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# DIMENSION OF SLICES THROUGH A SELF-SIMILAR SET WITH INITIAL CUBIC PATTERN

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**Abstract.** In this paper, we give a sufficient condition to ensure that the typical Hausdorff dimension of slices through a self-similar set in a fixed direction takes the value in Marstrand's theorem, i.e., the dimension of the self-similar set minus one.

## 1. Introduction

The intersections of Borel sets in $\mathbf{R}^n$ with $(n-m)$-dimensional subspaces in random directions are studied in many publications. The following Marstrand's theorem [8] (also see [9] Chapter 10) is well known: suppose $A \subset \mathbf{R}^n$ is a Borel set with $0 < \mathcal{H}^s(A) < \infty$ and $m < s < n$, then for $\gamma_{n,n-m}$-almost all $(n-m)$-dimensional subspace $V$ and $\mathcal{H}^s$-almost all $x \in A$,

$$\dim_H[A \cap (V + x)] = s - m.$$

This theorem was first proved by Marstrand [8] for the intersection of planar sets with lines. Later, Mattila [9] proved its higher-dimensional version. In particular, when $m = 1$, we call $(s-1)$ the *Marstrand's value* of Hausdorff dimension for the slices.

Wen and Xi [11] studied the slices of scaling self-similar set $E = \cup_i(r_i E + b_i)$ ($r_i \in (0,1)$) in $\mathbf{R}^n$, and obtained that for a fixed $(n-m)$-dimensional subspace $V$, $\dim_H[E \cap (V + a)]$ is constant for $\mathcal{H}^m$-almost all $a \in V^\perp$ with $E \cap (V + a) \neq \emptyset$.

Kenyon and Peres proved in [5] that given two Cantor set $X$ and $Y$ in $[0, 1)$, invariant under the map $x \mapsto bx \,(\mathrm{mod}\ 1)$, the Hausdorff dimension of $(X + t) \cap Y$ is constant almost everywhere.

The following results showed that for some special planar sets, their typical dimension of the slices in a fixed direction are strictly less than the *Marstrand's value*. Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbf{R}$.

- Hawkes obtained in [3] that

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$$\dim_H[(C\times C)\cap\{(x,y)\colon y=x+b\}]=\tfrac{\log 2}{3\log 3}<\dim_H[C\times C]-1$$
for $\mathcal{L}$-almost all $b\in[-1,1]$.

- Benjamini and Peres [2] gave a class of fractal sets in the unit square $[0,1]\times[0,1]$ with $\dim_H F=\frac{\log 3}{\log 2}$ and proved that for any $F$ in this class,
$$\dim_H\{y\in[0,1]\colon (x,y)\in F\}\geq 1/2$$
for $\mathcal{L}$-almost all $x\in[0,1]$.
- Liu, Xi and Zhao [6] investigated the Sierpinski carpet with $\dim_H E=\frac{\log 8}{\log 3}$. Denote by $L_{k,b}$ the line $y=kx+b$. For any fixed $k\in\mathbf{Q}^+$, they proved that
$$\dim_H[E\cap L_{k,b}]=\dim_B[E\cap L_{k,b}]=c_k$$
for $\mathcal{L}$-almost all $b\in[-k,1]$, where the constant $c_k$ depends only on the rational slope $k$ and
$$c_k\leq \log 8/\log 3-1=\dim_H E-1.$$
They posed a conjecture that $c_k<\log 8/\log 3-1$.
- Manning and Simon [7] proved the conjecture: for any fixed $k\in\mathbf{Q}^+$,
$$\dim_H[E\cap L_{k,b}]=\dim_B[E\cap L_{k,b}]=c_k<\dim_H E-1$$
for $\mathcal{L}$-almost all $b\in[-k,1]$.
- Bárány, Ferguson and Simon [1] studied the Sierpinski gasket in $\mathbf{R}^2$. Let $E_{\theta,b}=E\cap\{(x,y)\colon y=\tan\theta\cdot x+b\}$, where $\theta\in(0,\frac{\pi}{3})$ and $\tan\theta=\frac{\sqrt{3}p}{2q+p}$ with $p,q\in\mathbf{N}$. Then there exists constant $\alpha(\theta)$ depending only on $\theta$ such that for $\mathcal{L}$-almost all $b\in\Delta_\theta$,
$$\dim_H E_{\theta,b}=\dim_B E_{\theta,b}=\alpha(\theta)<\dim_H E-1.$$

In the above literatures, the typical dimensions of the corresponding slices do *not* take the Marstrand's value. Naturally, we will ask: when slicing the self-similar set, can we ensure that the typical dimension of the slices takes the Marstrand's value?

Wu and Xi [12] discussed the intersections of a class of generalized Sierpinski sponges in $\mathbf{R}^n$ with an $(n-1)$-dimensional hyperplane of the form
$$\{x\in\mathbf{R}^n\colon a\cdot x=b\}$$
where $a\in\mathbf{Z}^n\backslash\{0\}$ and $b\in\mathbf{Z}$. They give a sufficient condition, under which the Hausdorff dimensions of slices takes the Marstrand's value.

In this paper, we discuss the slices for $b\in\mathbf{R}$. When $b\in\mathbf{Z}$, after certain modification the slices satisfy the graph-directed construction. When $b\in\mathbf{R}$, the nested structure, which the slices have, is different to the graph-directed construction. Thus we use the method in [5] and [6], and show that the typical Hausdorff dimension of slices in a suitable fixed rational direction takes the Marstrand's value.

Let $m\geq 2$ be an integer and $\Omega\subset\{0,1,\cdots,(m-1)\}^n$ $(n\geq 2)$. For all $v\in\Omega$, let $f_v(x)=(x+v)/m$. The self-similar set

$$E=\bigcup_{v\in\Omega}f_v(E)=\bigcup_{v\in\Omega}\frac{1}{m}(E+v) \tag{1.1}$$

is called a self-similar set with initial *cubic pattern* $\{\frac{1}{m}([0,1]^n+v)\}_{v\in\Omega}$. Then $\dim_H E=\log(\#\Omega)/\log m$ (see e.g. [4]).

Fix $a = (a_1, a_2, \cdots, a_n) \in \mathbf{Z}^n \backslash \{0\}$. For $b \in \mathbf{R}$, let $\Pi_{a,b}$ be the hyperplane in $\mathbf{R}^n$ defined by

$$\Pi_{a,b} = \{x \in \mathbf{R}^n \colon a \cdot x = b\},$$

and the slice $E_{a,b} = E \cap \Pi_{a,b}$.

