Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 38, 2013, 229–244

OPTIMAL WEAK TYPE ESTIMATES FOR DYADIC-LIKE MAXIMAL OPERATORS

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Abstract. We provide sharp weak estimates for the distribution function of $\mathcal{M}\phi$ when on ϕ we impose L^1, L^q and $L^{p,\infty}$ restrictions. Here $\mathcal{M}$ is the dyadic maximal operator associated to a tree $\mathcal{T}$ on a non-atomic probability measure space. As a consequence we produce that the inequality $||\mathcal{M}_\mathcal{T}\phi||_{p,\infty} \leq |||\phi|||_{p,\infty}$ is sharp allowing every possible value for the L^1 and the L^q norm for a fixed q such that $1 < q < p$, where $|||\cdot|||_{p,\infty}$ is the integral norm on and $||\cdot||_{p,\infty}$ the usual quasi norm on $L^{p,\infty}$.

1. Introduction

The dyadic maximal operator on $\mathbf{R}^n$ is defined by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du\colon x \in Q,\ Q \subseteq \mathbf{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbf{R}^n)$ where the dyadic cubes are those formed by the grids $2^{-N}\mathbf{Z}^n$ for $N = 1, 2, \ldots$ and $|A|$ is the Lesbesgue measure of any measurable subset A of R^n. It is easy to prove by using the definition of $\mathcal{M}_d$ that it satisfies the following weak type $(1,1)$ inequality

$$|\{x \in \mathbf{R}^n\colon \mathcal{M}_d\phi(x) \geq \lambda\}| \leq \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi \geq \lambda\}} |\phi(u)|\,du \tag{1.2}$$

for every $\phi \in L^1(\mathbf{R}^n)$ and every $\lambda > 0$. Tis inequality is sharp as can be easily seen by considering characteristic functions over dyadic cubes. Using the fact that

$$||\mathcal{M}_d\phi||_p^p = \int_0^\infty p\lambda^{p-1}|\{\mathcal{M}_d\phi \geq \lambda\}|\,d\lambda$$

and in the sequel inequality (1.2) along with Fubini's theorem we easily get the following L^p inequality known as Doob's inequality

$$||\mathcal{M}_d\phi||_p \leq \frac{p}{p-1}||\phi||_p \tag{1.3}$$

for every $p > 1$ and every $\phi \in L^p(\mathbf{R}^n)$, which is proved to be best possible (see [2, 3] for the general martingales and [10] for the dyadic ones).

A way of studying the dyadic maximal operator is the introduction of the so called Bellman functions (see [8]). Actually, we define for every $p > 1$

$$B_p(f,F) = \sup\left\{\frac{1}{|Q|}\int_Q (\mathcal{M}_d\phi)^p\colon \frac{1}{|Q|}\int_Q \phi^p = F,\ \frac{1}{|Q|}\int_Q \phi = f\right\} \tag{1.4}$$

doi:10.5186/aasfm.2013.3817
2010 Mathematics Subject Classification: Primary 47A30.
Key words: Dyadic, maximal.

where Q is a fixed dyadic cube, ϕ is nonnegative in $L^p(Q)$ and f, F are such that $0 < f^p \leq F$. $B_p(f,F)$ has been computed in [5]. In fact it has been shown that $B_p(f,F) = F\omega_p(f^p/F)^p$ where $\omega_p\colon [0,1] \to \left[1, \frac{p}{p-1}\right]$ is the inverse function of

$$H_p(z) = -(p-1)z^p + pz^{p-1}.$$

This has been proved in a much more general setting of tree like maximal operators on non-atomic probability spaces. The result turns out to be independent of the choice of the measure space.The study of these operators has been continued in [7] where the Bellman functions of them in the case $p < 1$ have been computed. As in [5] and [7] we will follow the moregeneral approach. So for a tree $\mathcal{T}$ on a non atomic probability measure space (X,μ), we define the associated dyadic maximal operator, namely

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi|\, d\mu\colon x \in I \in \mathcal{T}\right\}$$

for every $\phi \in L^1(X,\mu)$.

As it can be seen in [9], $\mathcal{M}_{\mathcal{T}}\colon L^{p,\infty} \to L^{p,\infty}$ is a continuous operator and satisfies the following inequality

$$||\mathcal{M}_{\mathcal{T}}\phi||_{p,\infty} \leq |||\phi|||_{p,\infty}. \tag{1.5}$$

where $||\cdot||_{p,\infty}$ is the usual quasi-norm on $L^{p,\infty}$ defined by

$$||\phi||_{p,\infty} = \sup\left\{\lambda\mu(\{\phi \geq \lambda\})^{1/p}\colon \lambda > 0\right\}.$$

and $|||\cdot|||_{p,\infty}$ is the integral norm on $L^{p,\infty}$ given by

$$|||\phi|||_{p,\infty} = \sup\left\{\mu(E)^{-1+\frac{1}{p}}\int_E |\phi|\, d\mu\colon E \text{ measurable subset of } X \text{ such that } \mu(E) > 0\right\}.$$

$|||\cdot|||_{p,\infty}$ and $||\cdot||_{p,\infty}$ are equivalent because of the following

$$||\phi||_{p,\infty} \leq |||\phi|||_{p,\infty} \leq \frac{p}{p-1}||\phi||_{p,\infty},\ \forall\ \phi \in L^{p,\infty},$$

which can be seen in [4]. In this paper we prove that inequality (1.5) is sharp and independent of the L^1 and L^q norm of ϕ, for a fixed q such that $1 < q < p$. In fact we prove a stronger result, by evaluating the following function of $\lambda > 0$

$$\begin{aligned} &S(f,A,F,\lambda) \\ &= \sup\left\{\mu(\{\mathcal{M}_{\mathcal{T}}\phi \geq \lambda\})\colon \phi \geq 0,\ \int_X \phi\, d\mu = f,\ \int_X \phi^q\, d\mu = A,\ |||\phi|||_{p,\infty} = F\right\}, \end{aligned} \tag{1.6}$$

where (f,A,F) is on the domain of the extremal problem. That is we prove the following

Theorem 1.1. *For f, A such that $f^q < A \leq \Gamma f^{p-q/p-1}F^{p(q-1)/p-1}$ and $0 < f \leq F$ the following hold*

$$S(f,A,F,\lambda) = \min\left\{1, G_{f,A}(\lambda), \frac{F^p}{\lambda^p}\right\}$$

where

$$G_{f,A}(\lambda) = \sup\left\{\mu(\{\mathcal{M}_{\mathcal{T}}\phi \geq \lambda\})\colon \phi \geq 0,\ \int_X \phi\, d\mu = f,\ \int_X \phi^q\, d\mu = A\right\}. \tag{1.7}$$

In fact, $G_{f,A}(\lambda)$ has been precisely computed in [6] by using sharp inequalities on a certain class of functions which is enough to describe the related problem. In this paper we avoid the technique used in [6] and refine this result by proving the theorem mentioned using a different approach. As a corrolary we obtain the following

Corollary 1.1. *The following is true*

$$\sup\left\{||\mathcal{M}_{\mathcal{T}}\phi||_{p,\infty}\colon \phi\geq 0,\ \int_X \phi\,d\mu = f,\ \int_X \phi^q\,d\mu = A,\ |||\phi|||_{p,\infty} = F\right\} = F, \tag{1.8}$$

that is, (1.5) is sharp allowing every value of the integral and the L^q*-norm of* ϕ.

