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MULTIPLIERS AND IMAGINARY POWERS OF THE SCHRÖDINGER OPERATORS CHARACTERIZING UMD BANACH SPACES

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Abstract. In this paper we establish L^p-boundedness properties for Laplace type transform spectral multipliers associated with the Schrödinger operator $\mathcal{L} = -\Delta + V$. We obtain for this type of multipliers pointwise representation as principal value integral operators. We also characterize the UMD Banach spaces in terms of the L^p-boundedness of the imaginary powers $\mathcal{L}^{i\gamma}$, $\gamma \in \mathbf{R}$, of $\mathcal{L}$.

1. Introduction

We study certain class of spectral multipliers, usually called Laplace transform type multipliers associated with the Schrödinger operator $\mathcal{L} = -\Delta + V$, where Δ represents the Laplacian operator and the potential $V > 0$ satisfies a reverse Hölder inequality. We prove that L^p-boundedness of some of those multipliers, the imaginary powers of $\mathcal{L}$, acting on Banach valued functions characterizes the UMD property for the Banach space.

We now recall some definitions and properties that will be useful in order to state and to prove our results.

We consider the Schrödinger operator $\mathcal{L} = -\Delta + V$ on $\mathbf{R}^n$, with $n \geq 3$. We assume that $V > 0$ is a locally integrable function on $\mathbf{R}^n$ belonging to the class B_q, that is, there exists $C > 0$ such that, for every ball B in $\mathbf{R}^n$,

$$\left(\frac{1}{|B|}\int_B V^q(x)\,dx\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(x)\,dx,$$

for some $q \geq n/2$.

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The operator $\mathcal{L}$, suitably understood (see, for instance, [2]), is a closed unbounded and positive operator in $L^2(\mathbf{R}^n)$. Then, there exists the spectral measure $E_{\mathcal{L}}$ associated with $\mathcal{L}$. It is not hard to see that 0 is not an eigenvalue of $\mathcal{L}$ in $L^2(\mathbf{R}^n)$, and hence $E_{\mathcal{L}}(\{0\}) = 0$. For every Borel measurable bounded function m on $[0,\infty)$, we define the spectral multiplier $T_m^{\mathcal{L}}$ by

$$T_m^{\mathcal{L}}(f) = \int_{(0,\infty)} m(\lambda) E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbf{R}^n).$$

It is well known that $T_m^{\mathcal{L}}$ defines a bounded operator from $L^2(\mathbf{R}^n)$ into itself.

We say that a function m on $(0,\infty)$ is of Laplace transform type when $m(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\phi(t)\,dt$, $\lambda \in (0,\infty)$, for a certain $\phi \in L^\infty(0,\infty)$. Note that in this case m is continuous and bounded on $(0,\infty)$. The spectral multiplier $T_m^{\mathcal{L}}$ is called of Laplace transform type when the function m is of Laplace transform type.

For every $t > 0$, we define the operator $W_t^{\mathcal{L}}$ by

$$W_t^{\mathcal{L}}(f) = \int_{(0,\infty)} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbf{R}^n).$$

The uniparametric family $\{W_t^{\mathcal{L}}\}_{t>0}$ is the semigroup of operators generated by $-\mathcal{L}$ in $L^2(\mathbf{R}^n)$. For every $t > 0$ and $1 \le p \le \infty$ the operator $W_t^{\mathcal{L}}(f)$ can be extended from $L^p(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ as a contraction from $L^p(\mathbf{R}^n)$ into itself. The semigroup of operators $\{W_t^{\mathcal{L}}\}_{t>0}$ is not conservative. Moreover, we can write, for every $t > 0$ and $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$,

$$W_t^{\mathcal{L}}(f)(x) = \int_{\mathbf{R}^n} W_t^{\mathcal{L}}(x,y) f(y)\,dy, \quad x \in \mathbf{R}^n,$$

where $W_t^{\mathcal{L}}(x,y)$, $x, y \in \mathbf{R}^n$ and $t \in (0,\infty)$, is a $C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times (0,\infty))$ such that, according to Feynman–Kac property,

$$|W_t^{\mathcal{L}}(x,y)| \le C W_t(x,y), \quad x, y \in \mathbf{R}^n \text{ and } t > 0,$$

being

$$W_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbf{R}^n \text{ and } t > 0.$$

We establish a pointwise representation of the Laplace transform type multiplier $T_m^{\mathcal{L}}$ as a principal value integral operator and we prove L^p-boundedness properties of $T_m^{\mathcal{L}}$. As usual by $C_c^\infty(\mathbf{R}^n)$ we denote the space of smooth functions with compact support in $\mathbf{R}^n$.

Theorem 1. *Suppose that* $m(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\phi(t)\,dt$, $\lambda \in (0,\infty)$, *where* $\phi \in L^\infty(0,\infty)$. *Then, for every* $f \in C_c^\infty(\mathbf{R}^n)$,

$$(1) \qquad T_m^{\mathcal{L}}(f)(x) = \lim_{\varepsilon \to 0^+} \left(\alpha(\varepsilon) f(x) + \int_{|x-y|>\varepsilon} K_\phi^{\mathcal{L}}(x,y) f(y)\,dy \right), \quad \text{a.e. } x \in \mathbf{R}^n,$$

where

$$K_\phi^{\mathcal{L}}(x,y) = -\int_0^\infty \phi(t) \frac{\partial}{\partial t} W_t^{\mathcal{L}}(x,y)\,dt, \quad x, y \in \mathbf{R}^n, x \ne y,$$

and α *is a certain measurable bounded function on* $(0,\infty)$. *Moreover, if there exists the limit* $\lim_{t\to 0^+} \phi(t) = \phi(0^+)$, *then*

$$(2) \qquad T_m^{\mathcal{L}}(f)(x) = \phi(0^+) f(x) + \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} K_\phi^{\mathcal{L}}(x,y) f(y)\,dy, \quad \text{a.e. } x \in \mathbf{R}^n.$$

Theorem 2. *Suppose that $m(\lambda) = \lambda\int_0^\infty e^{-\lambda t}\phi(t)\,dt$, $\lambda \in (0,\infty)$, where $\phi \in L^\infty(0,\infty)$. Then $T_m^{\mathcal{L}}$ can be extended to $L^p(\mathbf{R}^n)$ as a bounded operator from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and as a bounded operator from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$. Moreover, this extension can be given by (1) and, when the limit $\lim_{t\to 0^+}\phi(t) = \phi(0^+)$ exists, by (2), for every $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$.*

Note that since the semigroup of operators $\{W_t^{\mathcal{L}}\}_{t>0}$ is not conservative, the L^p-boundedness, $1 < p < \infty$, of the Laplace transform type multipliers $T_m^{\mathcal{L}}$ can not be deduced from the result established in [17, p. 121]. The harmonic analysis operators (maximal operators, Riesz transforms and Littlewood–Paley g-functions) in the Schrödinger setting have been studied in L^p-spaces by several authors in last years (see for instance, [1], [8] and [16]). In order to show Theorems 1 and 2, inspired in the procedure developed by Shen [16] to analyze Riesz transforms, we take advantage that $\mathcal{L}$ is a "nice" perturbation of the Laplacian operator $-\Delta$. This fact allows us to write the multipliers in the Schrödinger setting, in some local sense, as perturbations of the corresponding multipliers associated to the Laplacian.

In the localization of our operators the function ρ defined in [16, p. 516] by

$$\rho(x) = \sup\Big\{r > 0\colon \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le 1\Big\}, \quad x \in \mathbf{R}^n,$$

plays an important role. The main properties of this function ρ can be encountered in [16, §1]. We also use several properties of the heat kernel $W_t^{\mathcal{L}}(x,y)$ associated to the Schrödinger operator $\mathcal{L}$ that can be found, for instance, in [8].

A Banach space B is said to be UMD when the Hilbert transform H defined in a natural way on $L^p(\mathbf{R}^n)\otimes B$ can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself for some (equivalently, for every) $1 < p < \infty$ (see [4] and [5]). Here, for every $1 \le p < \infty$, by $L^p(\mathbf{R}^n, B)$ we represent the Bochner–Lebesgue space of exponent p. UMD property is related to geometric properties of Banach spaces ([6]). In last years several authors have established connections between geometry of Banach spaces and harmonic analysis. In particular, characterizations of UMD, convexity or smoothness properties of a Banach space have been given in terms of L^p-boundedness of certain singular integrals or Littlewood–Paley g-functions ([1], [12], [13], [14] and [20]). Guerre-Delabrière ([11]) characterized the Banach spaces having the UMD property by the L^p-boundedness of the imaginary power $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$, $\gamma \in \mathbf{R}$, of the Laplacian in one dimension. Here, inspired in [11], we characterize the Banach spaces having the UMD property by the L^p-boundedness of the imaginary power $\mathcal{L}^{i\gamma}$, $\gamma \in \mathbf{R}$, of the Schrödinger operator.

