Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 38, 2013, 489–514

# REGULARITY AND IRREGULARITY OF FIBER DIMENSIONS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

**Volker Mayer, Bartłomiej Skorulski and Mariusz Urbański**

Université de Lille I, UFR de Mathématiques, UMR 8524 du CNRS
59655 Villeneuve d'Ascq Cedex, France; volker.mayer@math.univ-lille1.fr

Universidad Católica del Norte, Departamento de Matemáticas
Avenida Angamos 0610, Antofagasta, Chile; bskorulski@ucn.cl

University of North Texas, Department of Mathematics
Denton, TX 76203-1430, U.S.A.; urbanski@unt.edu

**Abstract.** This note concerns non-autonomous dynamics of rational functions and, more precisely, the fractal behavior of the Julia sets under perturbation of non-autonomous systems. We provide a necessary and sufficient condition for holomorphic stability which leads to Hölder continuity of dimensions of hyperbolic non-autonomous Julia sets with respect to the $l^\infty$-topology on the parameter space. On the other hand we show that, for some particular family, the Hausdorff and packing dimension functions are not differentiable at any point and that these dimensions are not equal on an open dense set of the parameter space still with respect to the $l^\infty$-topology.

## 1. Introduction

Let $\mathcal{F} = \{f_\tau\,;\ \tau \in \Lambda_0\}$ be a holomorphic family of rational functions depending analytically on a parameter $\tau \in \Lambda_0$, $\Lambda_0$ being some open and connected subset of $\mathbf{C}^d$, $d \geq 2$. We investigate the dynamics of functions

$$f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_1}, \quad n \geq 1,$$

where each $f_{\lambda_j}$ is an arbitrarily chosen function of the family $\mathcal{F}$. Such a dynamical system is usually called *non-autonomous*. They generalize *deterministic dynamics* (where all the functions $f_{\lambda_j}$ equal one fixed rational map) and *random dynamics* (where the functions $f_{\lambda_j}$ are chosen according to some probability law) that first have been considered by Fornaess and Sibony [FS91]. Non-autonomous maps are also deeply related with the skew-products studied by Jonsson [Jon99, Jon00], Sester [Ses99] and Sumi [Sum01].

If $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda_0^{\mathbf{N}}$, then it is convenient to denote

$$f_\lambda^n = f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_1}.$$

Like in deterministic dynamics, the normal family behavior of $(f_\lambda^n)_n$ splits the sphere into two subsets. The Fatou set $\mathcal{F}_\lambda$, i.e. the set of points for which $(f_\lambda^n)_n$ is normal on some neighborhood, and its complement the Julia set $\mathcal{J}_\lambda$. We are going to investigate

doi:10.5186/aasfm.2013.3829
2010 Mathematics Subject Classification: Primary 30D05.
Key words: Holomorphic dynamics, holomorphic motions, meromorphic functions.
The research of the third named author was supported in part by the NSF Grant DMS 1001874.

the fractal nature of the Julia set $\mathcal{J}_\lambda$ and, more precisely, the dependence of the fractal dimensions of $\mathcal{J}_\lambda$ on the parameter $\lambda \in \Lambda_0^{\mathbf{N}}$.

The deterministic hyperbolic case is completely understood by now. Indeed in 1979, Bowen [Bow79] showed that the Hausdorff dimension of the Julia set can be expressed by the zero of a pressure function. The picture was completed by Ruelle [Rue82] who showed that this dimension depends real analytically on the function. More recently, random dynamics became an active area and both Bowen's formula and Ruelle's real analyticity result have its counterparts in random dynamics. Bowen's formula has been established for various random dynamical systems (see e.g. [MUS11] and the corresponding references in this monograph) and Rugh [Rug] established real analyticity for random repellers. We will see in this note that the situation is completely different in the non-autonomous setting.

Bowen's and Ruelle's results are valid for hyperbolic deterministic functions and hyperbolic functions are so called *stable* functions of the parameter space. There are several notions of stability. We consider *holomorphic stability* that is based on the concept of holomorphic motions and the $\lambda$-Lemma, which has its origin in the fundamental paper [MSS83] by Mané, Sad and Sullivan. Let $\Lambda \subset \Lambda^{\mathbf{N}}$ be a complex Banach manifold. A parameter $\eta \in \Lambda$ is called *holomorphically stable* if there exists a family of holomorphic motions $\{h_{\sigma^n(\lambda)}\}_n$ over some neighborhood $V_\eta \subset \Lambda$ of $\eta$ such that the following diagram commutes. In here, $\sigma(\lambda_1, \lambda_2, \dots) = (\lambda_2, \lambda_3, \dots)$ is the usual shift map.

$$\begin{array}{ccccccccc} \mathcal{J}_\eta & \xrightarrow{f_{\eta_1}} & \mathcal{J}_{\sigma(\eta)} & \xrightarrow{f_{\eta_2}} & \mathcal{J}_{\sigma^2(\eta)} & \xrightarrow{f_{\eta_3}} & \mathcal{J}_{\sigma^3(\eta)} & \cdots \\ \downarrow{\scriptstyle h_\lambda} & & \downarrow{\scriptstyle h_{\sigma(\lambda)}} & & \downarrow{\scriptstyle h_{\sigma^2(\lambda)}} & & \downarrow{\scriptstyle h_{\sigma^3(\lambda)}} & \\ \mathcal{J}_\lambda & \xrightarrow{f_{\lambda_1}} & \mathcal{J}_{\sigma(\lambda)} & \xrightarrow{f_{\lambda_2}} & \mathcal{J}_{\sigma^2(\lambda)} & \xrightarrow{f_{\lambda_3}} & \mathcal{J}_{\sigma^3(\lambda)} & \cdots \end{array} \tag{1.1}$$

Comerford in [Com08] proved stability for certain hyperbolic non-autonomous polynomial maps. We establish the following characterization of holomorphic stability. We would like to mention that the usual theory developed by Mané, Sad and Sullivan [MSS83] is based on the stability of repelling periodic points. Such points do not exist at all in the non autonomous setting. Another remark is that the parameter space $\Lambda_0^{\mathbf{N}}$ is infinite dimensional.

As usual we denote by $\mathcal{C}_g = \{g' = 0\}$ the critical set of a function $g$. The definition of topological exactness is given in Definition 2.2.

**Theorem 1.1.** *Suppose that $\Lambda \subset \Lambda_0^{\mathbf{N}}$ is a complex Banach manifold. Let $f_\eta$, $\eta \in \Lambda$, have perfect Julia sets and suppose that $f_\lambda$ is topologically exact for $\lambda$ in a neighborhood of $\eta$. Then, the map $f_\eta$ is holomorphically stable if and only if there exist an open neighborhood $V$ of $\eta$ and three holomorphic functions $\alpha_i^n \colon V \to \hat{\mathbf{C}}$, $i = 1, 2, 3$, such that*

$$\alpha_i^n(\lambda) \in \mathcal{J}_{\sigma^n(\lambda)} \quad \text{and} \quad \alpha_i^n(\lambda) \neq \alpha_j^n(\lambda) \quad \text{for all } \lambda \in V \text{ and } i \neq j. \tag{1.2}$$

$$f_\lambda^n\big(\mathcal{C}_{f_\lambda^n}\big) \cap \{\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)\} = \emptyset \quad \text{for all } \lambda \in V \text{ and } n \geq 1. \tag{1.3}$$

$$\text{If } \alpha_i^{n+k}(\lambda) = f_{\sigma^n(\lambda)}^k(\alpha_j^n(\lambda)) \text{ for some } \lambda \in V, \text{ then this equality holds for all } \lambda \in V. \tag{1.4}$$

**Remark 1.2.** The dynamical assumptions perfectness of the Julia set and topological mixing are necessary in order to exclude some pathological examples. In general these are very natural and mild assumptions since they hold in most cases. Indeed, Sumi [Sum01] improving Jonsson [Jon00] has shown that $\mathcal{J}_\lambda$ is perfect if $\{f_{\lambda_j}\}_j$ is a equicontinuous family on $\hat{\mathbf{C}}$ ([Sum06] even contains a *uniformly perfectness* result). Details on the mixing property are in Section 2.

**Remark 1.3.** Throughout the whole scope of this paper we could have chosen in each fiber $j \geq 0$ the map $f_{\lambda_j}$ in a different family $\mathcal{F}_j$ of rational maps. In particular, Theorem 1.1 and the whole Section 3 on holomorphic stability does hold without any restrictions on these families $\mathcal{F}_j$, $j \geq 0$. Only starting from Section 4 we need some further control like, for example, a uniform bound on the degree of the functions. We do not insist for such a generalization simply because the notations are already involved enough.

This characterization is in the spirit of the stability of critical orbits in the deterministic case, i.e. the stability of orbits

$$c_\lambda \mapsto f_\lambda(c_\lambda) \mapsto \cdots \mapsto f_\lambda^n(c_\lambda) \mapsto \cdots$$

where $c_\lambda$ is a critical point of $f_\lambda$. By Montel's Theorem, such an orbit is stable if it avoids three values $\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)$ depending holomorphically on $\lambda$ and staying some definite spherical distance apart. Such a condition appears in Lyubich's paper [Lyu86] which itself is based on the previous work by Levin [Lev81]. It turns out that this is the right point of view for generalizing the characterization of stability to the non-autonomous setting.

Hyperbolic random and non-autonomous polynomials have been studied by Comerford [Com06], Jonsson [Jon00], Sester [Ses99] and Sumi [Sum01, Sum06, Sum10b]. H. Sumi also considered in [Sum97] related hyperbolic semi-groups. The definition of hyperbolicity is based on a uniform expanding property, and this is the reason why we will call such maps *uniformly hyperbolic.* We will consider hyperbolic and uniformly hyperbolic non-autonomous maps. Later in the course of the paper we will see that they have normal critical orbits and are therefore holomorphically stable provided we equip the parameter space with the $l^\infty$-topology. Using standard properties of quasiconformal mappings we get the following Hölder continuity result of the dimensions.

**Theorem 1.4.** *For every uniformly hyperbolic map $f_\eta$ there is a neighborhood $V$ of $\eta$ in $l^\infty(\Lambda_0)$ such that the functions*

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda) \quad \textit{and} \quad \lambda \mapsto \mathrm{PD}(\mathcal{J}_\lambda)$$

*(in fact all fractal dimensions) are Hölder continuous on $V$ with Hölder exponent $\alpha(\lambda) \to 1$ if $\lambda$ converges to the base point $\eta$.*

As already mentioned before, in deterministic as well as in random dynamics one has much more, namely, real analytic dependence of the dimension [Rue82, Rug]. Surprisingly it turned out that in the non-autonomous setting the Hölder continuity obtained in Theorem 1.4 is best possible. Indeed we show the following.

**Theorem 1.5.** *Consider the quadratic family*

$$\mathcal{F} = \left\{ f_\tau(z) = \tau/2(z^2 - 1) + 1, \ \tau \in \Lambda_0 \right\} \ \textit{where} \ \Lambda_0 = \{|\tau| > 40\}$$

*and let $\Lambda$ be the interior of $\Lambda_0^{\mathbf{N}} \cap l^\infty(\Lambda_0)$ for the $l^\infty$–topology. Then $\Lambda = \Lambda^{uHyp}$ (see Definition 4.2 ) and the functions*

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda) \quad \textit{and} \quad \lambda \mapsto \mathrm{PD}(\mathcal{J}_\lambda)$$

*are not differentiable at any point $\eta \in \Lambda$ when equipped with the $l^\infty$-topology.*

In order to prove this result we first produce conformal measures, introduce and study fiber pressures and establish an appropriate version of Bowen's formula. Considering the family $\mathcal{F}$ in greater detail we also show that generically the different fractal dimensions are not identical.

**Theorem 1.6.** *Let $\mathcal{F}$ and $\Lambda$ be like in Theorem 1.5. Then, there exists an open and dense set $\Omega \subset \Lambda$ such that*

$$\mathrm{HD}(\mathcal{J}_\lambda) < \mathrm{PD}(\mathcal{J}_\lambda) \quad \textit{for every} \quad \lambda \in \Omega.$$

The authors whish to thank the referee of our paper who provided us with a very detailed report containing important remarks, comments and suggestions. In particular, this influenced the final formulation of Theorem 3.6.

## 2. Non-autonomous dynamics

Rational functions are holomorphic endomorphisms of the Riemann sphere $\hat{\mathbf{C}}$ and the spherical geometry is the natural setting to work with. Therefore, all distances, disks and derivatives will be understood with respect to the spherical metric. For example, $D(z,r)$ will be the spherical disk centered at $z \in \hat{\mathbf{C}}$ and of radius $r > 0$.

We always assume that $\Lambda_0$ is an open and connected subset of $\mathbf{C}^d$ for some $d \geq 2$ and that $\mathcal{F} = \{f_\tau\,;\ \tau \in \Lambda_0\}$ is a holomorphic family of rational functions which means that $f_\tau$ is a rational function for every $\tau \in \Lambda_0$ and that $(\tau, z) \mapsto f_\tau(z)$ is a holomorphic map from $\Lambda_0 \times \hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$. We are interested in the dynamics of

$$f_{\lambda_n} \circ \cdots \circ f_{\lambda_2} \circ f_{\lambda_1}, \quad n \geq 1$$

where the $f_{\lambda_j} \in \mathcal{F}$ or, equivalently, the $\lambda_j \in \Lambda_0$ are arbitrarily chosen.