This paper is organized as follows: In Section 2 we provide some lemmas and facts concerning non-atomic probability measure spaces and trees on them. In Section 3 we find the domain of the extremal problem for the case $F = 1$. This is done by finding sharp inequalities relating the L^1 and L^q norm of a measurable function ϕ under the weak condition $|||\phi|||_{p,\infty} = 1$. Krein–Milman theorem is a tool for us in order to find these sharp inequalities. At last in section 4 we precisely evaluate $S(f, A, 1, \lambda)$. We need also to mention that all the estimates are independent of the measure space (X, μ) and the tree $\mathcal{T}$.

2. Preliminaries

Let (X, μ) be a non-atomic probability measure space. We state the following lemma which can be found in [1].

Lemma 2.1. *Let* $\phi\colon (X, \mu) \to \mathbf{R}^+$ *and* ϕ^* *the decreasing rearrangement of* ϕ, *defined on* $[0, 1]$. *Then*

$$\int_0^t \phi^*(u)\,du = \sup\left\{\int_E \phi\,d\mu\colon E \text{ measurable subset of } X \text{ with } \mu(E) = t\right\}$$

for every $t \in [0, 1]$, *with the supremum attained.*

We prove now the following

Lemma 2.2. *Let* $\phi\colon X \to \mathbf{R}^+$ *be measurable and* $I \subseteq X$ *be measurable with* $\mu(I) > 0$. *Suppose that* $\frac{1}{\mu(I)}\int_I \phi\,d\mu = s$. *Then for every* t *such that* $0 < t \leq \mu(I)$ *there exists a measurable set* $E_t \subseteq I$ *with* $\mu(E_t) = t$ *and* $\frac{1}{\mu(E_t)}\int_{E_t} \phi\,d\mu = s$.

Proof. Consider the measure space $(I, \mu/I)$ and let $\psi\colon I \to \mathbf{R}^+$ be the restriction of ϕ on I that is $\psi = \phi/I$. Then, if $\psi^*\colon [0, \mu(I)] \to \mathbf{R}^+$ is the decreasing rearrangement of ψ, we have that

$$\frac{1}{t}\int_0^t \psi^*(u)\,du \geq \frac{1}{\mu(I)}\int_0^{\mu(I)} \psi^*(u)\,du = s \geq \frac{1}{t}\int_{\mu(I)-t}^{\mu(I)} \psi^*(u)\,du. \tag{2.1}$$

Since ψ^* is decreasing, we get the inequalities in (2.1), while the equality is obvious since

$$\int_0^{\mu(I)} \psi^*(u)\,du = \int_I \phi\,d\mu.$$

From (2.1) it is easily seen that there exists $r \geq 0$ such that $t + r \leq \mu(I)$ with

$$\frac{1}{t}\int_r^{t+r} \psi^*(u)\,du = s. \tag{2.2}$$

It is also easily seen that there exists E_t measurable subset of I such that

$$\mu(E_t) = t \quad \text{and} \quad \int_{E_t} \phi \, d\mu = \int_r^{t+r} \psi^*(u)\, du, \tag{2.3}$$

since (X, μ) is non-atomic. From (2.2) and (2.3) we get the conclusion of the lemma. □

We now call two measurable subsets of X almost disjoint if $\mu(A \cap B) = 0$. We give now the following

Definition 2.1. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that
  (a) the elements of $C(I)$ are pairwise almost disjoint subsets of I,
  (b) $I = \cup C(I)$.

(iii) $\mathcal{T} = \bigcup\limits_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup\limits_{I \in \mathcal{T}_{(m)}} C(I)$.

(iv) $\lim\limits_{m \to +\infty} \sup\limits_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$.

From [5] we get the following

Lemma 2.3. *For every $I \in \mathcal{T}$ and every α such that $0 < \alpha < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\Bigg(\bigcup_{J \in \mathcal{F}(I)} J \Bigg) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1 - \alpha)\mu(I).$$

Let now (X, μ) be a non-atomic probability measure space and $\mathcal{T}$ a tree as in Definition 1.1. We define the associated maximal operator to the tree $\mathcal{T}$ as follows: For every $\phi \in L^1(X, \mu)$ and $x \in X$, then

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu \colon x \in I \in \mathcal{T} \right\}.$$

3. The domain of the extremal problem

Our aim is to find the exact allowable values of (f, A, F) for which there exists $\phi\colon (X, \mu) \to \mathbf{R}^+$ measurable such that

$$\int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A \quad \text{and} \quad |||\phi|||_{p,\infty} = F. \tag{3.1}$$

We find it in the case where $F = 1$. For the beginning assume that (f, A) are such that there exist ϕ as in (3.1). We set $g = \phi^*\colon [0, 1] \to \mathbf{R}^+$. Then

$$\int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad |||g|||_{p,\infty}^{[0,1]} = 1$$

where

$$|||g|||_{p,\infty}^{[0,1]} = \sup \left\{ |E|^{-1+\frac{1}{p}} \int_E g \colon E \subset [0, 1] \text{ Lebesque measurable such that } |E| > 0 \right\}.$$

This is true because of the definition of the decreasing rearrangement of ϕ and Lemma 2.1. In fact since g is decreasing $|||g|||_{p,\infty}$ is equal to

$$\sup\left\{t^{-1+\frac{1}{p}}\int_0^t g\colon 0<t\le 1\right\}.$$

Of course, we should have that $0<f\le 1$ and $f^q\le A$. We give now the following

Definition 3.1. If $n\in\mathbf{N}$, and $h\colon[0,1)\to\mathbf{R}^+$, h will be called $\frac{1}{2^n}$-step if it is constant on each interval

$$\left[\frac{i-1}{2^n},\frac{i}{2^n}\right),\quad i=1,2,\dots,2^n.$$

Now for $n\in\mathbf{N}$ and $0<f\le 1$ fixed, we set

$$\varDelta_n(f)=\left\{h\colon[0,1]\to\mathbf{R}^+\colon h \text{ is a } \frac{1}{2^n}\text{-step function}, \int_0^1 h=f,\ |||h|||_{p,\infty}^{[0,1]}\le 1\right\}.$$