Let $\gamma \in \mathbf{R}$. We denote by m_γ the function $m_\gamma(\lambda) = \lambda^{i\gamma}$, $\lambda \in (0,\infty)$. It is clear that $m_\gamma(\lambda) = \lambda\int_0^\infty e^{-\lambda t}\phi_\gamma(t)\,dt$, $\lambda \in (0,\infty)$, where $\phi_\gamma(t) = t^{-i\gamma}/\Gamma(1-i\gamma)$, $t \in (0,\infty)$. We define, as usual, the imaginary power $\mathcal{L}^{i\gamma}$ of $\mathcal{L}$ by

$$\mathcal{L}^{i\gamma} = T_{m_\gamma}^{\mathcal{L}}.$$

For every $\varepsilon > 0$ and every $f \in L^p(\mathbf{R}^n, B)$, $1 \le p < \infty$, we consider the truncation

$$\mathcal{L}_\varepsilon^{i\gamma}(f)(x) = \alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} K_{\phi_\gamma}^{\mathcal{L}}(x,y)f(y)\,dy, \tag{3}$$

where α and $K^{\mathcal{L}}_{\phi_\gamma}(x,y)$ are as in Theorem 1, and the maximal operator associated with $\mathcal{L}^{i\gamma}$ is defined by

$$\mathcal{L}^{i\gamma}_*(f) = \sup_{\varepsilon>0} \|\mathcal{L}^{i\gamma}_\varepsilon(f)\|_B.$$

According to Theorem 2, $\mathcal{L}^{i\gamma}$ can be extended to $L^p(\mathbf{R}^n)$ as a bounded operator from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$. If B is a Banach space we define $\mathcal{L}^{i\gamma}$ on $L^p(\mathbf{R}^n) \otimes B$, $1 < p < \infty$, in a natural way.

Theorem 3. *Let B be a Banach space. Then, the following properties are equivalent:*

(i) *B is a UMD space.*
(ii) *For every $\gamma \in \mathbf{R}$ and for some (equivalently, for any) $1 < p < \infty$, the operator $\mathcal{L}^{i\gamma}$ can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself.*
(iii) *For every $\gamma \in \mathbf{R}$, the operator $\mathcal{L}^{i\gamma}$ can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$.*
(iv) *For every $\gamma \in \mathbf{R}$ and every $f \in L^1(\mathbf{R}^n, B)$, $\mathcal{L}^{i\gamma}_*(f)(x) < \infty$, a.e. $x \in \mathbf{R}^n$.*
(v) *For every $\gamma \in \mathbf{R}$ and every $f \in L^1(\mathbf{R}^n, B)$, there exists the limit $\lim_{\varepsilon\to 0^+} \mathcal{L}^{i\gamma}_\varepsilon(f)(x)$, a.e. $x \in \mathbf{R}^n$.*

In order to prove Theorem 3 we need to establish previously the following extension of [11, Theorem, p. 402]. The maximal operator $(-\Delta)^{i\gamma}_*$ is defined on $L^1(\mathbf{R}^n, B)$ as follows

$$(-\Delta)^{i\gamma}_*(f) = \sup_{\varepsilon>0} \|(-\Delta)^{i\gamma}_\varepsilon(f)\|_B, \quad f \in L^1(\mathbf{R}^n, B),$$

and, for every $\varepsilon > 0$, the truncation $(-\Delta)^{i\gamma}_\varepsilon$ is defined as in (3) by replacing $\mathcal{L}$ by the Laplacian operator $-\Delta$.

Theorem 4. *Let B be a Banach space. Then, the following assertions are equivalent:*

(i) *B is a UMD space.*
(ii) *For every $\gamma \in \mathbf{R}$ and for some (equivalently, for any) $1 < p < \infty$, the operator $(-\Delta)^{i\gamma}$ can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself.*
(iii) *For every $\gamma \in \mathbf{R}$, the operator $(-\Delta)^{i\gamma}$ can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$.*
(iv) *For every $\gamma \in \mathbf{R}$ and every $f \in L^1(\mathbf{R}^n, B)$, $(-\Delta)^{i\gamma}_*(f)(x) < \infty$, a.e. $x \in \mathbf{R}^n$.*
(v) *For every $\gamma \in \mathbf{R}$ and every $f \in L^1(\mathbf{R}^n, B)$, there exists the limit $\lim_{\varepsilon\to 0^+} (-\Delta)^{i\gamma}_\varepsilon(f)(x)$, a.e. $x \in \mathbf{R}^n$.*

It is convenient to note that the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 4 extends [11, Theorem, p. 402] to higher dimensions, and the equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are, as far we know, news.

This paper is organized as follows. In Section 2 we present a proof of Theorem 1. Theorems 2, 3 and 4 are proved in Section 3. Finally, we present in the Appendix, for the sake of completeness, a proof of a version of Theorem 1 in the Laplacian (classical) case.

Throughout this paper by C and c we always denote positive constants that can change in each occurrence.

2. Proof of Theorem 1

Assume that $\phi \in L^\infty(0,\infty)$ and define the function m as follows

$$m(\lambda) = \lambda \int_0^\infty e^{-\lambda v}\phi(v)\,dv, \quad \lambda \in (0,\infty).$$

As it was commented m is continuous and bounded in $(0,\infty)$. The spectral multiplier $T_m^{\mathcal{L}}$ in the Schrödinger setting associated with m is defined by

$$T_m^{\mathcal{L}}(f) = \int_{(0,\infty)} m(\lambda) E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbf{R}^n),$$

where $E_{\mathcal{L}}$ represents the spectral measure for the Schrödinger operator $\mathcal{L}$. It is well known that $T_m^{\mathcal{L}}$ is a bounded operator from $L^2(\mathbf{R}^n)$ into itself.

We are going to prove Theorem 1. Let $f, g \in C_c^\infty(\mathbf{R}^n)$. We can write

$$\langle T_m^{\mathcal{L}}(f), g\rangle = \left\langle \int_{(0,\infty)} m(\lambda) E_{\mathcal{L}}(d\lambda) f, g \right\rangle = \int_{(0,\infty)} m(\lambda)\, d\mu_{f,g;\mathcal{L}}(\lambda),$$

where by $\mu_{f,g;\mathcal{L}}$ we denote the measure defined by

$$\mu_{f,g;\mathcal{L}}(A) = \langle E_{\mathcal{L}}(A) f, g\rangle,$$

for every Borel set $A \subset (0,\infty)$. The set function $\mu_{f,g;\mathcal{L}}$ is a complex measure on $(0,\infty)$ satisfying that $|\mu_{f,g;\mathcal{L}}|((0,\infty)) \leq \|f\|_2\|g\|_2$, where $|\mu_{f,g;\mathcal{L}}|$ represents the total variation measure of $\mu_{f,g;\mathcal{L}}$.

We have that

$$\begin{aligned}\langle T_m^{\mathcal{L}}(f), g\rangle &= \int_{(0,\infty)} \lambda \int_0^\infty e^{-\lambda v}\phi(v)\,dv\,d\mu_{f,g;\mathcal{L}}(\lambda) = \int_0^\infty \phi(v) \int_{(0,\infty)} \lambda e^{-\lambda v} d\mu_{f,g;\mathcal{L}}(\lambda)\,dv \\ &= \int_0^\infty \phi(v) \int_{(0,\infty)} \left(-\frac{\partial}{\partial v}\right)(e^{-\lambda v})\,d\mu_{f,g;\mathcal{L}}(\lambda)\,dv.\end{aligned}$$

Here, we can interchange the order of integration because

$$\int_{(0,\infty)} \int_0^\infty \lambda e^{-\lambda v}|\phi(v)|\,dv\,d|\mu_{f,g;\mathcal{L}}|(\lambda) \leq \|\phi\|_\infty |\mu_{f,g;\mathcal{L}}|((0,\infty)) < \infty.$$