Let $\pi\colon \Lambda_0^{\mathbf{N}} \to \Lambda_0$ be the canonical projection on the first coordinate and let $\sigma\colon \Lambda_0^{\mathbf{N}} \to \Lambda_0^{\mathbf{N}}$ be the shift map $\sigma(\lambda_1, \lambda_2, \dots) = (\lambda_2, \lambda_3, \dots)$. To $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda$ we associate a non-autonomous dynamical system by first identifying $f_\lambda$ with $f_{\pi(\lambda)} = f_{\lambda_1}$ and then by setting

$$f_\lambda^n = f_{\sigma^{n-1}(\lambda)} \circ \cdots \circ f_{\sigma(\lambda)} \circ f_\lambda := f_{\lambda_n} \circ \cdots \circ f_{\lambda_2} \circ f_{\lambda_1}, \quad n \geq 1.$$

A straightforward generalization of the deterministic case leads to the following definitions. The *Fatou set* of $(f_\lambda^n)_n$ is

$$\mathcal{F}(f_\lambda) = \Big\{ z \in \hat{\mathbf{C}}\ ;\ (f_\lambda^n)_n \text{ is a normal family near } z \Big\}$$

and the *Julia set* $\mathcal{J}(f_\lambda) = \hat{\mathbf{C}} \setminus \mathcal{F}(f_\lambda)$. Most often there will be only one non-autonomous map $f_\lambda$ associated to the parameter $\lambda$. Then we will use the simpler notations $\mathcal{F}_\lambda$ and $\mathcal{J}_\lambda$. For these sets we have the invariance property

$$f_{\lambda_j}^{-1}(\mathcal{J}_{\sigma^{j+1}(\lambda)}) = \mathcal{J}_{\sigma^j(\lambda)} \quad \text{and} \quad f_{\lambda_j}^{-1}(\mathcal{F}_{\sigma^{j+1}(\lambda)}) = \mathcal{F}_{\sigma^j(\lambda)}, \quad j \geq 1. \tag{2.1}$$

The *critical set* of $f_\lambda$ is $\mathcal{C}_{f_\lambda} = \{f'_\lambda = 0\}$.

**Lemma 2.1.** *The Julia set $\mathcal{J}_\lambda$ of a non-autonomous map $f_\lambda$ is either infinite or there exists $N \geq 0$ such that $\mathcal{J}_{\sigma^n(\lambda)}$ consists in at most two points for every $n \geq N$.*

*Proof.* From the invariance property (2.1) it is clear that either all the sets $\mathcal{J}_{\sigma^n(\lambda)}$, $n \geq 0$, are simultaneously infinite or finite and that the sequence $n_\lambda = \#\mathcal{J}_{\sigma^n(\lambda)}$ is decreasing hence stabilising when finite. Suppose that $\#\mathcal{J}_\lambda < \infty$ and let $N$ be the first integer such that

$$n_\lambda = (n+1)_\lambda \quad \text{for every} \quad n \geq N.$$

Since, by assumption, the functions of $\mathcal{F}$ are not injective, it follows that every point of $\mathcal{J}_{\sigma^N(\lambda)}$ is a totally ramified point of $f_{\sigma^N(\lambda)}$. Therefore we are done since a rational map of degree at least two has at most two such points. □

As usually, $\mathcal{J}_\lambda$ is called *perfect* if it does not have isolated points. In the case where $\mathcal{J}_\lambda$ is an infinite set then it is automatically perfect provided the map satisfies the following mixing property.

**Definition 2.2.** A map $f_\lambda$ is topologically exact if, for every open set $U$ that intersects $\mathcal{J}_\lambda$, there exists $N \geq 1$ such that $f_\lambda^N(U) \supset \mathcal{J}_{\sigma^N(\lambda)}$.

As we will see in Example 2.3, non-autonomous maps need not be topologically exact. However, this mixing property is satisfied in most natural settings and is a mild natural dynamical condition. Büger [Büg97] showed that polynomial non-autonomous maps with bounded coefficients are topologically mixing. This results suggest most likely that $f_\lambda$ is topologically exact if $\{\lambda_j\}_j$ is pre-compact in $\Lambda_0$.

Non-autonomous maps are very general and many of the basic properties valid in the deterministic case are no longer true here. For example, in the deterministic case a point is in the Julia set if *no* subsequence of the iterates is normal. Also, deterministic Julia sets are known to be perfect sets. Both these properties are no longer true in the non-autonomous setting. To illustrate this and some other particularities we provide here two simple examples.

**Example 2.3.** Let $f(z) = z^2$ and $h_j(z) = \alpha_j z$ for some $\alpha_j > 0$, $j \geq 0$. There are numbers $\lambda_j > 0$ such that for every $j \geq 1$

$$h_j \circ f = f_{\lambda_j} \circ h_{j-1} \quad \text{where} \quad f_{\lambda_j}(z) = \lambda_j z^2. \tag{2.2}$$

In other words, the deterministic map $f$ is conjugated by the similarities $(h_j)_j$ to the non-autonomous map $f_\lambda$. The numbers $\alpha_j$ can be chosen such that $f_\lambda^n(z) = f^n(z) = z^{2^n}$ for even $n$ and $f_\lambda^n(z) = r_n f^n(z) = r_n z^{2^n}$ for odd $n$. In here the coefficients $r_n$ are chosen to decrease to zero so fast that the sequence $(f_\lambda^n)_{n \text{ odd}}$ is normal at every finite point $z \in \mathbf{C}$. Notice that then $(f_\lambda^n)_{n \text{ odd}}$ is not normal at infinity from which easily follows that

$$\mathcal{J}_\lambda = \mathcal{S}^1 \cup \{\infty\}.$$

In particular, this example shows that the conjugation (2.2) does not preserve the Julia sets. Also, the initial system is perfect and topologically exact whereas the new non-autonomous map has neither of these properties.

**Example 2.4.** Consider $f$ a hyperbolic rational function such that the Fatou set of $f$ has infinitely many distinct connected components $U_1, U_2, \ldots$. For example, one might take $f(z) = z^2 + c$ where $c = -0.123 + 0.745i$ and where the associated Julia set $\mathcal{J}(f)$ is Douady's rabbit. Now, similarly to the first example, we will modify this deterministic map by conjugating it to a non-autonomous map $f_\lambda$ where

$$f_{\lambda_n} = \mathcal{M}_{n+1} \circ f \circ \mathcal{M}_n^{-1}.$$

This times, $\mathcal{M}_n = Id$ for odd $n$ and, for even $n$, $\mathcal{M}_n$ is a Möbius transformation of the Riemann sphere such that $\mathcal{M}_n(U_n) \supset \hat{\mathbf{C}} \setminus D(0, r_n)$ where $r_n \to 0$.

Notice that $f^2_{\sigma^{2k}(\lambda)} = f_{\lambda_{2k+2}} \circ f_{\lambda_{2k+1}} = f^2$ for every $k \geq 0$. It follows that the deterministic set $\mathcal{J}(f)$ is a subset of the non-autonomous set $\mathcal{J}_\lambda$. On the other hand, it is easy to see that $\mathcal{F}(f) \subset \mathcal{F}_\lambda$. Therefore, both systems have the same Julia set $\mathcal{J}(f) = \mathcal{J}_\lambda$.

In this example, the conjugation preserves the Julia and Fatou sets. However, although we started from a hyperbolic hence expanding function $f$, for the non-autonomous map $f_\lambda$ we have that

$$|(f_\lambda^{2k+1})'| \to 0 \quad \text{on } \mathcal{J}_\lambda$$

provided the numbers $r_{2k} \to 0$ sufficiently fast.

Further examples with pathological properties can be found e.g. in [Brü01] and especially in the very interesting papers [Sum10a, Sum10b, Sum11] by Sumi.

Both above examples are obtained in conjugating a deterministic map. The reason why in both cases the resulting dynamics differ from the original ones is the the lack of equicontinuity of the conjugating family of similarities or Möbius transformations respectively. Given this observation it is natural to introduce the following definition.

**Definition 2.5.** Two non-autonomous maps $f_\lambda$ and $f_\mu$ are conjugated if there are homeomorphisms $h_j\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, $j \geq 1$, such that

$$h_{j+1} \circ f_{\lambda_j} = f_{\mu_j} \circ h_j \quad \text{holds on } \hat{\mathbf{C}} \text{ for every } j \geq 1\,. \tag{2.3}$$

If in addition the families $\{h_j\}_j$ and $\{h_j^{-1}\}_j$ are equicontinuous then $f_\lambda$ and $f_\mu$ are called *bi-equicontinuous conjugated.* In the case the homeomorphisms $h_j$ being (quasi)-conformal then we say that the maps are (quasi)-conformally conjugated or (quasi)-conformally bi-equicontinuous conjugated.

The notion of bi-equicontinuous conjugation is consistent with the notion of affine conjugations used by Comerford in [Com03].

Often it is necessary to consider conjugations that do only hold on the Julia sets. But, in order to do so, it is necessary to first ensure that the conjugating maps do identify the Julia sets. Clearly, bi-equicontinous conjugations have this property.

**Lemma 2.6.** *If the non-autonomous maps $f_\lambda$ and $f_\mu$ are bi-equicontinuously conjugated, then the conjugating homeomorphisms identify the corresponding Julia sets.*

*Proof.* Suppose that $h_j\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, $j \geq 1$, are the homeomorphisms such that (2.3) holds and such that $\{h_j\}_j$ and $\{h_j^{-1}\}_j$ are equicontinuous. Then $f_\lambda^n = h_{n+1}^{-1} \circ f_\mu^n \circ h_1$ for every $n \geq 1$ which implies that $h_1^{-1}(\mathcal{F}(f_\mu)) \subset \mathcal{F}(f_\lambda)$ or, equivalently, $\mathcal{F}(f_\mu) \subset h_1(\mathcal{F}(f_\lambda))$. On the other hand, $f_\mu^n = h_{n+1} \circ f_\lambda^n \circ h_1^{-1}$ for every $n \geq 1$ from which follows that $h_1(\mathcal{F}(f_\lambda)) \subset \mathcal{F}(f_\mu)$. □

As we have seen in Example 2.3, general conjugations may not identify Julia sets. Nevertheless, in some special cases like in the Example 2.4 Julia sets are preserved. Here is a more general statement where this also holds.

**Lemma 2.7.** (Rescaling Lemma) *Suppose that $f_\lambda$ is a topologically exact non-autonomous map such that all the Julia sets $\mathcal{J}(f_{\sigma^n(\lambda)})$, $n \geq 0$, contain at least three* *distinct points. Suppose that $h_n$ are homeomorphisms of $\hat{\mathbf{C}}$ such that $0, 1, \infty \in h_{n+1}(\mathcal{J}(f_{\sigma^n(\lambda)}))$ and such that $(h_n)_n$ conjugates $f_\lambda$ to the non-autonomous map $g_\lambda$. Then*

$$\mathcal{J}(g_{\sigma^n(\lambda)}) = h_{n+1}(\mathcal{J}(f_{\sigma^n(\lambda)})) \quad \textit{for every } n \geq 0.$$

*Proof.* It suffices to establish the required identity for $n = 0$, i.e. we have to show that $\mathcal{J}(g_\lambda) = \tilde{\mathcal{J}}_\lambda$ if $\tilde{\mathcal{J}}_\lambda = h_0(\mathcal{J}(f_\lambda))$. Let $\alpha_1^n, \alpha_2^n, \alpha_3^n \in \mathcal{J}_{\sigma^n(\lambda)}$ be the points that are mapped by $h_n$ onto $0, 1, \infty$ respectively. If $\tilde{z} \notin \tilde{\mathcal{J}}_\lambda$ then it is easy to see from the conjugations that $\tilde{z}$ has an open neighborhood $U$ such that $g_\lambda^n(U)$ does not contain any of the points $0, 1, \infty$. Therefore, Montel's Theorem yields that $\hat{\mathbf{C}} \setminus \tilde{\mathcal{J}}_\lambda \subset \mathcal{F}(g_\lambda)$ or, equivalently, that $\mathcal{J}(g_\lambda) \subset \tilde{\mathcal{J}}_\lambda$.

Suppose now that there exists $\tilde{z} \in \tilde{\mathcal{J}}_\lambda \cap \mathcal{F}(g_\lambda)$. Then there exists an open neighborhood $U$ of $\tilde{z}$ such that $(g_\lambda^n)_n$ is normal on $U$. Let $\varphi$ be the limit on $U$ of a convergent subsequence of $(g_\lambda^n)_n$. Shrinking $U$ if necessary, we may assume that one of the points $0, 1, \infty$ is not in $\varphi(U)$. Let $\tilde{W}$ be an open neighborhood of $\tilde{z}$ such that $\tilde{W}$ is relatively compact in $U$. Since $z = h_0^{-1}(\tilde{z}) \in \mathcal{J}(f_\lambda)$, the open set $W = h_0^{-1}(\tilde{W})$ intersects $\mathcal{J}(f_\lambda)$. By assumption, the map $f_\lambda$ is topologically exact. Therefore, there is $N > 0$ such that $f_\lambda^n(W) \supset \mathcal{J}_{\sigma^n(\lambda)}$ for every $n \geq N$. It follows that $g_\lambda^n(\tilde{W}) \supset \{0, 1, \infty\}$ for every $n \geq N$. But then we get the contradiction that $\{0, 1, \infty\} \subset \varphi(U)$. We showed that $\tilde{\mathcal{J}}_\lambda \subset \mathcal{J}(g_\lambda)$ and thus both sets coincident. □

## 3. Stability and normality of critical orbits

In this section we study holomorphic stability and establish, in particular, Theorem 1.1. We would like to mention that Comerford in [Com08] has a partial result in this direction. He shows holomorphic stability for certain polynomial non-autonomous systems provided they are hyperbolic. Our result is an if and only if condition for the stability of a general non-autonomous rational map. The condition relies on the dynamics of the critical orbits and, due to the great generality of non-autonomous systems, we are lead to consider two different conditions of normal critical orbits. In the Proposition 3.5 and in Theorem 3.6 we relate them to holomorphic stability and they yield Theorem 1.1.