Let $\mathbf{T}^n$ be the $n$-dimensional torus and $P\colon \mathbf{R}^n \to \mathbf{T}^n$ the map defined by $P(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \in \mathbf{T}^n$ where $y_i = \{x_i\}$ the fractional part of $x_i$ for every $i$. Set $F = P(E)$, then $F$ is an invariant set with respect to the expansive mapping $\tau(y) = my$. For $b \in \mathbf{R}$, write

$$F_{a,b} = F \cap \left\{ y = (y_1, \cdots, y_n) \in \mathbf{T}^n \colon \sum\nolimits_{i=1}^{n} a_i y_i \equiv b \pmod 1 \right\}.$$

The main results of the paper are stated as follows.

**Theorem 1.** *If there exists a positive integer $s$ such that*

$$(\star) \qquad \#\{v \in \Omega \colon a \cdot v \equiv t \pmod m\} = s$$

*for all $t \in \{0, 1, \cdots, m-1\}$, then*

(1) *for all $b \in \mathbf{R}$,*

$$\dim_B F_{a,b} = \dim_B \left( \bigcup_{i \in \mathbf{Z}} E_{a,b+i} \right) = \frac{\log s}{\log m} = \dim_H E - 1;$$

(2) *for $\mathcal{L}$-a.e. $b \in \mathbf{R}$,*

$$\dim_H F_{a,b} = \dim_H \left( \bigcup_{i \in \mathbf{Z}} E_{a,b+i} \right) = \frac{\log s}{\log m} = \dim_H E - 1.$$

**Remark 1.** Under the assumption $(\star)$, Theorem 1 in [12] showed that for $b \in \mathbf{Z}$,

$$\dim_B(\cup_{i \in \mathbf{Z}} E_{a,b+i}) = \dim_H(\cup_{i \in \mathbf{Z}} E_{a,b+i}) = \dim_H E - 1.$$

When $b \in \mathbf{R} \backslash \mathbf{Z}$, it is not clear that whether the Hausdorff dimension of slices $\cup_{i \in \mathbf{Z}} E_{a,b+i}$ equals to its box dimension or not.

**Remark 2.** For $b \in \mathbf{Z}$, the slices have the graph-directed construction (see [12]). To compute the box dimension, we only need to evaluate the spectral radius of a fixed adjacency matrix. When $b$ is not an integer, the slices fail to have the graph-directed construction. To evaluate the box dimension, we need to compute the Lyapunov exponent $\lim_{k \to \infty} \frac{\log ||M(b)M(mb) \cdots M(m^k b)||}{k}$, where the matrix $M(t)$ is variable for $t$ (see Section 3).

Let $\Lambda := \{b \in \mathbf{R} \colon E \cap \Pi_{a,b} \neq \emptyset\}$. Then $\Lambda$ is self-similar. We have $\mathcal{L}(\Lambda) \geq 1$ under the assumption $(\star)$ (see Proposition 4).

**Theorem 2.** *Preserve the assumption of Theorem 1. Then for $\mathcal{L}$-a.e. $b \in \Lambda$,*

$$\dim_H E_{a,b} = \dim_B E_{a,b} = \frac{\log s}{\log m} = \dim_H E - 1.$$

**Remark 3.** Under the assumption $(\star)$, Theorem 2 answer the question on typical dimension of slices.

For $\Omega \subset \{0, 1, \cdots, m-1\}^n$, $n \geq 2$, then the self-similar set $E = \bigcup_{v \in \Omega} \frac{E+v}{m}$ has Hausdorff dimension $\dim_H E = \frac{\log \#\Omega}{\log m}$. Then $\dim_H E > 1$ if and only if

$$\#\Omega > m.$$

From Theorem 2 and the results listed above, we pose the following conjecture.

**Conjecture 1.** *Suppose $E = \bigcup_{v\in\Omega} \frac{E+v}{m}$ with $\#\Omega > m$. Let $a \in \mathbf{Z}^n\backslash\{0\}$ and $d_a$ the typical value of $\dim_H E_{a,b}$ for $\mathcal{L}$-almost all $b \in \Lambda$. Then*

$$d_a = \dim_H E - 1$$

*if and only if the assumption $(\star)$ holds for $\Omega$.*

In this paper, we face the following difficulty: in the Sierpinski carpet, we have

$$\Pi_{a,b} \cap E \neq \emptyset \text{ iff } \Pi_{a,b} \cap [0,1]^n \neq \emptyset, \tag{1.2}$$

note that $[0,1]^n \cap \Pi_{a,b} \neq \emptyset$ iff $b \in [A^-, A^+]$, where

$$A^- = \sum_{a_i<0} a_i, \;\; A^+ = \sum_{a_i>0} a_i,$$

and set $A^- = 0$ if $\{i\colon a_i < 0\} = \emptyset$ and $A^+ = 0$ if $\{i\colon a_i > 0\} = \emptyset$. Thus we obtain that

$$\Pi_{a,b} \cap E \neq \emptyset \quad \text{iff} \quad b \in [A^-, A^+]; \tag{1.3}$$

whereas for the self-similar set with initial cubic pattern, the property (1.2) does not hold as in the following example and thus the claim (1.3) also fails. That means we do not know when the slice $E \cap \Pi_{a,b} \neq \emptyset$. We overcome this difficulty by introducing an equivalent condition ([12, Lemma 1] or Lemma 6 in Section 3) to determine that when the slice is not empty.

**Example 1.** Let $a = (3, -4)$ and $\Pi_{a,b}$ the line $3x - 4y = b$.

(1) When $m = 3$ and $\Omega = \{0,1,2\}^2\backslash\{(1,1)\}$, $E$ is the Sierpinski carpet and $E \cap \Pi_{a,b} \neq \emptyset$ if $\Pi_{a,b}$ touches $[0,1]^2$ (Fig. 1).
(2) When $m = 4$ and $\Omega = \{(0,1),(0,2),(1,2),(1,3),(2,0),(2,3),(3,0),(3,1)\}$, $E$ is a self-similar set with initial cubic pattern. In this case, the assumption $(\star)$ holds and $E \cap \{(x,y)\colon 3x - y = 0\} = \emptyset$ (Fig. 2).