Since

$$\left|\frac{e^{-\lambda(v+h)} - e^{-\lambda v}}{h}\right| \leq \lambda e^{\lambda(|h|-v)} \leq \lambda e^{-\lambda v/2}, \quad v, \lambda > 0 \text{ and } |h| < \frac{v}{2},$$

and

$$\int_{(0,\infty)} \lambda e^{-\lambda v/2}\,d|\mu_{f,g;\mathcal{L}}|(\lambda) \leq \frac{2}{v}|\mu_{f,g;\mathcal{L}}|((0,\infty)) < \infty, \quad v > 0,$$

we can differentiate under the integral sign and write

$$\begin{aligned}\langle T_m^{\mathcal{L}}(f), g\rangle &= \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right)\int_{(0,\infty)} e^{-\lambda v}\,d\mu_{f,g;\mathcal{L}}(\lambda)\,dv \\ &= \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right)\langle W_v^{\mathcal{L}}(f), g\rangle\,dv \\ &= \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} W_v^{\mathcal{L}}(x,y) f(y)\,dy\,\overline{g(x)}\,dx\,dv.\end{aligned}$$

We have that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |W_v^{\mathcal{L}}(x,y)-W_v(x,y)||f(y)||g(x)|\,dy\,dx<\infty, \quad v\in(0,\infty),$$

and

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \left|\frac{\partial}{\partial v}(W_v^{\mathcal{L}}(x,y)-W_v(x,y))\right| |f(y)||g(x)|\,dy\,dx<\infty, \quad v\in(0,\infty).$$

Indeed, according to [8, (2.2) and (2.7)] it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |W_v^{\mathcal{L}}(x,y)-W_v(x,y)||f(y)||g(x)|\,dy\,dx \\
&\qquad+\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \left|\frac{\partial}{\partial v}(W_v^{\mathcal{L}}(x,y)-W_v(x,y))\right| |f(y)||g(x)|\,dy\,dx \\
&\leq C\frac{1+v}{v^{n/2+1}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-c\frac{|x-y|^2}{v}}|f(y)||g(x)|\,dy\,dx \\
&\leq C\frac{1+v}{v^{n/2+1}}\int_{\mathbf{R}^n}|f(y)|\,dy\int_{\mathbf{R}^n}|g(x)|\,dx<\infty, \quad v\in(0,\infty).
\end{aligned}$$

Hence, the function

$$\psi(v)=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(W_v^{\mathcal{L}}(x,y)-W_v(x,y))f(y)\,\overline{g(x)}\,dy\,dx, \quad v\in(0,\infty),$$

is differentiable in $(0,\infty)$ and

$$\frac{d}{dv}\psi(v)=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{\partial}{\partial v}(W_v^{\mathcal{L}}(x,y)-W_v(x,y))f(y)\,\overline{g(x)}\,dy\,dx, \quad v\in(0,\infty).$$

We can write

$$\begin{aligned}
\langle T_m^{\mathcal{L}}(f),g\rangle &= \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} W_v(x,y)f(y)\,dy\,\overline{g(x)}\,dx\,dv \\
&\quad+\int_0^\infty \phi(v)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\left(-\frac{\partial}{\partial v}\right)\big(W_v^{\mathcal{L}}(x,y)-W_v(x,y)\big)\,f(y)\,\overline{g(x)}\,dy\,dx\,dv. \qquad (4)
\end{aligned}$$

Also, we have that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_0^\infty |\phi(v)|\left|\frac{\partial}{\partial v}\big(W_v^{\mathcal{L}}(x,y)-W_v(x,y)\big)\right| |f(y)||\overline{g(x)}|\,dv\,dx\,dy<\infty, \qquad (5)$$

and

$$\int_{\mathbf{R}^n}\int_0^\infty \left|\frac{\partial}{\partial v}\big(W_v^{\mathcal{L}}(x,y)-W_v(x,y)\big)\right| |f(y)||\phi(v)|\,dv\,dy<\infty. \qquad (6)$$

Indeed, to see (6) we write

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_0^\infty \left|\frac{\partial}{\partial v}\big(W_v^{\mathcal{L}}(x,y)-W_v(x,y)\big)\right| |f(y)||\phi(v)|\,dv\,dy \\
&\leq \|\phi\|_\infty\left(\int_{\mathbf{R}^n}\int_0^{\rho(x)^2}+\int_{\mathbf{R}^n}\int_{\rho(x)^2}^\infty\right)\left|\frac{\partial}{\partial v}\big(W_v^{\mathcal{L}}(x,y)-W_v(x,y)\big)\right| |f(y)|\,dv\,dy \\
&= B_1(x)+B_2(x).
\end{aligned}$$

According to [8, (2.7)] we get

$$B_2(x) \leq C \int_{\mathbf{R}^n} \int_{\rho(x)^2}^{\infty} \frac{e^{-c\frac{|x-y|^2}{v}}}{v^{\frac{n}{2}+1}} |f(y)|\, dv\, dy \leq \frac{C}{\rho(x)^n}, \quad x \in \mathbf{R}^n. \tag{7}$$

Since $0 < \rho(x) < \infty$, it follows that $B_2(x) < \infty$, $x \in \mathbf{R}^n$.

By proceeding as in [3, p. 15–17] we can obtain that

$$B_1(x) \leq C\|f\|_\infty, \quad x \in \mathbf{R}^n. \tag{8}$$

Thus, we have proved (6). Moreover, by using [16, Lemma 1.4], (7) and (8) imply also (5). Then (4) can be written

$$\begin{aligned} \langle T_m^{\mathcal{L}}(f), g\rangle &= \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right) \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} W_v(x,y) f(y)\, dy\, \overline{g(x)}\, dx\, dv \\ &\quad + \int_{\mathbf{R}^n} \left(\lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \left(K_\phi^{\mathcal{L}}(x,y) - K_\phi(x,y)\right) f(y)\, dy\right) \overline{g(x)}\, dx, \end{aligned} \tag{9}$$

where

$$K_\phi^{\mathcal{L}}(x,y) = -\int_0^\infty \phi(v) \frac{\partial}{\partial v} W_v^{\mathcal{L}}(x,y)\, dv, \quad x, y \in \mathbf{R}^n,\ x \neq y,$$

and

$$K_\phi(x,y) = -\int_0^\infty \phi(v) \frac{\partial}{\partial v} W_v(x,y)\, dv, \quad x, y \in \mathbf{R}^n,\ x \neq y.$$

On the other hand, as above we can see that

$$\langle T_m(f), g\rangle = \int_0^\infty \phi(v)\left(-\frac{d}{dv}\right) \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} W_v(x,y) f(y)\, dy\, \overline{g(x)}\, dx\, dv. \tag{10}$$

where T_m represents the spectral multiplier associated with $-\Delta$ defined by m. Moreover, we can write

$$T_m(f)(x) = \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon) f(x) + \int_{|x-y|>\varepsilon} K_\phi(x,y) f(y)\, dy\right), \quad \text{a.e. } x \in \mathbf{R}^n. \tag{11}$$

where

$$\alpha(\varepsilon) = \frac{1}{\Gamma(\frac{n}{2})} \int_0^\infty \phi\Big(\frac{\varepsilon^2}{4u}\Big) e^{-u} u^{\frac{n}{2}-1}\, du, \quad \varepsilon > 0. \tag{12}$$

Also, if there exists the limit $\lim_{t\to 0^+} \phi(t) = \phi(0^+)$, then $\lim_{\varepsilon\to 0^+} \alpha(\varepsilon) = \phi(0^+)$, and

$$T_m(f)(x) = \phi(0^+) f(x) + \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} K_\phi(x,y) f(y)\ dy, \quad \text{a.e. } x \in \mathbf{R}^n. \tag{13}$$

Although we are sure that the properties (11) and (13) are known, we include in the appendix complete proofs for these properties of T_m, for the sake the interested reader.

By combining (9), (10), (11) and (13) we obtain that

$$T_m^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon)\ f(x) + \int_{|x-y|>\varepsilon} K_\phi^{\mathcal{L}}(x,y) f(y)\, dy\right), \quad \text{a.e. } x \in \mathbf{R}^n,$$

and

$$T_m^{\mathcal{L}}(f)(x) = \phi(0^+)\ f(x) + \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} K_\phi^{\mathcal{L}}(x,y) f(y)\, dy, \quad \text{a.e. } x \in \mathbf{R}^n,$$

provided that there exists the limit $\phi(0^+) = \lim_{t\to 0^+} \phi(t)$.

3. Proof of Theorems 2, 3 and 4

Guerre-Delabrière [11, Theorem, p. 402] established that a Banach space B is UMD if, only if, for every $\gamma \in \mathbf{R}$, the operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ can be extended to $L^p(\mathbf{R}, B)$ into itself, for some (equivalently, for any) $1 < p < \infty$. In the proof of [11, Theorem, p. 402] a vector valued version of a classical transference result was used.

