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ON BANK–LAINE TYPE FUNCTIONS

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Abstract. We continue our previous study on the Bank–Laine type functions: meromorphic functions f that satisfy $f(z)=0 \iff f'(z)\in\{a,b\}$ on the plane, where a,b are two distinct nonzero values. Using quasi-normality, we prove that there is no transcendental meromorphic function with this property when the quotient a/b is a positive integer. Moreover, we prove a quasi-normal criterion for families of such functions. This completes our previous results.

1. Introduction

A Bank–Laine function f is an entire function that has the following property: $f(z)=0 \Longrightarrow f'(z)\in\{-1,1\}$. The Bank–Laine functions arise in connection with solutions of second order homogeneous linear differential equations [1], (see also [6]). In our previous paper [2], we studied the meromorphic functions f on the plane $\mathbf{C}$ that satisfy $f(z)=0 \iff f'(z)\in\{a,b\}$, where a,b are two distinct nonzero values. We call such functions *Bank–Laine type functions*. We constructed there some transcendental meromorphic functions with this property when the quotient a/b is a negative rational number, and proved the following results.

Theorem 1.1. *For two distinct nonzero values a and b that satisfy $a/b\in\mathbf{N}$ (positive integers), there is no transcendental meromorphic function f of finite order that satisfies $f(z)=0 \iff f'(z)\in\{a,b\}$.*

A rational function f satisfying $f(z)=0 \iff f'(z)\in\{a,b\}$ exists if and only if a/b or b/a is an integer. In fact, all such rational functions have been classified completely [2, Lemma 10, Lemma 11].

It remains a question: whether there are transcendental meromorphic functions f of *infinite order* that satisfy $f(z)=0 \iff f'(z)\in\{a,b\}$ for $a/b\in\mathbf{N}$. In this paper we answer this question completely by making use of quasi-normality.

Theorem 1.2. *For two distinct nonzero values a and b that satisfy $a/b\in\mathbf{N}$, there is no transcendental meromorphic function f that satisfies $f(z)=0 \iff f'(z)\in\{a,b\}$.*

In order to prove Theorem 1.2, we first study the normality or quasi-normality of the family $\mathcal{F}_{a,b}(D)$ which consists of all meromorphic functions f in a plane domain $D\subset\mathbf{C}$ that satisfy $f(z)=0 \iff f'(z)\in\{a,b\}$.

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The idea of proving the results in function theory by making use of quasi-normality first appears in [7] where it was proved that the derivative of a transcendental meromorphic function with finitely many simple zeros takes every non-zero values infinitely often.

Recall [3, 9] that a family $\mathcal{F}$ of meromorphic functions defined in a plane domain $D \subset \mathbf{C}$ is said to be normal (quasi-normal) on D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally uniformly in D (minus a set E that has no accumulation point in D). The set E may depend on the subsequence. If there exists an integer $\nu \in \mathbf{N}$ such that the set E always can be chosen to contain at most ν points, then $\mathcal{F}$ is said to be quasi-normal of order ν. Also, we say that the family $\mathcal{F}$ is normal (quasi-normal) at a point $z_0 \in D$, if there exists a neighborhood $U \subset D$ of z_0 such that $\mathcal{F}$ is normal (quasi-normal) on U. An useful fact, which can be proved by making use of the diagonal method, is that $\mathcal{F}$ is normal (quasi-normal) on D if and only if $\mathcal{F}$ is normal (quasi-normal) at every point in D. Another fact is that if $\mathcal{F}$ is not quasi-normal of order ν in D, then there exist a sequence $\{f_n\} \subset \mathcal{F}$ and $\nu+1$ points $z_1, z_2, \cdots, z_{\nu+1} \in D$ such that no subsequence of $\{f_n\}$ is normal at each z_j.

The family $\mathcal{F}_{a,b}(D)$ is not quasi-normal in general as showed by the following example.

Example 1. Let for each $n \in \mathbf{N}$

$$f_n(z) = \frac{\sin(nz)}{n} = \frac{e^{inz} - e^{-inz}}{2in}, \quad (i = \sqrt{-1}).$$

Then we have $f_n(z) = 0 \Longleftrightarrow f_n'(z) \in \{-1, 1\}$. However, it is not difficult to see that no subsequence of $\{f_n\}$ can be normal at every point on the real axis. In fact, for $x_0 \in \mathbf{R}$, $f_n(x_0) \to 0$ while $\left| f_n\left(x_0 - \frac{i}{\sqrt{n}}\right)\right| \geq (e^{\sqrt{n}} - e^{-\sqrt{n}})/(2n) \to \infty$. Hence each subsequence of $\{f_n\}$ fails to be equicontinuous in any neighborhood of x_0, and so $\mathcal{F}_{-1,1}(\mathbf{C})$ is not quasi-normal.

However, we prove in this paper the following quasi-normality criterion.

Theorem 1.3. *For two distinct nonzero values a and b that satisfy $a/b \in \mathbf{N}$, the family $\mathcal{F}_{a,b}(D)$ is quasi-normal on D of order 1.*

We remark that if $|\arg a/b| \leq \pi/3$ and neither a/b nor b/a is an integer, then the family $\mathcal{F}_{a,b}(D)$ is normal in D. This can be seen from [2]. We do not know whether the number $\pi/3$ can be replaced by a larger one.

The following example shows that the conclusion of Theorem 1.3 is sharp.

Example 2. Let $k \in \mathbf{N}$ and for each $n \in \mathbf{N}$

$$f_n(z) = z - \frac{1}{nz^k}.$$

Then $\{f_n\}$ is quasi-normal of order 1 on $\mathbf{C}$, since $\{f_n\}$ converges locally uniformly to z on the punctured plane $\mathbf{C} \setminus \{0\}$ and no subsequence is normal at 0. We see that $f_n(z) = 0 \Longleftrightarrow f_n'(z) = k+1$ and $f_n'(z) \neq 1$. Hence $\{f_n\} \subset \mathcal{F}_{k+1,1}(\mathbf{C})$.

Our proof of Theorem 1.3 is inspired by [7, Theorem 1]. The structure of the present paper is in certain sense similar to [7], and this can be seen by comparing our Lemmas 2.4, 2.7 and 2.8 with Lemmas 4, 7 and 8 in [7], respectively. However, there are many different points and the proofs are different to a large extent.

2. Notations and preliminary results

Throughout in this paper, we denote by $\mathbf{C}$ the complex plane, by $\mathbf{C}^*$ the punctured complex plane $\mathbf{C}\setminus\{0\}$, by $\Delta(z_0,r)$ the open disk $\{z\colon |z-z_0|<r\}$, by $\Delta^\circ(z_0,r)$ the punctured disk $\Delta(z_0,r)\setminus\{z_0\}=\{z\colon 0<|z-z_0|<r\}$, and by $\overline{\Delta}(z_0,r)$ the closed disk $\{z\colon |z-z_0|\le r\}$, where $z_0\in\mathbf{C}$ and $r>0$.

In Lemmas 2.8, 2.10 and their proofs, and in the proof of Theorem 1.3, we shall use frequently the following auxiliary function

$$F_{f,a,b}(z) := \frac{f\left(\frac{a+b}{2}+(a-b)z\right)}{a-b}$$

for a function f and two distinct constants a and b.

For a sequence $\{f_n\}$ of functions, we say that they are locally uniformly holomorphic (meromorphic) on D if for each compact subset $E\subset D$, there exists $N\in\mathbf{N}$ such that f_n for every $n>N$ is holomorphic (meromorphic) on E.

Also, we write $f_n\stackrel{\chi}{\to} f$ on D to indicate that the sequence $\{f_n\}$ converges spherically locally uniformly to f on D, and $f_n\to f$ on D if the convergence is already in Euclidean metric.

Let f be a function meromorphic on D. Then for each closed disk $\overline{\Delta}(z_0,r)\subset D$, define

$$A(z_0,r;f) := \frac{1}{\pi}\iint_{\overline{\Delta}(z_0,r)} [f^{\#}(z)]^2\,d\sigma.$$

Here, as usual, $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$ is the spherical derivative. An important fact is that $A(z_0,r;,f)$ is the normalized spherical area of the image of $\overline{\Delta}(z_0,r)$ under f [5].

We also use $n(r,f)$ to denote the number of poles of f on $\Delta(0,r)$, counting multiplicity. Similarly, $n(r,1/f)$ denotes the number of zeros of f on $\Delta(0,r)$.

For a meromorphic function f on $\mathbf{C}$, its Ahlfors–Shimizu characteristic [5, 10] is defined by

$$T_0(r,f)=\int_0^r \frac{A(0,t;f)}{t}\,dt,$$

and the order of f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T_0(r,f)}{\log r}.$$

Thus, each meromorphic function with bounded spherical derivative has order at most 2.

To prove our results, we require some preliminary results.