Then

$$\varDelta_n=\varDelta_n(f)\subset L^{p,\infty}([0,1])$$

where we use the $|||\cdot|||_{p,\infty}^{[0,1]}$ norm for functions defined on $[0,1]$. $\varDelta_n$ is also convex, that is,

$$h_1,h_2\in\varDelta_n \implies \frac{h_1+h_2}{2}\in\varDelta_n.$$

Additionally, we have the following

Lemma 3.1. *$\varDelta_n$ is compact subset of $L^{p,\infty}([0,1])=Y$ where the topology on Y is that endowed by $|||\cdot|||_{p,\infty}^{[0,1]}$.*

Proof. $(Y,|||\cdot|||_{p,\infty})$ is a Banach space. So, especially a metric space. As a consequence we just need to prove that $\varDelta_n$ is sequentially compact.Let now $(h_i)_i\subset\varDelta_n$. It is now easy to see by a finite diagonal argument that there exists $(h_{i_j})_j$ subsequence and $h\colon[0,1]\to\mathbf{R}^+$. such that $h_{i_j}\to h$ uniformly on $[0,1]$. Then obviously $\int_0^1 h=f$, $|||h|||_{p,\infty}^{[0,1]}\le 1$, so $h\in\varDelta_n$. Additionally

$$\begin{aligned}|||h_{i_j}-h|||_{p,\infty}^{[0,1]}&=\sup\left\{|E|^{-1+\frac{1}{p}}\int_E|h_{i_j}-h|\colon |E|>0\right\}\\&\le\sup\left\{|(h_{i_j}-h)(t)|,\ t\in[0,1]\right\}\to 0\end{aligned}$$

as $j\to\infty$. That is $h_{i_j}\xrightarrow{Y} h\in\varDelta_n$. Consequently, $\varDelta_n$ is a compact subset of $L^{p,\infty}([0,1])$. □

We give now the following known

Definition 3.2. For a closed convex subset K of a topological vector space Y, and for a $y\in K$ we say that y is an extreme point of K, if whenever $y=\frac{x+z}{2}$, with $x,z\in K$ it is implied that $y=x=z$. We write $y\in\operatorname{ext}(K)$.

Definition 3.3. For a subset A of a topological vector space Y we set

$$\operatorname{conv}(A)=\left\{\sum_{i=1}^n\lambda_i x_i\colon \lambda_i\ge 0,\ x_i\in A,\ n\in\mathbf{N}^*,\ \sum_{i=1}^n\lambda_i=1\right\}.$$

We call $\operatorname{conv}(A)$ the convex hull of A.

We state now the following well known

Theorem 3.1. (Krein–Milman) *Let K be a convex, compact subset of a locally convex topological vector space Y. Then $K = \overline{\mathrm{conv}(\mathrm{ext}(K))}^{Y}$, that is, K is the closed convex hull of its extreme points.*

According now to Lemma 3.1 we have that

$$\varDelta_n = \overline{\mathrm{conv}[\mathrm{ext}(\varDelta_n)]}^{L^{p,\infty}([0,1])}.$$

We find now the set $\mathrm{ext}(\varDelta_n)$.

Lemma 3.2. *Let $g \in \mathrm{ext}(\varDelta_n)$. Then for every $i \in \{1, 2, \ldots, 2^n\}$ such that $\left(\frac{i}{2^n}\right)^{1-\frac{1}{p}} \leq f$, we have that*

$$\sup\left\{|E|^{-1+\frac{1}{p}} \int_E g \colon |E| = \frac{i}{2^n}\right\} = 1.$$

Proof. We prove it first when $i = 1$ and $\left(\frac{1}{2^n}\right)^{1-\frac{1}{p}} \leq f$. It is now easy to see that $g \in \mathrm{ext}(\varDelta_n)$ if and only if $g^* \in \mathrm{ext}(\varDelta_n)$. So we just need to prove that $\int_0^{1/2^n} g^* = \left(\frac{1}{2^n}\right)^{1-\frac{1}{p}}$. We write

$$g^* = \sum_{i=1}^{2^n} \alpha_i \xi_{I_i} \quad \text{with} \quad I_i = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$$

and $\alpha_i \geq \alpha_{i+1}$ for every $i \in \{1, 2, \ldots, 2^n - 1\}$. Suppose now that $\alpha_1 < 2^{n/p}$, and that $\alpha_1 > \alpha_2$ (the case $\alpha_1 = \alpha_2$ is handled in an analogous way). For a suitable $\varepsilon > 0$ we set

$$g_1 = \sum_{i=1}^{2^n} \alpha_i^{(1)} \xi_{I_i}, \quad g_2 = \sum_{i=1}^{2^n} \alpha_i^{(2)} \xi_{I_i}, \quad \text{where} \quad \left.\begin{array}{l} \alpha_1^{(1)} = \alpha_1 + \varepsilon, \quad \alpha_2^{(1)} = \alpha_2 - \varepsilon \\ \alpha_1^{(2)} = \alpha_1 - \varepsilon, \quad \alpha_2^{(2)} = \alpha_2 + \varepsilon \end{array}\right\}$$

and $\alpha_k^{(1)} = \alpha_k^{(2)} = \alpha_k$ for every $k > 2$. Since $\alpha_1 < 2^{n/p}$, we can find small enough $\varepsilon > 0$ such that g_i satisfy $|||g_i|||_{p,\infty}^{[0,1]} \leq 1$, for $i = 1, 2$. Indeed, for $i = 1$, we need to prove that for small enough $\varepsilon > 0$

$$\int_0^t g_1 \leq t^{1-\frac{1}{p}} \tag{3.2}$$

for every $t \in [0, 1)$, since g_1 is decreasing. (3.2) is now obviously true for $t \geq \frac{2}{2^n}$ since

$$\int_0^t g_1 = \int_0^t g^* \quad \text{for every such } t. \tag{3.3}$$

(3.2) is also true for $t = 0, \frac{1}{2^n}$ for a suitable $\varepsilon > 0$. But then it remains true for every $t \in \left(0, \frac{1}{2^n}\right)$ since the function $t \mapsto \int_0^t g_1$ represents a straight line on $\left[0, \frac{1}{2^n}\right]$ and $t^{1-\frac{1}{p}}$ is concave there, analogously for the interval $\left[\frac{1}{2^n}, \frac{2}{2^n}\right]$. That is we proved $|||g_1|||_{p,\infty}^{[0,1]} \leq 1$. For $i = 2$ we use the same arguments and the hypothesis $\alpha_1 > \alpha_2$ in order to ensure that for small enough $\varepsilon > 0$, g_2 is decreasing. Obviously now,