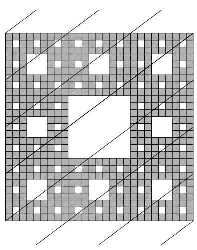

Figure 1. Slices of Sierpinski carpet.

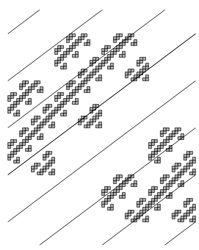

Figure 2. The case that empty slices occur.

The rest of the paper is organized as follows. In Section 2, we give some basic results on discrete structure $\{b + i\colon i \in \mathbf{Z}\} \cap [A^-, A^+]$ and images of slices by map $P$. In Section 3, we investigate the box dimensions of $F_{a,b}$ and $\bigcup_{i\in\mathbf{Z}} E_{a,b+i}$ to prove Theorem 1(1). Section 4 is devoted to Theorem 1(2) on Hausdorff dimension. In Section 5, we discuss the Hausdorff and box dimension of $E_{a,b}$ for $b \in \Lambda$ and prove Theorem 2.

## 2. Preliminaries

Fix $\Omega \subset \{0, 1, \cdots, m-1\}^n$ and

$$a = (a_1, \cdots, a_n) \in \mathbf{Z}^n \backslash \{0\}$$

satisfying for every $t \in \{0, 1, \cdots, (m-1)\}$,

$$\#\{v \in \Omega\colon a \cdot v \equiv t \ (\mathrm{mod}\ m)\} = s.$$

For all $v \in \Omega$, let

$$f_v(x) = (x+v)/m.$$

Let $\Pi_{a,b}$ ($b \in \mathbf{R}$) be the hyperplane in $\mathbf{R}^n$ defined by

$$\Pi_{a,b} = \{x \in \mathbf{R}^n\colon a \cdot x = b\},$$

and the slice $E_{a,b} = E \cap \Pi_{a,b}$, the intersection of the self-similar set with the hyperplane. For any $v \in \Omega$, set

$$T_v(x) = mx - a \cdot v, \quad S_v(x) = T_v^{-1}(x) = (x + a \cdot v)/m.$$

Note that

$$f_v^{-1}(\Pi_{a,b}) = \Pi_{a,T_v(b)}.$$

Using the above formula, we can check that

$$f_{v_1 \cdots v_k}(E) \cap \Pi_{a,z} = f_{v_1 \cdots v_k}(E \cap \Pi_{a,T_{v_k \cdots v_1}(z)}). \tag{2.1}$$

Recall that

$$A^- = \sum_{a_i<0} a_i, \quad A^+ = \sum_{a_i>0} a_i, \tag{2.2}$$

and set $A^- = 0$ if $\{i\colon a_i < 0\} = \emptyset$ and $A^+ = 0$ if $\{i\colon a_i > 0\} = \emptyset$. Then $A^+ - A^- = \|a\|_1$, where the norm of a row vector is given by

$$\|(x_1, \cdots, x_n)\|_1 = \sum_{i=1}^n |x_i|.$$

It is easy to see that

$$[0,1]^n \cap \Pi_{a,b} \neq \emptyset \quad \text{if and only if} \quad b \in [A^-, A^+]. \tag{2.3}$$

We also need the following lemma, for a proof, refer to [12, equation (1.4)].

**Lemma 1.** [12] *For any* $v \in \Omega$,

$$S_v([A^-, A^+]) \subset [A^-, A^+], \tag{2.4}$$

$$T_v([A^-, A^+]^c) \subset [A^-, A^+]^c. \tag{2.5}$$

### 2.1. Discrete Structure $\Gamma_b$.

Let $D = \{b \in \mathbf{R}\colon m^k b \notin \mathbf{Z}$ for any integer $k \geq 0\}$. Let $\tau_0\colon \mathbf{R} \to \mathbf{R}$ be the map $\tau_0(x) = mx$. It is easy to see that for any integer $k \geq 0$,

$$\tau_0^k(D) \subset D \quad \text{and} \quad \tau_0^k(D^c) \subset D^c. \tag{2.6}$$

For any $b \in \mathbf{R}$, let

$$\Gamma_b = \{b + i \in [A^-, A^+]\colon i \in \mathbf{Z}\} \subset [A^-, A^+].$$

Then $\Gamma_{b_1} = \Gamma_{b_2}$ if $b_1 \equiv b_2 (\text{mod } 1)$ and

$$\#\Gamma_b = \begin{cases} A^+ - A^- + 1, & \text{if } b \in \mathbf{Z}, \\ A^+ - A^-, & \text{otherwise.} \end{cases}$$

In particular,

$$\#\Gamma_{m^k b} = A^+ - A^- \quad \text{for any } b \in D \text{ and } k \geq 0,$$

and

$$\Gamma_b \equiv \{A^-, (A^- + 1), \cdots, A^+\} \quad \text{for all } b \in \mathbf{Z}.$$

Given $b \in \mathbf{R}$, we arrange the element of $\Gamma_b$ in ascending order, i.e.,

$$\Gamma_b(1) < \Gamma_b(2) < \cdots < \Gamma_b(\#\Gamma_b).$$

Set

$$I_i := [A^- - 1 + i, A^- + i)(1 \leq i \leq A^+ - A^-).$$

When $i \leq A^+ - A^-$, we have

$$\Gamma_b(i) \in I_i \quad \text{for all } b \in \mathbf{R}. \tag{2.7}$$

Whereas $i = A^+ - A^- + 1$ we have

$$\Gamma_b(A^+ - A^- + 1) = A^+ \quad \text{for all } b \in \mathbf{Z}. \tag{2.8}$$

Since $a \cdot v \in \mathbf{Z}$, we obtain the following lemma.