Lemma 2.1. [4] *Let $\mathcal{F}=\{f\}$ be a family of meromorphic functions on D such that $f(z)\ne 0$ and $f'(z)\ne 1$. Then $\mathcal{F}$ is normal on D.*

Lemma 2.2. [8, Lemma 2] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and suppose that there exists $A\ge 1$ such that $|f'(z)|\le A$ whenever $f(z)=0$ and $f\in\mathcal{F}$. Then if $\mathcal{F}$ is not normal at z_0, there exist,*

(a) *points $z_n\in D$, $z_n\to z_0$;*
(b) *functions $f_n\in\mathcal{F}$; and*
(c) *positive numbers $\rho_n\to 0$*

such that $g_n(\zeta) := \rho_n^{-1} f_n(z_n + \rho_n\zeta) \stackrel{\chi}{\to} g(\zeta)$ *on* $\mathbf{C}$, *where* g *satisfies* $g^{\#}(\zeta) \le g^{\#}(0) = A + 1$, *and hence is nonconstant and of finite order.*

The following lemma is a direct corollary to the maximum modulus principle.

Lemma 2.3. *Let* z_0 *be a point in* D *and* $\{f_n\}$ *be a sequence of meromorphic functions on* D *such that* $f_n \stackrel{\chi}{\to} f$ *on* $D \setminus \{z_0\}$, *where* f *may be identically* ∞.

(a) *If* f_n *are holomorphic on* D *and* $f \not\equiv \infty$, *then* $f_n \to f$ *on* D, *and* f *is holomorphic on* D;

(b) *If* f_n *are zero-free on* D *and* $f \not\equiv 0$, *then* $f_n \stackrel{\chi}{\to} f$ *on* D, *and* $f \neq 0$ *on* D;

(c) *If* f_n *are holomorphic and zero-free on* D, *then* $f_n \to f$ *on* D.

Proof. Since $f_n \stackrel{\chi}{\to} f$ on $D \setminus \{z_0\}$, f is also meromorphic on $D \setminus \{z_0\}$ or $f \equiv \infty$ [9, Corollary 3.1.4].

Now we turn to prove (a). Since f_n is holomorphic, we get $f_n \to f$ on $D\setminus\{z_0\}$ [9, Proposition 3.1.6]. Since $f \not\equiv \infty$, f is also holomorphic on $D\setminus\{z_0\}$ [9, Corollary 3.1.5]. Fix a bounded subdomain U of D such that $z_0 \in U$ and $\overline{U} \subset D$. Thus for arbitrary given number $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for $n > N$, $|f_n(z) - f(z)| < \varepsilon$ on the boundary ∂U of U. Hence for $m, n > N$, $|f_m(z) - f_n(z)| < 2\varepsilon$ on ∂U. Since f_n are holomorphic on D, by the maximum modulus principle, we get $|f_m(z) - f_n(z)| < 2\varepsilon$ on the domain $\overline{U}$. Thus, by Cauchy's criterion, $\{f_n\}$ converges uniformly to a function ϕ which is holomorphic on U. Uniqueness of the limit function shows that $f \equiv \phi$ on $U \setminus \{z_0\}$. This shows that f can be extended holomorphicly to z_0 and hence f is holomorphic on D, and $f_n \to f$ on D.

Applying (a) to the sequence $\{1/f_n\}$ and $1/f$, we prove (b). The (c) is a direct corollary to (a) and (b). □

Lemma 2.4. [2, Theorem 4(ii)] *Let* k *be a positive integer. Then the nonconstant meromorphic functions* $f \in \mathcal{F}_{k+1,1}(\mathbf{C})$ *which are of finite order must be rational functions with the form*

$$f(z) = z - z_0 - \frac{d}{(z - z_0)^k}, \tag{1}$$

where d *and* z_0 *are constants with* $d \neq 0$.

Lemma 2.5. *Let* k *be a positive integer. If the rational function* f *defined by* (1) *has two zeros* $\pm 1/2$, *then there exists a* $K = K(k) > k + 1$ *which only depends on* k *such that*

$$\sup_{z \in \mathbf{C}} f^{\#}(z) \le K.$$

Proof. Since $f(\pm 1/2) = 0$, we have

$$\left(\frac{1}{2} - z_0\right)^{k+1} = \left(-\frac{1}{2} - z_0\right)^{k+1} = d. \tag{2}$$

Since $d \neq 0$, we have $z_0 \neq \pm 1/2$. Let $c = (\frac{1}{2} - z_0)/(-\frac{1}{2} - z_0)$. Then $c \neq 1$ and by (2), $c^{k+1} = 1$, so that

$$c \in \left\{ c_j = \exp\left(\frac{2j\pi}{k+1} i\right) : j = 1, 2, \cdots, k \right\}. \tag{3}$$

By $c = (\frac{1}{2} - z_0)/(-\frac{1}{2} - z_0)$, we get $z_0 = \frac{1+c}{2(1-c)}$, and hence by (2),

$$d = \left(-\frac{1}{2} - z_0\right)^{k+1} = \frac{(-1)^{k+1}}{(1-c)^{k+1}}.$$

Thus by (1) and (3), $f \in \{f_j \colon j = 1, 2, \cdots, k\}$, where

$$f_j(z) = z - \frac{1+c_j}{2(1-c_j)} + \frac{(-1)^k}{(1-c_j)^{k+1}\left(z - \frac{1+c_j}{2(1-c_j)}\right)^k}. \tag{4}$$

Since $f_j(z) \to \infty$ and $f_j'(z) \to 1$ as $z \to \infty$, we have $f_j^{\#}(z) \to 0$ as $z \to \infty$. Hence, the conclusion follows from the continuity of each $f_j^{\#}(z)$ on $\mathbf{C}$. □

Lemma 2.6. *Let k be a positive integer. Then the sub-family $\mathcal{F}$ of $\mathcal{F}_{k+1,1}(D)$ which consists of holomorphic functions is normal on D.*

Proof. Suppose that $\mathcal{F}$ is not normal at some point $z_0 \in D$. Then by Lemma 2.2, there exist functions $\{f_n\} \subset \mathcal{F}$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0$ such that $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ on $\mathbf{C}$, where g is a nonconstant entire function of finite order and satisfy $g^{\#}(\zeta) \leq g^{\#}(0) = k+2$.

We claim that $g(\zeta) = 0 \Longleftrightarrow g'(\zeta) \in \{k+1, 1\}$.

To prove $g(\zeta) = 0 \Longrightarrow g'(\zeta) \in \{k+1, 1\}$, let ζ_0 be a zero of g. Then as $g \not\equiv 0$, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $g_n(\zeta_n) = 0$, and hence $f_n(z_n + \rho_n\zeta_n) = 0$. Since $f_n \in \mathcal{F}_{k+1,1}(D)$, we get $f_n'(z_n + \rho_n\zeta_n) \in \{k+1, 1\}$, so that $g_n'(\zeta_n) \in \{k+1, 1\}$. Since $g_n' \to g'$ on $\mathbf{C}$, we get $g'(\zeta_0) \in \{k+1, 1\}$. This proves $g(\zeta) = 0 \Longrightarrow g'(\zeta) \in \{k+1, 1\}$.

Now we turn to prove $g'(\zeta) \in \{k+1, 1\} \Longrightarrow g(\zeta) = 0$. let ζ_0 be a point such that $g'(\zeta_0) \in \{k+1, 1\}$. Suppose first that $g'(\zeta_0) = k+1$. Since $g^{\#}(0) = k+2$, we have $g'(\zeta) \not\equiv k+1$. Thus by $g_n'(\zeta) - (k+1) \to g'(\zeta) - (k+1)$ and Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $g_n'(\zeta_n) = k+1$, and hence $f_n'(z_n + \rho_n\zeta_n) = k+1$. Since $f_n \in \mathcal{F}_{k+1,1}(D)$, we get $f_n(z_n + \rho_n\zeta_n) = 0$, so that $g_n(\zeta_n) = 0$. Thus by $g_n \to g$, we get $g(\zeta_0) = 0$. A similar argument yields that for $g'(\zeta_0) = 1$, we also have $g(\zeta_0) = 0$. This proves $g'(\zeta) \in \{k+1, 1\} \Longrightarrow g(\zeta) = 0$.

Hence the claim is proved. However, by Lemma 2.4, such an entire function g must be a constant. This contradiction shows that $\mathcal{F}$ is normal on D. □

Lemma 2.7. *Let k be a positive integer, and $\{f_n\}$ be a sequence in $\mathcal{F}_{k+1,1}(D)$. Let $z_0 \in D$. Suppose that*

(i) *no subsequence of $\{f_n\}$ is normal at z_0;*
(ii) *there exists $\delta > 0$ such that f_n for sufficiently large n has at most one single (simple or multiple) pole in $\Delta(z_0, \delta)$; and*
(iii) *$\{f_n\}$ is normal on $D \setminus \{z_0\}$.*

Then there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that

(I) *$f_n(z) \xrightarrow{\chi} z - z_0$ on $D \setminus \{z_0\}$;*
(II) *there exists $\delta_0 > 0$ such that every f_n takes each value $w \in \overline{\mathbf{C}}$ at most $k+1$ times on $\Delta(z_0, \delta_0)$, counting multiplicity.*