Assume that B is a Banach space and $\gamma \in \mathbf{R}$. The operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ takes the form

$$\left(-\frac{d^2}{dx^2}\right)^{i\gamma} f = (|y|^{2i\gamma}\hat{f})\check{}, \quad f \in L^2(\mathbf{R}),$$

where $\hat{f}$ denotes the Fourier transform of f and $\check{f}$ the inverse Fourier transform of f. If $f \in L^1(\mathbf{R})$ we define

$$\hat{f}(y) = \int_{\mathbf{R}} e^{-ixy} f(x)\, dx, \quad y \in \mathbf{R},$$

and

$$\check{f}(y) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{ixy} f(x)\, dx, \quad y \in \mathbf{R}.$$

As it is well known the Fourier transform can be extended from $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ as a bijective bounded operator from $L^2(\mathbf{R})$ into itself. The operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ is bounded from $L^p(\mathbf{R})$ into itself, for every $1 < p < \infty$. If $1 < p < \infty$ and $f \in L^p(\mathbf{R}) \otimes B$, that is, $f = \sum_{j=1}^r \beta_j f_j$, where $\beta_j \in B$, $f_j \in L^p(\mathbf{R})$, $j = 1, \dots, r \in \mathbf{N}$, we define, as usual,

$$\left(-\frac{d^2}{dx^2}\right)^{i\gamma}(f) = \sum_{j=1}^r \beta_j \left(-\frac{d^2}{dx^2}\right)^{i\gamma}(f_j).$$

We also consider the operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}$, where $\mathbf{T} = [0, 2\pi)$ denotes the one-dimensional torus, defined by

$$\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}(g)(x) = \sum_{j\in\mathbf{Z},\, j\neq 0} |j|^{2i\gamma} c_j(g) e^{ijx},\ x \in (0, 2\pi) \quad \text{and} \quad g \in L^p(\mathbf{T}),\ 1 < p < \infty,$$

being $c_j(g) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(\theta) e^{-ij\theta}\, d\theta$, $j \in \mathbf{Z}$. The operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}$ is bounded from $L^p(\mathbf{T})$ into itself, $1 < p < \infty$. If $1 < p < \infty$ and $g \in L^p(\mathbf{T}) \otimes B$, that is, $g = \sum_{j=1}^r \beta_j g_j$, where $\beta_j \in B$, $g_j \in L^p(\mathbf{T})$, $j = 1, \dots, r \in \mathbf{N}$, we define

$$\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}(g) = \sum_{j=1}^r \beta_j \left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}(g_j).$$

Guerre-Delabrière ([11, p. 402]) showed that if $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}$ can be extended to $L^2(\mathbf{T}, B)$ as a bounded operator from $L^2(\mathbf{T}, B)$ into itself, then B is UMD. Moreover, she used a vector valued transference result (see [7] for the scalar result) that implies that $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}$ can be extended to $L^2(\mathbf{T},B)$ as a bounded operator from $L^2(\mathbf{T},B)$ into itself, provided that $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ can be extended to $L^2(\mathbf{R},B)$ as a bounded operator from $L^2(\mathbf{R},B)$ into itself. Note that, by using vector valued Calderón–Zygmund theory ([15]) we can see that $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ can be extended to $L^p(\mathbf{R},B)$ as a bounded operator from $L^p(\mathbf{R},B)$ into itself, for some $1<p<\infty$, if and only if $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}$ can be extended to $L^2(\mathbf{R},B)$ as a bounded operator from $L^2(\mathbf{R},B)$ into itself.

The operators $(-\Delta)^{i\gamma}$ (respectively, $(-\Delta)^{i\gamma}_{|\mathbf{T}^n}$) are defined on $L^p(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n)$ $\otimes B$ (respectively, on $L^p(\mathbf{T}^n)$ and $L^p(\mathbf{T}^n)\otimes B$), $1<p<\infty$, in the natural way.

3.1. Proof of Theorem 4. (i) $\Rightarrow$ (ii). It is a consequence of [21, Proposition 3].

(ii) $\Rightarrow$ (i). We show this part by adapting standard transference arguments to a vector valued setting. For the sake of completeness we include the proof.

Let $\gamma\in\mathbf{R}$ and $1<p<\infty$. Suppose that the operator $(-\Delta)^{i\gamma}$ can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself. We choose an even smooth function on $\mathbf{R}$ such that $\phi(x)=1$, $|x|\le 1/4$, and $\phi(x)=0$, $|x|\ge 1/2$. We split the operator $(-\Delta)^{i\gamma}$ as follows

$$\begin{aligned}(-\Delta)^{i\gamma}(f)&=(\phi(|x|^2)|x|^{2i\gamma}\hat{f})\check{}+((1-\phi(|x|^2))|x|^{2i\gamma}\hat{f})\check{}\\&=A_1(f)+A_2(f),\quad f\in C_c^\infty(\mathbf{R}^n)\otimes B.\end{aligned}$$

Here, $\hat{h}$ denotes the Fourier transform of h and $\check{h}$ the inverse Fourier transform of h in $\mathbf{R}^n$, defined, for every $h\in L^1(\mathbf{R}^n)$, by

$$\hat{h}(y)=\int_{\mathbf{R}^n}e^{-ixy}h(x)\,dx,\quad y\in\mathbf{R}^n,$$

and

$$\check{h}(y)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}e^{ixy}h(x)\,dx,\quad y\in\mathbf{R}^n.$$

Also, we consider the function $\varphi(x)=\phi(|x|^2)$, $x\in\mathbf{R}^n$, and the Fourier multiplier T_φ defined by

$$T_\varphi(f)=(\varphi\hat{f})\check{},\quad f\in C_c^\infty(\mathbf{R}^n)\otimes B,$$

in a natural way. Since $\hat{\varphi}\in L^1(\mathbf{R}^n)$, T_φ can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself. Then, A_1 and therefore A_2 can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself.

We denote by $\mathcal{P}(\mathbf{T}^n,X)$ the space of trigonometric polynomials of period 2π on $\mathbf{T}^n$ with coefficients in a Banach space X. Let $P\in\mathcal{P}(\mathbf{T}^n,B)$ and $Q\in\mathcal{P}(\mathbf{T}^n,B')$, where B' is the dual space of B. Since A_2 can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself, by proceeding as in the proof of [18, Theorem 3.8, p. 260] we have that

$$\left|\int_{\mathbf{T}^n}\langle(-\Delta)^{i\gamma}_{|\mathbf{T}^n}(P)(x),Q(x)\rangle\,dx\right|\le C\|P\|_{L^p(\mathbf{T}^n,B)}\|Q\|_{L^{p'}(\mathbf{T}^n,B')},$$

where p' is the exponent conjugated to p.

By using [10, Lemma 2.3] we get

$$\|(-\Delta)^{i\gamma}_{|\mathbf{T}^n}(P)\|_{L^p(\mathbf{T}^n,B)} \le C\|P\|_{L^p(\mathbf{T}^n,B)}.$$

Hence, $(-\Delta)^{i\gamma}_{|\mathbf{T}^n}$ can be extended to $L^p(\mathbf{T}^n, B)$ as a bounded operator from $L^p(\mathbf{T}^n, B)$ into itself.

In order to see that the operator $\left(-\frac{d^2}{dx^2}\right)^{i\gamma}_{|\mathbf{T}}$ can be extended to $L^p(\mathbf{T}, B)$ as a bounded operator from $L^p(\mathbf{T}, B)$ into itself, it is sufficient to use that $(-\Delta)^{i\gamma}_{|\mathbf{T}^n}$ can be extended to $L^p(\mathbf{T}^n, B)$ as a bounded operator from $L^p(\mathbf{T}^n, B)$ into itself, and to extend every function $f \in L^p(\mathbf{T})$ to $\mathbf{T}^n$ in the natural way, that is, defining $\tilde{f}(x_1, \dots, x_n) = f(x_1)$, $(x_1, \dots, x_n) \in \mathbf{T}^n$.

According to [11, Theorem, p. 402] the above arguments allow us to conclude that (ii) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). These properties are true because, for every $\gamma \in \mathbf{R}$, the operator $(-\Delta)^{i\gamma}$ is a Calderón–Zygmund operator (see [15]).