In the following we assume that $\Lambda \subset \Lambda_0^{\mathbf{N}}$ is a complex Banach manifold. The most relevant example is the $l^\infty$-topology. Given any function $\omega\colon \mathbf{N} \to ]0, \infty[$, let $\Lambda := l_\omega^\infty(\Lambda_0)$ be the interior of $\Lambda_0^{\mathbf{N}} \cap l_\omega^\infty(\mathbf{C}^d)$ in $l_\omega^\infty(\mathbf{C}^d)$ (remember that $\Lambda_0 \subset \mathbf{C}^d$) where the weighted sup-norm is given by $\|\lambda\|_{\omega,\infty} := \sup_j |\omega(j)\lambda_j|$. Then a sequence $\lambda \in \Lambda_0^{\mathbf{N}}$ belongs to $l_\omega^\infty(\Lambda_0)$ if and only if $(\omega(1)\lambda_1, \omega(2)\lambda_2, \ldots)$ is a bounded sequence such that $\inf_j \omega(j) \operatorname{dist}(\lambda_j, \partial\Lambda_0) > 0$.

Starting from Section 4 we most often deal with uniform hyperbolic maps (see Definition 4.2). Then the natural associated parameter space is $\Lambda = l^\infty(\Lambda_0)$, i.e. the space $l_\omega^\infty(\Lambda_0)$ with weight function $\omega \equiv 1$.

**3.1. Holomorphic motions.** Since this section relies on quasiconformal mappings and holomorphic motions, we start by summarizing some facts from this theory. Let $\eta \in \Lambda$ be a base point.

**Definition 3.1.** A *holomorphic motion* of a set $E \subset \hat{\mathbf{C}}$ over $\Lambda$ is a mapping $h\colon \Lambda \times E \to \hat{\mathbf{C}}$ having the following three properties.

- $h_\eta = id_E$,
- for every $\lambda \in \Lambda$, the map $z \mapsto h_\lambda(z)$ is injective on $E$ and
- for every $z \in E$, $\lambda \mapsto h_\lambda(z)$ is a holomorphic map on $\Lambda$ .

As already mentioned in the introduction, Mané, Sad and Sullivan [MSS83] initially established a $\lambda$-Lemma stating that any holomorphic motion of a set $E \subset \hat{\mathbf{C}}$ over the unit disk of $\mathbf{C}$ can be extended to a holomorphic motion of the closure of $E$. Since then, this $\lambda$-Lemma has been extensively studied and generalized. Most notably, Slodkowski [Slo95] showed that every holomorphic motion over the unit disk is the restriction of a holomorphic motion of the whole sphere. Hubbard [Hub76] discovered that this is false for holomorphic motions over higher-dimensional parameter spaces and [JM07] contains a simpler example. Nevertheless, we dispose in the following $\lambda$-Lemma due to Mitra [Mit00] and Jiang–Mitra [JM07].

**Theorem 3.2.** ($\lambda$-Lemma) *A holomorphic motion $h$ of a set $E \subset \hat{\mathbf{C}}$ over a simply connected complex Banach manifold $V$ with basepoint $\eta \in V$ extents to a holomorphic motion $H$ of $\overline{E}$ over $V$ such that*

(1) *for every $\lambda \in V$, the map $H_\lambda$ is a global quasiconformal map of $\hat{\mathbf{C}}$ with dilatation bounded by $\exp(2\rho_V(\eta,\lambda))$ where $\rho_V$ is the Kobayashi pseudometric on $V$,*
(2) *the map $(\lambda, z) \mapsto H_\lambda(z)$ is continuous.*

**3.2. Holomorphic stability and normal critical orbits.** Here is the precise definition of the stability we use. Notice that, in this definition, the conjugating maps $h_{\sigma^n(\lambda)}$ are not necessarily bi-equicontinuous. We therefore have to include here that the conjugating maps identify the Julia sets.

**Definition 3.3.** A map $f_\eta$, $\eta \in \Lambda$, is holomorphically stable if there is an open neighborhood $V \subset \Lambda$ of $\eta$ and a family of holomorphic motions $\{h_{\sigma^n(\lambda)}\}_n$ of $\{\mathcal{J}_{\sigma^n(\eta)}\}_n$ over $V$ such that, for every $\lambda \in V$, $h_{\sigma^n(\lambda)}(\mathcal{J}_{\sigma^n(\eta)}) = \mathcal{J}_{\sigma^n(\lambda)}$ and

$$h_{\sigma^{n+1}(\lambda)} \circ f_{\sigma^n(\eta)} = f_{\sigma^n(\lambda)} \circ h_{\sigma^n(\lambda)} \quad \text{on} \quad \mathcal{J}_{\sigma^n(\eta)} \text{ for every } n \geq 0.$$

The set of holomorphic stable parameters is denoted by $\Lambda^{stable}$.

In the theory by Mané, Sad and Sullivan [MSS83] and, independently, Lyubich [Lyu86], showing in particular density of stable parameters in any deterministic holomorphic family of rational functions, appear several equivalent characterizations of stability. Most of this theory relies heavily on the stability of repelling cycles which, in the present non-autonomous setting, do not exist at all. There is one criterion of stability in [Lyu86] which turns out to be appropriate for generalization to the present setting. This criterion exploits the dynamics of the critical orbits $c_\lambda \mapsto f_\lambda(c_\lambda) \mapsto \ldots \mapsto f_\lambda^n(c_\lambda) \mapsto \ldots$ under perturbation of $\lambda$. Indeed, stability coincides with the normality of these orbits and, as already mentioned in the introduction, Montel's Theorem implies that such an orbit is stable if it avoids three values $\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)$ depending holomorphically on $\lambda$ and staying some definite distance apart. It is therefore natural to make the following definition.

**Definition 3.4.** A map $f_\eta$ has *normal critical orbits* if there exists an open neighborhood $V$ of $\eta$, $\kappa > 0$ and, for each $n \geq 0$, three holomorphic functions

$\alpha_i^n \colon V \to \hat{\mathbf{C}}$, $i = 1, 2, 3$, such that

$$\text{(3.1)} \qquad \mathrm{dist}_S(\alpha_i^n(\lambda), \alpha_j^n(\lambda)) \geq \kappa \quad \text{for all } \lambda \in V \text{ and } i \neq j.$$

$$\text{(3.2)} \qquad f_\lambda^n(\mathcal{C}_{f_\lambda^n}) \cap \{\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)\} = \emptyset \quad \text{for all } \lambda \in V \text{ and } n \geq 1.$$

$$\text{(3.3)} \qquad \text{If } \alpha_i^{n+k}(\lambda) = f_{\sigma^n(\lambda)}^k(\alpha_j^n(\lambda)) \text{ for some } \lambda \in V, \text{ then this equality holds for all } \lambda \in V.$$

Notice that (3.2) is precisely (1.3) and the compatibility condition (3.3) is also exactly the condition (1.3) of Theorem 1.1. Only the first condition (3.1) differs from the corresponding one in Theorem 1.1. It is a normalized version of condition (1.2) in which we allow the functions $\alpha_j^n$ to have values not only in the corresponding Julia set but in the whole Riemann sphere. If, in this definition, the condition (3.1) is replaced by (1.2), then we will say that $f_\eta$ has *normal critical orbits in the sense of Theorem 1.1* on $V$.

**Proposition 3.5.** *Suppose that $\eta \in \Lambda^{stable}$ is a holomorphic stable parameter and that $\mathcal{J}_\eta$ is a perfect set. Then $f_\eta$ has normal critical orbits in the sense of Theorem 1.1.*

*Proof.* Consider first the map $f_\eta$ and let us define the points $\alpha_j^n(\eta)$ by induction. Since $\mathcal{J}_\eta$ is perfect, there exist three distinct points $\alpha_1^0(\eta), \alpha_2^0(\eta), \alpha_3^0(\eta) \in \mathcal{J}_\eta$. Suppose that all the points $\alpha_j^k(\eta)$ are defined for $0 \leq k < n$. The set $\mathcal{J}_{\sigma^n(\eta)}$ is also perfect and so there are distinct points

$$\alpha_1^n(\eta), \alpha_2^n(\eta), \alpha_3^n(\eta) \in \mathcal{J}_{\sigma^n(\eta)} \setminus \left[ f_\eta^n(\mathcal{C}_{f_\eta^n}) \cup \bigcup_{k=0}^{n-1} f_{\sigma^k(\eta)}^{n-k}(\alpha_j^k(\eta)) \right].$$

By assumption there are holomorphic motions $\{h_{\sigma^n(\lambda)}\}_n$ such that Definition 3.3 is satisfied. It suffices now to set

$$\alpha_j^n(\lambda) := h_{\sigma^n(\lambda)}(\alpha_j^n(\eta)) \quad \text{for every } \lambda \in V \text{ and all } n, j.$$

□

The following main result of this section goes in the opposite direction.

**Theorem 3.6.** *Suppose that $U$ is an open subset of $\Lambda$ such that, for every $\lambda \in U$, $f_\lambda$ is topologically exact and*

$$\limsup_{n \to \infty} \mathrm{diam}\, \mathcal{J}_{\sigma^n(\lambda)} > 0.$$

*Suppose that $\eta \in U$ such that $f_\eta$ has normal critical orbits. Then $f_\eta$ is holomorphically stable, i.e. $\eta \in \Lambda^{\mathrm{stable}}$. Moreover, the corresponding family of holomorphic motions is bi-equicontinuous; it gives rise to a bi-equicontinuous conjugation on the Julia sets.*

Before giving a proof of it, let us first explain how Theorem 1.1 results.

*Proof of Theorem 1.1.* Given Proposition 3.5 we only have to show that normality of critical orbits in the sense of Theorem 1.1 implies holomorphic stability. Let $f_\eta$ be a map such that there exist functions $\alpha_1^n, \alpha_2^n, \alpha_3^n$ defined and holomorphic on some neighborhood $V$ of $\eta$ such that the conditions (1.2), (1.3) and (1.4) are satisfied. Let $\mathcal{M}_{\sigma^n(\lambda)}$ be a Möbius transformation sending the points $\alpha_j^n(\lambda)$, $j = 1, 2, 3$, to $0, 1, \infty$ and consider $\tilde{f}_{\sigma^n(\lambda)}$ defined by

$$\text{(3.4)} \qquad \tilde{f}_{\sigma^n(\lambda)} \circ \mathcal{M}_{\sigma^n(\lambda)} = \mathcal{M}_{\sigma^{n+1}(\lambda)} \circ f_{\sigma^n(\lambda)} \quad \text{for every } \lambda \in V \text{ and } n \geq 0.$$

By assumption, $f_\lambda$ is topologically exact near $\eta$, say on $V$. Therefore, Lemma 2.7 applies and yields that

$$\mathcal{J}(\tilde{f}_{\sigma^n(\lambda)}) = \mathcal{M}_{\sigma^n(\lambda)}\Big(\mathcal{J}(f_{\sigma^n(\lambda)})\Big) \quad \text{for all } \lambda, n.$$

Since the functions $\lambda \mapsto \alpha_j^n(\lambda)$ are holomorphic on $V$, it suffices to establish holomorphic stability of $\tilde{f}_\eta$. This new function $\tilde{f}_\eta$ has normal critical orbits (with functions $\tilde{\alpha}_j^n$ constant $0, 1$ or $\infty$) and so we would like to conclude by applying Theorem 3.6. However, on every fiber the map $\tilde{f}_{\sigma^j(\lambda)}$, $j \geq 0$, belongs to a different holomorphic family $\mathcal{F}_j = \{\tilde{f}_{\sigma^j(\lambda)}\,;\ \lambda \in V\}$. But, as already mentioned in Remark 1.3, the whole paper and especially Theorem 3.6 does hold in this generality with the same proof. Therefore $\tilde{f}_\eta$ is holomorphically stable. □

The remainder of this section is devoted to the proof of Theorem 3.6. In order to do so, we now suppose that the conditions of Theorem 3.6 are satisfied and that, in particular, $\eta \in U \subset \Lambda$ is such that $f_\eta$ has normal critical orbits: there are $V \subset U$, an open neighborhood of $\eta$, and holomorphic functions $\alpha_j^n$ such that the conditions of Definition 3.4 are satisfied. We can and will suppose that $V$ is simply connected. Consider the sets

$$E_{\sigma^j(\lambda),n} = f_{\sigma^j(\lambda)}^{-(n-j)}\big(\{\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)\}\big), \quad j \leq n$$

and

(3.5) $$\mathcal{E}_{\sigma^j(\lambda)} = \bigcup_{n \geq j} E_{\sigma^j(\lambda),n}, \quad \lambda \in V \text{ and } j \geq 0.$$

**Proposition 3.7.** *For every $j \geq 0$, there are holomorphic motions $h_{\sigma^j(\lambda)}\colon \mathcal{E}_{\sigma^j(\eta)} \to \mathcal{E}_{\sigma^j(\lambda)}$ over $V$ such that*

(3.6) $$h_{\sigma^j(\lambda)}(\alpha_i^j(\eta)) = \alpha_i^j(\lambda) \quad \textit{for all } \lambda \in V \textit{ and } i \in \{1,2,3\}, \textit{ and}$$

(3.7) $$h_{\sigma^{j+1}(\lambda)} \circ f_{\sigma^j(\eta)} = f_{\sigma^j(\lambda)} \circ h_{\sigma^j(\lambda)} \quad \textit{on } \mathcal{E}_{\sigma^j(\eta)},\ \lambda \in V.$$

*Proof.* We explain how to obtain the motions in the case $j = 0$. The general case is proven exactly the same way.