Proof. Say $z_0 = 0$. Since $\{f_n\}$ is not normal at $z_0 = 0$, by Lemma 2.2, there exist a subsequence of $\{f_n\}$ which we continue to call $\{f_n\}$, points $z_n \to 0$ and positive numbers $\rho_n \to 0$ such that $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta) \overset{\chi}{\to} g(\zeta)$ on $\mathbf{C}$, where g is a nonconstant meromorphic function of finite order and satisfies $g^{\#}(\zeta) \le g^{\#}(0) = k+2$. Further, by an argument similar as showed in the proof of Lemma 2.6, we have $g(\zeta) = 0 \Longleftrightarrow g'(\zeta) \in \{k+1, 1\}$. Hence by Lemma 2.4,

$$g(\zeta) = \zeta - \zeta_0 - \frac{d}{(\zeta - \zeta_0)^k} = \frac{(\zeta - \zeta_0)^{k+1} - d}{(\zeta - \zeta_0)^k} \tag{5}$$

with $d, \zeta_0 \in \mathbf{C}$ and $d \ne 0$. Let $\zeta_0^{(j)}$, $j = 0, 1, \cdots, k$, be $k+1$ distinct zeros of g. Then by (5), $\prod_{j=0}^{k} (\zeta - \zeta_0^{(j)}) = P(\zeta) := (\zeta - \zeta_0)^{k+1} - d$, and hence for a given $j_0 \in \{0, 1, \cdots, k\}$, $\prod_{j \ne j_0} (\zeta_0^{(j_0)} - \zeta_0^{(j)}) = P'(\zeta_0^{(j_0)}) = (k+1)(\zeta_0^{(j_0)} - \zeta_0)^k$. Thus

$$\frac{\prod_{j \ne j_0} (\zeta_0^{(j_0)} - \zeta_0^{(j)})}{(\zeta_0^{(j_0)} - \zeta_0)^k} = k+1 \tag{6}$$

for a given $j_0 \in \{0, 1, \cdots, k\}$.

Since $g_n \overset{\chi}{\to} g$ on $\mathbf{C}$, g_n has a pole $\zeta_{n,\infty}$ with $\zeta_{n,\infty} \to \zeta_0$ and $k+1$ zeros $\zeta_{n,0}^{(j)}$, $(j = 0, 1, \cdots, k)$, with $\zeta_{n,0}^{(j)} \to \zeta_0^{(j)}$. Thus f_n has a pole $z_{n,\infty} = z_n + \rho_n \zeta_{n,\infty} \to 0$ and $k+1$ zeros $z_{n,0}^{(j)} = z_n + \rho_n \zeta_{n,0}^{(j)} \to 0$, $(j = 0, 1, \cdots, k)$.

By the assumption (ii), the multiplicity of the pole $z_{n,\infty}$ is k (for sufficiently large n). Let

$$f_n^*(z) = \frac{f_n(z)}{R_n(z)}, \text{ where } R_n(z) = \frac{\prod_{j=0}^{k} (z - z_{n,0}^{(j)})}{(z - z_{n,\infty})^k}. \tag{7}$$

Then by (ii), f_n^* for sufficiently large n is holomorphic in $\Delta(0, \delta)$. Let $g_n^*(\zeta) = f_n^*(z_n + \rho_n \zeta)$. Then $\{g_n^*\}$ are locally uniformly holomorphic on $\mathbf{C}$. Since

$$\rho_n^{-1} R_n(z_n + \rho_n \zeta) = \frac{\prod_{j=0}^{k} (\zeta - \zeta_{n,0}^{(j)})}{(\zeta - \zeta_{n,\infty})^k} \overset{\chi}{\to} g(\zeta)$$

on $\mathbf{C}$, and $\rho_n^{-1} R_n(z_n + \rho_n \zeta) g_n^*(\zeta) = g_n(\zeta) \overset{\chi}{\to} g(\zeta)$, we see that $\{g_n^*\}$ are locally uniformly zero-free on $\mathbf{C}$, i.e., for each $r > 0$, there exists $N \in \mathbf{N}$ such that for $n > N$, $g_n^*(\zeta) \ne 0$ on $\overline{\Delta}(0, r)$. We also see that

$$g_n^*(\zeta) = f_n^*(z_n + \rho_n \zeta) \to 1 \tag{8}$$

on $\mathbf{C} \setminus \{\zeta_0, \zeta_0^{(j)} \colon 0 \le j \le k\}$, and hence on $\mathbf{C}$ by Lemma 2.3(c). In particular, $f_n^*(z_n) \ne 0$.

We claim that there exists η, $0 < \eta \le \delta$, such that f_n^* for sufficiently large n has no zero in $\Delta(0, \eta)$.

Suppose not. Then there exist a subsequence of $\{f_n^*\}$, which we continue to call $\{f_n^*\}$, and a sequence $\{z_{n,0}\}$ of points satisfying $z_{n,0} \to 0$ such that $f_n^*(z_{n,0}) = 0$. Since $f_n^*(z_n) \ne 0$, we have $z_{n,0} \ne z_n$. We may say that $z_{n,0}$ is the nearest zero of f_n^* away from z_n, so that $f_n^*(z) \ne 0$ on $\Delta(z_n, |z_{n,0} - z_n|)$. By (8), we have

$$\zeta_{n,0} = \frac{z_{n,0} - z_n}{\rho_n} \to \infty. \tag{9}$$

Set

$$\hat{f}_n^*(z) = f_n^*(z_n + (z_{n,0} - z_n) z). \tag{10}$$

Then, $\{\hat{f}_n^*\}$ are locally uniformly holomorphic on $\mathbf{C}$, $\hat{f}_n^*(z) \neq 0$ on $\Delta(0,1)$, and $\hat{f}_n^*(1) = 0$. Further, let

$$\hat{f}_n(z) = \frac{f_n(z_n + (z_{n,0} - z_n)z)}{z_{n,0} - z_n}. \tag{11}$$

Then $\hat{f}_n(z) = 0 \Longleftrightarrow \hat{f}_n'(z) \in \{k+1, 1\}$, since $f_n(z) = 0 \Longleftrightarrow f_n'(z) \in \{k+1, 1\}$.

By (7), (10) and (11), we see that $\hat{f}_n(z) = \hat{R}_n(z)\hat{f}_n^*(z)$, where

$$\hat{R}_n(z) = \frac{R_n(z_n + (z_{n,0} - z_n)z)}{z_{n,0} - z_n} = \frac{\prod_{j=0}^{k}(z - \frac{\rho_n}{z_{n,0}-z_n}\zeta_{n,0}^{(j)})}{(z - \frac{\rho_n}{z_{n,0}-z_n}\zeta_{n,\infty})^k}. \tag{12}$$

Hence by (9) and since $\{\hat{f}_n^*\}$ are locally uniformly holomorphic on $\mathbf{C}$, we see that $\{\hat{f}_n\}$ are locally uniformly holomorphic on $\mathbf{C}^*$. By (9) and (12), we also see that $\hat{R}_n(z) \to z$ on $\mathbf{C}^*$.

Thus, since $\hat{f}_n(z) = 0 \Longleftrightarrow \hat{f}_n'(z) \in \{k+1, 1\}$ and $\{\hat{f}_n\}$ are locally uniformly holomorphic on $\mathbf{C}^*$, by Lemma 2.6, $\{\hat{f}_n\}$ and hence $\{\hat{f}_n^*\}$ is normal on $\mathbf{C}^*$. Since $\hat{f}_n^*(1) = 0$, by taking a subsequence and renumbering, we may assume that $\hat{f}_n^* \to \hat{f}^*$ on $\mathbf{C}^*$ with $\hat{f}^*(1) = 0$. As $\{\hat{f}_n^*\}$ are locally uniformly holomorphic on $\mathbf{C}$, by Lemma 2.3(a), we get $\hat{f}_n^* \to \hat{f}^*$ on $\mathbf{C}$ and $\hat{f}^*$ is an entire function. Hence, by $\hat{f}_n^*(0) = f_n^*(z_n) = g_n^*(0) \to 1$, we get $\hat{f}^*(0) = 1$. Thus $(z\hat{f}^*(z))' - 1 \not\equiv 0$. For otherwise, we would have $z\hat{f}^*(z) = z + c$ for some constant c, which contradicts that $\hat{f}^*(1) = 0$ and $\hat{f}^*(0) = 1$.

We claim that $\hat{f}_n'(z) \neq 1$ on $\Delta(0,1)$ for sufficiently large n. Suppose not. Then there exist a subsequence of $\{\hat{f}_n\}$, which we continue to call $\{\hat{f}_n\}$, and a sequence $\{z_n^*\}$ of points contained in $\Delta(0,1)$ such that $\hat{f}_n'(z_n^*) = 1$. Then, by $\hat{f}_n(z) = 0 \Longleftrightarrow \hat{f}_n'(z) \in \{k+1, 1\}$, we get $\hat{f}_n(z_n^*) = 0$. Since $\hat{f}_n^*(z) \neq 0$ on $\Delta(0,1)$, we get from $\hat{f}_n(z) = \hat{R}_n(z)\hat{f}_n^*(z)$ that $\hat{R}_n(z_n^*) = 0$. Thus by (12),

$$z_n^* = \frac{\rho_n}{z_{n,0} - z_n}\zeta_{n,0}^{(j_0)} \to 0$$

for some $j_0 \in \{0, 1, \cdots, k\}$. Hence $\hat{f}_n^*(z_n^*) \to \hat{f}^*(0) = 1$ and by (6)

$$\hat{R}_n'(z_n^*) = \frac{\prod_{j\neq j_0}\left(\zeta_{n,0}^{(j_0)} - \zeta_{n,0}^{(j)}\right)}{\left(\zeta_{n,0}^{(j_0)} - \zeta_{n,\infty}\right)^k} \to \frac{\prod_{j\neq j_0}\left(\zeta_0^{(j_0)} - \zeta_0^{(j)}\right)}{\left(\zeta_0^{(j_0)} - \zeta_0\right)^k} = k+1.$$

Thus by $\hat{f}_n(z) = \hat{R}_n(z)\hat{f}_n^*(z)$, we get a contradiction:

$$1 = \hat{f}_n'(z_n^*) = \hat{R}_n'(z_n^*)\hat{f}_n^*(z_n^*) + \hat{R}_n(z_n^*)\,{\hat{f}_n^*}'(z_n^*) = \hat{R}_n'(z_n^*)\hat{f}_n^*(z_n^*) \to k+1.$$

This contradiction shows that $\hat{f}_n'(z) \neq 1$ on $\Delta(0,1)$.