(iv) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v). Assume that (iv) holds. Let $\gamma \in \mathbf{R}$. For every $0 < \varepsilon < 1$, we define the operator $(-\Delta)^{i\gamma}_{(\varepsilon,1/\varepsilon)}$ on $L^1(\mathbf{R}^n, B)$ by

$$(-\Delta)^{i\gamma}_{(\varepsilon,1/\varepsilon)}(f)(x) = \alpha(\varepsilon)f(x) + \int_{\varepsilon<|x-y|<1/\varepsilon} K_{\phi_\gamma}(x,y)f(y)\,dy, \quad f \in L^1(\mathbf{R}^n, B),$$

that is bounded from $L^1(\mathbf{R}^n, B)$ into itself. By using (iv) and continuity Banach principle ([9, Proposition 1.4, p. 529]) the maximal operator $(-\Delta)^{i\gamma}_{**}$ defined by

$$(-\Delta)^{i\gamma}_{**}(f) = \sup_{\varepsilon\in(0,1)} \|(-\Delta)^{i\gamma}_{(\varepsilon,1/\varepsilon)}(f)\|_B, \quad f \in L^1(\mathbf{R}^n, B),$$

is continuous from $L^1(\mathbf{R}^n, B)$ into $L^0(\mathbf{R}^n)$, where the space $L^0(\mathbf{R}^n)$ of all measurable functions is endowed with the local convergence in measure ([9, p. 528]). Since the function α is given by (12), the maximal operator $(-\Delta)^{i\gamma}_*$ is dilation and translation invariant. Moreover, by taking into account that $|K_{\phi_\gamma}(x,y)| \le C|x-y|^{-n}$, $x, y \in \mathbf{R}^n$, $x \ne y$, $(-\Delta)^{i\gamma}_*$ is bounded from $L^1(\mathbf{R}^n, B)$ into $L^0(\mathbf{R}^n)$. Then, since every Banach space is of Rademacher type 1, according to [14, Lemma 7.3], we deduce that $(-\Delta)^{i\gamma}_*$ is bounded from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n)$.

For every $f \in C_c^\infty(\mathbf{R}^n) \otimes B$, there exists the limit $\lim_{\varepsilon\to 0^+}(-\Delta)^{i\gamma}_\varepsilon(f)(x)$, a.e. $x \in \mathbf{R}^n$. Since $C_c^\infty(\mathbf{R}^n) \otimes B$ is a dense subspace of $L^1(\mathbf{R}^n, B)$, for every $f \in L^1(\mathbf{R}^n, B)$, there exists the limit $\lim_{\varepsilon\to 0^+}(-\Delta)^{i\gamma}_\varepsilon(f)(x)$, a.e. $x \in \mathbf{R}^n$, and $(-\Delta)^{i\gamma}$ can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$.

(v) $\Rightarrow$ (iv). It is clear. □

3.2. Proof of Theorem 3. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Let $\gamma \in \mathbf{R}$. The imaginary power $\mathcal{L}^{i\gamma}$ of $\mathcal{L}$ (respectively, $(-\Delta)^{i\gamma}$ of $-\Delta$) is the spectral multiplier associated with $\mathcal{L}$ (respectively, $-\Delta$) defined by the function $m_\gamma(\lambda) = \lambda^{i\gamma}$, $\lambda \in (0, \infty)$. Note that $m_\gamma(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\phi_\gamma(t)\,dt$, $\lambda \in (0, \infty)$, where $\phi_\gamma(t) = \frac{t^{-i\gamma}}{\Gamma(1-i\gamma)}$, $t \in (0, \infty)$.

Assume that B is a Banach space and that $f = \sum_{j=1}^d \beta_j f_j$, where $f_j \in C_c^\infty(\mathbf{R}^n)$ and $\beta_j \in B$, $j = 1, \dots, d$. By Theorem 1 and (11), we have that

$$T^{\mathcal{L}}_{m_\gamma}(f)(x) = \sum_{j=1}^d \beta_j \lim_{\varepsilon\to 0^+}\left(\alpha(\varepsilon)f_j(x) + \int_{|x-y|>\varepsilon} K^{\mathcal{L}}_{\phi_\gamma}(x,y)f_j(y)\,dy\right), \quad \text{a.e. } x \in \mathbf{R}^n,$$

and

$$T_{m_\gamma}(f)(x)=\sum_{j=1}^{d}\beta_j\lim_{\varepsilon\to 0^+}\left(\alpha(\varepsilon)f_j(x)+\int_{|x-y|>\varepsilon}K_{\phi_\gamma}(x,y)f_j(y)\,dy\right),\quad \text{a.e. } x\in\mathbf{R}^n.$$

We split the operator T_{m_γ} as follow

$$T_{m_\gamma}=T_{m_\gamma,g}+T_{m_\gamma,\ell},$$

where $T_{m_\gamma,g}(f)(x)=\int_{|x-y|\geq\rho(x)}K_{\phi_\gamma}(x,y)f(y)\,dy$, $x\in\mathbf{R}^n$.

The operator $S_{m_\gamma}=T^{\mathcal{L}}_{m_\gamma}-T_{m_\gamma,\ell}$ can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself, for every $1<p<\infty$. Indeed, we can write

$$\begin{aligned}S_{m_\gamma}(f)(x)&=\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-y|<\rho(x)}f(y)\left(K^{\mathcal{L}}_{\phi_\gamma}(x,y)-K_{\phi_\gamma}(x,y)\right)dy\\&\quad-\int_{|x-y|\geq\rho(x)}f(y)\int_0^\infty\phi_\gamma(t)\frac{\partial}{\partial t}W^{\mathcal{L}}_t(x,y)\,dt\,dy\\&=S_{m_\gamma,1}(f)(x)+S_{m_\gamma,2}(f)(x).\end{aligned}$$

By proceeding as in [3, p. 15–17] we can get

$$\|S_{m_\gamma,1}(f)(x)\|_B\leq\int_{|x-y|<\rho(x)}\|f(y)\|_B\left|K^{\mathcal{L}}_{\phi_\gamma}(x,y)-K_{\phi_\gamma}(x,y)\right|dy\leq C\mathcal{M}(\|f\|)(x),$$

$x\in\mathcal{R}^n$, and

$$\begin{aligned}\|S_{m_\gamma,2}f(x)\|_B&\leq\int_{|x-y|\geq\rho(x)}\|f(y)\|_B\int_0^{\rho(x)^2}|\phi_\gamma(t)|\left|\frac{\partial}{\partial t}W^{\mathcal{L}}_t(x,y)\right|dt\,dy\\&\quad+\int_{|x-y|\geq\rho(x)}\|f(y)\|_B\int_{\rho(x)^2}^\infty|\phi_\gamma(t)|\left|\frac{\partial}{\partial t}W^{\mathcal{L}}_t(x,y)\right|dt\,dy\\&\leq C\mathcal{M}(\|f\|_B)(x),\quad x\in\mathbf{R}^n,\end{aligned}$$

because $\|\phi_\gamma\|_\infty=1/|\Gamma(1-i\gamma)|$. Here and in the sequel $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. Hence, by using the well known Maximal Theorem we conclude that the operator S_{m_γ} can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself, for every $1<p<\infty$.

Suppose now that B is a UMD Banach space. According to Theorem 4 the operator T_{m_γ} can be extended to $L^p(\mathbf{R}^n,B)$ as a bounded operator from $L^p(\mathbf{R}^n,B)$ into itself, for every $1<p<\infty$, and to $L^1(\mathbf{R}^n,B)$ as a bounded operator from $L^1(\mathbf{R}^n,B)$ into $L^{1,\infty}(\mathbf{R}^n,B)$. For every $f\in L^\infty_c(\mathbf{R}^n)\otimes B$ we have that

$$T_{m_\gamma}(f)(x)=\int_{\mathbf{R}^n}K_{\phi_\gamma}(x,y)f(y)\,dy,\quad \text{a.e. } x\notin\operatorname{supp} f.$$

Moreover, K_{ϕ_γ} is a standard Calderón–Zygmund kernel, that is, there exists $C>0$ such that

$$|K_{\phi_\gamma}(x,y)|\leq\frac{C}{|x-y|^n},\quad x\neq y,$$

and

$$\sum_{j=1}^{n}\left(\left|\frac{\partial K_{\phi_\gamma}(x,y)}{\partial x_j}\right|+\left|\frac{\partial K_{\phi_\gamma}(x,y)}{\partial y_j}\right|\right)\leq\frac{C}{|x-y|^{n+1}},\quad x\neq y.$$

Then, by proceeding as in the scalar case (see [17, p. 34]), we can show that the maximal operator

$$T^*_{m_\gamma}(f)(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} K_{\phi_\gamma}(x,y) f(y)\, dy \right\|_B ,$$

is bounded from $L^p(\mathbf{R}^n, B)$ into $L^p(\mathbf{R}^n)$, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n)$.

It is clear that $\|T_{m_\gamma,g}(f)(x)\|_B \le T^*_{m_\gamma}(f)(x),\ x \in \mathbf{R}^n$. Then, $T_{m_\gamma,g}$ is bounded from $L^p(\mathbf{R}^n, B)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$. Hence, since $T_{m_\gamma,\ell} = T_{m_\gamma} - T_{m_\gamma,g}$, we conclude that, $T_{m_\gamma,\ell}$, and then also $T^{\mathcal{L}}_{m_\gamma}$, are bounded from $L^p(\mathbf{R}^n, B)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$.