Let $z_\eta \in \mathcal{E}_\eta$ and let $n \geq 0$ be minimal such that $z_\eta \in E_{\eta,n}$. A point $z_\eta \in E_{\eta,n}$ if $f_\eta^n(z_\eta) = \alpha_i^n(\eta)$ for some $i \in \{1,2,3\}$. Hence, we have to consider the equation

(3.8) $$f_\lambda^n(z) = \alpha_i^n(\lambda).$$

We want to apply the implicit function theorem to this equation and get $z$ as a function of $\lambda$. This is possible as long as $(f_\lambda^n)'(z) \neq 0$. If $(f_\lambda^n)'(z) = 0$, then the point $\alpha_i^n(\lambda)$ is a critical value of $f_\lambda^n$. However, the assumption (3.2) implies that this is not the case for $\lambda \in V$. The set $V$ being simply connected it follows that there is a uniquely defined holomorphic function $\lambda \mapsto z_\lambda$, $\lambda \in V$, starting at the given point $z_\eta$, if $\lambda = \eta$, and such that $(\lambda, z_\lambda)$ is solution of (3.8). Therefore, we can define

$$h_\lambda(z_\eta) = z_\lambda, \quad \lambda \in V.$$

If ever $z_\lambda \in E_{\lambda,k} \cap E_{\lambda,n}$ for some $\lambda \in V$ and $1 \leq k \leq n$, then there are $i, j \in \{1,2,3\}$ such that $\alpha_i^n(\lambda) = f_{\sigma^k(\lambda)}^{n-k}(\alpha_j^k(\lambda))$. But then the compatibility condition (3.3) implies that the last equation holds for all $\lambda \in V$ and that it does not matter for the definition of the function $\lambda \mapsto z_\lambda$ if we start with $\alpha_i^n(\eta)$ or with $\alpha_j^k(\eta)$.

The normalization (3.6) and the conjugating relation (3.7) are clearly satisfied simply by the way we constructed the holomorphic motions. Hence, the proof is complete. □

*End of the proof of Theorem 3.6.* We are now able to conclude the proof of Theorem 3.6 by using Mitra's version of the $\lambda$-Lemma. Indeed, Theorem 3.2 asserts that the motions $h_{\sigma^j(\lambda)}$ extend to holomorphic motions of the closure $\overline{\mathcal{E}}_{\sigma^j(\lambda)}$. We continue to denote these extended motions by $h_{\sigma^j(\lambda)}$. These maps $h_{\sigma^j(\lambda)}$ are global quasiconformal homeomorphisms with dilatation bounded by $\exp(2\rho_V(\eta,\lambda))$. Therefore, for every fixed $\lambda \in V$ the family $(h_{\sigma^j(\lambda)})_j$ is uniformly quasiconformal and normalized by (3.6). Since the points $\alpha_i^j(\lambda)$, $i = 1, 2, 3$, are at definite spherical distance (see Condition (3.1)), it results from standard properties of families of uniformly quasiconformal mappings that the conjugation by $\{h_{\sigma^j(\lambda)}\}_j$ is bi-equicontinuous.

Up to now we showed that Theorem 3.6 holds but with the Julia sets $\mathcal{J}_{\sigma^j(\lambda)}$ replaced by the sets $\overline{\mathcal{E}}_{\sigma^j(\lambda)}$. However, it is not hard to see that $\mathcal{J}_{\sigma^j(\lambda)} \subset \overline{\mathcal{E}}_{\sigma^j(\lambda)}$. Indeed, for every open set $U \subset \hat{\mathbf{C}} \setminus \overline{\mathcal{E}}_{\sigma^j(\lambda)}$ we have that

$$f^n_{\sigma^j(\lambda)}(U) \cap \left\{\alpha_1^{j+n}(\lambda), \alpha_2^{j+n}(\lambda), \alpha_3^{j+n}(\lambda)\right\} = \emptyset \quad \text{for every } n \geq 0.$$

Hence, Montel's Theorem along with Condition (3.1) imply that $U \subset \mathcal{F}_{\sigma^j(\lambda)}$. Consequently, $\mathcal{J}_{\sigma^j(\lambda)} \subset \overline{\mathcal{E}}_{\sigma^j(\lambda)}$ for every $j \geq 0$.

It remains to show that

$$h_{\sigma^j(\lambda)}(\mathcal{J}_{\sigma^j(\eta)}) = \mathcal{J}_{\sigma^j(\lambda)} \quad \text{for every } j \geq 0 \text{ and } \lambda \in V.$$

This would be immediate (see Lemma 2.6) if the maps $\{h_{\sigma^j(\lambda)}\}_j$ where global conjugacies since they form a bi-equicontinuous family.

It suffices to consider the case $j = 0$. Let $\lambda \in V$. We show by contradiction that $h_\lambda(\mathcal{J}_\eta) \subset \mathcal{J}_\lambda$. The proof of the converse inclusion is the same. Suppose that there is $z_\eta \in \mathcal{J}_\eta$ such that $z_\lambda = h_\lambda(z_\eta) \in \mathcal{F}_\lambda \cap \overline{\mathcal{E}}_\lambda$. Let $r_0 > 0$ such that $(f_\lambda^n)_n$ is normal on $D(z_\lambda, r_0)$. For $0 < r \leq r_0$, consider the neighborhood $U_r = h_\lambda^{-1}(D(z_\lambda, r))$ of $z_\eta$. Since $f_\eta$ is topologically exact there exists, for every $0 < r \leq r_0$, an integer $N_r$ such that $f_\eta^n(U_r) \supset \mathcal{J}_{\sigma^n(\eta)}$, $n \geq N_r$. We even have

$$\mathcal{J}_{\sigma^n(\eta)} \subset f_\eta^n(U_r \cap \overline{\mathcal{E}}_\eta) = h^{-1}_{\sigma^n(\lambda)} \circ f_\lambda^n(U_r \cap \overline{\mathcal{E}}_\eta), \quad n \geq N_r.$$

But then it follows from (3.7) with $\mathcal{E}_{\sigma^j(\eta)}$ replaced by $\overline{\mathcal{E}}_{\sigma^j(\eta)}$, from normality of $(h^{-1}_{\sigma^n(\lambda)} \circ f_\lambda^n)_n$ on $D(z_\lambda, r)$ and by letting $r \to 0$, that

$$\lim_{n\to\infty} \operatorname{diam} \mathcal{J}_{\sigma^n(\eta)} = 0$$

contrary to the assumptions of Theorem 3.6. The proof is complete. □

From this study of holomorphic stability we get first informations concerning our initial problem, namely the behavior of the variation of the Julia sets and of their dimensions.

**Corollary 3.8.** *Suppose that $\Lambda \subset \Lambda_0^{\mathbf{N}}$ is a complex Banach manifold and let $\eta \in \Lambda^{\text{stable}}$. Then, in some neighborhood of $\eta$ in $\Lambda$, the function $\lambda \mapsto \mathcal{J}_\lambda$ is continuous and $\lambda \mapsto HD(\mathcal{J}_{\sigma^j(\lambda)})$ as well as $\lambda \mapsto BD(\mathcal{J}_{\sigma^j(\lambda)})$ are Hölder continuous with Hölder constants depending on $\lambda$ only.*

*Proof.* The assertion on the Hölder continuity directly results from known properties of quasiconformal mappings along with the fact that the distortions of the quasiconformal mappings $h_{\sigma^j(\lambda)}$ do only depend on $\lambda$ and not on $j \geq 0$. Concerning the continuity of the Julia sets, this is a consequence of the continuity of the function $(\lambda, z) \mapsto h_{\sigma^j(\lambda)}(z)$ (see property (2) of Theorem 3.2). □

## 4. Hyperbolic non-autonomous systems

In deterministic dynamics a hyperbolic function is stable. But if we perturb a deterministic hyperbolic function to a non-autonomous map then the stability depends on the topology we use on the parameter space. As an illustration we first consider the simple Tychonov convergence and explain that, for this topology, every map is unstable (see Proposition 4.1).

Then we investigate non-autonomous hyperbolic and uniform hyperbolic functions and will see that the later are stable provided the parameter space is $\Lambda = l^\infty(\Lambda_0)$. In order to prove their stability it suffices to use Theorem 3.6. Indeed, the normal critical orbits condition is best appropriated since it is easy to check for hyperbolic maps.

**4.1. Stability and Tychonov topology.** Up to here, the parameter space $\Lambda$ was equipped with any arbitrary complex manifold structure. Let us first consider the case when the space $\Lambda = \Lambda_0^{\mathbf{N}}$ is equipped with the Tychonov topology.

**Proposition 4.1.** *Suppose that $\mathcal{F}$ contains at least two deterministic hyperbolic maps having Julia sets with different Hausdorff dimension. Suppose further that $\Lambda = \Lambda_0^{\mathbf{N}}$ and that $\Lambda$ is equipped with the Tychonov topology induced by the simple convergence. Then the function $\lambda \mapsto HD(\mathcal{J}_\lambda)$ is discontinuous at every point of $\Lambda$.*

*Proof.* Let $\eta \in \Lambda$ and set $\delta = HD(\mathcal{J}_\eta)$. By hypothesis there exists $f_{\lambda_0} \in \mathcal{F}$ a deterministic hyperbolic map with $\delta' = HD(\mathcal{J}(f_{\lambda_0}) \neq \delta$. Consider then

$$\lambda^{(n)} = (\eta_1, \eta_2, \dots, \eta_n, \lambda_0, \lambda_0, \lambda_0, \dots).$$

On the one hand we have that $\lambda^{(n)} \to \eta$ point wise. On the other hand we have $HD(\mathcal{J}_{\lambda^{(n)}}) = \delta'$ for every $n \geq 1$ and hence $HD(\mathcal{J}_{\lambda^{(n)}}) \not\to HD(\mathcal{J}_\eta)$ as $n \to \infty$. □

**4.2. Hyperbolicity.** Hyperbolic random systems have been studied in various papers (see e.g. [Com06, Ses99, Sum01] and also [Sum97] where hyperbolic semigroups are considered). In these papers, normalized most often polynomial families are considered and the definitions of hyperbolicity rely on uniform conditions. We therefore call such functions *uniformly* hyperbolic.

In the following we make the standard convention that all the derivatives are taken with respect to the spherical metric.

**Definition 4.2.** A map $f_\lambda$ is *uniformly hyperbolic* if the family $\{f_{\lambda_j}\,;\ j \geq 1\}$ is equicontinuous (which, for example, is the case if $\{\lambda_j,\ j \geq 1\}$ is relatively compact in $\Lambda_0$ or, equivalently, if $\lambda \in l^\infty(\Lambda_0)$), if $\#\mathcal{J}_{\sigma^j(\lambda)} \geq 2$ and if there exist $c > 0$ and $\gamma > 1$ such that for every $j \geq 0$ we have

$$|(f^n_{\sigma^j(\lambda)})'(z)| \geq c\gamma^n \quad \text{for all } z \in \mathcal{J}_{\sigma^j(\lambda)} \text{ and } n \geq 1. \tag{4.1}$$

The set of parameters of uniformly hyperbolic random maps is denoted by $\Lambda^{u\mathrm{Hyp}}$.

For general families of non-autonomous maps this definition is not entirely satisfactory. For instance, in the Example 2.4 we have conjugated a deterministic hyperbolic function by Möbius maps. The resulting non-autonomous map does not satisfy the requirements of Definition 4.2 although it shares many properties of maps that should be called hyperbolic. It is uniformly expanding "up to a conformal change of coordinates". Moreover, it is topologically exact which, as we will see (Lemma 4.8), is a property that uniform hyperbolic maps always have.

A natural candidate for the class of hyperbolic maps is to take all the maps that are Möbius conjugate to uniform hyperbolic maps. However, one has to be careful since the map given in Example 2.3, obtained by conjugation by similarities of a deterministic hyperbolic function, should really not be called hyperbolic. Given these examples and Lemma 2.7 which ensures that the Julia sets are identified provided the dynamics are topologically exact, it is natural to introduce the following definition.

**Definition 4.3.** A non-autonomous map $f_\lambda$ is *hyperbolic* if it is topologically exact, if $\#\mathcal{J}_{\sigma^j(\lambda)} \geq 2$ for all $j \geq 0$ and if there are Möbius transformations conjugating $f_\lambda$ to a uniformly hyperbolic map.