Since $\hat{f}_n(z) = \hat{R}_n(z)\hat{f}_n^*(z) \to z\hat{f}^*(z)$ on $\mathbf{C}^*$, we get $\hat{f}_n'(z) - 1 \to (z\hat{f}^*(z))' - 1$ on $\mathbf{C}^*$. Since $(z\hat{f}^*(z))' - 1 \not\equiv 0$ and $\hat{f}_n'(z) - 1 \neq 0$ on $\Delta(0,1)$, by Lemma 2.3(b), we have $\hat{f}_n'(z) - 1 \overset{\chi}{\to} (z\hat{f}^*(z))' - 1$ on $\Delta(0,1)$, and $(z\hat{f}^*(z))' - 1 \neq 0$ on $\Delta(0,1)$. However, we have $(z\hat{f}^*(z))'\Big|_{z=0} = \hat{f}^*(0) = 1$. This is a contradiction.

Thus f_n^* for sufficiently large n is zero-free and holomorphic in $\Delta(0,\eta)$. Since $\{f_n^*\}$ is normal on $D \setminus \{0\}$, it follows from Lemma 2.3(c) that $\{f_n^*\}$ is normal at 0 and hence is normal on D. Further, as $f_n^*(z_n) = g_n^*(0) \to 1$, there exists a subsequence of $\{f_n^*\}$, which we continue to call $\{f_n^*\}$, such that $f_n^* \overset{\chi}{\to} f^*$ on D with $f^*(0) = 1$.

Now we claim that there exists $\mu > 0$ such that $f_n'(z) \neq 1$ in $\Delta(0, \mu)$ for sufficiently large n. Suppose not. Then there exist a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, and a sequence $\{w_n\}$ of points satisfying $w_n \to 0$ such that $f_n'(w_n) = 1$. Then by the condition $f_n(z) = 0 \iff f_n'(z) \in \{k+1, 1\}$, we see that $f_n(w_n) = 0$ and hence $R_n(w_n) = 0$ since $f_n = R_n f_n^*$ and f_n^* is zero-free in $\Delta(0, \eta)$. Thus by (7), $w_n = z_{n,0}^{(j_0)}$ for some $j_0 \in \{0, 1, \cdots, k\}$. Hence we have $f_n^*(w_n) \to f^*(0) = 1$, and by (6)

$$
\begin{aligned}
R_n'(w_n) = R_n'(z_{n,0}^{(j_0)}) &= \frac{\prod_{j\neq j_0}(z_{n,0}^{(j_0)} - z_{n,0}^{(j)})}{(z_{n,0}^{(j_0)} - z_{n,\infty})^k} \\
&= \frac{\prod_{j\neq j_0}\left(\zeta_{n,0}^{(j_0)} - \zeta_{n,0}^{(j)}\right)}{\left(\zeta_{n,0}^{(j_0)} - \zeta_{n,\infty}\right)^k} \to \frac{\prod_{j\neq j_0}\left(\zeta_{0}^{(j_0)} - \zeta_{0}^{(j)}\right)}{\left(\zeta_{0}^{(j_0)} - \zeta_{0}\right)^k} = k+1.
\end{aligned} \tag{13}
$$

Thus we get by $f_n = R_n f_n^*$ the following contradiction:

$$1 = f_n'(w_n) = R_n'(w_n) f_n^*(w_n) + R_n(w_n)\, {f_n^*}'(w_n) = R_n'(w_n) f_n^*(w_n) \to k+1.$$

This proved the claim that for some $\mu > 0$, $f_n'(z) \neq 1$ in $\Delta(0, \mu)$ for sufficiently large n.

Next we prove that $(zf^*(z))' \equiv 1$. Suppose not. Since f_n^* for sufficiently large n is zero-free and holomorphic in $\Delta(0, \eta)$, we have by Lemma 2.3(a) that $f_n^* \to f^*$ on $\Delta(0, \eta)$ with $f^*(0) = 1$. It follows that $f_n(z) = R_n(z) f_n^*(z) \to z f^*(z)$ on $\Delta^\circ(0, \eta)$. Thus $f_n'(z) \to (zf^*(z))'$ and hence $f_n'(z) - 1 \to (zf^*(z))' - 1$ on $\Delta^\circ(0, \eta)$. Since $f_n' \neq 1$ in $\Delta(0, \mu)$, we may say that $f_n' - 1 \neq 0$ on $\Delta(0, \eta)$. Thus by Lemma 2.3(b), we get $f_n'(z) - 1 \overset{\chi}{\to} (zf^*(z))' - 1$ on $\Delta(0, \eta)$, and $(zf^*(z))' - 1 \neq 0$ on $\Delta(0, \eta)$. This contradicts that $(zf^*(z))'|_{z=0} = f^*(0) = 1$.

Thus $(zf^*(z))' \equiv 1$ and hence $zf^*(z) = z + c$ for some constant c. Since $f^*(0) = 1$, we get $c = 0$, and hence $f^*(z) \equiv 1$. Thus, by $R_n(z) \to z$ in $\mathbf{C}^*$ and $f_n^* \overset{\chi}{\to} 1$ in D, we get $f_n(z) = R_n(z) f_n^*(z) \overset{\chi}{\to} z$ in $D \setminus \{0\}$. The assertion (I) for the special case $z_0 = 0$ is proved. In general case, one can consider the sequence $\{F_n\}$, where $F_n(z) = f_n(z_0 + z)$ and $z \in U = \{z - z_0 \colon z \in D\}$, and obtain that $F_n(z) \overset{\chi}{\to} z$ in $U \setminus \{0\}$ by the special case. Then by the fact $f_n(z) = F_n(z - z_0)$, the assertion (I) follows.

Now we turn to prove the assertion (II). Fix a number $\delta_0 > 0$ such that $\overline{\Delta}(0, \delta_0) \subset D$. Then as $f_n = R_n f_n^*$ and $f_n^* \to f^* \equiv 1$ on $\overline{\Delta}(0, \delta_0)$, there exists $N_1 \in \mathbf{N}$ such that for $n > N_1$, f_n has exactly one pole with multiplicity k on $\overline{\Delta}(0, \delta_0) \subset D$ and $f_n \neq 0, \infty$ on the circle $|z| = \delta_0$. In other words, every f_n for $n > N_1$ takes the value ∞ exactly k times on $\overline{\Delta}(0, \delta_0)$.

For $w \neq \infty$, we consider two cases. Suppose first that $|w| \leq \delta_0/2$. By (I), we have $f_n(z) \overset{\chi}{\to} z$ on $D \setminus \{0\}$, and hence on the circle $|z| = \delta_0$, $f_n(z) \to z$, $f_n'(z) - 1 \to 0$ and $zf_n'(z) - f_n(z) \to 0$. Thus there exists $N_2 \in \mathbf{N}$, $N_2 \geq N_1$, such that for $n > N_2$ and the points z on $|z| = \delta_0$, $|f_n(z)| \geq 3\delta_0/4$, $|f_n'(z) - 1| \leq \delta_0/(16 + 8\delta_0)$ and $|zf_n'(z) - f_n(z)| \leq \delta_0/(16 + 8\delta_0)$. It follows from the argument principle that for

$n > N_2$,

$$\begin{aligned}&\left|n\left(\delta_0, \frac{1}{f_n - w}\right) - n\left(\delta_0, f_n - w\right) - 1\right| \\ &= \left|\frac{1}{2\pi i}\int_{|z|=\delta_0} \frac{f_n'(z)}{f_n(z) - w}\, dz - \frac{1}{2\pi i}\int_{|z|=\delta_0} \frac{1}{z-w}\, dz\right| \\ &= \frac{1}{2\pi}\left|\int_{|z|=\delta_0} \frac{zf_n'(z) - f_n(z) - w(f_n'(z)-1)}{(f_n(z)-w)(z-w)}\, dz\right| \\ &\le \frac{1}{2\pi}\int_{|z|=\delta_0} \frac{\frac{\delta_0}{16+8\delta_0}(1+\frac{\delta_0}{2})}{(\frac{3}{4}\delta_0 - \frac{1}{2}\delta_0)(\delta_0 - \frac{1}{2}\delta_0)}\, |dz| = \frac{1}{2},\end{aligned}$$

and hence $n\left(\delta_0, \frac{1}{f_n - w}\right) - n\left(\delta_0, f_n - w\right) - 1 = 0$ since it is an integer. Thus $n\left(\delta_0, \frac{1}{f_n-w}\right) = n\left(\delta_0, f_n - w\right) + 1 = n\left(\delta_0, f_n\right) + 1 = k+1$. That is to say, every f_n for $n > N_2$ takes each value w satisfying $|w| \le \delta_0/2$ exactly $k+1$ times on $\overline{\Delta}(0,\delta_0) \subset D$.