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). Assume firstly that for a certain $1 < p < \infty$ and every $\gamma \in \mathbf{R}$ the operator $T^{\mathcal{L}}_{m_\gamma}$ can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself. Then, for every $\gamma \in \mathbf{R}$ the operator $T_{m_\gamma,\ell}$ can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself. According to Theorem 4 in order to show that B is a UMD Banach space it is sufficient to see that for every $\gamma \in \mathbf{R}$, T_{m_γ} can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself.

Let $\gamma \in \mathbf{R}$. Suppose that $f \in C^\infty_c(\mathbf{R}^n)$ and $\operatorname{supp} f \subset B(0, M)$ for a certain $M > 0$. For every $R > 0$ we define $f_R(x) = f(\sqrt{R}x)$, $x \in \mathbf{R}^n$. It is clear that, $\operatorname{supp} f_R \subset B(0, \frac{M}{\sqrt{R}})$, $R > 0$.

In the following our arguments are inspired in the ones developed by Abu-Falahah, Stinga and Torrea in [1]. We are going to show that for every $\lambda > 0$ there exists $R > 0$ such that $\operatorname{supp} f_R \subset B(\frac{x}{R}, \rho(\frac{x}{R}))$, provided that $|x| < \lambda$. According to [16, Lemma 1.1] there exists $C_1 > 0$ for which

$$\frac{1}{C_1}\rho(y) \le \rho(x) \le C_1 \rho(y), \quad |x-y| \le \rho(x).$$

Let $\lambda > 0$. From [1, Lemma 3.5] we can find $R_\lambda > 0$ such that $|y - \frac{x}{R}| < \rho(\frac{x}{R})$, when $|y| < \frac{C_1^2 \rho(0)}{2}$, $|x| < \lambda$ and $R \ge R_\lambda$. We can take $R \ge \max\{R_\lambda, \left(\frac{2M}{\rho(0) C_1^2}\right)^2\}$. Then $\operatorname{supp} f_R \subseteq B(\frac{x}{R}, \rho(\frac{x}{R}))$, $|x| < \lambda$.

We can write, for every $R > 0$,

$$\begin{aligned}
&T_{m_\gamma}(f_R)\left(\frac{x}{\sqrt{R}}\right) \\
&= \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon) f_R\left(\frac{x}{\sqrt{R}}\right) + \int_{|\frac{x}{\sqrt{R}}-y|>\varepsilon} f_R(y) \int_0^\infty \phi_\gamma(t) \left(-\frac{\partial}{\partial t}\right) W_t\left(\frac{x}{\sqrt{R}}, y\right) dt\, dy \right) \\
&= \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon) f(x) + \int_{|x-u|>\varepsilon\sqrt{R}} f(u) \int_0^\infty \phi_\gamma(t) \left(-\frac{\partial}{\partial t}\right) W_t\left(\frac{x}{\sqrt{R}}, \frac{u}{\sqrt{R}}\right) dt\, \frac{du}{R^{n/2}} \right) \\
&= \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon) f(x) + R \int_{|x-u|>\varepsilon\sqrt{R}} f(u) \int_0^\infty \phi_\gamma(t) \left(-\frac{\partial}{\partial s} W_s(x,u)\right)\Big|_{s=Rt} dt\, du \right) \\
&= \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon) f(x) + \int_{|x-u|>\varepsilon\sqrt{R}} f(u) \int_0^\infty \phi_\gamma\left(\frac{s}{R}\right) \left(-\frac{\partial}{\partial s}\right) W_s(x,u)\, ds\, du \right),
\end{aligned}$$

a.e. $x \in \mathbf{R}^n$. Here $\alpha(\varepsilon) = \dfrac{1}{\Gamma(\frac{n}{2})} \displaystyle\int_0^\infty e^{-u} u^{\frac{n}{2}-1} \phi_\gamma\left(\frac{\varepsilon^2}{4u}\right) du, \ \ \varepsilon \in (0,1)$.

Since $\phi_\gamma(as) = a^{-i\gamma}\phi_\gamma(s)$, $a, s > 0$, it follows that, for every $R > 0$,

$$\begin{aligned}
&T_{m_\gamma}(f_R)\left(\frac{x}{\sqrt{R}}\right)\\
&= R^{i\gamma} \lim_{\varepsilon\to 0^+} \left(\alpha(\varepsilon\sqrt{R}) f(x) + \int_{|x-u|>\varepsilon\sqrt{R}} f(u) \int_0^\infty \phi_\gamma(s)\left(-\frac{\partial}{\partial s}\right) W_s(x,u)\, ds\, du\right)\\
&= R^{i\gamma} T_{m_\gamma}(f)(x), \quad \text{a.e. } x \in \mathbf{R}^n.
\end{aligned}$$

As it was proved above, for every $N \in \mathbf{N}$, there exists $R_N > 0$ such that $\operatorname{supp} f_{R_N} \subseteq B\left(\frac{x}{R_N}, \rho(\frac{x}{R_N})\right)$, $|x| \le N$, and $R_N \le R_{N+1}$. Then, it follows that

$$\begin{aligned}
T_{m_\gamma}(f)(x) &= R_N^{-i\gamma} T_{m_\gamma}\left(f_{R_N}\chi_{B(\frac{x}{R_N},\rho(\frac{x}{R_N}))}\right)\left(\frac{x}{\sqrt{R_N}}\right)\\
&= R_N^{-i\gamma} T_{m_\gamma,\ell}(f_{R_N})\left(\frac{x}{\sqrt{R_N}}\right), \quad |x| \le N, \ N \in \mathbf{N}.
\end{aligned} \tag{14}$$

We deduce that,

$$\begin{aligned}
\int_{B(0,N)} |T_{m_\gamma}(f)(x)|^p\, dx &\le R_N^{n/2} \int_{\mathbf{R}^n} |T_{m_\gamma,\ell}(f_{R_N})(x)|^p\, dx\\
&\le C R_N^{n/2} \int_{\mathbf{R}^n} |f_{R_N}(x)|^p\, dx \le C\|f\|_p^p, \quad N \in \mathbf{N}.
\end{aligned}$$

Note that C does not depend on N.

We conclude that

$$\|T_{m_\gamma}(f)\|_p \le C\|f\|_p. \tag{15}$$

Also, (15) holds for every $f \in C_c^\infty(\mathbf{R}^n)\otimes B$. Hence, T_{m_γ} can be extended to $L^p(\mathbf{R}^n, B)$ as a bounded operator from $L^p(\mathbf{R}^n, B)$ into itself.

Suppose now that for every $\gamma \in \mathbf{R}$ the operator $T^{\mathcal{L}}_{m_\gamma}$ can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$. Then, for every $\gamma \in \mathbf{R}$, the operator $T_{m_\gamma,\ell}$ can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$.

Let $\gamma \in \mathbf{R}$ and $f \in C_c^\infty(\mathbf{R}^n)\otimes B$. By (14), for every $N \in \mathbf{N}$, there exists $R_N > 0$ such that

$$T_{m_\gamma}(f)(x) = R_N^{-i\gamma} T_{m_\gamma,\ell}(f_{R_N})\left(\frac{x}{\sqrt{R_N}}\right), \quad |x| \le N.$$

Hence, for every $N \in \mathbf{N}$ and $\lambda > 0$, we get

$$\begin{aligned}
\left|\{x \in B(0,N) \colon \|T_{m_\gamma}(f)(x)\|_B > \lambda\}\right| &\le R_N^{n/2} \left|\{x \in \mathbf{R}^n \colon \|T_{m_\gamma,\ell}(f_{R_N})(x)\|_B > \lambda\}\right|\\
&\le C\frac{R_N^{n/2}}{\lambda} \int_{\mathbf{R}^n} \|f_{R_N}(x)\|_B\, dx\\
&\le C\frac{1}{\lambda} \int_{\mathbf{R}^n} \|f(x)\|_B\, dx.
\end{aligned}$$

Then, by letting $N \to \infty$, we deduce that

$$|\{x \in \mathbf{R}^n \colon \|T_{m_\gamma}(f)(x)\|_B > \lambda\}| \le C\frac{1}{\lambda}\int_{\mathbf{R}^n} \|f(x)\|_B \, dx, \quad \lambda > 0.$$

Since $C_c^\infty(\mathbf{R}^n) \otimes B$ is dense in $L^1(\mathbf{R}^n, B)$, T_{m_γ} can be extended to $L^1(\mathbf{R}^n, B)$ as a bounded operator from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$, and by Theorem 4 we conclude that B is UMD.