$\int_0^1 g_i = f$, so that $g_i \in \Delta_n$, for $i = 1, 2$. But $g^* = \frac{g_1+g_2}{2}$, with $g_i \neq g$ and $g_i \in \Delta_n$, $i = 1, 2,$, a contradiction since $g^* \in \operatorname{ext}(\Delta_n)$. So,

$$\alpha_1 = 2^{n/p} \quad \text{and} \quad \int_0^{1/2} g^* = \left(\frac{1}{2^n}\right)^{1-\frac{1}{p}},$$

that is what we wanted to prove. In the same way we prove that for $i \in \{1, 2, \ldots, 2^n - 1\}$ such that

$$\left(\frac{i+1}{2^n}\right)^{1-\frac{1}{p}} \leq f, \quad \text{if} \quad \int_0^{i/2^n} g^* = \left(\frac{i}{2^n}\right)^{1-\frac{1}{p}}, \quad \text{then} \quad \int_0^{(i+1)/2^n} g^* = \left(\frac{i+1}{2^n}\right)^{1-\frac{1}{p}}.$$

The lemma is now proved by induction. □

Let now $g \in \operatorname{ext}(\Delta_n)$ and $k = \max\left\{i \leq 2^n \colon \left(\frac{i}{2^n}\right)^{1-\frac{1}{p}} \leq f\right\}$, so if we suppose that $f < 1$, we have that

$$\left(\frac{k}{2^n}\right)^{1-\frac{1}{p}} \leq f < \left(\frac{k+1}{2^n}\right)^{1-\frac{1}{p}}.$$

By Lemma 3.2,

$$\int_0^{k/2^n} g^* = \left(\frac{k}{2^n}\right)^{1-\frac{1}{p}}.$$

But by using the reasoning of the previous lemma it is easy to see that

$$\int_0^{(k+1)/2^n} g^* = f,$$

which gives

$$\int_{k/2^n}^{k+1/2^n} g^* = f - \left(\frac{k}{2^n}\right)^{1-\frac{1}{p}} \implies \alpha_{k+1} = 2^n \cdot f - 2^{n/p} \cdot k^{1-\frac{1}{p}}.$$

Additionally, $\alpha_i = 0$ for $i > k+1$. From the above we obtain the following

Corollary 3.1. *Let $g \in \operatorname{ext}(\Delta_n)$. Then $g^* = \sum_{i=1}^{2^n} \alpha_i \xi_{I_i}$, where*

$$\alpha_i = 2^{n/p}\left(i^{1-\frac{1}{p}} - (i-1)^{1-\frac{1}{p}}\right) \quad \text{for } i = 1, 2, \ldots, k$$

and

$$\alpha_{k+1} = 2^n f - 2^{n/p} \cdot k^{1-\frac{1}{p}}, \quad \alpha_i = 0, \quad i > k+1,$$

where

$$k = \max\left\{i \leq 2^n \colon \left(\frac{i}{2^n}\right)^{1-\frac{1}{p}} \leq f\right\}.$$

We estimate now the L^q-norm of every $g \in \operatorname{ext}(\Delta_n)$. We state it as

Lemma 3.3. *Let $g \in \operatorname{ext}(\Delta_n)$ and $A = \int_0^1 g^q$. Then $A \leq \Gamma f^{p-q/p-1} + \mathcal{E}_n(f)$, where*

$$\Gamma = \left(\frac{p-1}{p}\right)^q \frac{p}{p-q} \quad \text{and} \quad \mathcal{E}_n(f) = \frac{\alpha_{k+1}^q}{2^n} = \frac{(2^n f - 2^{n/p} k^{1-\frac{1}{p}})^q}{2^n}.$$

Proof. For g we write $g^* = \sum_{i=1}^{2^n} \alpha_i \xi_{I_i}$, where α_i are given in Corollary 3.1. Then

$$A = \int_0^1 (g^*)^q = \left[\left(\sum_{i=1}^{k} \alpha_i^q\right) + \alpha_{k+1}^q\right] \cdot \frac{1}{2^n}. \tag{3.4}$$

Now for $i \in \{1, 2, \dots, k\}$

$$\begin{aligned}\alpha_i^q &= \left[2^{n/p}\left(i^{1-\frac{1}{p}} - (i-1)^{1-\frac{1}{p}}\right)\right]^q = \left\{2^n\left[\left(\frac{i}{2^n}\right)^{1-\frac{1}{p}} - \left(\frac{i-1}{2^n}\right)^{1-\frac{1}{p}}\right]\right\}^q \\ &= \left[2^n \int_{i-1/2^n}^{i/2^n} \psi\right]^q,\end{aligned} \tag{3.5}$$

where $\psi\colon (0,1] \to \mathbf{R}^+$ is defined by $\psi(t) = \frac{p-1}{p} t^{-1/p}$. By (3.5) and in view of Hölder's inequality we have that for $i \in \{1, 2, \dots, k\}$

$$\alpha_i^q \le 2^n \int_{i-1/2^n}^{i/2^n} \psi^q. \tag{3.6}$$

Summing up relations (3.6) we have that

$$\sum_{i=1}^{k} \alpha_i^q \le 2^n \int_0^{k/2^n} \psi^q = 2^n \cdot \Gamma \cdot \left(\frac{k}{2^n}\right)^{1-\frac{q}{p}}. \tag{3.7}$$

Additionally from the definition of k we have that

$$\left(\frac{k}{2^n}\right)^{1-\frac{1}{p}} \le f \implies k^{1-\frac{q}{p}} \le (2^n)^{1-\frac{q}{p}} \cdot f^{p-q/p-1}. \tag{3.8}$$

From (3.4), (3.7) and (3.8) we obtain

$$A \le \left[2^n \cdot \Gamma \cdot f^{p-q/p-1} + \alpha_{k+1}^q\right] \frac{1}{2^n} = \Gamma f^{p-q/p-1} + \mathcal{E}_n(f)$$

and Lemma 3.3 is proved. □

Corollary 3.2. *For every $g \in \Delta_n$,*

$$A \le \Gamma f^{p-q/p-1} + \mathcal{E}_n(f), \quad \textit{where } A = \int_0^1 g^q.$$