**Lemma 2.** *If $z \in \Gamma_b$ and $T_v(z) \in [A^-, A^+]$, then $T_v(z) \in \Gamma_{mb}$.*

**2.2. Projection from $\mathbf{R}^n$ to $\mathbf{T}^n$.** Let

$$\mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$$

be the $n$-dimensional torus and $P$ the corresponding natural mapping from $\mathbf{R}^n$ to $\mathbf{T}^n$. For $y, y' \in \mathbf{T}^n$, the metric $d$ on $\mathbf{T}^n$ is defined as follow

$$d(y, y') = \left( \sum_{i=1}^{n} (\min\{|y_i - y_i'|, 1 - |y_i - y_i'|\})^2 \right)^{1/2} = \min_{\substack{P(x)=y \\ P(x')=y'}} |x - x'|.$$

Now, we will map the slice $E_{a,b}$ into $\mathbf{T}^n$. Let $\tau \colon \mathbf{T}^n \to \mathbf{T}^n$ be the map $\tau(y) = my$. Set

$$K = P\left( \bigcup_{v \in \Omega} f_v([0,1]^n) \right) \subset \mathbf{T}^n.$$

Suppose $F = \{y \in \mathbf{T}^n \colon \tau^k(y) \in K, \forall\, k \geq 0\}$, then

$$F = P(E)$$

where $E$ is the self-similar set defined by (1.1).

Recall that $a = (a_1, \cdots, a_n) \in \mathbf{Z}^n \backslash \{0\}$. For $b \in \mathbf{R}$, let

$$F_{a,b} = F \cap \left\{ y = (y_1, \cdots, y_n) \in \mathbf{T}^n \colon \sum_{i=1}^{n} a_i y_i \equiv b \ (\text{mod } 1) \right\}.$$

**Lemma 3.** *For any $b \in \mathbf{R}$, we have*

$$P\left( \bigcup_{z \in \Gamma_b} E_{a,z} \right) = F_{a,b}.$$

*Proof.* For any $x \in \bigcup_{z\in\Gamma_b} E_{a,z}$, we have $\sum_{i=1}^n a_i x_i \equiv b \pmod 1$. Suppose $y = P(x)$, then $y \in F$ and $\sum_{i=1}^n a_i y_i \equiv b \pmod 1$. Hence

$$P\Bigl(\bigcup_{z\in\Gamma_b} E_{a,z}\Bigr) \subset F_{a,b}.$$

On the other hand, suppose $y \in F_{a,b}$ with $\sum_{i=1}^n a_i y_i \equiv b \pmod 1$. Since $P(E) = F$, there exists $x \in E$ such that $P(x) = y$. Then $\sum_{i=1}^n a_i x_i = z \in \{b+i\colon i \in \mathbf{Z}\}$. Note that $x \in E \subset [0,1]^n$, we have $\Pi_{a,z} \cap [0,1]^n \neq \emptyset$, which implies $z \in [A^-, A^+]$ by (2.3), i.e., $z \in \Gamma_b = \{b+i \in [A^-, A^+]\colon i \in \mathbf{Z}\}$. Hence $y = P(x)$, where $x \in E \cap \Pi_{a,z} = E_{a,z}$ with $z \in \Gamma_b$. Therefore

$$F_{a,b} \subset P\Bigl(\bigcup_{z\in\Gamma_b} E_{a,z}\Bigr). \qquad \square$$

Notice that $E \subset [0,1]^n$ and there exists a constant $\delta > 0$ such that

$$d(P(x), P(x')) = |x - x'|$$

whenever $|x - x'| < \delta$. Therefore, we have the following result.

**Lemma 4.** *For any $b \in \mathbf{R}$, we have*

$$\dim \bigcup_{z\in\Gamma_b} E_{a,z} = \dim F_{a,b},$$

*where 'dim' stands for any one of $\dim_H, \underline{\dim}_B$ and $\overline{\dim}_B F_{a,b}$.*

## 3. Box dimension of sections

The *nested structure* of the slices $\{E_{a,b}\}_{b\in[A^-,A^+]}$ is characterized in the following Lemma, which is similar to Proposition 1 in [12].

**Lemma 5.** *For any $b \in [A^-, A^+]$,*

$$E_{a,b} = \bigcup_{v\in\Omega} f_v(E_{a,T_v(b)}) = \bigcup_{v\in\Omega} \frac{E_{a,T_v(b)} + v}{m}. \tag{3.1}$$

*Further, for $k \geq 1$,*

$$\begin{aligned} E \cap \Pi_{a,b} &= \bigcup_{v_1\cdots v_k\in\Omega^k} f_{v_1\cdots v_k}(E \cap \Pi_{a,T_{v_k\cdots v_1}(b)}) \\ &= \bigcup_{T_{v_k\cdots v_1}(b)\in\Gamma_{m^k b}} f_{v_1\cdots v_k}(E \cap \Pi_{a,T_{v_k\cdots v_1}(b)}). \end{aligned} \tag{3.2}$$

*Proof.* The equation (3.1) follows from Proposition 1 in [12].

For (3.2), using induction, we have

$$E \cap \Pi_{a,b} = \bigcup_{v_1\cdots v_k\in\Omega^k} f_{v_1\cdots v_k}(E \cap \Pi_{a,T_{v_k\cdots v_1}(b)}).$$

Further, if $T_{v_k\cdots v_1}(b) \in [A^-, A^+]^c$, then $[0,1]^n \cap \Pi_{a,T_{v_k\cdots v_1}(b)} = \emptyset$ by (2.3), and thus

$$E \cap \Pi_{a,T_{v_k\cdots v_1}(b)} = \emptyset.$$

Combine this fact and Lemma 2, we have

$$E \cap \Pi_{a,b} = \bigcup_{T_{v_k\cdots v_1}(b)\in\Gamma_{m^k b}} f_{v_1\cdots v_k}(E \cap \Pi_{a,T_{v_k\cdots v_1}(b)}). \qquad \square$$

**Remark 4.** Some $E_{a,T_v(b)}$ in (3.1) may be empty.

The following lemma describes when the intersection $E\cap\Pi_{a,b}$ is not empty. This has been done in [12] for integer $b\in[A^-,A^+]$. In fact the proof in [12] holds for all $b\in[A^-,A^+]$.

**Lemma 6.** [12, Lemma 1] *For any* $b\in[A^-,A^+]$,

$$E\cap\Pi_{a,b}\neq\emptyset \iff \exists\ v_1\cdots v_k\cdots\in\Omega^\infty,\ \textit{s.t. } T_{v_k\cdots v_1}(b)\in[A^-,A^+]\ \textit{ for all } k>0.$$

Lemma 5 and Lemma 2 implies the following fact.

**Claim 1.** For any $z\in\Gamma_b$, $E_z$ is composed of some reduced (with ratio $1/m$) copies of $E_{z'}$ for some $z'\in\Gamma_{mb}$.