(i) $\Leftrightarrow$ (iv). Let $f \in L^1(\mathbf{R}^n, B)$. For every $x \in \mathbf{R}^n$ and $0 < \varepsilon < \rho(x)$ we can write

$$\begin{aligned}
&\int_{|x-y|>\varepsilon} f(y) K^{\mathcal{L}}_{\phi_\gamma}(x,y)\, dy - \int_{|x-y|>\varepsilon} f(y) K_{\phi_\gamma}(x,y)\, dy \\
&= \int_{\varepsilon<|x-y|<\rho(x)} f(y)(K^{\mathcal{L}}_{\phi_\gamma}(x,y) - K_{\phi_\gamma}(x,y))\, dy \\
&\quad + \int_{|x-y|\ge\rho(x)} f(y) K^{\mathcal{L}}_{\phi_\gamma}(x,y)\, dy - \int_{|x-y|\ge\rho(x)} f(y) K_{\phi_\gamma}(x,y)\, dy \\
&= H^\varepsilon_{m_\gamma,1}(f)(x) + H_{m_\gamma,2}(f)(x) + H_{m_\gamma,3}(f)(x).
\end{aligned}$$

By proceeding as above we have that

$$\|H^\varepsilon_{m_\gamma,1}(f)(x)\|_B \le C\mathcal{M}(\|f\|_B)(x), \quad 0 < \varepsilon < \rho(x), \ \ x \in \mathbf{R}^n,$$

and

$$\|H_{m_\gamma,2}(f)(x)\|_B \le C\mathcal{M}(\|f\|_B)(x), \quad x \in \mathbf{R}^n.$$

Also, since $|K_{\phi_\gamma}(x,y)| \le C|x-y|^{-n}$, $x, y \in \mathbf{R}^n$, $x \ne y$, it follows that

$$\|H_{m_\gamma,3}(f)(x)\|_B \le C\int_{|x-y|\ge\rho(x)} \frac{\|f(y)\|_B}{|x-y|^n}\, dy \le C\frac{1}{\rho(x)^n}\int_{\mathbf{R}^n} \|f(y)\|_B dy, \quad x \in \mathbf{R}^n.$$

Moreover, if $x \in \mathbf{R}^n$ and $\varepsilon \ge \rho(x)$, we have

$$\begin{aligned}
&\Big\| \int_{|x-y|>\varepsilon} f(y) K^{\mathcal{L}}_{\phi_\gamma}(x,y)\, dy - \int_{|x-y|>\varepsilon} f(y) K_{\phi_\gamma}(x,y)\, dy \Big\|_B \\
&\le \int_{|x-y|\ge\rho(x)} \|f(y)\|_B |K^{\mathcal{L}}_{\phi_\gamma}(x,y)|\, dy + \int_{|x-y|\ge\rho(x)} \|f(y)\|_B |K_{\phi_\gamma}(x,y)|\, dy \\
&\le C\Big(\mathcal{M}(\|f\|_B)(x) + \frac{1}{\rho(x)^n}\int_{\mathbf{R}^n} \|f(y)\|_B\, dy\Big).
\end{aligned}$$

Putting together all the above estimations we deduce that

$$|\mathcal{L}^{i\gamma}_*(f)(x) - (-\Delta)^{i\gamma}_*(f)(x)| < \infty, \quad \text{a.e. } x \in \mathbf{R}^n.$$

Hence, $\mathcal{L}^{i\gamma}_*(f)(x) < \infty$, a.e. $x \in \mathbf{R}^n$, if and only if, $(-\Delta)^{i\gamma}_*(f)(x) < \infty$, a.e. $x \in \mathbf{R}^n$, and Theorem 4 implies that (i) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (v). By Theorems 1 and 2, for every $f \in L^1(\mathbf{R}^n) \otimes B$, there exists the limit

$$\lim_{\varepsilon\to 0^+} \mathcal{L}^{i\gamma}_\varepsilon(f)(x), \quad \text{a.e. } x \in \mathbf{R}^n.$$

Assume that B is UMD. According to Theorem 4 the maximal operator $(-\Delta)^{i\gamma}_*$ is bounded from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$. By proceeding as above we get

$$\mathcal{L}^{i\gamma}_*(f)(x) \le C(\mathcal{M}(\|f\|_B)(x) + \|f(x)\|_B + (-\Delta)^{i\gamma}_*(f)(x)), \quad \text{a.e. } x \in \mathbf{R}^n.$$

The Maximal Theorem allows us to conclude that $\mathcal{L}_*^{i\gamma}$ is bounded from $L^1(\mathbf{R}^n, B)$ into $L^{1,\infty}(\mathbf{R}^n, B)$. Then, since $C_c^\infty(\mathbf{R}^n) \otimes B$ is dense in $L^1(\mathbf{R}^n, B)$, for every $f \in L^1(\mathbf{R}^n, B)$, there exists the limit

$$\lim_{\varepsilon\to 0^+} \mathcal{L}_\varepsilon^{i\gamma}(f)(x), \quad \text{a.e. } x \in \mathbf{R}^n.$$

(v) $\Rightarrow$ (iv). It is clear. □

3.3. Proof of Theorem 2. This proof follows the same way that the one of the L^p-boundedness of the imaginary power $\mathcal{L}^{i\gamma}$ of $\mathcal{L}$, $\gamma \in \mathbf{R}^n$, when B is a UMD space.

Suppose that $m(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\phi(t)\,dt$, $\lambda \in (0,\infty)$, where $\phi \in L^\infty(0,\infty)$. Let $f \in C_c^\infty(\mathbf{R}^n)$. According to Theorem 1

$$T_m^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+}\left(\alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} f(y)K_\phi^{\mathcal{L}}(x,y)\,dy\right), \quad \text{a.e. } x \in \mathbf{R}^n.$$

Also, by (11),

$$T_m(f)(x) = \lim_{\varepsilon\to 0^+}\left(\alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} f(y)K_\phi(x,y)\,dy\right), \quad \text{a.e. } x \in \mathbf{R}^n.$$

Here $\alpha \in L^\infty(0,\infty)$.

The operator T_m is bounded from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$. Moreover T_m is a Calderón–Zygmund operator. Hence, the maximal operator T_m^* defined by

$$T_m^*(f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} f(y)K_\phi(x,y)\,dy\right|$$

is bounded from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$. Also, the same L^p-boundedness properties are satisfied by the operators

$$T_{m,\ell}(f)(x) = \lim_{\varepsilon\to 0}\left(\alpha(\varepsilon)f(x) + \int_{\varepsilon<|x-y|<\rho(x)} f(y)K_\phi(x,y)\,dy\right)$$

and

$$T_{m,g}(f)(x) = \int_{|x-y|\geq\rho(x)} f(y)K_\phi(x,y)\,dy.$$

The difference $T_m^{\mathcal{L}}(f) - T_{m,\ell}(f)$ can be written as

$$\begin{aligned} &T_m^{\mathcal{L}}(f)(x) - T_{m,\ell}(f)(x) \\ &= \int_{|x-y|<\rho(x)} (K_\phi^{\mathcal{L}}(x,y) - K_\phi(x,y))f(y)\,dy + \int_{|x-y|\geq\rho(x)} K_\phi^{\mathcal{L}}(x,y)f(y)\,dy. \end{aligned}$$

By proceeding as in the proof of Theorem 3 we can see that the operator $T_m^{\mathcal{L}} - T_{m,\ell}$ is bounded from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$.

Hence we conclude that $T_m^{\mathcal{L}}$ can be extended to $L^p(\mathbf{R}^n)$, $1 < p < \infty$, as a bounded operator from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$.

Moreover, we can deduce that the maximal operator

$$T_m^{\mathcal{L},*}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} K_\phi^{\mathcal{L}}(x,y)f(y)\,dy\right|, \quad x \in \mathbf{R}^n,$$

is bounded from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$.

Hence, for every $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$, there exists the limit

$$\lim_{\varepsilon\to 0^+}\left(f(x)\alpha(\varepsilon) + \int_{|x-y|>\varepsilon} f(y)K_\phi^{\mathcal{L}}(x,y)\,dy\right), \quad \text{a.e. } x \in \mathbf{R}^n,$$

and, for every $f \in L^2(\mathbf{R}^n)$,

$$T_m^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+}\left(f(x)\alpha(\varepsilon) + \int_{|x-y|>\varepsilon} f(y)K_\phi^{\mathcal{L}}(x,y)\,dy\right), \quad \text{a.e. } x \in \mathbf{R}^n.$$

We conclude that the operator $T_m^{\mathcal{L}}$ can be extended from $L^2(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, $1 \le p < \infty$, as a bounded operator from $L^p(\mathbf{R}^n)$ into itself, for every $1 < p < \infty$ and from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$.