Proof. This is true, of course, for $g \in \mathrm{ext}(\Delta_n)$, and so also for $g \in \mathrm{conv}(\mathrm{ext}\,\Delta_n)$, since $t \mapsto t^q$ is convex for $q > 1$ on $\mathbf{R}^+$. It remains true for $g \in \overline{\mathrm{conv}(\mathrm{ext}(\Delta_n))}^{L^{p,\infty}([0,1])}$ using a simple continuity argument. In fact, we just need the continuity of the identity operator if it is viewed as $I\colon L^{p,\infty}([0,1]) \to L^q([0,1])$. See [4]. Using now Krein–Milman Theorem the Corollary is proved. □

We have now the following

Corollary 3.3. *Let $\phi\colon (X,\mu) \to \mathbf{R}^+$ such that*

$$\int_X \phi\, d\mu = f, \quad \int_X \phi^q\, d\mu = A, \quad |||\phi|||_{p,\infty} \le 1.$$

Then

$$f^q \le A \le \Gamma f^{p-q/p-1}.$$

Proof. Let $g = \phi^*\colon [0,1] \to \mathbf{R}^+$. There exist a sequence (g_n) of $\frac{1}{2^n}$-simple functions, such that $g_n \le g_{n+1} \le g$ and g_n converges almost everywhere to g. But then by defining

$$f_n = \int_0^1 g_n, \quad A_n = \int_0^1 g_n^q$$

we have that

$$g_n \in \Delta_n(f_n) \ \text{ so that } \ A_n \le \Gamma f_n^{p-q/p-1} + \mathcal{E}_n(f_n). \tag{3.9}$$

By the monotone convergence theorem $f_n \to f$, $A_n \to A$. Moreover,

$$\mathcal{E}_n(f_n) = \frac{(2^n f_n - k_n^{1-\frac{1}{p}} 2^{n/p})^q}{2^n},$$

where k_n satisfy

$$\left(\frac{k_n}{2^n}\right)^{1-\frac{q}{p}} \le f_n < \left(\frac{k_n+1}{2^n}\right)^{1-\frac{1}{p}}.$$

As a consequence

$$\begin{aligned}\mathcal{E}_n(f_n) &= (2^n)^{q-1}\left[f_n - \left(\frac{k_n}{2^n}\right)^{1-\frac{1}{p}}\right]^q < (2^n)^{q-1}\left[\left(\frac{k_n+1}{2^n}\right)^{1-\frac{1}{p}} - \left(\frac{k_n}{2^n}\right)^{1-\frac{1}{p}}\right]^q \\ &\le (2^n)^{q-1}\left[\left(\frac{1}{2^n}\right)^{1-\frac{1}{q}}\right]^q = \left(\frac{1}{2^{1-\frac{q}{p}}}\right)^n \to 0, \ \text{ as } \ n \to \infty\end{aligned}$$

where in the second inequality we used the known

$$(t+s)^\alpha \le t^\alpha + s^\alpha \quad \text{for } t, s \ge 0, \ 0 < \alpha < 1.$$

Now (3.9) gives the corollary. □

In fact the converse of Corollary 3.3 is also true.

Theorem 3.2. *For $0 < f \le 1$, $A > 0$ the following are equivalent:*

i) $f^q \le A \le \Gamma f^{p-q/p-1}$,

ii) $\exists\, \phi\colon (X,\mu) \to \mathbf{R}^+$ *such that*

$$\int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A, \quad |||\phi|||_{p,\infty} \le 1.$$

We prove first the following

Lemma 3.4. *Let $\alpha \in (0,1)$ and (f, A) such that*

$$f \lneq \alpha^{1-\frac{1}{p}}, \tag{3.10}$$

$$f^q \lneq \alpha^{q-1} A, \tag{3.11}$$

$$A \le \Gamma f^{p-q/p-1}. \tag{3.12}$$

Then there exists $g\colon [0,\alpha] \to \mathbf{R}^+$ such that

$$\int_0^\alpha g = f, \quad \int_0^\alpha g^q = A, \quad \textit{and} \quad |||g|||_{p,\infty}^{[0,\alpha]} = 1,$$

where

$$|||g|||_{p,\infty}^{[0,\alpha]} = \sup\left\{|E|^{-1+\frac{1}{p}} \int_E g\colon E \textit{ measurable subset of } [0,\alpha] \textit{ such that } |E| > 0\right\}.$$

Proof. We search for a g of the form

$$g := \begin{cases} \frac{p-1}{p} t^{-1/p}, & 0 < t \le c_1, \\ \mu_2, & c_1 < t \le \alpha, \end{cases}$$

for suitable constant $c_1 \mu_2$. We must have that

$$\int_0^\alpha g = f \iff c_1^{1-\frac{1}{p}} + \mu_2(\alpha - c_1) = f. \tag{3.13}$$

Additionally, g must satisfy

$$\int_0^\alpha g^q = A \iff \Gamma c_1^{1-\frac{q}{p}} + \mu_2^q(\alpha - c_1) = A. \tag{3.14}$$

(3.13) gives

$$\mu_2 = \frac{f - c_1^{1-\frac{1}{p}}}{a - c_1}, \tag{3.15}$$

so (3.14) becomes

$$\Gamma c_1^{1-\frac{q}{p}} + \frac{(f - c_1^{1-\frac{1}{P}})^q}{(\alpha - c_1)^{q-1}} = A. \tag{3.16}$$

That is we search for a $c_1 \in (0, \alpha)$ such that

$$T(c_1) = A \quad \text{where} \quad T\colon [0, \alpha) \to \mathbf{R}^+$$

is defined by

$$T(t) = \Gamma t^{1-\frac{q}{p}} + \frac{(f - t^{1-\frac{1}{p}})^q}{(\alpha - t)^{q-1}}.$$

Observe that $T(0) = \frac{f^q}{\alpha^{q-1}} \lneq A$ because of (3.11) and that $T(f^{p/p-1}) = \Gamma f^{p-q/p-1} \ge A$. Now because of the continuity of T, there exists $c_1 \in (0, f^{p/p-1}]$ such that $T(c_1) = A$. Then $c_1 \in (0, \alpha)$ because of (3.10), and if we define μ_2 by (3.15), we guarantee (3.13) and (3.14). We need to prove now that $|||g|||_{p,\infty}^{[0,\alpha]} = 1$. Obviously, because of the form of g, $|||g|||_{p,\infty}^{[0,\alpha]} \ge 1$. So we have to prove that

$$\int_0^t g \le t^{1-\frac{1}{p}}, \quad \forall\, t \in (0, \alpha]. \tag{3.17}$$

This is of course true for $t \in [0, c_1]$. For $t \in (c_1, \alpha]$,

$$\int_0^t g = c_1^{1-\frac{1}{p}} + \mu_2(t - c_1) =: G(t).$$

Since $G(c_1) = c_1^{1-\frac{1}{p}}$, $G(\alpha) = f < \alpha^{1-\frac{1}{p}}$ and $t \mapsto t^{1-\frac{1}{p}}$ is concave on $(c_1, \alpha]$, (3.17) is true. Thus Lemma 3.4 is proved. □

We have now the

Proof of Theorem 3.2. We have to prove the direction i) ⇒ ii). Indeed, if $f^q \lneq A \le \Gamma f^{p-q/p-1}$ and $f < 1$, we apply Lemma 3.4. If $f^q = A$ with $0 < f \le 1$, we set g by $g(t) = f$, for every $t \in [0, 1]$, while if $f = 1 \le A \le \Gamma$ a simple modification of Lemma 3.4 gives the result. □

We conclude Section 3 with the following theorem which can be proved easily using all the above.