We record the number of copies with a non-negative integer matrix.

For $b\in\mathbf{R}$, the integer matrix $M(b)=(c_{i,j})_{1\le i\le\#\Gamma_b,1\le j\le\#\Gamma_{mb}}$ is defined by

$$c_{i,j}=\#\{v\in\Omega\colon T_v(\Gamma_b(i))=\Gamma_{mb}(j)\}, \tag{3.3}$$

where $c_{i,j}$ is the number of reduced copies of $E_{\Gamma_{mb}(j)}$ contained in $E_{\Gamma_b(i)}$.

- If $b\in D$, then $M(b)$ is an $(A^+-A^-)\times(A^+-A^-)$ matrix.
- If $b\in\mathbf{Z}$, then $M(b)$ is an $(A^+-A^-+1)\times(A^+-A^-+1)$ matrix.
- For $b\in D^c\backslash\mathbf{Z}$, suppose $m^kb\in\mathbf{Z}$ and $m^{k-1}b\notin\mathbf{Z}$, then $M(m^{k-1}b)$ is an $(A^+-A^-)\times(A^+-A^-+1)$ matrix and $M(m^tb)$ is a square matrix for any non-negative integer $t\neq k-1$.

Let $\mathbf{1}_b=(1,\cdots,1)$ be a vector in $\mathbf{R}^{\#\Gamma_b}$ with every coordinate 1.

**Lemma 7.** *For any* $b\in\mathbf{R}$, *every column sum of the matrix* $M(b)$ *equals to* $s$, *i.e.,*

$$\mathbf{1}_bM(b)=s\mathbf{1}_{mb}. \tag{3.4}$$

*Proof.* For all $1\le j\le\#\Gamma_{mb}$, we need to show that

$$\sum_{i=1}^{\#\Gamma_b}c_{i,j}=s.$$

By (3.3), we have

$$\begin{aligned}\sum_{i=1}^{\#\Gamma_b}c_{i,j}&=\sum_{i=1}^{\#\Gamma_b}\#\{v\in\Omega\colon T_v(\Gamma_b(i))=\Gamma_{mb}(j)\}\\&=\#\{v\in\Omega\colon\exists\ z\in\Gamma_b,\ \text{s.t. } T_v(z)=\Gamma_{mb}(j)\}.\end{aligned}$$

Suppose that $z=b+i',\ \Gamma_{mb}(j)=mb+j'$ where $i',j'\in\mathbf{Z}$. Then

$$\begin{aligned}T_v(z)=\Gamma_{mb}(j)\quad&\Longrightarrow\quad mb+mi'-a\cdot v=mb+j'\\&\Longrightarrow\quad a\cdot v\equiv -j'\pmod m.\end{aligned}$$

Hence

$$s=\#\{v\in\Omega\colon a\cdot v\equiv -j'\pmod m\}\ge\#\{v\in\Omega\colon\exists\ z\in\Gamma_b,\ \text{s.t. } T_v(z)=\Gamma_{mb}(j)\}.$$

Conversely, for any $v\in\Omega$ satisfying $a\cdot v\equiv -j'\pmod m$, we have

$$S_v(\Gamma_{mb}(j))=S_v(mb+j')=b+\frac{j'+a\cdot v}{m}\in b+\mathbf{Z}.$$

By (2.4), we know that $S_v(\Gamma_{mb}(j)) \in [A^-, A^+]$. That means

$$S_v(\Gamma_{mb}(j)) \in \Gamma_b,$$

i.e., $\exists\ z \in \Gamma_b$, s.t. $T_v(z) = \Gamma_{mb}(j)$. Therefore

$$s = \#\{v \in \Omega\colon a \cdot v \equiv -j' \pmod m\} \le \#\{v \in \Omega\colon \exists\ z \in \Gamma_a,\ \text{s.t. } T_v(z) = \Gamma_{mb}(j)\}.$$

This completes the proof. □

Write $f_{v_1\cdots v_k} = f_{v_1} \circ \cdots \circ f_{v_k}$. Let

$$N_k(b) = \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k : f_{v_1\cdots v_k}([0,1]^n) \cap \Pi_{a,z} \neq \emptyset\}. \tag{3.5}$$

We also have

$$N_k(b) = \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k : T_{v_k\cdots v_1}(z) \in [A^-, A^+]\}, \tag{3.6}$$

since

$$\begin{aligned} f_{v_1\cdots v_k}([0,1]^n) \cap \Pi_{a,z} \neq \emptyset \quad &\Longleftrightarrow \quad [0,1]^n \cap \Pi_{a,T_{v_k\cdots v_1}(z)} \neq \emptyset \\ &\Longleftrightarrow \quad T_{v_k\cdots v_1}(z) \in [A^-, A^+] \end{aligned}$$

due to (2.1) and (2.3).

The following proposition shows us how to compute $N_k(b)$.

**Lemma 8.** *For any $k > 0$, we have*

$$N_k(b) = \|\mathbf{1}_b M(b) M(mb) \cdots M(m^{k-1}b)\|_1. \tag{3.7}$$

*Proof.* First, by (3.6), for any $k > 0$, we have

$$N_k(b) = \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k\colon T_{v_k\cdots v_1}(z) \in \Gamma_{m^k b}\}. \tag{3.8}$$

Now we will show that

$$\sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k\colon T_{v_k\cdots v_1}(z) \in \Gamma_{m^k b}\} = \|\mathbf{1}_b M(b) M(mb) \cdots M(m^{k-1}b)\|_1.$$