We now consider uniform hyperbolicity greater in detail. Let $\mathcal{V}_\delta(E) = \{z\ ;\ \mathrm{dist}(z, E) < \delta\}$ be the $\delta$-neigborhood of the set $E$.

**Lemma 4.4.** *The map $f_\lambda$ is uniformly hyperbolic if and only if the family $\left\{f_{\lambda_j}\ ;\ j \geq 1\right\}$ is equicontinuous and there exist $\delta > 0$, $N \geq 1$ and $\tau > 1$ such that*

$$|(f^N_{\sigma^j(\lambda)})'(z)| \geq \tau > 1 \quad \text{for all } z \in \mathcal{V}_\delta(\mathcal{J}_{\sigma^j(\lambda)}) \text{ and } j \geq 0. \tag{4.2}$$

*In particular, if $f_\lambda$ is uniformly hyperbolic then there exist $\delta > 0$ such that for all $n \geq 1$, $j \geq 0$ and $z \in \mathcal{J}_{\sigma^{n+j}(\lambda)}$ all holomorphic inverse branches of $f^n_{\sigma^j(\lambda)}$ are well defined on $D(z, \delta)$ have uniform distortion and are uniformly contracting.*

*Proof.* Suppose that $f_\lambda$ is uniformly hyperbolic and fix $N \geq 1$ such that $c\gamma^N > 1$. Suppose that (4.2) does not hold. More precisely, suppose that for any $\delta > 0$ and any $1 < \tau < c\gamma^N$ there exist $w = w_{\delta,\tau} \in \mathcal{V}_\delta\left(\mathcal{J}_{\sigma^j(\lambda)}\right)$ for some $j = j_{\delta,\tau} \geq 0$ such that

$$|(f^N_{\sigma^j(\lambda)})'(w)| \leq \tau.$$

Let $z_{\delta,\tau} \in \mathcal{J}_{\sigma^j(\lambda)}$ such that $|z_{\delta,\tau} - w_{\delta,\tau}| < \delta$. Due to the equicontinuity of the family $\left\{f^N_{\sigma^j(\lambda)}\ ,\ j \geq 0\right\}$ we can choose sequences $\delta_n \to 0$, $\tau_n \to 1$ such that the corresponding functions $f^N_{\sigma^{j(n)}(\lambda)} \to \varphi$ and points $w_{\delta_n,\tau_n} \to \xi$, $z_{\delta_n,\tau_n} \to \xi$ converge as $n \to \infty$. But then it is easy to see that $|\varphi'(\xi)| \leq 1$ and, in the same time, $|\varphi'(\xi)| \geq c\gamma^N > 1$. This contradiction shows that uniform hyperbolicity implies (4.2). The other assertion results now from standard arguments. □

In the case of deterministic iteration of rational functions there are several equivalent conditions for hyperbolicity. One of them is the *expanding* condition, another condition demands that critical orbits are captured by attracting domains. Here is a version in the non-autonomous case which in fact is an adaption of [Ses99].[1]

[1]The paper [Ses99] seems to be inspired by [Jon99] and in [Sum01] are more general results in this direction. All these papers deal with *fibered maps* or *skew-products* which are deeply related to our non-autonomous setting in the following sense. To a non-autonomous map $f_\lambda$, $\lambda \in \Lambda$, one can associate $F\colon \Lambda \times \hat{\mathbf{C}} \to \Lambda \times \hat{\mathbf{C}}$ defined by $F(\lambda, z) = (\sigma(\lambda), f_{\lambda_1}(z))$. This is a skew-product in the sense of the papers mentionned excepted that the base space $\Lambda$ is not necessarily compact. However,

**Proposition 4.5.** *A map $f_\lambda$ is uniformly hyperbolic if and only if the family $\{f_{\lambda_j}\}_j$ is equicontinuous and if there exist $m_0 > 0$ and open sets $U_j$ such that, for every $j \geq 0$,*

(1) $\overline{f_{\sigma^j(\lambda)}(U_j)} \subset U_{j+1}$ *and* $\mathrm{dist}_S(f_{\sigma^j(\lambda)}(U_j), \partial U_{j+1}) \geq m_0$,
(2) $D(z, m_0) \cap U_j = \emptyset$, *for every* $z \in \mathcal{J}_{\sigma^j(\lambda)}$, *and*
(3) *the critical points of* $f_{\sigma^j(\lambda)}$ *are contained in* $U_j$.

*Proof.* Since most of the proof is standard we only give a brief outline of it. Especially, finding the sets $U_j$ knowing that $f_\lambda$ is uniformly hyperbolic is a straightforward adaption of Sester's arguments [Ses99, pp. 414–415] which themselves are based on the deterministic case. The main idea is to build a metric in which all the functions $f_{\sigma^j(\lambda)}$ have a derivative greater than some constant $\gamma > 1$ on $\mathcal{V}_\delta(\mathcal{J}_{\sigma^j(\lambda)})$ for some $\delta > 0$.

The proof of the opposite implication is based on hyperbolic geometry. Suppose the sets $U_j$ are given, set $V_{j+1} = f_{\sigma^j(\lambda)}(U_j)$ and $\tilde{U}_j = f^{-1}_{\sigma^j(\lambda)}(V_{j+1})$. Then $f_{\sigma^j(\lambda)}\colon \tilde{U}_j \to V_{j+1}$ is a proper map and, the critical orbits being captured by the domains $U_j$ (see (3)), $f_{\sigma^j(\lambda)}\colon \omega_j \to \Omega_{j+1}$ is a covering map where $\omega_j, \Omega_{j+1}$ is the complement of the closure of $\tilde{U}_j, V_{j+1}$ respectively. Therefore this map is a local hyperbolic isometry with respect to the hyperbolic distances of these domains. Property (1) implies that there is $0 < c < 1$ such that the inclusion map $i\colon \omega_{j+1} \to \Omega_{j+1}$ is a hyperbolic $c$–contraction for all $j \geq 0$. Combining these properties it follows that $f_{\sigma^j(\lambda)}$ is a $1/c$–expansion on $\mathcal{J}_{\sigma^j(\lambda)} \subset \omega_j \cap f^{-1}_{\sigma^j(\lambda)}(\omega_{j+1})$ with respect to the hyperbolic distances of $\omega_j$ and $\omega_{j+1}$. Finally, it results from property (2) that it is possible to compare the hyperbolic and spherical distance for points in $\mathcal{J}_{\sigma^j(\lambda)} \subset \omega_j$, $j \geq 0$, and to conclude. □

The topological characterization of Propositon 4.5 and especially the uniform control due to the constant $m_0$ implies the following.

**Corollary 4.6.** *Uniform hyperbolicity is an open condition for the $l^\infty$-topology on $\Lambda$ (but not for the Tychonov topology). Moreover, if $\eta \in \Lambda^{\mathrm{hyp}}$, then there is an open neighborhood $V \subset \Lambda^{\mathrm{hyp}}$ of $\eta$ such that the open sets $U_j$ and the number $\delta = \delta(\lambda) > 0$ given by Lemma 4.4 can be chosen to be the same for all the maps $f_\lambda$, $\lambda \in V$.*

This result immediatley implies the following continuity property of non-autonomous Julia sets which, in various versions, is well known to the specialists (see for example [Brü00, Ses99, Sum01, Com06]).

**Proposition 4.7.** *Every $\eta \in \Lambda^{\mathrm{uHyp}}$ has an open neighborhood $V \subset l^\infty(\Lambda_0)$ such that the map*

$$\lambda \longmapsto \mathcal{J}_\lambda$$

*from* ($V$, Tychonov topology) *into* ($\mathcal{K}(\hat{\mathbf{C}})$, Hausdorff topology) *is continuous.*

*Proof.* Let $\eta \in \Lambda^{\mathrm{uHyp}}$ and let the open neighborhood $V$ of $\eta$ be sufficiently small in $\Lambda$ with respect to the $l^\infty$-topology and chosen according to Corollary 4.6, i.e. there are open sets $U_j$ such that every map $f_\lambda$, $\lambda \in V$, satisfies the conditions (1), (2) and (3) of Proposition 4.5 with these sets $U_j$. Denote

$$\tilde{U}_j = \{z \in U_j\ ;\ \mathrm{dist}_S(z, \partial U_j) > m_0/2\}.$$

---

one is in the setting of [Jon99, Ses99, Sum01] if one restricts the base space. For example, if one replaces $\Lambda$ by $K^{\mathbf{N}}$ equipped with the Tychonov topology where $K$ is any compact subset of $\Lambda_0$.

Shrinking the neighborhood $V$ if necessary and replacing $m_0$ by a smaller constant we may assume that the open sets $\tilde{U}_j$ satisfy also the conditions (1), (2) and (3) of Proposition 4.5 for every $\lambda \in V$. Moreover, all inverse branches exist and are uniformly contracting on the complement of $\tilde{U}_j$, $j \geq 1$.

Define

$$\mathcal{A}_\lambda^n = \{z \in \hat{\mathbf{C}}\ ;\ f_\lambda^n(z) \notin U_n\} \quad \text{and} \quad \tilde{\mathcal{A}}_\lambda^n = \{z \in \hat{\mathbf{C}}\ ;\ f_\lambda^n(z) \notin \tilde{U}_n\}.$$

Clearly $\mathcal{J}_\lambda \subset \bigcap_n \mathcal{A}_\lambda^n \subset \bigcap_n \tilde{\mathcal{A}}_\lambda^n$. On the other hand, since all inverse branches exists and are uniformly contracting on the complement of $\tilde{U}_j$, $j \geq 1$, we have first of all that $\mathcal{J}_\lambda = \bigcap_n \mathcal{A}_\lambda^n = \bigcap_n \tilde{\mathcal{A}}_\lambda^n$ and, secondly, that for every $\varepsilon > 0$ there exist $n = n_\varepsilon \geq 1$ such that $\mathcal{A}_\lambda^n \subset \tilde{\mathcal{A}}_\lambda^n \subset \mathcal{V}_\varepsilon(\mathcal{J}_\lambda)$ for every $\lambda \in V$.

Fix $\varepsilon > 0$ and let $n = n_\varepsilon$. Notice that the sets $\mathcal{A}_\lambda^n$ and $\tilde{\mathcal{A}}_\lambda^n$ do only depend on the $n$ functions $f_{\lambda_1}, \dots, f_{\lambda_n}$. A standard compactness argument shows now that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\mathcal{A}_\lambda^n \subset \tilde{\mathcal{A}}_{'}^n \ \text{ for every } \ \lambda,' \in V \ \text{ such that } \sup_{i=1,\dots,n} |\lambda_i - {}'_i| < \delta.$$

Therefore, for every $\lambda,' \in V$ such that $\sup_{i=1,\dots,n} |\lambda_i - {}'_i| < \delta$ we have that

$$\mathcal{J}_\lambda \subset \mathcal{A}_\lambda^n \subset \tilde{\mathcal{A}}_{'}^n \subset \mathcal{V}_\varepsilon(\mathcal{J}_{'})$$

This proves the proposition. □

We conclude the discussion on uniform hyperbolicity with the following uniform mixing property.

**Lemma 4.8.** *Let $\lambda \in \Lambda^{\mathrm{uHyp}}$ and let $\delta = \delta(\lambda)$. Then, for every $r_1 > 0$ and $0 < r_2 \leq \delta$, there exist $N = N(r_1, r_2)$ such that for all $j \geq 0$, $z_1 \in \mathcal{J}_{\sigma^j(\lambda)}$ and $z_2 \in \mathcal{J}_{\sigma^{j+N}(\lambda)}$ we have that*

$$f_{\sigma^j(\lambda)}^N\big(D(z_1, r_1)\big) \supset D(z_2, r_2)\ .$$

*In particular, $f_\lambda$ is (uniformly) topologically exact: for every $r_1 > 0$ there exist $N = N(r_1)$ such that for $j \geq 0$ and $z_1 \in \mathcal{J}_{\sigma^j(\lambda)}$ we have that $f_{\sigma^j(\lambda)}^N\big(D(z_1, r_1)\big) \supset \mathcal{J}_{\sigma^{j+N}(\lambda)}$.*

*Proof.* Suppose to the contrary that there exist $r_1 > 0$ and $0 < r_2 \leq \delta$ and, for every $N$, $j_N \geq 0$, $z_{1,N} \in \mathcal{J}_{\sigma^j(\lambda)}$ and $z_{2,N} \in \mathcal{J}_{\sigma^{j_N+N}(\lambda)}$ such that

$$D(z_{2,N}, r_2) \setminus f_{\sigma^{j_N}(\lambda)}^N\,(D(z_{1,N}, r_1)) \neq \emptyset. \tag{4.3}$$

Consider then $\varphi_N(z) = f_{\sigma^{j_N}(\lambda)}^N(r_1 z + z_{1,N})$, $z \in \mathbf{D}$. Since $f_\lambda$ is expanding on the Julia set the family $(\varphi_N)_N$ is not normal at the origin. Therefore there are infinitely many $N$ such that

$$\varphi_N(\mathbf{D}(0, 1/2)) \cap D(z_{2,N}, r_2) \neq \emptyset. \tag{4.4}$$

Since $r_2 \leq \delta$, all inverse branches of $f_{\sigma^{j_N}(\lambda)}^N$ are well defined and have bounded distortion on $D(z_{2,N}, r_2)$. It suffices then to choose $N$ big enough and to deduce from expanding along with (4.4) that

$$f_{\sigma^{j_N}(\lambda),*}^{-N}\,(D(z_{2,N}, r_2)) \subset D(z_{1,N}, r_1)$$

where $f_{\sigma^{j_N}(\lambda),*}^{-N}$ is some well chosen inverse branch. This contradicts (4.3). □

**4.3. Hyperbolicity and stability.** The definition of hyperbolic map is based on uniform controls, e.g. the iterated maps $f^n_{\sigma^j(\lambda)}$ are expanding uniformly in $j$. With respect to this and in order to deal with perturbations of hyperbolic functions it is natural to equip the parameter space $\Lambda$ with the sup-norm, i.e. to work with the space $\Lambda = l^\infty(\Lambda_0)$. Throughout the rest of this paper we suppose that $\Lambda$ is this particular Banach manifold.