Suppose now that $|w| \ge \delta_0/2$. Similar as showed above, we have $f_n(z) \to z$, $f_n'(z) - 1 \to 0$ and $zf_n'(z) - f_n(z) \to 0$ on the circle $|z| = \delta_0/4$. Thus there exists $N_3 \in \mathbf{N}$, $N_3 \ge N_1$, such that for $n > N_3$ and the points z on $|z| = \delta_0/4$, $|f_n(z)| \le 3\delta_0/8$, $|f_n'(z) - 1| \le \delta_0/(16+8\delta_0)$ and $|zf_n'(z) - f_n(z)| \le \delta_0/(16+8\delta_0)$. It follows from the argument principle that for $n > N_3$,

$$\begin{aligned}&\left|n\left(\frac{\delta_0}{4}, \frac{1}{f_n - w}\right) - n\left(\frac{\delta_0}{4}, f_n - w\right)\right| \\ &= \left|\frac{1}{2\pi i}\int_{|z|=\frac{\delta_0}{4}} \frac{f_n'(z)}{f_n(z) - w}\, dz - \frac{1}{2\pi i}\int_{|z|=\frac{\delta_0}{4}} \frac{1}{z-w}\, dz\right| \\ &= \frac{1}{2\pi}\left|\int_{|z|=\frac{\delta_0}{4}} \frac{zf_n'(z) - f_n(z) - w(f_n'(z)-1)}{(f_n(z)-w)(z-w)}\, dz\right| \\ &= \frac{1}{2\pi}\left|\int_{|z|=\frac{\delta_0}{4}} \frac{\frac{1}{w}(zf_n'(z) - f_n(z)) - (f_n'(z)-1)}{(w-f_n(z))(1-\frac{z}{w})}\, dz\right| \\ &\le \frac{1}{2\pi}\int_{|z|=\frac{\delta_0}{4}} \frac{\frac{\delta_0}{16+8\delta_0}(\frac{2}{\delta_0}+1)}{(\frac{1}{2}\delta_0 - \frac{3}{8}\delta_0)(1-\frac{1}{2})}\, |dz| = \frac{1}{2},\end{aligned}$$

and hence $n\left(\frac{\delta_0}{4}, \frac{1}{f_n-w}\right) = n\left(\frac{\delta_0}{4}, f_n - w\right) = n\left(\frac{\delta_0}{4}, f_n\right) \le n\left(\delta_0, f_n\right) = k$. That is to say, every f_n for $n > N_3$ takes each value w satisfying $|w| \ge \delta_0/2$ at most k times on $\overline{\Delta}(0,\delta_0/4) \subset D$.

Thus for $n > N = \max\{N_1, N_2, N_3\}$, every f_n takes each value $w \in \overline{\mathbf{C}}$ on $\overline{\Delta}(0,\delta_0/4)$ at most $k+1$ times.

The proof of Lemma 2.7 is completed. □

Lemma 2.8. *Let k be a positive integer, and $\{f_n\}$ be a sequence in $\mathcal{F}_{k+1,1}(D)$. Let $z_0 \in D$. Suppose that*

(i) *no subsequence of $\{f_n\}$ is normal at z_0; and*
(ii) *for every $\delta > 0$, there exists $N \in \mathbf{N}$ such that f_n for $n > N$ has at least two distinct poles in $\Delta(z_0, \delta)$.*

Then there exist a subsequence of $\{f_n\}$ *which we continue to call* $\{f_n\}$*, and a sequence* $\{\eta_n\}$ *of positive numbers satisfying* $\eta_n \to 0$ *such that* f_n *has at least two distinct zeros* a_n *and* b_n *in* $\Delta(z_0, \eta_n)$ *such that*

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,a_n,b_n}(z) \to \infty. \tag{14}$$

Proof. Say $z_0 = 0$. As showed in the proof of Lemma 2.7, since $\{f_n\}$ is not normal at 0, there exist a subsequence of $\{f_n\}$ which we continue to call $\{f_n\}$, points $z_n \to 0$ and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n\zeta) \overset{\chi}{\to} g(\zeta) = \zeta - \zeta_0 - \frac{d}{(\zeta-\zeta_0)^k} = \frac{(\zeta-\zeta_0)^{k+1} - d}{(\zeta-\zeta_0)^k} \tag{15}$$

on $\mathbf{C}$, where $d, \zeta_0 \in \mathbf{C}$ and $d \neq 0$.

As showed in the proof of Lemma 2.7, it follows that f_n has a pole $z_{n,\infty} = z_n + \rho_n\zeta_{n,\infty} \to 0$ with $\zeta_{n,\infty} \to \zeta_0$ and $k+1$ zeros $z_{n,0}^{(j)} = z_n + \rho_n\zeta_{n,0}^{(j)} \to 0$, $(j = 0, 1, \cdots, k)$, with $\zeta_{n,0}^{(j)} \to \zeta_0^{(j)}$, where $\zeta_0^{(j)}$ are $k+1$ distinct zeros of g. Note that, we also have (6).

We claim that $g_n'(\zeta) \neq 1$ locally uniformly on $\mathbf{C}$, i.e., for each $r > 0$, there exists $N \in \mathbf{N}$ such that for $n > N$, $g_n'(\zeta) \neq 1$ on $\overline{\Delta}(0, r)$. Suppose not, then there exist $r_0 > 0$, a subsequence of $\{g_n\}$, which we continue to call $\{g_n\}$, and a sequence $\{\zeta_n\}$ of points contained in $\overline{\Delta}(0, r_0)$ such that $g_n'(\zeta_n) = 1$. By taking a subsequence, we may say $\zeta_n \to c \in \overline{\Delta}(0, r_0)$. By $g_n'(\zeta_n) = 1$, we get $f_n'(z_n + \rho_n\zeta_n) = 1$, and hence by the condition, $f_n(z_n + \rho_n\zeta_n) = 0$. Thus $g_n(\zeta_n) = 0$ and hence $g(c) = 0$ by $g_n \overset{\chi}{\to} g$, so that g and hence g_n (for sufficiently large n) are holomorphic in a neighborhood of c. It follows that $g_n \to g$ and hence $g_n' \to g'$ on this neighborhood. Thus by $g_n'(\zeta_n) = 1$, we get $g'(c) = 1$. This is a contradiction, since $g'(\zeta) \neq 1$ on $\mathbf{C}$.

Since $g_n \overset{\chi}{\to} g$ on $\mathbf{C}$ and ζ_0 is a pole of g with exact multiplicity k, every g_n (for sufficiently large n) has exactly k poles tending to ζ_0.

We claim that the k poles of g_n coincide. Suppose not. Then g_n has $s \geq 2$ distinct poles $\zeta_{n,\infty}^{(j)}, 1 \leq j \leq s$ with multiplicity m_j such that $\sum_{j=1}^{s} m_j = k$ and $\zeta_{n,\infty}^{(j)} \to \zeta_0$. By taking subsequence, we may assume that the number s and the multiplicities m_j are all independent of n. Further, as ζ_0 is the unique pole of g, the poles of g_n other than $\zeta_{n,\infty}^{(j)}, 1 \leq j \leq s$, if exist, must tend to ∞. Also, the zeros of g_n other than $\zeta_{n,0}^{(j)}, 0 \leq j \leq k$, if exist, must tend to ∞. It follows that the sequence $\{g_n^*\}$ of functions defined by

$$g_n^*(\zeta) := \frac{\prod_{j=1}^{s}(\zeta - \zeta_{n,\infty}^{(j)})^{m_j}}{\prod_{j=0}^{k}(\zeta - \zeta_{n,0}^{(j)})} g_n(\zeta) \tag{16}$$

is locally uniformly zero-free and holomorphic on $\mathbf{C}$. Since $g_n \overset{\chi}{\to} g$ on $\mathbf{C}$, we see from (16) that

$$g_n^*(\zeta) \to 1 \tag{17}$$

on $\mathbf{C} \setminus \{\zeta_0, \zeta_0^{(j)} : 0 \leq j \leq k\}$, and hence on $\mathbf{C}$ by Lemma 2.3(c).

