-stretch maps. - J. Anal. Math. 113, 2011, 265–291.

[BHT02] Balogh, Z. M., I. Holopainen, and J. T. Tyson: Singular solutions, homogeneous norms, and quasiconformal mappings in Carnot groups. - Math. Ann. 324:1, 2002, 159–186.

[BKR07] Balogh, Z. M., and P. Koskela, and S. Rogovin: Absolute continuity of quasiconformal mappings on curves. - Geom. Funct. Anal. 17:3, 2007, 645–664.

[Bra10] Brakalova, M. A.: Sufficient and necessary conditions for conformality at a point. I. Geometric viewpoint. - Complex Var. Elliptic Equ. 55:1-3, 2010, 137–155.

[Dai00] Dairbekov, N. S.: On mappings with bounded distortion on the Heisenberg group. - Sibirsk. Mat. Zh. 41:1, 2000, 49–59.

[Fug57] Fuglede, B.: Extremal length and functional completion. - Acta Math. 98, 1957, 171–219.

[GL00] Gardiner, F. P. and N. Lakic: Quasiconformal Teichmüller theory. - Math. Surveys Monogr. 76, Amer. Math. Soc., 2000.

[Geh73] Gehring, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. - Acta Math. 130, 1973, 265–277.

[Gol99] Goldman, W. M.: Complex hyperbolic geometry. - Oxford Math. Monogr., The Clarendon Press, Oxford Univ. Press, New York, 1999.

[Grö32] Grötzsch, H.: Über möglichst konforme Abbildungen von schlichten Bereichen. - Ber. Math.-phys. Kl. Sachs. Akad. Wiss. Leipzig 84, 1932, 114–120.

[GM01] Gutlyanskiĭ, V., and O. Martio: Rotation estimates and spirals. - Conform. Geom. Dyn. 5, 2001, 6–20 (electronic).

[Hei95a] Heinonen, J.: Calculus on Carnot groups. - In: Fall School in Analysis (Jyväskylä, 1994), Report 68, Univ. Jyväskylä, Jyväskylä, 1995, 1–31.

[Hei95b] Heinonen, J.: A capacity estimate on Carnot groups. - Bull. Sci. Math. 119:5, 1995, 475–484.

[HKST01] Heinonen, J., P. Koskela, N. Shanmugalingam, and J. T. Tyson: Sobolev classes of Banach space-valued functions and quasiconformal mappings. - J. Anal. Math. 85, 2001, 87–139.

[Kor85] Korányi, A.: Geometric properties of Heisenberg-type groups. - Adv. Math. 56:1, 1985, 28–38.

[KR85] Korányi, A., and H. M. Reimann: Quasiconformal mappings on the Heisenberg group. - Invent. Math. 80:2, 309–338.

[KR87] Korányi, A., and H. M. Reimann: Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group. - Bull. Sci. Math. (2) 111:1, 1987, 3–21.

[KR95] Korányi, A., and H. M. Reimann: Foundations for the theory of quasiconformal mappings on the Heisenberg group. - Adv. Math. 111:1, 1995, 1–87.

[Lee09] Lee, J. M.: Manifolds and differential geometry. - Grad. Stud. Math., Amer. Math. Soc., 2009.

[LV73] Lehto, O., and K. I. Virtanen: Quasiconformal mappings in the plane. - Grundlehren Math. Wiss. 126, 2nd edition, Springer-Verlag, New York, 1973.

[MV06] Markina, I., and S. Vodopyanov: On value distributions for quasimeromorphic mappings on **H**-type Carnot groups. - Bull. Sci. Math. 130:6, 2006, 467–523.

[Mar09] Martin, G. J.: The Teichmüller problem for mean distortion. - Ann. Acad. Sci. Fenn. Math. 34:1, 2009, 233–247.

[Min94] Miner, R. R.: Quasiconformal equivalence of spherical CR manifolds. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19:1, 1994, 83–93.

[Mos73] Mostow, G. D.: Strong rigidity of locally symmetric spaces. - Ann. of Math. Stud. 78, Princeton Univ. Press, Princeton, N.J., 1973.

[Pan89a] Pansu, P.: Dimension conforme et sphère à l'infini des variétés à courbure négative. - Ann. Acad. Sci. Fenn. Ser. A I Math. 14:2, 1989, 177–212.

[Pan89b] Pansu, P.: Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. - Ann. of Math. (2) 129:1, 1989, 1–60.

[Pla09] Platis, I. D.: The geometry of complex hyperbolic packs. - Math. Proc. Cambridge Philos. Soc. 147:1, 2009, 205–234.

[Sha00] Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. - Rev. Mat. Iberoam. 16:2, 2000, 243–279.

[Str86] Strebel, K.: Extremal quasiconformal mappings. - Results Math. 10:1-2, 1986, 168–210.

[Tan96] Tang, P.: Regularity and extremality of quasiconformal homeomorphisms on CR 3-manifolds. - Ann. Acad. Sci. Fenn. Math. 21:2, 1996, 289–308.

[Tei40] Teichmüller, O.: Extremale quasikonforme Abbildungen und quadratische Differentiale. - Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1939:22, 1940, 197.

[Tei40] Teichmüller, O.: Bestimmung der extremalen quasikonformen Abbildung bei geschlossenen orientierbaren Riemannschen Flächen. - Abh. Preuss. Akad. Wiss. Math.-Naturw. 4, 1943, 1–42.

[Vod96] Vodop'yanov, S. K.: Monotone functions and quasiconformal mappings on Carnot groups. - Sibirsk. Mat. Zh. 37:6, 1996, 1269–1295.

Received 10 February 2012 • Accepted 3 September 2012