As already mentioned, in order to establish stability of uniformly hyperbolic maps, the condition of normal critical orbits as defined in Definition 3.4 is perfectly adapted since easy to verify for such functions.

**Proposition 4.9.** *If $f_\eta$ is a uniform hyperbolic map, then $f_\eta$ has normal singular orbits on some open neighborhood $V \subset \Lambda$ of $\eta$.*

*Proof.* By Corollary 4.6, there is an open neighborhood $V \subset \Lambda$ such that the open sets $U_n$ in Proposition 4.5 can be chosen independently on $\lambda \in V$. Since we know that

$$\mathrm{dist}_S(f_{\lambda_n}(U_n), \partial U_{n+1}) \geq m_0,$$

we can find three points $a_i^0 \in U_0$ and, if if $n > 0$,

$$a_i^n \in U_n \setminus \bigcup_{\lambda \in V} f_{\lambda_{n-1}}(\overline{U}_{n-1})$$

such that $\mathrm{dist}_S(a_i^n, a_j^n) \geq c_0$ for some $c_0 > 0$ and for all $n \geq 0$ and $i \neq j$. Since $\mathcal{C}_{f_{\lambda_j}} \subset U_j$, $j \geq 1$, we have the inclusion $f^n_\lambda(\mathcal{C}_{f^n_\lambda}) \subset f_{\lambda_n}(U_{n-1}) \subset U_n$. The constant functions $\lambda \mapsto \alpha_i^n(\lambda) = z_i^n$, $\lambda \in V$, therefore satisfy the conditions (1) and (2) of Definition 3.4 and appropriate perturbations of these constant functions if necessary yield that Condition (3) of this definition is also satisfied. Therefore, $f_\lambda$ has normal critical orbits on $V$. □

The following statement follows now from Theorem 3.6.

**Corollary 4.10.** $\Lambda^{\mathrm{uHyp}} \subset \Lambda^{\mathrm{stable}}$ *when equipped with the $l^\infty$-topology.*

## 5. Conformal measures, pressure and dimensions

In this section we consider a single non-autonomous uniformly hyperbolic map $f_\lambda$, $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda^{\mathrm{uHyp}}$. Remember that all the derivatives are taken with respect to the spherical metric. Since $\{\lambda_n\}_n$ is relatively compact in the set $\Lambda_0$ and since the rational maps are Lipschitz with respect to the spherical metric [Bea91, Theorem 2.3.1], there is a constant $A < \infty$ such that $|f'_{\sigma^j(\lambda)}(z)| \leq A$ for all $z \in \hat{\mathbf{C}}$ and $j \geq 1$. Combining this with uniform hyperbolicity and setting $a = c\gamma$ we get

$$a \leq |f'_{\sigma^j(\lambda)}(z)| \leq A \quad \text{for all } z \in \mathcal{J}_{\sigma^j(\lambda)} \text{ and } j \geq 1. \tag{5.1}$$

**5.1. Conformal measures.** Denote by $\mathcal{C}(\mathcal{J})$ the set of continuous functions $\varphi\colon \mathcal{J} \to \mathbf{R}$. Let $t \geq 0$ and consider the operators $\mathcal{L}_{\sigma^j(\lambda),t}\colon \mathcal{C}(\mathcal{J}_{\sigma^j(\lambda)}) \to \mathcal{C}(\mathcal{J}_{\sigma^{j+1}(\lambda)})$ defined by

$$\mathcal{L}_{\sigma^j(\lambda),t}\, g(w) = \sum_{f_{\sigma^j(\lambda)}(z)=w} |f'_{\sigma^j(\lambda)}(z)|^{-t} g(z), \quad w \in \mathcal{J}_{\sigma^{j+1}(\lambda)}. \tag{5.2}$$

For $n \geq 1$, we denote $\mathcal{L}^n_{\sigma^j(\lambda),t} = \mathcal{L}_{\sigma^{j+n-1}(\lambda),t} \circ \cdots \circ \mathcal{L}_{\sigma^j(\lambda),t}$.

**Proposition 5.1.** *For every $t \geq 0$ there exist a sequence of probability measures $m_{\sigma^j(\lambda),t}$ supported on $\mathcal{J}_{\sigma^j(\lambda)}$ and positive numbers $\rho_{\sigma^j(\lambda),t}$ such that*

$$\mathcal{L}^*_{\sigma^j(\lambda),t}(m_{\sigma^{j+1}(\lambda),t}) = \rho_{\sigma^j(\lambda),t} m_{\sigma^j(\lambda),t} \quad \textit{for all } j \geq 0. \tag{5.3}$$

*Moreover, there exist a sequence $N_k \to \infty$ and points $w_k \in \mathcal{J}_{\sigma^{N_k}(\lambda)}$ such that*

$$\rho_{\sigma^j(\lambda),t} = \lim_{k\to\infty} \frac{\mathcal{L}^{N_k-j}_{\sigma^j(\lambda),t}\mathbb{1}(w_k)}{\mathcal{L}^{N_k-j-1}_{\sigma^{j+1}(\lambda),t}\mathbb{1}(w_k)} \quad \textit{for all } j \geq 0. \tag{5.4}$$

Measures, actually a sequence of measures, satisfying (5.3) are called *t-conformal*. To simplify the notations we will use often in this section the following shorthands

$$m_{j,t} = m_{\sigma^j(\lambda),t} \quad \text{and} \quad \rho_{j,t} = \rho_{\sigma^j(\lambda),t}.$$

This does not lead to confusions since the parameter $\lambda \in \Lambda^{\mathrm{uHyp}}$ is fixed.

*Proof.* Choose for every $N \geq 0$ arbitrarily a point $w_N \in \mathcal{J}_{\sigma^N(\lambda)}$ and consider the probability measures

$$m_j^N = \beta_j^N \left(\mathcal{L}^{N-j}_{\sigma^j(\lambda),t}\right)^* \delta_{w_N} \text{ where } \beta_j^N = \left(\mathcal{L}^{N-j}_{\sigma^j(\lambda),t}\mathbb{1}(w_N)\right)^{-1}, \;\; 0 \leq j \leq N.$$

Observe that

$$\mathcal{L}^*_{\sigma^j(\lambda),t}(m^N_{j+1}) = \frac{\mathcal{L}^{N-j}_{\sigma^j(\lambda),t}\mathbb{1}(w_N)}{\mathcal{L}^{N-j-1}_{\sigma^{j+1}(\lambda),t}\mathbb{1}(w_N)} m_j^N \quad \text{for all } 0 \leq j \leq N-1. \tag{5.5}$$

Let $N_k \to \infty$ be a sequence such that all the measures $m_j^{N_k}$ converge weakly as $k \to \infty$ and denote $m_{j,t} = \lim_{k\to\infty} m_j^{N_k}$. It follows then from (5.5) that, for every $j \geq 0$, the limit (5.4) also exists and that we have (5.3). □

**Remark 5.2.** It is a standard observation (see [DU91]) that (5.3) is equivalent with

$$dm_{j+1,t} \circ f_{\lambda_j} = \rho_{j,t} |f'_{\lambda_j}|^t dm_{j,t}. \tag{5.6}$$

The explicit expression (5.4) for the generalized eigenvalue $\rho_{\sigma^j(\lambda),t}$ leads to the following very useful bounds.

**Lemma 5.3.** *With the notations of Proposition 5.1 and of* (5.1)*, we have for every $j \geq 0$ and $t \geq 0$ that*

$$A^{-t}\deg(f_\lambda) \leq \rho_{j,t} \leq a^{-t}\deg(f_\lambda).$$

*Proof.* Since $\mathcal{L}^{N_k-j}_{\sigma^j(\lambda),t}\mathbb{1}(w_k) = \mathcal{L}^{N_k-j-1}_{\sigma^{j+1}(\lambda),t}\left(\mathcal{L}_{\sigma^j(\lambda),t}\mathbb{1}\right)(w_k)$ and since by (5.1)

$$A^{-t}\deg(f_\lambda) \leq \mathcal{L}_{\sigma^j(\lambda),t}\mathbb{1}(z) \leq a^{-t}\deg(f_\lambda) \quad \text{for all } z \in \mathcal{J}_{\sigma^{j+1}(\lambda)} \tag{5.7}$$

the lemma follows from the expression (5.4). □

Remember that $\delta = \delta(\lambda)$ is such that all inverse branches are well defined and have bounded distortion on disks of radius $\delta$ centered on Julia sets.

**Lemma 5.4.** *For every $t \geq 0$, there exist a constant $C_t \geq 1$ such that for every t-conformal measure $m_{j,t}$ and associated $\rho_{j,t}$ and for all $r > 0$ and $z \in \mathcal{J}_{\sigma^j(\lambda)}$ we have*

$$C_t^{-1}\rho_{j,t}^{-n} \leq \frac{m_{j,t}(D(z,r))}{r^t} \leq C_t \rho_{j,t}^{-n}$$

*where* $\rho_{j,t}^n = \rho_{j,t}\rho_{j+1,t}\dots\rho_{j+n-1,t}$ *and* $\rho_{j,t}^{-n} = \left(\rho_{j,t}^n\right)^{-1}$ *and where* $n \geq 1$ *is maximal such that* $|(f_{\sigma^j(\lambda)}^n)'(z)|^{-1} \geq \frac{r}{\delta}$.

*Proof.* First of all, since $f_\lambda$ is expanding we have a lower bound of the derivatives $|f'_{\sigma_j(\lambda)}|$ on Julia sets. Together with the Lipschitz estimation (5.1) it follows that there is $a > 0$ such that

$$a \leq |f'_{\sigma^j(\lambda)}(z)| \leq A \quad \text{for all } z \in \mathcal{J}_{\sigma^j(\lambda)} \text{ and } j \geq 1\,. \tag{5.8}$$

Therefore, if $z \in \mathcal{J}_{\sigma^j(\lambda)}$ and if we put $r_n = |f_{\sigma^j(\lambda)}^n(z)|^{-1}$ then for every $r > 0$ there exist $n$ such that

$$r \asymp r_n \tag{5.9}$$

which signifies that $\frac{r}{r_n}$ is bounded below and above by implicit constants that do not depend on $z$, $j$. Therefore it suffices to establish Lemma 5.4 for radii of the form $r = r_n = |f_{\sigma^j(\lambda)}^n(z)|^{-1}$. But this follows from a standard zooming argument along with the conformality of the measures. More precisely from formula (5.6) provided we can prove the following claim.

**Claim 5.5.** *There is a constant* $c > 0$ *such that for every sequence of* $t$*–conformal measures* $m_{j,t}$ *we have that*

$$m_{j,t}(D(z,\delta)) \geq c \quad \textit{for all } j \geq 0 \textit{ and } z \in \mathcal{J}_{\sigma^j(\lambda)}. \tag{5.10}$$

In order to establish this lower bound we first make the following general observation. The sphere having finite spherical volume and the number $\delta$ being fixed, there is an absolute number $M$ such that every Julia set $\mathcal{J}_{\sigma^n(\lambda)}$ can be covered by no more than $M$ disks of radius $\delta$. Consequently there exist, for every $n \geq 0$, a disk $D_n = D(z, \delta)$, $z \in \mathcal{J}_{\sigma^n(\lambda)}$, having measure $m_{n,t}(D_n) \geq 1/M$.

The mixing property of Lemma 4.8 with $r_1 = r_2 = \delta$ asserts that there is a number $N = N(\delta)$ such that

$$f_{\sigma^j(\lambda)}^N(D(z,\delta)) \supset D_{j+N} \quad \text{for every } j \geq 0 \text{ and } z \in \mathcal{J}_{\sigma^j(\lambda)}. \tag{5.11}$$

Therefore, there is $\Omega \subset D(z,\delta)$ such that $f_{\sigma^j(\lambda)}^N : \Omega \to D_{j+N}$ is a conformal bijection with bounded distortion. With $\xi \in \Omega$ an arbitrarily chosen point we get

$$m_{j,t}(D(z,\delta)) \geq m_{j,t}(\Omega) \asymp |(f_{\sigma^j(\lambda)}^N)'(\xi)|^{-t}\rho_{j,t}^{-N} m_{j+N,t}(D_{j+N}) \geq A^{-tN}\rho_j^{-N}/M$$

with $\rho_{j,t}$ the eigenvalues associated to $m_{j,t}$ by (5.3).