Theorem 3.3. *For* f, A *such that* $0 < f < 1$, $A > 0$ *the following are equivalent:*

i) $f^q \lneq A \le \Gamma f^{p-q/p-1}$,

ii) $\exists\, \phi\colon (X,\mu) \to \mathbf{R}^+$ *such that* $\int_X \phi\, d\mu = f$, $\int_X \phi^q\, d\mu = A$, $|||\phi|||_{p,\infty} = 1$.

Remark 3.1. Theorem 3.3 is completed if we mention that for $f = 1$ the following are equivalent:

i) $f = 1 \le A \le \Gamma$,

ii) $\exists\, \phi\colon (X,\mu) \to \mathbf{R}^+$ such that $\int_X \phi\, d\mu = 1$, $\int_X \phi^q\, d\mu = A$, $|||\phi|||_{p,\infty} = 1$.

4. The extremal problem

Let $\mathcal{M}_{\mathcal{T}} = \mathcal{M}$ the dyadic maximal operator associated to the tree $\mathcal{T}$, on the probability non-atomic measure space (X,μ). Our aim is to find

$$T_{f,A,F}(\lambda) = \sup\left\{\mu(\{\mathcal{M}\phi \ge \lambda\})\colon \phi \ge 0, \int_X \phi\, d\mu = f, \int_X \phi^q\, d\mu = A,\ |||\phi|||_{p,\infty} = F\right\}$$

for all the allowable values of f, A, F. We find it in the case where $F = 1$. We write $T_{f,A}(\lambda)$ for $T_{f,A,1}(\lambda)$. In order to find $T_{f,A}(\lambda)$ we find first the following

$$T^{(1)}_{f,A}(\lambda) = \sup\left\{\mu(\{\mathcal{M}\phi \ge \lambda\})\colon \phi \ge 0, \int_X \phi\, d\mu = f, \int_X \phi^q\, d\mu = A,\ |||\phi|||_{p,\infty} \le 1\right\}.$$

The domain of this extremal problem is the following

$$D = \Big\{(f,A)\colon 0 < f \le 1,\ f^q \le A \le \Gamma f^{p-q/p-1}\Big\}.$$

Obviously, $T^{(1)}_{f,A}(\lambda) = 1$, for $\lambda \le f$. Let now $\lambda > f$ and $(f,A) \in D$. Let ϕ be as in the definition of $T^{(1)}_{f,A}(\lambda)$. Consider the decreasing rearrangement of ϕ, $g = \phi^*\colon [0,1] \to \mathbf{R}^+$. Then

$$\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad |||g|||^{[0,1]}_{p,\infty} \le 1.$$

Consider also $E = \{\mathcal{M}\phi \ge \lambda\} \subseteq X$. Then E is the almost disjoint union of elements of $\mathcal{T}$, let $(I_j)_j$. In fact, we just need to consider the elements I of $\mathcal{T}$, maximal under the condition

$$\frac{1}{\mu(I)} \int_I \phi\, d\mu \ge \lambda. \tag{4.1}$$

We then have $E = \bigcup_j I_j$ and $\int_E \phi\, d\mu \ge \lambda\mu(E)$ because of (4.1). Then according to Lemma 2.1 we have that $\int_0^\alpha g \ge \alpha\lambda$ where $\alpha = \mu(E)$. That is

$$T^{(1)}_{f,A}(\lambda) \le \Delta_{f,A}(\lambda), \tag{4.2}$$

where

$$\begin{aligned}\Delta_{f,A}(\lambda) = \sup\Big\{&\alpha \in (0,1]\colon \exists\, g\colon [0,1] \to \mathbf{R}^+\colon \\ &\int_0^1 g = f,\ \int_0^1 g^q = A,\ |||g|||^{[0,1]}_{p,\infty} \le 1, \int_0^\alpha g \ge \alpha\lambda\Big\}.\end{aligned} \tag{4.3}$$

We prove now the converse inequality in (4.2) by proving the following

Lemma 4.1. *Let g be as in (4.3) for a fixed $\alpha \in (0,1]$. Then there exists $\phi\colon (X,\mu) \to \mathbf{R}^+$ such that*

$$\int_X \phi\, d\mu = f, \quad \int_X \phi^q\, d\mu = A, \quad |||\phi|||_{p,\infty} \le 1 \quad \text{and} \quad \mu(\{\mathcal{M}\phi \ge \lambda\}) \ge \alpha.$$

Proof. Lemma 2.3 guarantees the existence of a sequence $(I_j)_j$ of pairwise almost disjoint elements of $\mathcal{T}$ such that

$$\mu\left(\bigcup I_j\right) = \sum \mu(I_j) = \alpha. \tag{4.4}$$

Consider now the finite measure space $([0,\alpha], |\cdot|)$, where $|\cdot|$ is the Lebesque measure. Then since $\int_0^\alpha g \ge \alpha\lambda$ and (4.4) holds, applying Lemma 2.2 repeatedly, we obtain the existence of a sequence (A_j) of Lebesque measurable subsets of $[0,\alpha]$ such that the following hold:

$$(A_j)_j \text{ is a pairwise disjoint family, } \bigcup A_j = [0,\alpha], \ |A_j| = \mu(I_j), \ \frac{1}{|A_j|}\int_{A_j} g \ge \lambda.$$