By (16), we have

$$g_n(\zeta) = \frac{\prod_{j=0}^{k}(\zeta - \zeta_{n,0}^{(j)})}{\prod_{j=1}^{s}(\zeta - \zeta_{n,\infty}^{(j)})^{m_j}} g_n^*(\zeta)$$

and hence

$$g_n'(\zeta) = \frac{\left(g_n^*(\zeta)\prod_{j=0}^{k}(\zeta-\zeta_{n,0}^{(j)})\right)'}{\prod_{j=1}^{s}(\zeta-\zeta_{n,\infty}^{(j)})^{m_j}} - g_n^*(\zeta)\prod_{j=0}^{k}(\zeta-\zeta_{n,0}^{(j)})\frac{\sum_{j=1}^{s} m_j\prod_{i\neq j}(\zeta-\zeta_{n,\infty}^{(i)})}{\prod_{j=1}^{s}(\zeta-\zeta_{n,\infty}^{(j)})^{m_j+1}},$$

so that

$$g_n'(\zeta)-1 = \frac{M_n(\zeta)}{\prod_{j=1}^{s}(\zeta-\zeta_{n,\infty}^{(j)})^{m_j+1}}, \tag{18}$$

where

$$\begin{aligned} M_n(\zeta) = {} & \left(g_n^*(\zeta)\prod_{j=0}^{k}(\zeta-\zeta_{n,0}^{(j)})\right)'\prod_{j=1}^{s}(\zeta-\zeta_{n,\infty}^{(j)}) \\ & - g_n^*(\zeta)\prod_{j=0}^{k}(\zeta-\zeta_{n,0}^{(j)})\sum_{j=1}^{s} m_j\prod_{i\neq j}(\zeta-\zeta_{n,\infty}^{(i)}) - \prod_{j=1}^{s}(\zeta-\zeta_{n,\infty}^{(j)})^{m_j+1}. \end{aligned} \tag{19}$$

Since $\zeta_{n,0}^{(j)} \to \zeta_0^{(j)}, (0\le j\le k)$ and $\zeta_{n,\infty}^{(j)} \to \zeta_0, (1\le j\le s)$, we see from (17) and (19) that

$$\begin{aligned} M_n(\zeta) &\to (k+1)(\zeta-\zeta_0)^{k+s} - [(\zeta-\zeta_0)^{k+1}-d]\cdot k(\zeta-\zeta_0)^{s-1} - (\zeta-\zeta_0)^{k+s} \\ &= kd(\zeta-\zeta_0)^{s-1} \end{aligned}$$

on $\mathbf{C}$. Thus, by Hurwitz's theorem, M_n has $s-1\ge 1$ zeros (counting multiplicity) tending to ζ_0.

On the other hand, we can see from (19) that for each $1\le \nu\le s$,

$$M_n(\zeta_{n,\infty}^{(\nu)}) = -g_n^*(\zeta_{n,\infty}^{(\nu)})\prod_{j=0}^{k}(\zeta_{n,\infty}^{(\nu)}-\zeta_{n,0}^{(j)})\cdot m_\nu\prod_{i\neq\nu}(\zeta_{n,\infty}^{(\nu)}-\zeta_{n,\infty}^{(i)}) \neq 0.$$

Hence, the $s-1\ge 1$ zeros of M_n are different from the points $\{\zeta_{n,\infty}^{(j)} : 1\le j\le s\}$, so that by (18), $g_n'(\zeta)-1$ has $s-1\ge 1$ zeros tending to ζ_0. This contradicts the above claim that $g_n'(\zeta)\neq 1$ locally uniformly on $\mathbf{C}$.

This contradiction shows that the k poles of g_n coincide, and hence $\zeta_{n,\infty}$ is a pole of g_n with exact multiplicity k. Thus by (15), $z_{n,\infty} = z_n + \rho_n\zeta_{n,\infty}$ is a pole of f_n with exact multiplicity k.

Now let

$$R_n(z) = \frac{\prod_{j=0}^{k}(z-z_{n,0}^{(j)})}{(z-z_{n,\infty})^k}, \tag{20}$$

$$f_n^*(z) = \frac{f_n(z)}{R_n(z)}, \tag{21}$$

$$g_n^*(\zeta) = f_n^*(z_n+\rho_n\zeta) = \frac{(\zeta-\zeta_{n,\infty})^k}{\prod_{j=0}^{k}(\zeta-\zeta_{n,0}^{(j)})}g_n(\zeta). \tag{22}$$

Then as showed above, by (15) and (22), we have

$$g_n^*(\zeta) \to 1 \tag{23}$$

on $\mathbf{C}\setminus\{\zeta_0, \zeta_0^{(j)} : 0\le j\le k\}$, and hence on $\mathbf{C}$ by Lemma 2.3(c).

We claim that for every $\delta > 0$, there exists $N \in \mathbf{N}$ such that f_n^* for $n > N$ has at least one zero in $\Delta(0,\delta)$.

Suppose not, then there exist $\delta_0 > 0$ and a subsequence of $\{f_n^*\}$, which we continue to call $\{f_n^*\}$, such that $f_n^* \neq 0$ on $\Delta(0,\delta_0)$. Since $f_n = R_n f_n^*$ and $f_n(z) = 0 \iff f_n'(z) \in \{1, k+1\}$, we see by (20) that $f_n(z) \neq 0$ and $f_n'(z) \neq 1$ for $z \in \Delta(0,\delta_0) \setminus \{z_{n,0}^{(j)} \colon 0 \le j \le k\}$. Since $z_{n,0}^{(j)} \to 0$, by Gu's criterion (Lemma 2.1), $\{f_n\}$ and hence $\{f_n^*\}$ is normal on $\Delta^\circ(0,\delta_0)$. So we may assume that $f_n^* \overset{\chi}{\to} f^*$ on $\Delta^\circ(0,\delta_0)$, where f^* may be ∞ identically.

We prove further that there exists $0 < \delta_1 \le \delta_0$ such that $f_n'(z) \neq 1$ on $\Delta(0,\delta_1)$ for sufficiently large n. Suppose not, then as showed in the proof of Lemma 2.7 (before (13)), there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that $f_n'(z_{n,0}^{(j_0)}) = 1$ for some j_0. This, combined with (21), (23) and (6), would lead to the following contradiction:

$$\begin{aligned} 1 = f_n'(z_{n,0}^{(j_0)}) &= R_n'(z_{n,0}^{(j_0)}) f_n^*(z_{n,0}^{(j_0)}) = \frac{\prod_{j\neq j_0}(z_{n,0}^{(j_0)} - z_{n,0}^{(j)})}{(z_{n,0}^{(j_0)} - z_{n,\infty})^k} f_n^*(z_{n,0}^{(j_0)}) \\ &= \frac{\prod_{j\neq j_0}(\zeta_{n,0}^{(j_0)} - \zeta_{n,0}^{(j)})}{(\zeta_{n,0}^{(j_0)} - \zeta_{n,\infty})^k} g_n^*(\zeta_{n,0}^{(j_0)}) \to \frac{\prod_{j\neq j_0}(\zeta_0^{(j_0)} - \zeta_0^{(j)})}{(\zeta_0^{(j_0)} - \zeta_0)^k} = k+1. \end{aligned}$$

We claim that $f^* \not\equiv 0$. For otherwise, we would have $f_n \overset{\chi}{\to} 0$ and hence $f_n' \to 0$, $f_n'' \to 0$ on $\Delta^\circ(0,\delta_1)$. Thus by the argument principle and $f_n'(z) \neq 1$ on $\Delta(0,\delta_1)$,

$$n\left(\frac{\delta_1}{2}, f_n' - 1\right) = \left| n\left(\frac{\delta_1}{2}, f_n' - 1\right) - n\left(\frac{\delta_1}{2}, \frac{1}{f_n' - 1}\right)\right| = \frac{1}{2\pi}\left|\int_{|z|=\frac{\delta_1}{2}} \frac{f_n''}{f_n' - 1}\, dz\right| \to 0.$$

It follows that f_n' has no poles in $\Delta(0,\delta_1/2)$. This is a contradiction, since $z_{n,\infty} \to 0$ is a pole of f_n.

Thus $f^* \not\equiv 0$. Hence by $f_n^* \neq 0$ on $\Delta(0,\delta_0)$ and $f_n^* \overset{\chi}{\to} f^*$ on $\Delta^\circ(0,\delta_0)$, it follows from Lemma 2.3(b) that $f_n^* \overset{\chi}{\to} f^*$ on $\Delta(0,\delta_0)$. Since $f_n^*(z_n) = g_n^*(0) \to 1$, we get $f^*(0) = 1$. This shows that f^* is holomorphic in some neighborhood $\Delta(0,\delta_2)$ of 0 and hence so is f_n^* in $\Delta(0,\delta_2/2)$ for sufficiently large n. Thus by $f_n = R_n f_n^*$, f_n has only one single pole (of multiplicity k) in $\Delta(0,\delta_2/2)$. This contradicts the assumption (ii).