It remains to estimate $\rho_{j,t}^N$. But this has already been done in Lemma 5.3 from which follows that $\rho_{j,t}^N \leq a^{-Nt}\deg(f_\lambda)^N$. Therefore, we get the final estimation

$$m_{j,t}(D(z,\delta)) \geq \frac{1}{M}\left(\frac{a}{A}\right)^{tN}\deg(f_\lambda)^{-N} \quad \text{for all } j \geq 0 \text{ and } z \in \mathcal{J}_{\sigma^j(\lambda)}.$$

□

As a first consequence of the previous result we get the following key estimation. Here and in the following we use the symbol $\mathbb{1}$ for the function that is constant equal to 1.

**Lemma 5.6.** *For every* $t \geq 0$*, there exists a constant* $D_t \geq 1$ *such that*

$$\frac{1}{D_t} \leq \rho_{j,t}^{-n}\mathcal{L}_{\sigma^j(\lambda),t}^n \mathbb{1}(w) \leq D_t \quad \textit{for every } j \geq 0,\ n \geq 1 \textit{ and } w \in \mathcal{J}_{\sigma^{j+n}(\lambda)}.$$

*Proof.* Let again $\delta = \delta(\lambda)$ and remember from the previous proof that there is an absolute number $M$ such that, for every $j, n$, the Julia set $\mathcal{J}_{\sigma^{j+n}(\lambda)}$ can be covered by at most $M$ disks $D_i = D(z_i, \delta)$, $i = 1, \dots, M$, of radius $\delta$. Let $j \geq 0$, $n \geq 1$ and let $U_{i,k}$ be the components of $f^{-n}_{\sigma^j(\lambda)}(D_i)$. Notice that $\{U_{i,k}\}_{i,k}$ is a Besicovitch covering of $\mathcal{J}_{\sigma^j(\lambda)}$, i.e. $z \in U_{i,k}$ can happen for at most $M$ indices $(i,k)$. Together with conformality of the measures we get that

$$1 \asymp \sum_{i,k} m_{j,t}(U_{i,k}) \asymp \rho^{-n}_{j,t} \sum_{i,k} |(f_\lambda)'(z_{i,k})|^{-t} m_{j+n,t}(D_i)$$

where $z_{i,k} \in U_{i,k}$ is such that $f^n_\lambda(z_{i,k}) = z_i$. Now, by Claim 5.5 we have that $m_{j+n,t}(D_i) \asymp 1$ from which follows that

$$1 \preceq \rho^{-n}_{j,t} M \max_{w \in \mathcal{J}_{\sigma^{j+n}(\lambda)}} \mathcal{L}^n_{\sigma^j(\lambda),t} 1\!\!1(w) \quad \text{and} \tag{5.12}$$

$$1 \succeq \rho^{-n}_{j,t} \mathcal{L}^n_{\sigma^j(\lambda),t} 1\!\!1(z_i) \quad \text{for every } i = 1, \dots, M, \tag{5.13}$$

where here and in the following $\preceq$, $\succeq$ stands for the corresponding inequality up to a multiplicative universal constant. The right-hand inequality of the lemma follows now easily from Koebe's distortion theorem and (5.13). For the other inequality we proceed as follows. Let again $N = N(\delta)$ be an integer such that the mixing property (5.11) holds. For all $n < N$ the required estimation is true (see (5.7) and Lemma 5.3). Let $n \geq N$ and $j \geq 0$. Denote then $w_{\max} \in \mathcal{J}_{\sigma^{j+n-N}(\lambda)}$ a point such that

$$\mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1(w_{\max}) = \|\mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1\|_\infty .$$

Then (5.12) yields $\mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1(w_{\max}) \succeq \rho^{n-N}_{j,t}$. Let $w \in \mathcal{J}_{\sigma^{j+n}(\lambda)}$ be any point. The choice of $N$ implies that there exists $b \in D(w_{\max}, \delta) \cap f^{-N}_{\sigma^{j+n-N}(\lambda)}(w_{\max})$. Therefore

$$\mathcal{L}^n_{\sigma^j(\lambda),t} 1\!\!1(w) \geq \left|\left(f^N_{\sigma^{j+n-N}(\lambda)}\right)'(b)\right|^{-t} \mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1(b).$$

Applying Koebe's Distortion Theorem yields $\mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1(b) \asymp \mathcal{L}^{n-N}_{\sigma^j(\lambda),t} 1\!\!1(w_{\max}) \succeq \rho^{n-N}_{j,t}$. Since, by Lemma 5.3, $\rho^N_{j+n-N,t} \leq a^{-Nt} \deg(f_\lambda)^N$ and since $\left|\left(f^N_{\sigma^{j+n-N}(\lambda)}\right)'(b)\right| \leq A^N$ we finally get

$$\mathcal{L}^n_{\sigma^j(\lambda),t} 1\!\!1(w) \succeq \left(\frac{a}{A}\right)^{Nt} \deg(f_\lambda)^{-N} \rho^n_{j,t}$$

which is the required inequality. □

We have not shown yet unicity of conformal measures. If $\tilde{m}_{j,t}$ are some other conformal measures and $\tilde{\rho}_{j,t}$ are the corresponding eigenvalues from (5.3) then they are uniformly close to the eigenvalues $\rho_{j,t}$ of $m_{j,t}$ in the following sense.

**Lemma 5.7.** *For every $t \geq 0$, there exist a constant $B_t \geq 1$ such that for all $j \geq 0$ and $n \geq 1$ we have*

$$\frac{1}{B_t} \leq \frac{\tilde{\rho}^n_{j,t}}{\rho^n_{j,t}} \leq B_t .$$

*Proof.* With the above notations we get from Lemma 5.4 that

$$m_{j,t}(D(z,r)) \asymp r^t \rho^{-n}_{j,t} \quad \text{and} \quad \tilde{m}_{j,t}(D(z,r)) \asymp r^t \tilde{\rho}^{-n}_{j,t}$$

for every $z \in \mathcal{J}_{\sigma^j(\lambda)}$ and $r = r(z,n) = |(f^n_{\sigma^j(\lambda)})'(z)|^{-1}$. Fix $n \geq 1$. Taking a Besicovitch covering of $\mathcal{J}_{\sigma^j(\lambda)}$ by disks $D_k = D(z_k, r(z_k, n))$ centered on $\mathcal{J}_{\sigma^j(\lambda)}$ we get that

$$1 \asymp \sum_k m_{j,t}(D_k) \asymp \sum_k \rho_{j,t}^{-n} \frac{\tilde{m}_{j,t}(D_k)}{\tilde{\rho}_{j,t}^{-n}} = \frac{\tilde{\rho}_{j,t}^n}{\rho_{j,t}^n} \sum_k \tilde{m}_{j,t}(D_{r,k}) \asymp \frac{\tilde{\rho}_{j,t}^n}{\rho_{j,t}^n}$$

for all $j \geq 0$ and $n \geq 1$. □

**5.2. Pressure.** To every $\lambda \in \Lambda^{\text{hyp}}$ and $t \geq 0$ we associate the lower and upper topological pressure

$$\underline{P}_\lambda(t) = \liminf_{n\to\infty} \frac{1}{n} \log \rho_{\lambda,t}^n \leq \limsup_{n\to\infty} \frac{1}{n} \log \rho_{\lambda,t}^n = \overline{P}_\lambda(t), \tag{5.14}$$

where we used the already introduced notation $\rho_{\lambda,t}^n = \rho_{\lambda,t}\rho_{\sigma(\lambda),t} \dots \rho_{\sigma^{n-1}(\lambda),t}$. Notice that these definitions do not dependent on the choice of conformal measures because of Lemma 5.7.

Since we have good estimations (Lemma 5.6) for the iterated operator $\mathcal{L}_{\lambda,t}^n$ we also have the following expression for the pressures.

$$\underline{P}_\lambda(t) = \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{\lambda,t}^n 1\!\!1(w_n) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{\lambda,t}^n 1\!\!1(w_n) = \overline{P}_\lambda(t) \tag{5.15}$$

for any arbitrary choice of points $w_n \in \mathcal{J}_{\sigma^n(\lambda)}$.

The pressures, seen as functions of $t$, have the following properties.

**Proposition 5.8.** $\underline{P}_\lambda(0) = \overline{P}_\lambda(0) = \log(\deg(f_\lambda))$ *and both pressures are continuous and strictly decreasing. More precisely, if* $0 \leq t_1 < t_2$*, then*

$$-(t_2 - t_1) \log A \leq \underline{P}_\lambda(t_2) - \underline{P}_\lambda(t_1) \leq -(t_2 - t_1) \log \gamma \tag{5.16}$$

*and the same relation is true for the upper pressure* $\overline{P}_\lambda$.

*Proof.* The statement about the evaluation of the pressures at zero is clear directly from (5.15). For the remaining part, in fact the proof of (5.16), we consider $t \mapsto \underline{P}_\lambda(t)$, the case of the upper pressure function is analogous.

Let $0 \leq t_1 < t_2$ and set $p_i = \underline{P}_\lambda(t_i)$, $i = 1, 2$. If $m_{\lambda,t_i}$ is a $t_i$-conformal measure then Lemma 5.4 yields that for every $z \in \mathcal{J}_\lambda$ and $n \geq 1$

$$m_{\lambda,t_i}(D(z,r)) \asymp r^{t_i} \rho_{\lambda,t_i}^{-n} \quad \text{where} \quad r = |(f_\lambda^n)'(z)|^{-1}.$$

The expanding property implies $r \preceq \gamma^{-n}$. Therefore,

$$m_{\lambda,t_2}(D(z,r)) \asymp r^{t_2 - t_1} \frac{\rho_{\lambda,t_1}^n}{\rho_{\lambda,t_2}^n} m_{\lambda,t_1}(D(z,r)) \preceq \gamma^{-(t_2-t_1)n} \frac{\rho_{\lambda,t_1}^n}{\rho_{\lambda,t_2}^n} m_{\lambda,t_1}(D(z,r)).$$

Choose now a sequence $n_j \to \infty$ such that $\frac{1}{n_j} \log \rho_{\lambda,t_1}^{n_j} \to \underline{P}_\lambda(t_1) = p_1$. Then, for every $\varepsilon > 0$,

$$\rho_{\lambda,t_1}^{n_j} \leq e^{n_j(p_1+\varepsilon)} \quad \text{and} \quad \rho_{\lambda,t_2}^{n_j} \geq e^{n_j(p_2-\varepsilon)}$$

provided $j$ is sufficiently large. For such $j$ and with $r_j = |(f_\lambda^{n_j})'(z)|^{-1}$ we get

$$\frac{m_{\lambda,t_2}(D(z,r_j))}{m_{\lambda,t_1}(D(z,r_j))} \preceq \exp\Big\{ n_j\big(p_1 - (t_2 - t_1)\log\gamma - p_2 + 2\varepsilon\big)\Big\}.$$

If $p_2 > p_1 - (t_2 - t_1)\log\gamma$ then there is $\varepsilon > 0$ sufficiently small such that for some sequence $r_j \to 0$ we get $\lim_{j\to\infty} \frac{m_{\lambda,t_2}(D(z,r_j))}{m_{\lambda,t_1}(D(z,r_j))} = 0$. This holds for every $z \in \mathcal{J}_\lambda$. Therefore it would follow from Besicovitch's covering theorem that $m_{t_2}(\mathcal{J}_\lambda) = 0$, a contradiction. Therefore, $p_2 \leq p_1 - (t_2 - t_1)\log\gamma$.

The second inequality can be proven in the same way replacing the estimation $r \preceq \gamma^{-n}$ by

$$r = |(f_\lambda^n)'(z)|^{-1} \geq A^{-n}. \qquad \square$$

**5.3. Dimensions.** Given the properties of the pressure functions in Proposition 5.8, there are uniquely defined zeros $\underline{h}_\lambda$ and $\overline{h}_\lambda$ of $\underline{P}_\lambda$ and $\overline{P}_\lambda$ respectively. With these numbers we get the following formula of Bowen's type.

**Theorem 5.9.** *$\underline{h}_\lambda = HD(\mathcal{J}_\lambda)$ and $\overline{h}_\lambda = PD(\mathcal{J}_\lambda)$.*

*Proof.* Given Lemma 5.4, the expression (5.14) of the pressures along with the properties of the pressure functions (Proposition 5.8) the proof of the theorem is by now a standard application of the Frostman type lemma Theorem 7.6.1 in [PU]. For more details we refer the reader to Chapter 7 of [PU] or to the, parallel but technically more involved, proof of Theorem 5.2 in [MUS11]. □

## 6. Irregularity of pressure and dimensions

Considering a particular family of quadratic polynomials greater in detail, we now establish that the Hölder-continuity of dimensions obtained in Theorem 1.4 is almost best possible, i.e. we prove Theorem 1.5. The key point is to show non-differentiability of the pressure functions. As a byproduct we get that generically there is a gap between the Hausdorff and the packing dimension as described in Theorem 1.6. We recall that these results concern the family of functions

$$\mathcal{F} = \Big\{ f_l(z) = l/2(z^2-1)+1,\ l \in \Lambda_0 \Big\} \quad \text{where } \Lambda_0 = \{|l| > 40\}. \tag{6.1}$$

Note that for $f_l \in \mathcal{F}$ we have $f_l'(z) = lz$. The inverse branches of $f_l$ have the form

$$f_l^{-1}(w) = \pm\sqrt{1 + \frac{2(w-1)}{l}}.$$

Let

$$U_0 = \{z \in \mathbf{C}\colon |z-1| < 1/3\} \text{ and } U_1 = \{z \in \mathbf{C}\colon |z+1| < 1/3\}$$

and denote $U := U_0 \cup U_1$. A simple calculation shows that $f_l(U_i) \supset \mathbf{D}(0,2)$ and that moreover $f_\lambda^{-1}(\overline{U}) \subset U$ for every $i = 0,1$ and $\lambda \in \Lambda = \Lambda_0^{\mathbf{N}}$. Consequently, the Julia set $\mathcal{J}_\lambda$ is a Cantor set

$$\mathcal{J}_\lambda = \bigcap_{n=0}^{\infty} f_\lambda^{-n}(U) \subset U \tag{6.2}$$

and all critical orbits of $(f_\lambda^n)_n$, $l \in \Lambda_0^{\mathbf{N}}$, do not intersect the set $U$. This last property means that every $\lambda \in l^\infty(\Lambda_0)$ gives rise to a uniformly hyperbolic map and that, in particular, $\lambda$ is a stable parameter. Let, in the following, $\Lambda = l^\infty(\Lambda_0)$. We have $\Lambda = \Lambda^{\mathrm{uHyp}} = \Lambda^{\mathrm{stable}}$.