Then we define $g_j\colon [0,|A_j|] \to \mathbf{R}^+$ by $g_j = (g/A_j)^*$. Define also for every j a measurable function $\phi_j\colon I_j \to \mathbf{R}^+$ so that $\phi_j^* = g_j$. The existence of such a function is guaranteed by the fact that $(I_j, \mu/I_j)$ is non-atomic. Since (I_j) is almost pairwise disjoint family we produce a $\phi^{(1)}\colon \cup I_j \to \mathbf{R}^+$ measurable such that $\phi^{(1)}/I_j = \phi_j$. We set now $Y = X \setminus \cup I_j$ and $h\colon [0, 1-\alpha] \to \mathbf{R}^+$ by $h = (g/[\alpha,1])^*$. Then since $\mu(Y) = 1-\alpha$ there exists $\phi^{(2)}\colon Y \to \mathbf{R}^+$ such that $(\phi^{(2)})^* = h$. Set now

$$\phi = \begin{cases} \phi^{(1)}, & \text{on } \cup I_j, \\ \phi^{(2)}, & \text{on } Y. \end{cases}$$

It is easy to see from the above construction that $\phi^* = g$ a.e. with respect to Lesbesgue measure, which gives $\int_X \phi\, d\mu = f$, $\int_X \phi^q\, d\mu = A$ and $|||\phi|||_{p,\infty} \le 1$. Additionally,

$$\frac{1}{\mu(I_j)}\int_{I_j} \phi\, d\mu = \frac{1}{|A_j|}\int_{A_j} g \ge \lambda \quad \text{for every } j,$$

that is,

$$\{\mathcal{M}\phi \ge \lambda\} \supseteq \cup I_j, \quad \text{so} \quad \mu(\{\mathcal{M}\phi \ge \lambda\}) \ge \alpha$$

and the lemma is proved. □

It is now not difficult to see that we can replace the inequality $\int_0^\alpha g \ge \alpha\lambda$ in the definition of $\Delta_{f,A}(\lambda)$ by equality, thus defining $S_{f,A}(\lambda)$, in such a way that

$$T^{(1)}_{f,A}(\lambda) = \Delta_{f,A}(\lambda) = S_{f,A}(\lambda). \tag{4.5}$$

This is true since if g is as in (4.3) and $\lambda > f$, there exists $\beta \ge \alpha$ such that $\int_0^\beta g = \beta\lambda$. For $(f,A) \in D$ we set

$$G_{f,A}(\lambda) = \sup\left\{\mu(\{\mathcal{M}\phi \ge \lambda\})\colon \phi \ge 0, \ \int_X \phi\, d\mu = f, \ \int_X \phi^q\, d\mu = A\right\}.$$

It is obvious that $T^{(1)}_{f,A}(\lambda) \le G_{f,A}(\lambda)$. As a matter of fact $G_{f,A}(\lambda)$ has been computed in [6] and was found to be

$$G_{f,A}(\lambda) = \begin{cases} 1, & \lambda \le f, \\ \frac{f}{\lambda}, & f < \lambda < \left(\frac{A}{f}\right)^{1/q-1}, \\ k, & \left(\frac{A}{f}\right)^{1/q-1} \le \lambda, \end{cases} \tag{4.6}$$

where k is the unique root of the equation

$$\frac{(f-\alpha\lambda)^q}{(1-\alpha)^{q-1}} + \alpha\lambda^q = A \quad \text{on } \alpha \in \left[0, \frac{f}{\lambda}\right], \quad \text{when } \lambda > \left(\frac{A}{f}\right)^{1/q-1}.$$

We have now the following

Proposition 4.1. *If $(f, A) \in D$, then*

$$T^{(1)}_{f,A}(\lambda) \le \min\left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}.$$

Proof. We just need to see that $\mu(\{\mathcal{M}\phi \ge \lambda\}) \le \frac{1}{\lambda^p}$ for every ϕ such that $|||\phi|||_{p,\infty} \le 1$. But if $E = \{\mathcal{M}\phi \ge \lambda\}$, we have by the definition of the norm $|||\cdot|||_{p,\infty}$ that $\int_E \phi \le \mu(E)^{1-\frac{1}{p}}$. But by (1.3) $\int_E \phi \ge \lambda\mu(E)$, so that

$$\lambda\mu(E) \le \mu(E)^{1-\frac{1}{p}} \implies \mu(E) \le \frac{1}{\lambda^p}.$$

So Proposition 4.1 is true. □

We prove now that in Proposition 4.1 we have equality.

Proposition 4.2. *Let $(f, A) \in D$ and λ such that*

$$\frac{f}{\lambda} = \min\left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}. \tag{4.7}$$

Then $T^{(1)}_{f,A}(\lambda) = \frac{f}{\lambda}$.

Proof. We use Lemma 3.4 and equations (4.5). Because of (4.5) we need to find $g\colon [0,1] \to \mathbf{R}^+$ such that

$$\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad |||g|||_{p,\infty} \le 1 \quad \text{and} \quad \int_0^{f/\lambda} g = \frac{f}{\lambda}\cdot\lambda = f,$$

that is, g should be defined on $[0, f/\lambda]$. We apply Lemma 3.4, with $\alpha = \frac{f}{\lambda}$. In fact, since (4.7) is true, we have that $G_{f,A}(\lambda) = \frac{f}{\lambda}$ so, $\lambda < \left(\frac{A}{f}\right)^{1/q-1}$ which gives (3.11), while $\frac{f}{\lambda} \le \frac{1}{\lambda^p}$ gives (3.10). In fact, Lemma 3.4 works even with equality on (3.10) as it is easily can be seen by continouity reasons. So, in view of (4.5) we have $T^{(1)}_{f,A}(\lambda) \ge f/\lambda$ and the proposition is proved. □

At the next step we have

Proposition 4.3. *Let $(f, A) \in D$ and λ such that*

$$k = \min\left\{1, G_{f,A}(\lambda)\frac{1}{\lambda^p}\right\}. \tag{4.8}$$

Then $T^{(1)}_{f,A}(\lambda) = k$.

Proof. Obviously, (4.8) gives $\lambda \geq \left(\frac{A}{f}\right)^{1/q-1}$. We prove that there exists $g\colon [0,1] \to \mathbf{R}^+$ such that

$$\int_0^k g = k\lambda, \quad \int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad |||g|||_{p,\infty} \leq 1. \tag{4.9}$$

For this purpose we define

$$g := \begin{cases} \lambda, & \text{on } [0,k], \\ \frac{f-k\lambda}{1-k}, & \text{on } (k,1]. \end{cases}$$

Then, obviously, the first two conditions in (4.9) are satisfied, while

$$\int_0^1 g^q = \frac{(f-k\lambda)^q}{(1-k)^{q-1}} + k\lambda^q = A,$$

by the definition of k. Moreover, $|||g|||_{p,\infty} \leq 1$. This is true since $k\lambda \leq k^{1-\frac{q}{p}}$, $f \leq 1$ and the fact that g is constant on each of the intervals $[0,k]$ and $(k,1]$. So the proposition is proved. □

At last we prove

Proposition 4.4. *Let $(f,A) \in D$ and λ such that*

$$\frac{1}{\lambda^p} = \min\left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}. \tag{4.10}$$

Then $T^{(1)}_{f,A}(\lambda) = \frac{1}{\lambda^p}$.