Thus, for every $\delta > 0$, there exists $N \in \mathbf{N}$ such that f_n^* for $n > N$ has at least one zero in $\Delta(0,\delta)$. It follows that there exists a subsequence of $\{f_n^*\}$, which we continue to call $\{f_n^*\}$, such that f_n^* has a zero b_n in $\Delta(0,1/n)$. By (22) and (23), we have

$$\zeta_n^* = \frac{b_n - z_n}{\rho_n} \to \infty. \tag{24}$$

Set $a_n = z_{n,0}^{(1)}$ and $\eta_n = \left|z_{n,0}^{(1)}\right| + 1/n$. Then by $z_{n,\infty} = z_n + \rho_n \zeta_{n,\infty}$ with $\zeta_{n,\infty} \to \zeta_0$ and $a_n = z_n + \rho_n \zeta_{n,0}^{(1)}$ with $\zeta_{n,0}^{(1)} \to \zeta_0^{(1)}$, we see that $\eta_n \to 0$, $a_n, b_n \in \Delta(0,\eta_n)$ and from (24) that $a_n \neq b_n$ and

$$\frac{z_{n,\infty} - \frac{a_n + b_n}{2}}{a_n - b_n} = \frac{2\zeta_{n,\infty} - \zeta_{n,0}^{(1)} - \zeta_n^*}{2(\zeta_{n,0}^{(1)} - \zeta_n^*)} \to \frac{1}{2}.$$

This, combined with $F_{f_n,a_n,b_n}(1/2) = f_n(a_n)/(a_n - b_n) = 0$ and

$$F_{f_n,a_n,b_n}\left(\frac{z_{n,\infty} - \frac{a_n+b_n}{2}}{a_n - b_n}\right) = \frac{f_n(z_{n,\infty})}{a_n - b_n} = \infty,$$

shows that each subsequence of $\{F_{f_n,a_n,b_n}\}$ fails to be equicontinuous in any neighborhood of $z = 1/2$ and hence fails to be normal at $1/2$. Now (14) follows from Marty's theorem. Lemma 2.8 is proved. □

Lemma 2.9. *If f is a meromorphic function of infinite order, then there exist points $z_n \to \infty$ and positive numbers $\varepsilon_n \to 0$ such that*

$$A(z_n, \varepsilon_n; f) = \frac{1}{\pi}\iint_{\overline{\Delta}(z_n,\varepsilon_n)} (f^{\#}(z))^2 d\sigma \to \infty. \tag{25}$$

Proof. See [7, p. 12]. □

Lemma 2.10. *Let k be a positive integer, and f be a meromorphic function in $\mathcal{F}_{k+1,1}(\mathbf{C})$. If f is of infinite order, then f has infinitely many pairs of distinct zeros $(z_{n,1}, z_{n,2})$ such that $z_{n,1} - z_{n,2} \to 0$ and*

$$\sup_{\overline{\Delta}(0,1)} F_{f,z_{n,1},z_{n,2}}(z) \to \infty. \tag{26}$$

Proof. Since f is of infinite order, by Lemma 2.9, we have (25) for some points $z_n \to \infty$ and positive numbers $\varepsilon_n \to 0$. It follows that there exist $w_n \in \overline{\Delta}(z_n, \varepsilon_n)$ such that $f^{\#}(w_n) \to \infty$. Thus, by Marty's theorem, no subsequence of $\{f_n\}$ is normal at 0, where

$$f_n(z) = f(w_n + z). \tag{27}$$

Thus, by Lemma 2.6, f_n for sufficiently large n has at least one pole w_n such that $w_n \to w_0$.

Suppose first that there exist $\delta > 0$ and $N \in \mathbf{N}$ such that f_n for $n > N$ has at most one single (simple or multiple) pole in $\Delta(0, \delta)$. Then f_n for $n > N$ is holomorphic on $\Delta(0, \delta) \setminus \{w_n\}$. Since $w_n \to w_0$, it follows from Lemma 2.6 that $\{f_n\}$ is normal on $\Delta^\circ(0, \delta)$. So by Lemma 2.7, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that for some $\delta_0 > 0$, f_n takes each value $w \in \overline{\mathbf{C}}$ at most $k + 1$ times on $\Delta(0, \delta_0)$, counting multiplicity. Thus for sufficiently large n,

$$A(z_n, \varepsilon_n; f) \le A(0, \delta_0; f_n) \le k + 1,$$

which contradicts (25).

Thus there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that for every $\delta > 0$, there exists $N \in \mathbf{N}$ such that f_n for $n > N$ has at least two distinct poles in $\Delta(0, \delta)$. Then by Lemma 2.8, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that each f_n has at least two distinct zeros a_n and b_n tending to 0 such that

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,a_n,b_n}(z) \to \infty. \tag{28}$$

Let $z_{n,1} = w_n + a_n$, $z_{n,2} = w_n + b_n$. Then $(z_{n,1}, z_{n,2})$ is a pair of distinct zeros of f satisfying $z_{n,1} - z_{n,2} \to 0$, and (26) follows from (28), since $F_{f,z_{n,1},z_{n,2}}(z) \equiv F_{f_n,a_n,b_n}(z)$ by (27). The lemma is proved. □

3. Proof of Theorem 1.3

By the definition, we have $\mathcal{F}_{a,b} = b\mathcal{F}_{a/b,1} = \{bf\colon f \in \mathcal{F}_{a/b,1}\}$ for $b \neq 0$. Thus by the assumption that a/b is a positive integer and $a \neq b$, we may assume that $a = k+1$ and $b = 1$, where k is a positive integer.

Let $E \subset D$ be the set of points at which $\mathcal{F}_{k+1,1}(D)$ is not normal. Then for every $z_0 \in E$, there exists at least one sequence $\{f_n\} \subset \mathcal{F}_{k+1,1}(D)$ such that no subsequence of $\{f_n\}$ is normal at z_0. We consider two cases.

Case 1. For every $z_0 \in E$ and every sequence $\{f_n\} \subset \mathcal{F}_{k+1,1}(D)$ which has no subsequence normal at z_0, there exist $\delta > 0$ and $N \in \mathbf{N}$ such that f_n for $n > N$ has at most one single (simple or multiple) pole in $\Delta(z_0, \delta)$.

We first show that $\mathcal{F}_{k+1,1}(D)$ is quasi-normal at every point $w_0 \in D$.

To prove this, let $\{f_n\} \subset \mathcal{F}_{k+1,1}(D)$ be a sequence. If $w_0 \notin E$, the by the definition of E, $\{f_n\}$ is normal at w_0; If $w_0 \in E$, there are two cases: one is that $\{f_n\}$ has subsequence which is normal at w_0, and the other is that no subsequence is normal at w_0. In the later case, by Lemma 2.6, f_n for sufficiently large n has at least one pole w_n such that $w_n \to w_0$. Hence, by the hypothesis, there exists $\delta > 0$ such that f_n for sufficiently large n is holomorphic in $\Delta(w_0, \delta) \setminus \{w_n\}$. Thus, again by Lemma 2.6, $\{f_n\}$ is normal on $\Delta^\circ(w_0, \delta)$. These discussions show that every sequence $\{f_n\}$ has a subsequence which is normal on $\Delta^\circ(w_0, \eta)$ for some $\eta > 0$. Thus, by the definition, $\mathcal{F}_{k+1,1}(D)$ is quasi-normal at every point in D. Hence, $\mathcal{F}_{k+1,1}(D)$ is quasi-normal on D.

Next, we show further that $\mathcal{F}_{k+1,1}(D)$ is quasi-normal of order 1.

To prove this, let $\{f_n\} \subset \mathcal{F}_{k+1,1}(D)$ be a sequence. Since $\mathcal{F}_{k+1,1}(D)$ is quasi-normal on D, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, and a set E_0 having no accumulation points in D such that $f_n \overset{\chi}{\to} \phi$ on $D \setminus E_0$.

We have to show that the set E_0 can be chosen to be a single-point set. Suppose not. Then there exist two distinct points z_1 and z_2 such that no subsequence of $\{f_n\}$ is normal at z_1 or z_2. Then by Lemma 2.7, the limit function ϕ coincides with $z - z_1$ in a punctured neighborhood of z_1 and with $z - z_2$ in a punctured neighborhood of z_2. It follows from the uniqueness of the limit function that $z - z_1 \equiv z - z_2$. This is impossible, since $z_1 \neq z_2$.

Case 2. There exist $z_0 \in E$ and a sequence $\{f_n\} \subset \mathcal{F}_{k+1,1}(D)$ which has no subsequence normal at z_0 such that for every $\delta > 0$, there exists $N \in \mathbf{N}$ such that f_n for $n > N$ has at least two distinct poles in $\Delta(z_0, \delta)$.

We argue by contradictions for showing that this case can not occur.

By Lemma 2.8, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that each f_n (for sufficiently large n) has at least two distinct zeros u_n and v_n tending to z_0 as $n \to \infty$ such that

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,u_n,v_n}(z) > K + 1, \tag{29}$$

where the constant $K > k+1$ is defined in Lemma 2.5.

Fix $\delta > 0$ such that $\overline{\Delta}(z_0, 3\delta) \subset D$. It guarantees that for each pair (a, b) of distinct zeros of f_n in $\Delta(z_0, \delta)$, the corresponding function $F_{f_n,a,b}(z)$ is meromorphic on $\overline{\Delta}(0,1)$. Since $f_n \not\equiv 0$, each f_n has finitely many zeros in $\Delta(z_0, \delta)$. This, combined with (29), shows that the set

$$E_n = \left\{ (a,b)\colon a,b \in f_n^{-1}(0) \cap \Delta(z_0,\delta) \text{ satisfying } \sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,a,b}(z) > K+1 \right\} \tag{30}$$

is non-empty and finite. For $(a,b) \in E_n$, define

$$\tau(a,b) := \frac{|a-b|}{\delta - \left|\frac{a+b}{2} - z_0\right|} > 0. \tag{31}$$

Then there exists $(a_n, b_n) \in E_n$ such that for every $(a,b) \in E_n$,

$$\tau_n := \tau(a_n, b_n) \le \tau(a,b). \tag{32}$$

Since $(u_n, v_n) \in E_n$, we have $\tau_n \le \tau(u_n, v_n) \to 0$ and hence $a_n - b_n \to 0$.