4. Appendix

In this section we present a pointwise representation of the multiplier T_m. We establish the properties (11) and (13).

For every $f \in L^2(\mathbf{R}^n)$ we have that

$$T_m(f) = (m(|y|^2)\hat{f})\check{}.$$

Let $f \in C_c^\infty(\mathbf{R}^n)$. We can write

$$\begin{aligned}
T_m(f)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ixy} m(|y|^2)\hat{f}(y)\,dy \\
&= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ixy}\hat{f}(y)|y|^2\int_0^\infty \phi(t)e^{-t|y|^2}\,dt\,dy \\
&= \frac{1}{(2\pi)^n}\int_0^\infty \phi(t)\int_{\mathbf{R}^n} e^{ixy}|y|^2 e^{-t|y|^2}\hat{f}(y)\,dy\,dt, \quad x \in \mathbf{R}^n.
\end{aligned}$$

The interchange in the order of integration is justified because

$$\int_{\mathbf{R}^n}|\hat{f}(y)||y|^2\int_0^\infty e^{-t|y|^2}|\phi(t)|\,dt\,dy \le \|\phi\|_\infty\int_{\mathbf{R}^n}|\hat{f}(y)|\,dy < \infty.$$

Then,

$$\begin{aligned}
T_m(f)(x) &= \frac{1}{(2\pi)^n}\int_0^\infty \phi(t)\int_{\mathbf{R}^n} e^{ixy}e^{-t|y|^2}\widehat{(-\Delta)f}(y)\,dy\,dt \\
&= \frac{-1}{(2\pi)^n}\int_0^\infty \phi(t)\int_{\mathbf{R}^n}\Delta f(z)\int_{\mathbf{R}^n} e^{-iy(z-x)}e^{-t|y|^2}\,dy\,dz\,dt \\
&= -\int_0^\infty \phi(t)\int_{\mathbf{R}^n}\Delta f(z)W_t(x,z)\,dz\,dt, \quad x \in \mathbf{R}^n.
\end{aligned}$$

We have taken into account that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\Delta f(z)|e^{-t|y|^2}\,dz\,dy < \infty, \quad t > 0,$$

and that

$$\int_{\mathbf{R}^n} e^{-iyz}e^{-t|y|^2}\,dy = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|z|^2}{4t}}.$$

Since $\displaystyle\int_{\mathbf{R}^n} \Delta f(z)\, dz = \widehat{\Delta f}(0) = -|y|^2 \hat{f}(y)_{|y=0} = 0$, we can write

$$T_m(f)(x) = -\int_0^\infty \phi(t) \int_{\mathbf{R}^n} \Delta f(z) \left(W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right) dz\, dt, \quad x \in \mathbf{R}^n.$$

It is not hard to see that

$$\left| W_t(x,z) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \right| \leq C \frac{|x-z|^2}{t^{\frac{n+2}{2}}}, \quad x, z \in \mathbf{R}^n \text{ and } t > 0.$$

Hence it follows that

$$\int_0^\infty \int_{\mathbf{R}^n} |\Delta f(z)| \left| W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right| dz\, dt$$
$$\leq C \left(\int_0^1 \int_{\mathbf{R}^n} W_t(x,z)\, dz\, dt + \int_1^\infty \int_{\operatorname{supp} f} \frac{|x-z|^2}{t^{\frac{n+2}{2}}}\, dz\, dt \right) \leq C(1+|x|^2), \quad x \in \mathbf{R}^n.$$

Then,

$$T_m(f)(x) = -\lim_{\varepsilon \to 0^+} \int_0^\infty \phi(t) \int_{|x-z|>\varepsilon} \Delta f(z) \left(W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right) dz\, dt, \quad x \in \mathbf{R}^n.$$

Let $0 < \varepsilon < 1$. The Green formula leads to,

$$\int_{|x-z|>\varepsilon} \Delta f(z) \left(W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right) dz$$
$$= \int_{|x-z|>\varepsilon} f(z) \Delta_z W_t(x,z) dz + \int_{|x-z|=\varepsilon} \partial_n f(z) \left(W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right) d\sigma(z)$$
$$- \int_{|x-z|=\varepsilon} f(z) \partial_{n,z} W_t(x,z)\, d\sigma(z), \quad x \in \mathbf{R}^n \text{ and } t > 0.$$

Here ∂_n represents the derivative in the direction normal exterior to the sphere $S_\varepsilon = \{z \in \mathbf{R}^n \colon |z-x| = \varepsilon\}$.

By using [19, Lemma 2.1] we have that

$$\left| \int_0^\infty \phi(t) \int_{|x-z|=\varepsilon} \partial_n f(z) \left(W_t(x,z) - \frac{\chi_{(1,\infty)}(t)}{(4\pi t)^{\frac{n}{2}}} \right) d\sigma(z)\, dt \right|$$
$$\leq C \int_{|x-z|=\varepsilon} \left(\int_0^1 \frac{e^{-\frac{|x-z|^2}{4t}}}{t^{\frac{n}{2}}} dt + \int_1^\infty \frac{|x-z|^2}{t^{\frac{n}{2}+1}}\, dt \right) d\sigma(z)$$
$$\leq C \int_{|x-z|=\varepsilon} \left(\frac{1}{|x-z|^{n-2}} + |x-z|^2 \right) d\sigma(z) \leq C\varepsilon, \quad x \in \mathbf{R}^n.$$

If $n(z)$ denotes a unitary vector in the direction exterior normal in $z \in S_\varepsilon$, we obtain

$$\partial_{n,z} W_t(x,z) = \langle \nabla_z W_t(x,z), n(z) \rangle = W_t(x,z) \langle \frac{x-z}{2t}, n(z) \rangle$$
$$= W_t(x,z) \frac{|x-z|}{2t} = \frac{e^{-\frac{\varepsilon^2}{4t}} \varepsilon}{2(4\pi)^{\frac{n}{2}} t^{\frac{n}{2}+1}}, \quad z \in S_\varepsilon.$$

Moreover, $\sigma(S_\varepsilon) = 2\varepsilon^{n-1}\dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Then we have that

$$\begin{aligned}
&\int_0^\infty \int_{|x-z|=\varepsilon} f(z)\partial_{n,z} W_t(x,z)\, d\sigma(z)\, \phi(t)\, dt \\
&= \varepsilon \int_{|x-z|=\varepsilon} f(z) \int_0^\infty \frac{e^{-\frac{\varepsilon^2}{4t}}}{2(4\pi)^{\frac{n}{2}}} \frac{\phi(t)}{t^{\frac{n}{2}+1}}\, dt\, d\sigma(z) \\
&= \frac{1}{\varepsilon^{n-1} 2\pi^{\frac{n}{2}}} \int_{|x-z|=\varepsilon} f(z) \int_0^\infty \phi\left(\frac{\varepsilon^2}{4u}\right) e^{-u} u^{\frac{n}{2}-1}\, du\, d\sigma(z) \\
&= \frac{1}{\varepsilon^{n-1} 2\pi^{\frac{n}{2}}} \int_{|x-z|=\varepsilon} (f(z)-f(x)) \int_0^\infty \phi\left(\frac{\varepsilon^2}{4u}\right) e^{-u} u^{\frac{n}{2}-1}\, du\, d\sigma(z) + f(x)\alpha(\varepsilon),
\end{aligned}$$

$x \in \mathbf{R}^n$, where $\alpha(\varepsilon) = \dfrac{1}{\Gamma(\frac{n}{2})}\displaystyle\int_0^\infty \phi\left(\frac{\varepsilon^2}{4u}\right) e^{-u} u^{\frac{n}{2}-1}\, du$, $0 < \varepsilon < 1$.

Since f is a continuous function we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{n-1} 2\pi^{\frac{n}{2}}} \int_{|x-z|=\varepsilon} (f(z)-f(x)) \int_0^\infty \phi\left(\frac{\varepsilon^2}{4u}\right) e^{-u} u^{\frac{n}{2}-1}\, du\, d\sigma(z) = 0.$$

It is clear that α is a bounded function on $(0,\infty)$. Moreover, if there exists $\phi(0^+) = \lim_{t\to 0^+} \phi(t)$, by using the dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0^+} \alpha(\varepsilon) = \phi(0^+).$$

Since $\Delta_z W_t(x,z) = \frac{\partial}{\partial t} W_t(x,z)$, $x, z \in \mathbf{R}^n$ and $t > 0$, the above arguments allow us to establish (11) and (13). $\square$

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