Suppose that $M(m^t b) = (c_{i_{t+1} i_{t+2}})_{i_{t+1}, i_{t+2}}$, then we obtain that

$$\begin{aligned} &\|\mathbf{1}_b M(b) M(mb) \cdots M(m^{k-1}b)\|_1 \\ &= \sum_{i_1\cdots i_{k+1}} c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_k i_{k+1}} \\ &= \sum_{i_1\cdots i_{k+1}} \prod_{t=1}^{k} \#\{v_t \in \Omega\colon T_{v_t}(\Gamma_{m^{t-1}b}(i_t)) = \Gamma_{m^t b}(i_{t+1})\} \\ &= \sum_{i_1\cdots i_{k+1}} \#\{v_1 \cdots v_k \in \Omega^k\colon T_{v_t}(\Gamma_{m^{t-1}b}(i_t)) = \Gamma_{m^t b}(i_{t+1}) \text{ for } 1 \le t \le k\} \\ &= \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k\colon T_{v_1}(z) \in \Gamma_{mb}, T_{v_2 v_1}(z) \in \Gamma_{m^2 b}, T_{v_3 v_2 v_1}(z) \in \Gamma_{m^3 b}, \\ &\qquad \cdots, T_{v_k\cdots v_1}(z) \in \Gamma_{m^k b}\} \\ &= \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k\colon T_{v_k\cdots v_1}(z) \in \Gamma_{m^k b}\}, \end{aligned}$$

where the last equality follows from (2.5). Therefore

$$N_k(b) = \|\mathbf{1}_b M(b)M(mb)\cdots M(m^{k-1}b)\|_1. \qquad \square$$

**Proposition 1.** *Every column sum of $M(b)M(mb)\cdots M(m^{k-1}b)$ is $s^k$. Further,*

$$N_k(b) = (\#\Gamma_{m^k b})s^k.$$

*Proof.* It follows from Lemma 7 that

$$\mathbf{1}_b M(b)M(mb)\cdots M(m^{k-1}b) = s(\mathbf{1}_{mb}M(mb))\cdots M(m^{k-1}b) = \cdots = s^k \mathbf{1}_{m^k b}.$$

By Lemma 8, we have

$$N_k(b) = s^k\|\mathbf{1}_{m^k b}\|_1 = (\#\Gamma_{m^k b})s^k. \qquad \square$$

Although some $E_{a,b}$ may be empty, the following corollary tells us the union $\bigcup_{z\in\Gamma_b} E_{a,z}$ is not empty.

**Corollary 1.** *For any $b \in \mathbf{R}$,*

$$\bigcup_{z\in\Gamma_b} E_{a,z} \neq \emptyset.$$

*Proof.* Suppose on the contrary that $\bigcup_{z\in\Gamma_b} E_{a,z} = \emptyset$. Then by Lemma 6, there exists integer $k$ such that

$$T_{v_k\cdots v_1}(z) \notin [A^-, A^+] \ \text{ for all } z \in \Gamma_b \text{ and } v_1\cdots v_k \in \Omega^k.$$

It follows from (3.6) that

$$N_k(b) = 0.$$

However, by Proposition 1, $N_k(b) = (\#\Gamma_{m^k b})s^k \geq (A^+ - A^-)s^k > 0$. It is a contradiction. $\square$

Let

$$V_k(b) = \sum_{z\in\Gamma_b} \#\{v_1\cdots v_k \in \Omega^k \colon f_{v_1\cdots v_k}(E) \cap \Pi_{a,z} \neq \emptyset\}.$$

Then we have the following estimate.

**Corollary 2.** *For any $b \in \mathbf{R}$ and $k > 0$,*

$$V_k(b) \geq s^k.$$

*Proof.* The last corollary implies there exists $z \in \Gamma_b$ such that $E \cap \Pi_{a,z} \neq \emptyset$. In the same way, there exists $z^* \in \Gamma_{m^k b}$ such that $E \cap \Pi_{a,z^*} \neq \emptyset$.

By Lemma 5, we obtain that

$$\begin{aligned}\bigcup_{z\in\Gamma_b} E \cap \Pi_{a,z} &= \bigcup_{z\in\Gamma_b} \bigcup_{v_1\cdots v_k\in\Omega^k} f_{v_1\cdots v_k}(E \cap \Pi_{a,T_{v_k\cdots v_1}(z)}) \\ &= \bigcup_{z'\in\Gamma_{m^k b}} \bigcup_{z\in\Gamma_b} \bigcup_{T_{v_k\cdots v_1}(z)=z'} f_{v_1\cdots v_k}(E \cap \Pi_{a,z'}).\end{aligned}$$

Then

$$f_{v_1\cdots v_k}(E \cap \Pi_{a,z^*}) = f_{v_1\cdots v_k}(E) \cap \Pi_{a,z} \neq \emptyset$$

for any $T_{v_k\cdots v_1}(z) = z^*$ with $z \in \Gamma_b$. Denote

$$\alpha_k = \#\{v_1\cdots v_k \colon f_{v_1\cdots v_k}(E \cap \Pi_{a,z^*}) \neq \emptyset \text{ with } T_{v_k\cdots v_1}(z) = z^* \text{ for some } z \in \Gamma_b\}.$$

Hence

$$V_k(b) \geq \alpha_k.$$

For $M(b)M(mb)\cdots M(m^{k-1}b)$, when we consider its column sum with respect to $z^*$, by Proposition 1 we have

$$\alpha_k \geq s^k,$$

which implies $V_k(b) \geq s^k$. □

Consider the union $\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]$ of slices, we have

**Proposition 2.** *For all* $b \in \mathbf{R}$,

$$\dim_B(\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]) = \frac{\log s}{\log m} = \dim_H E - 1.$$

*Proof.* Let $U_k(b)$ be the number of $m$-adic squares of side length $m^{-k}$ intersecting $\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]$. By the definition of the box dimension, we have

$$\overline{\dim}_B(\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]) = \limsup_{k\to\infty}\frac{\log U_k(b)}{k\log m},$$

$$\underline{\dim}_B(\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]) = \liminf_{k\to\infty}\frac{\log U_k(b)}{k\log m}.$$

We notice that

$$U_k(b) \geq V_k(b) \geq s^k. \tag{3.9}$$

On the other hand, it suffices to verify

$$U_k(b) \leq m^n N_k(b) \leq m^n(\#\Gamma_{m^kb})s^k \leq m^n(A^+ - A^- + 1)s^k. \tag{3.10}$$

In fact, by (3.9) and (3.10), we have

$$\dim_B(\bigcup_{z\in\Gamma_b}[E\cap\Pi_{a,z}]) = \lim_{k\to\infty}\frac{\log U_k(b)}{k\log m} = \frac{\log s}{\log m}.$$

To verify (3.10), given an $m$-adic cube of side length $m^{-k}$ intersecting $E\cap\Pi_{a,z}$, denote by $B$, then $B\cap E\cap\Pi_{a,z}\neq\emptyset$, i.e.,

$$B\cap\left(\bigcup_{v_1\cdots v_k\in\Omega^k} f_{v_1\cdots v_k}(E)\right)\cap\Pi_{a,z}\neq\emptyset.$$