Let $h_n(z) = F_{f_n,a_n,b_n}(z)$. Then by $a_n - b_n \to 0$, the sequence $\{h_n\}$ are locally uniformly meromorphic on $\mathbf{C}$. And by the definition of the set E_n, we have

$$\sup_{\overline{\Delta}(0,1)} h_n^{\#}(z) > K+1. \tag{33}$$

We claim that no subsequence of $\{h_n\}$ is normal on $\mathbf{C}$. Suppose not, by taking a subsequence, we may say $h_n \overset{\chi}{\to} h$ on $\mathbf{C}$. Since $h_n(\pm 1/2) = 0$, we have $h(\pm 1/2) = 0$ so that $h \not\equiv \infty$. Thus by (33),

$$\sup_{\overline{\Delta}(0,1)} h^{\#}(z) \ge K+1. \tag{34}$$

It then follows that h is nonconstant and from $K > k+1$ that $h'(z) \not\equiv 1$ and $h'(z) \not\equiv k+1$.

Since $f_n(z) = 0 \Longleftrightarrow f_n'(z) \in \{1, k+1\}$, we have $h_n(z) = 0 \Longleftrightarrow h_n'(z) \in \{1, k+1\}$. Hence, by Hurwitz's theorem, $h(z) = 0 \Longleftrightarrow h'(z) \in \{1, k+1\}$.

We claim that h is of infinite order. Suppose not, then as h is nonconstant, it follows from Lemma 2.4 that h is a rational function with the form (1). Hence, as $h(\pm 1/2) = 0$, we get by Lemma 2.5 that $\sup_{\overline{\Delta}(0,1)} h^{\#}(z) \le K$. This contradicts (34).

Thus, h is of infinite order. Hence, by Lemma 2.10, there exist two distinct zeros α and β of h which are not 0 such that $|\alpha - \beta| < 1$ and

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{h,\alpha,\beta}(z) > K+2. \tag{35}$$

Since $h_n \overset{\chi}{\to} h$ on $\mathbf{C}$, there exist points $\alpha_n \to \alpha$ and $\beta_n \to \beta$ such that $h_n(\alpha_n) = h_n(\beta_n) = 0$, and by (35), for sufficiently large n,

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{h_n,\alpha_n,\beta_n}(z) > K+1. \tag{36}$$

Now set

$$\hat{a}_n = \frac{a_n+b_n}{2} + (a_n - b_n)\alpha_n, \quad \hat{b}_n = \frac{a_n+b_n}{2} + (a_n - b_n)\beta_n. \tag{37}$$

Then $\hat{a}_n$ and $\hat{b}_n$ are two distinct zeros of f_n by the definition of h_n. Since $\tau_n \to 0$ and $\alpha_n \to \alpha$, we have

$$|\hat{a}_n - z_0| \le \left| \frac{a_n+b_n}{2} - z_0 \right| + |a_n - b_n||\alpha_n| = \delta - \left(\frac{1}{\tau_n} - |\alpha_n| \right) |a_n - b_n| < \delta$$

for sufficiently large n. Thus $\hat{a}_n \in \Delta(z_0, \delta)$. Similarly, $\hat{b}_n \in \Delta(z_0, \delta)$.

Further, as $F_{f_n,\hat{a}_n,\hat{b}_n}(z) \equiv F_{h_n,\alpha_n,\beta_n}(z)$, we get by (36) that

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,\hat{a}_n,\hat{b}_n}(z) > K + 1.$$

Thus $(\hat{a}_n, \hat{b}_n) \in E_n$ by (30), and hence $\tau(a_n, b_n) \le \tau(\hat{a}_n, \hat{b}_n)$ by (32).

However, by (31) and (37), we have

$$\begin{aligned} \frac{\tau(\hat{a}_n, \hat{b}_n)}{\tau(a_n, b_n)} &= \frac{\delta - \left|\frac{a_n+b_n}{2} - z_0\right|}{\delta - \left|\frac{a_n+b_n}{2} - z_0 + \frac{\alpha_n+\beta_n}{2}(a_n - b_n)\right|} |\alpha_n - \beta_n| \\ &\le \frac{\delta - \left|\frac{a_n+b_n}{2} - z_0\right|}{\delta - \left|\frac{a_n+b_n}{2} - z_0\right| - \left|\frac{\alpha_n+\beta_n}{2}\right| |a_n - b_n|} |\alpha_n - \beta_n| \\ &= \frac{|\alpha_n - \beta_n|}{1 - \left|\frac{\alpha_n+\beta_n}{2}\right| \tau_n} \to |\alpha - \beta| < 1, \end{aligned} \tag{38}$$

so that for sufficiently large n, $\tau(\hat{a}_n, \hat{b}_n) < \tau(a_n, b_n)$. This contradicts that $\tau(a_n, b_n) \le \tau(\hat{a}_n, \hat{b}_n)$.

Thus no subsequence of $\{h_n\}$ is normal on $\mathbf{C}$. Now let H be the set of points at which $\{h_n\}$ is not normal. Then H is non-empty.

Suppose first that for every $\zeta_0 \in H$ and every subsequence $\{h_{n_j}\}$ of $\{h_n\}$ which has no subsequence normal at ζ_0, there exist $\eta > 0$ and $J \in \mathbf{N}$ such that h_{n_j} for $j > J$ has at most one single (simple or multiple) pole in $\Delta(\zeta_0, \eta)$.

Then by an argument similar to that in Case 1, $\{h_n\}$ is quasinormal of order 1 on $\mathbf{C}$, and further, there exists a subsequence of $\{h_n\}$, which we continue to call $\{h_n\}$, and a point $\zeta_0^* \in H$ such that $h_n(z) \overset{\chi}{\to} z - \zeta_0^*$ on $\mathbf{C} \setminus \{\zeta_0^*\}$. But, this contradicts $h_n(\pm 1/2) = 0$.

Thus there exist $\zeta_0^{**} \in H$ and a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ which has no subsequence normal at ζ_0^{**} such that for every $\eta > 0$, there exists $J \in \mathbf{N}$ such that h_{n_j} for $j > J$ has at least two distinct poles in $\Delta(\zeta_0^{**}, \eta)$. We rewrite the subsequence $\{h_{n_j}\}$ by $\{h_n\}$.

Then by Lemma 2.8, there exists a subsequence of $\{h_n\}$, which we continue to call $\{h_n\}$, such that each h_n has at least two distinct zeros a_n^* and b_n^* tending to ζ_0^{**} as $n \to \infty$ such that

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{h_n,a_n^*,b_n^*}(z) > K + 1. \tag{39}$$

Now let

$$A_n = \frac{a_n + b_n}{2} + (a_n - b_n)a_n^*, \quad B_n = \frac{a_n + b_n}{2} + (a_n - b_n)b_n^*.$$

Then as showed above, A_n and B_n are two distinct zeros of f_n in $\Delta(z_0, \delta)$, and by (39) with the fact $F_{f_n,A_n,B_n}(z) \equiv F_{h_n,a_n^*,b_n^*}(z)$,

$$\sup_{\overline{\Delta}(0,1)} F^{\#}_{f_n,A_n,B_n}(z) > K + 1.$$

It follows that $(A_n, B_n) \in E_n$ and hence $\tau(a_n, b_n) \le \tau(A_n, B_n)$ by (32). However, an argument similar to (38) yields that $\tau(A_n, B_n) < \tau(a_n, b_n)$ for sufficiently large n, which contradicts that $\tau(a_n, b_n) \le \tau(A_n, B_n)$.

The proof of Theorem 1.3 is completed. □

4. Proof of Theorem 1.2

As a/b is a positive integer and a, b are distinct and nonzero, we may assume that $\{a, b\} = \{k+1, 1\}$, where k is a positive integer.

Suppose that there is a transcendental meromorphic function f in $\mathcal{F}_{k+1,1}(\mathbf{C})$. By Theorem A, the function f is of infinite order.

Thus by Lemma 2.9, there exist points $z_n \to \infty$ and positive numbers $\varepsilon_n \to 0$ such that

$$A(z_n, \varepsilon_n; f) = \frac{1}{\pi} \iint_{\overline{\Delta}(z_n,\varepsilon_n)} (f^{\#}(z))^2 \, d\sigma \to \infty. \tag{40}$$

It follows that there exist $w_n \in \overline{\Delta}(z_n, \varepsilon_n)$, $w_n \to \infty$, such that $f^{\#}(w_n) \to \infty$, and hence by Marty's theorem, no subsequence of $\{f_n\}$ is normal at 0, where $f_n(z) = f(w_n + z)$.

Since $f \in \mathcal{F}_{k+1,1}(\mathbf{C})$, we see that $\{f_n\} \subset \mathcal{F}_{k+1,1}(\mathbf{C})$. Thus by Theorem 1.3, $\{f_n\}$ is quasi-normal of order 1 on $\mathbf{C}$. Since no subsequence of $\{f_n\}$ is normal at 0, $\{f_n\}$ is normal on $\mathbf{C}^*$. Further, by the proof of Theorem 1.3, only the Case 1 can occur. It follows that there exist $\delta > 0$ and $N \in \mathbf{N}$ such that f_n for $n > N$ has at most one single (simple or multiple) pole in $\Delta(0, \delta)$. So by Lemma 2.7(II), there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that on some neighborhood $\overline{\Delta}(0, \delta_0)$ of 0, each f_n takes each value $a \in \overline{\mathbf{C}}$ at most $k+1$ times, counting multiplicity. Thus $A(z_n, \varepsilon_n; f) \le A(0, \delta_0; f_n) \le k+1$. This contradicts (40).

The proof of Theorem 1.2 is completed.

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