Proof. As before we search for a function g such that

$$\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad |||g|||_{p,\infty} \leq 1 \quad \text{and} \quad \int_0^{1/\lambda^p} g = \frac{1}{\lambda^p} \cdot \lambda = \frac{1}{\lambda^{p-1}}. \tag{4.11}$$

We define

$$\vartheta_\lambda = \frac{\varGamma}{\lambda^{p-q}} + \frac{\left(f - \frac{1}{\lambda^{p-1}}\right)^q}{\left(1 - \frac{1}{\lambda^p}\right)^{q-1}},$$

and we consider two cases:

i) $\vartheta_\lambda > A$. We search for a function of the form

$$g := \begin{cases} \left(1 - \frac{1}{p}\right) t^{-1/p}, & 0 < t \leq c_1, \\ \mu_2, & c_1 < t \leq \frac{1}{\lambda^p}, \\ \mu_3, & \frac{1}{\lambda^p} < t < 1, \end{cases} \tag{4.12}$$

for suitable constants $c_1 \leq \frac{1}{\lambda^p}$, μ_2, μ_3. Then in view of (4.11) the following must hold:

$$c_1^{1-\frac{1}{p}} + \mu_2\left(\frac{1}{\lambda^p} - c_1\right) = \frac{1}{\lambda^{p-1}}, \tag{4.13}$$

$$c_1^{1-\frac{1}{p}} + \mu_2\left(\frac{1}{\lambda^p} - c_1\right) + \mu_3\left(1 - \frac{1}{\lambda^p}\right) = f, \tag{4.14}$$

$$\varGamma c_1^{1-\frac{q}{p}} + \mu_2^q\left(\frac{1}{\lambda^p} - c_1\right) + \mu_3^q\left(1 - \frac{1}{\lambda^p}\right) = A. \tag{4.15}$$

Notice that the condition $|||g|||_{p,\infty} \le 1$ is automatically satisfied because of the form of g and the previous stated relations. Now (4.13) and (4.14) give

$$\mu_3 = \frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}} \tag{4.16}$$

and

$$\mu_2 = \frac{\frac{1}{\lambda^{p-1}} - c_1^{1-\frac{1}{p}}}{\frac{1}{\lambda^p} - c_1}, \tag{4.17}$$

while (4.15) gives $T(c_1) = A$ where T is defined on $\left[0, \frac{1}{\lambda^p}\right)$ by

$$T(c) = \Gamma c^{1-\frac{q}{p}} + \frac{\left(\frac{1}{\lambda^{p-1}} - c^{1-\frac{1}{p}}\right)^q}{\left(\frac{1}{\lambda^p} - c\right)^{q-1}} + \frac{\left(f - \frac{1}{\lambda^{p-1}}\right)^q}{\left(1 - \frac{1}{\lambda^p}\right)^{q-1}}.$$

Then

$$T(0) = \frac{1}{\lambda^{p-q}} + \frac{\left(f - \frac{1}{\lambda^{p-1}}\right)^q}{\left(1 - \frac{1}{\lambda^p}\right)^{q-1}}.$$

It is now easy to see that $T(0) \le A$ by using that $F\colon [0, f/\lambda] \to \mathbf{R}^+$ defined by

$$F(t) = \frac{(f - t\lambda)^q}{(1-t)^{q-1}} + t\lambda^q$$

is increasing, and the definition of $G_{f,A}(\lambda)$. Moreover $\lim_{c \to \frac{1^-}{\lambda^p}} T(c) = \vartheta_\lambda > A$, so by continuity of the function t, we end case i). Now for

ii) $\vartheta_\lambda \le A$. We search for a function of the form

$$g := \begin{cases} \left(1 - \frac{1}{p}\right) t^{1-1/p}, & 0 < t \le c_1, \\ \mu_2, & c_1 < t \le 1, \end{cases}$$

where $\frac{1}{\lambda^p} < c_1$. Similar arguments as in case i) give the result. □

From Propositions 4.1–4.4 we have now

Theorem 4.1. *For $(f, A) \in D$,*

$$T^{(1)}_{f,A}(\lambda) = \min\left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}.$$

Remark 4.1. Notice that $T_{f,A}(\lambda) = T^{(1)}_{f,A}(\lambda)$ for every f, A such that $f^q < A \le \Gamma f^{p-q/p-1}$ and $0 < f \le 1$. Indeed, suppose that $\alpha = T^{(1)}_{f,A}(\lambda)$. Then there exists $g\colon [0,1] \to \mathbf{R}^+$ such that

$$\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \int_0^\alpha g = \alpha\lambda \;\text{ and }\; |||g|||_{p,\infty} \le 1. \tag{4.18}$$

It is easy to see that for every $\varepsilon > 0$, small enough we can produce from g a function g_ε satisfying

$$\int_0^{\alpha-\varepsilon} g_\varepsilon \ge (\alpha - \varepsilon)\lambda, \quad \int_0^1 g_\varepsilon = f, \quad \int_0^1 g_\varepsilon = A + \delta_\varepsilon \quad\text{and}\quad |||g_\varepsilon|||_{p,\infty} = 1,$$

where $\lim_{\varepsilon\to 0^+} \delta\varepsilon = 0$. This and continuity reasons shows $T_{f,A}(\lambda) = \alpha$.

iii) The case $A = f^q$ can be worked out separately because there is essentially unique function g satisfying $\int_0^1 g = f$, $\int_0^1 g^q = f^q$, namely the constant function with value f.

Scaling all the above we have that

Theorem 4.2. *For f, A such that $f^q < A \le \Gamma f^{p-q/p-1} F^{p(q-1)/(p-1)}$ and $0 < f \le F$ the following hold*

$$\begin{aligned} &\sup\left\{\mu(\{\mathcal{M}\phi \ge \lambda\})\colon \phi \ge 0,\ \int_X \phi\, d\mu = f,\ \int_X \phi^q\, d\mu = A,\ |||\phi|||_{p,\infty} = F\right\} \\ &= \min\left\{1, G_{f,A}(\lambda), \frac{F^p}{\lambda^p}\right\} \end{aligned} \tag{4.19}$$

and

$$\sup\left\{||\mathcal{M}\phi||_{p,\infty}\colon \phi \ge 0,\ \int_X \phi\, d\mu = f,\ \int_X \phi^q\, d\mu = A,\ |||\phi|||_{p,\infty} = F\right\} = F.$$

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Received 24 February 2012 • Accepted 12 October 2012