Then for some $v_1\cdots v_k\in\Omega^k$,

$$\begin{aligned} & B\cap f_{v_1\cdots v_k}(E)\cap\Pi_{a,z}\neq\emptyset \\ \Longrightarrow\quad & B\cap f_{v_1\cdots v_k}(E)\neq\emptyset \text{ and } f_{v_1\cdots v_k}(E)\cap\Pi_{a,z}\neq\emptyset \\ \Longrightarrow\quad & B\cap f_{v_1\cdots v_k}([0,1]^n)\neq\emptyset \text{ and } f_{v_1\cdots v_k}([0,1]^n)\cap\Pi_{a,z}\neq\emptyset, \end{aligned}$$

which implies

$$U_k(b) \leq m^n N_k(b).$$

Hence (3.10) holds. □

Using Lemma 4 and the above proposition, we have

**Proposition 3.** *For all* $b \in \mathbf{R}$,

$$\dim_B(F_{a,b}) = \frac{\log s}{\log m} = \dim_H E - 1.$$

Then Theorem 1(1) follows from Proportions 2 and 3.

## 4. Proof of Theorem 1(2)

In this section, we will show that the Hausdorff dimension of the slice equals its box dimension almost everywhere. During the proof we will use the following result provided by Ledrappier (see [5] Proposition 2.6).

**Lemma 9.** (Ledrappier) *Let $T_m$ denote the endomorphism $T_m x = mx \pmod 1$ of the $(n-1)$-dimensional torus $\mathbf{T}^{n-1}$, and let $S$ be a continuous transformation of a metric space $Y$. Assume that $\Lambda \subset \mathbf{T}^{n-1} \times Y$ is compact and invariant under the map $T_m \times S$, and that $\nu$ is an $S$-invariant probability measure on $Y$. Then for $\nu$-a.e. $y$, we have*

$$\dim_H \left[\pi^{-1}(y)\right] = \dim_B \left[\pi^{-1}(y)\right],$$

*where $\pi\colon \Lambda \to Y$ is the projection onto the second coordinate.*

Recall that $\tau\colon \mathbf{T}^n \to \mathbf{T}^n$ is the endomorphism

$$\tau(y) = my \pmod 1.$$

*Proof of Theorem 1(2).* Suppose $a = (a_1, \cdots, a_n) \in \mathbf{Z}^n \backslash \{0\}$, without loss of generality, we may assume

$$a_n \neq 0.$$

For any fixed $b \in \mathbf{R}$, we have

$$F_{a,b} = F \cap \{x = (x_1, \cdots, x_n) \in \mathbf{T}^n \colon a \cdot x \equiv b \pmod 1\}.$$

Let $T_m$ denote the endomorphism $T_m x = mx (\mathrm{mod}\ \ 1)$ of the $(n-1)$-dimensional torus $\mathbf{T}^{n-1}$, $S(x) = mx (\mathrm{mod}\ 1)$ the map on one-dimensional torus $\mathbf{T}$, and $g\colon \mathbf{T}^n \to \mathbf{T}^n$ the map

$$g(x) = (x_1, \cdots, x_{n-1}, a \cdot x (\mathrm{mod}\ 1)).$$

Then

$$\tau = T_m \times S$$

and both $F$ and $g(F)$ are $\tau$-invariant, i.e., $F = \tau(F)$ and

$$\tau(g(F)) = g(\tau(F)) = g(F)$$

since $g \circ \tau = \tau \circ g$.

Since $|a_n| \geq 1$, for $i = 0, \cdots, (3|a_n| - 1)$, we let

$$B_i = \left\{ y \in \left[ \frac{i}{3|a_n|}, \frac{i+1}{3|a_n|} \right] \pmod 1 \right\} \subset \mathbf{T}.$$

Let

$$h(x) = a_n x$$

which is defined on the torus $\mathbf{T}$. Then $h|_{B_i}$ is a bi-Lipschitz map. Set $a' = (a_1, \cdots, a_{n-1})$, we have

$$g(x, y) = (x, a' \cdot x + a_n y) \quad \text{for all } x \in \mathbf{T}^{n-1} \text{ and } y \in \mathbf{T}.$$

It is easy to check that

$$g|_{\mathbf{T}^{n-1} \times B_i} \quad \text{is a bi-Lipschitz endomorphism.} \tag{4.1}$$

Since $\mathbf{T}^n = \cup_i (\mathbf{T}^{n-1} \times B_i)$, by (4.1), we obtain that

$$\dim g(F_{a,b}) = \dim F_{a,b}, \tag{4.2}$$

where 'dim' stands any one of $\dim_H, \underline{\dim}_B$ and $\overline{\dim}_B$.

Now, let $Y = \mathbf{T}$ equipped with a normalized Lebesgue measure $\nu$. Since

$$\pi^{-1}[b \,(\mathrm{mod}\ 1)] = g(F_{a,b}),$$

then by the previous lemma, for $\nu$-almost all $b \in \mathbf{T}$,

$$\dim_H g(F_{a,b}) = \dim_B g(F_{a,b}).$$

Therefore, it follows from (4.2) that for $\mathcal{L}$-almost all $b \in \mathbf{R}$,

$$\dim_H F_{a,b} = \dim_B F_{a,b}.$$

This completes the proof of Theorem 1(2). □

## 5. Proof of Theorem 2

Recall that

$$\Lambda = \{b \in [A^-, A^+] \colon E \cap \Pi_{a,b} \neq \emptyset\}.$$

In this section, we will give the Hausdorff and box dimension of $E_{a,b}$ for $\mathcal{L}$-almost all $b \in \Lambda$.

The following lemma is useful (for details, see [10] and [11]).

**Lemma 10.** *The following functions*

$$g_1(b) = \dim_H[E \cap \Pi_{a,b}], \;\; g_2(b) = \underline{\dim}_B[E \cap \Pi_{a,b}] \;\; \text{and} \;\; g_3(b) = \overline{\dim}_B[E \cap \Pi_{a,b}]$$

*are Borel measurable.*

We will show some properties about $\Lambda$.

**Proposition 4.** *$\Lambda$ is a self-similar set with positive Lebesgue measure, i.e.,*

$$\Lambda = \bigcup_{v \in \Omega} S_v(\Lambda) \;\; \text{and} \;\; \mathcal{L}(\Lambda) \geq 1. \tag{5.1}$$

*Further,*

$$\mathcal{L}(\{b \in \Lambda \colon \dim_H E_{a,b} = \frac{\log s}{\log m}\}) \geq 1. \tag{5.2}$$

*Proof.* First, we will show that $\Lambda$ is compact. In fact, we only need to show that $\Lambda$ is closed, since $\Lambda \subset [A^-, A^+]$ is bounded. For any $b \in \Lambda^c$, $E \cap \Pi_{a,b} = \emptyset$, which implies for sufficiently small $\varepsilon > 0$,

$$E \cap \{x \in \mathbf{R}^n \colon a \cdot x \in (b - \varepsilon, b + \varepsilon)\} = \emptyset.$$

That means $\Lambda^c$ is open. Hence $\Lambda$ is closed and thus compact.

By (3.1) in Lemma 5, we know that

$$E \cap \Pi_{a,b} = \bigcup_{v \in \Omega} f_v(E \cap \Pi_{a,T_v(b)}).$$

Then for any $b \in \Lambda$, the above formula implies

$$\begin{aligned} E \cap \Pi_{a,b} \neq \emptyset \quad &\Longleftrightarrow \quad \exists\, v \in \Omega, \text{ s.t. } E \cap \Pi_{a,T_v(b)} \neq \emptyset \\ &\Longleftrightarrow \quad \exists\, v \in \Omega, \text{ s.t. } T_v(b) \in \Lambda \\ &\Longleftrightarrow \quad \exists\, v \in \Omega, \text{ s.t. } b \in S_v(\Lambda), \end{aligned}$$

which means

$$\Lambda = \bigcup_{v \in \Omega} S_v(\Lambda).$$

Therefore $\Lambda$ is a self-similar set.

Now we will show that

$$\mathcal{L}(\Lambda) \geq 1.$$

By Theorem 1(2), we know that

$$\max_{z\in\Gamma_b}(\dim_H E_{a,b}) = \dim_H \bigcup_{z\in\Gamma_b} E_{a,b} = \frac{\log s}{\log m} \quad \text{for } \mathcal{L}\text{-a.e. } b \in [A^-, A^+].$$

Hence for $\mathcal{L}$-a.e. $b \in [A^-, A^+]$,

$$\exists\, b' \in [A^-, A^+] \cap (b + \mathbf{Z})(=\Gamma_b) \ \text{ s.t. } \dim_H E_{a,b'} = \frac{\log s}{\log m}. \tag{5.3}$$

Let

$$K := \{b \in \Lambda\colon \dim_H E_{a,b} = \frac{\log s}{\log m}\}.$$

Then $K$ is Borel measurable from Lemma 10. We have $K \subset \Lambda$ and

$$\mathcal{L}(K) = \mathcal{L}\left(\bigcup_{i=1}^{A^+-A^-} (K \cap I_i)\right) = \sum_{i=1}^{A^+-A^-} \mathcal{L}\left(K \cap I_i\right).$$

It follows from (5.3) that

$$b \in \bigcup_{i=1}^{(A^+-A^-)} \bigcup_{j=-i+1}^{(A^+-A^--i)} (K \cap I_i + j) \ \text{ for } \mathcal{L}\text{-a.e. } b \in [A^-, A^+].$$

By the above formula, we obtain that

$$\begin{aligned}(A^+ - A^-) = \mathcal{L}([A^-, A^+]) &\leq \sum_{i=1}^{(A^+-A^-)} \sum_{j=-i+1}^{(A^+-A^--i)} \mathcal{L}(K \cap I_i + j) \\ &\leq \sum_{i=1}^{A^+-A^-} (A^+ - A^-)\mathcal{L}(K \cap I_i) \leq (A^+ - A^-)\mathcal{L}(K)\end{aligned}$$

which implies

$$\mathcal{L}(\Lambda) \geq \mathcal{L}(K) \geq 1. \tag{5.4}$$

This completes the proof. □

We will also use the following key technique in [11].

**Lemma 11.** [11] *If $B \subset \Lambda$ is a Borel measurable set such that*

$$\bigcup_{v\in\Omega} S_v(B) \subset B$$

*then $\mathcal{L}(B) = \mathcal{L}(\Lambda)$ or 0.*

Now we will give the proof of Theorem 2.

*Proof of Theorem 2.* Let $K_1 = \{b \in \Lambda\colon g_1(b) = \dim_H E_{a,b} \geq \frac{\log s}{\log m}\}$. Then $K_1$ is Borel measurable with $\mathcal{L}(K_1) \geq \mathcal{L}(K) \geq 1$ due to Lemma 10 and (5.2).

According to Lemma 5, for any $v \in \Omega$ and $b \in K_1$,

$$E \cap \Pi_{a,S_v(b)} \supset \frac{E \cap \Pi_{a,b} + v}{m} = f_v(E \cap \Pi_{a,b}),$$

then

$$\dim_H(E \cap \Pi_{a,S_v(b)}) \geq \dim_H(f_v(E \cap \Pi_{a,b})) = \dim_H(E \cap \Pi_{a,b}) \geq \frac{\log s}{\log m},$$

which implies $S_v(b) \in K_1$. Hence $S_v(K_1) \subset K_1$ for any $v \in \Omega$, i.e.,

$$\bigcup_{v \in \Omega} S_v(K_1) \subset K_1.$$

Notice that $K_1 \subset \Lambda$. Then it follows from Lemma 11 that

$$\mathcal{L}(K_1) = \mathcal{L}(\Lambda).$$

That means

$$\dim_H E_{a,b} \geq \frac{\log s}{\log m} \quad \text{for } \mathcal{L}\text{-a.e. } b \in \Lambda.$$

Whereas for all $b \in \mathbf{R}$,

$$\dim_H E_{a,b} \leq \underline{\dim}_B E_{a,b} \leq \overline{\dim}_B E_{a,b} \leq \dim_B\Big(\bigcup_{z \in \Gamma_b} E_{a,z}\Big) = \frac{\log s}{\log m},$$

hence

$$\dim_H E_{a,b} = \dim_B E_{a,b} = \frac{\log s}{\log m} \quad \text{for } \mathcal{L}\text{-a.e. } b \in \Lambda. \qquad \square$$

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