Let $\eta \in \Lambda$ and let $\{h_{\sigma^n(\lambda)}\}_n$ be a family of holomorphic motions over $V$ neighborhood of $\eta$ such that (1.1) holds. We first investigate the speed of these motions.

**Lemma 6.1.** *Let $\eta \in \Lambda$ and let $V_\eta$ and $\{h_{\sigma^n(\lambda)}\}_n$ be as above. Then, with $\Delta = \sup_{k\geq 1} \frac{|\lambda_k - \eta_k|}{|\eta_k|}$,*

$$e^{-\Delta/6} \leq \frac{|h_{\sigma^n(\lambda)}(z)|}{|z|} \leq e^{\Delta/6} \quad \textit{for every } z \in \mathcal{J}_{\sigma^n(\eta)} \textit{ and } n \geq 0.$$

*Proof.* We give a proof for the case $n = 0$, the general case follows exactly in the same way. Since $z, h_\lambda(z) \in U$, a simple calculation shows that it is sufficient to establish

$$|z - h_\lambda(z)| \le \frac{\Delta}{9} \quad \text{for every } z \in \mathcal{J}_\eta. \tag{6.3}$$

For the sake of proving this inequality we recall that the holomorphic motions are first constructed on the set $\mathcal{E}_\eta$ defined in (3.5) and that the Julia set $\mathcal{J}_\eta$ is in the closure of $\mathcal{E}_\eta$. Consequently, it suffices to establish (6.3) for all points $z \in \mathcal{E}_\eta$.

For points $z \in \mathcal{E}_\eta$ the holomorphic motion $h_\lambda$ is given by

$$h_\lambda(z) = f_\lambda^{-n}(\alpha_i^n) \tag{6.4}$$

for some $i \in \{1, 2, 3\}$ and $n \ge 0$ and where $f_\lambda^{-n}$ is a certain inverse branch of $f_\lambda^n$ which has been determined by the implicit function theorem in (3.8). Therefore, we now consider in detail the behavior of these inverse branches under variation of the parameter $\lambda$.

Fix $i \in \{0, 1\}$ and $k \ge 1$ and consider inverse branches $f_{\lambda_k}^{-1}$, $f_{\eta_k}^{-1}$ both sending the euclidean disk $\mathbf{D}(0, 2)$ into $U_i$. Our first step is to show that for every $w_1, w_2 \in U_i$ with $|w_1 - w_2| \le \frac{\Delta}{9}$ we have

$$|f_{\lambda_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_2)| \le \frac{\Delta}{9}. \tag{6.5}$$

Since

$$|f_{\lambda_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_2)| \le |f_{\lambda_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_1)| + |f_{\eta_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_2)|, \tag{6.6}$$

it suffices to estimate separately these two terms. Concerning the first one, observe that

$$\Big|\frac{df_l^{-1}(w)}{dl}\Big| = \frac{|w-1|}{\Big|\sqrt{1 + \frac{2(w-1)}{l}}\Big|} \frac{1}{|l^2|} \le \frac{3}{|l|^2} \tag{6.7}$$

for all $w \in U$ and $|l| \ge 40$. It follows that for $|\lambda_k - \eta_k| < 1$, $\lambda_k, \eta_k \in \Lambda_0$,

$$|f_{\lambda_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_1)| \le \frac{3}{(|\eta_k| - 1)^2} |\lambda_k - \eta_k| \le \frac{3}{(|\eta_k| - 1)} \frac{40}{39} \Delta.$$

Concerning the second term, we have that

$$|f_{\eta_k}^{-1}(w_1) - f_{\eta_k}^{-1}(w_2)| \le \frac{|w_1 - w_2|}{\sqrt{5/6}(|\eta_k| - 1)} \le \frac{1}{(|\eta_k| - 1)} \Delta.$$

Adding both estimations and using again that $|\eta_k| - 1 \ge 39$ we obtain (6.5).

It suffices now to proceed by induction and to get, with the notation of (6.4), that

$$|h_\lambda(z) - z| = |f_\lambda^{-n}(\alpha_i^n) - f_\eta^{-n}(\alpha_i^n)| \le \frac{\Delta}{9}. \qquad \square$$

Having analyzed the speed of holomorphic motions we now use this tool in order to study the variation of the lower and upper pressure $\underline{P}_\lambda(t)$, $\overline{P}_\lambda(t)$ defined in (5.15). In order to do so, fix $\eta \in \Lambda$. We will choose later on for every $t > 0$ an element $(s_0, s_1 \dots) \in \{-1, 1\}^{\mathbf{N}}$ and consider, for $x \in (-r, r)$, the parameter $\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots)$ defined by

$$\lambda_k(x) = e^{x s_k} \eta_k, \quad k \ge 1.$$

Since $\eta \in \Lambda = l^\infty(\Lambda_0)$, there is a number $r \in (0,1]$ such that $\lambda(x) \in \Lambda$ for all $x \in (-r,r)$. Moreover, the map $x \mapsto \lambda(x)$ is differentiable from $(-r,r)$ into $\Lambda$. Clearly, $\lambda(0) = \eta$. Hence, for every $t > 0$, we consider a particular choice of perturbation of $f_\eta \in \mathcal{F}$.

**Proposition 6.2.** *For every $t > 0$ there is a choice of numbers $s_j = s_j(t) \in \{-1,1\}$ such that, with the preceding notation, we have for every $x \in (-r,r)$*

$$\overline{P}_{\lambda(x)}(t) \geq \overline{P}_\eta(t) + \frac{t}{2}|x| \tag{6.8}$$

*and*

$$\underline{P}_{\lambda(x)}(t) \leq \underline{P}_\eta(t) - \frac{t}{2}|x|. \tag{6.9}$$

*In particular, the functions $\lambda \mapsto \underline{P}_\lambda(t)$ and $\lambda \mapsto \overline{P}_\lambda(t)$ are not differentiable at any point $\eta \in \Lambda$.*

*Proof.* The particular choice of the functions in the family $\mathcal{F}$ leads to the following expressions. First of all, for every $n \geq 1$,

$$\left(f_\eta^n\right)'(z) = \prod_{k=1}^n \eta_k f_\eta^{k-1}(z).$$

Now, using again holomorphic stability and the notation $z_x = h_{\lambda(x)}(z)$, $z \in \mathcal{J}_\eta$, we also have that

$$\left(f_{\lambda(x)}^n\right)'(z_x) = \prod_{k=1}^n \lambda_k(x) f_{\lambda(x)}^{k-1}(z_x) = \prod_{k=1}^n e^{xs_k}\eta_k h_{\sigma^{k-1}(\lambda(x))} \circ f_\eta^{k-1}(z).$$

If we now apply Lemma 6.1 then we get the estimation

$$\left|(f_{\lambda(x)}^n)'(z_x)\right| \leq \prod_{k=1}^n e^{xs_k}|\eta_k| e^{\Delta/6}|f_\eta^{k-1}(z)| = e^{n\Delta/6}\left(\prod_{k=1}^n e^{xs_k}\right)\left|(f_\eta^n)'(z)\right|$$

and, similarly,

$$\left|(f_{\lambda(x)}^n)'(z_x)\right| \geq e^{-n\Delta/6}\left(\prod_{k=1}^n e^{xs_k}\right)\left|(f_\eta^n)'(z)\right| \quad \text{for every } z \in \mathcal{J}_\eta.$$

For the particular perturbation we have chosen we have

$$\Delta = \Delta(x) = \sup_{k\geq 1}\frac{|\lambda_k(x) - \eta_k|}{|\eta_k|} = \sup_{k\geq 1}\left|e^{s_k x} - 1\right| \leq e \sup_{k\geq 1}|s_k x| = e|x|.$$

Replacing $\Delta$ by this estimation in the preceding inequalities leads to

$$e^{-tn|x|/2}\left(\prod_{k=1}^n e^{-txs_k}\right)\left|(f_\eta^n)'(z)\right|^{-t} \leq \left|(f_{\lambda(x)}^n)'(z_x)\right|^{-t} \leq e^{tn|x|/2}\left(\prod_{k=1}^n e^{-txs_k}\right)\left|(f_\eta^n)'(z)\right|^{-t}$$

for every $z \in \mathcal{J}_\eta$ and $t > 0$.

The operators $\mathcal{L}_{\lambda,t}$ have been defined in (5.2). The previous inequality yields

$$\begin{aligned} e^{-tn|x|/2}\left(\prod_{k=1}^{n} e^{-txs_k}\right)\mathcal{L}^n_{\eta,t}1\!\!1(w) &\le \mathcal{L}^n_{\lambda(x),t}1\!\!1(w_x) \\ &\le e^{tn|x|/2}\left(\prod_{k=1}^{n} e^{-txs_k}\right)\mathcal{L}^n_{\eta,t}1\!\!1(w) \end{aligned} \tag{6.10}$$

for every $n \ge 0$, $w \in \mathcal{J}_{\sigma^n(\eta)}$ and with $w_x = h_{\sigma^n(\lambda(x))}(w)$. Avoiding long notation, we have just shown this inequality for the first fiber. But it is clear that one can replace here the parameters $\eta$ and $\lambda(x)$ by their images by $\sigma^j$, $j \ge 1$, and one still has the corresponding estimation.

We can now study the behavior of the pressures. Let us recall that we have the expression (5.15) of $\underline{P}_\lambda(t)$ and of $\overline{P}_\lambda(t)$ in terms of the iterated operators $\mathcal{L}^n_{\lambda,t}1\!\!1$. Inequality (6.10) implies that, for all $x \in (-r,r)$ and $t > 0$,

$$\begin{aligned} -t\frac{|x|}{2} + \frac{1}{n}\log \mathcal{L}^n_{\lambda(x),t}1\!\!1(w_x) &\le \frac{1}{n}\log \mathcal{L}^n_{\eta,t}1\!\!1(w) - t\frac{x}{n}\sum_{k=1}^{n} s_k \\ &\le t\frac{|x|}{2} + \frac{1}{n}\log \mathcal{L}^n_{\lambda(x),t}1\!\!1(w_x). \end{aligned} \tag{6.11}$$

For the conclusion of the proof let $t > 0$ again be fixed. There are a sequence $n_j \to \infty$ such that $\overline{P}_\eta(t) = \lim_{j\to\infty} \frac{1}{n_j}\log \mathcal{L}^{n_j}_{\eta,t}1\!\!1(w_{n_j})$ and a sequence $m_j \to \infty$ such that $\underline{P}_\eta(t) = \lim_{j\to\infty} \frac{1}{m_j}\log \mathcal{L}^{m_j}_{\eta,t}1\!\!1(w_{m_j})$ Choose now numbers $s_k = s_k(t) \in \{-1,1\}$ such that

$$\liminf_j \frac{1}{n_j}\sum_{k=1}^{n_j} s_k = -1 \quad \text{and} \quad \limsup_j \frac{1}{n_j}\sum_{k=1}^{n_j} s_k = 1.$$

There is then a sequence $n_j \to \infty$ such that $\overline{P}_\eta(t) = \lim_{j\to\infty} \frac{1}{n_j}\log \mathcal{L}^{n_j}_{\eta,t}1\!\!1(w_{n_j})$. Choose now the numbers $s_k = s_k(t) \in \{-1,1\}$ such that

$$\begin{aligned} &\liminf_j \frac{1}{n_j}\sum_{k=1}^{n_j} s_k = -1, \quad \limsup_j \frac{1}{n_j}\sum_{k=1}^{n_j} s_k = 1, \\ &\liminf_j \frac{1}{m_j}\sum_{k=1}^{m_j} s_k = -1 \quad \text{and} \quad \limsup_j \frac{1}{m_j}\sum_{k=1}^{m_j} s_k = 1. \end{aligned}$$

This choice makes that either $\limsup_j -t\frac{x}{n}\sum_{k=1}^{n_j} s_k = t|x|$ or $\liminf_j -t\frac{x}{n}\sum_{k=1}^{n_j} s_k = t|x|$. It follows now from (6.11) that

$$\overline{P}_\eta(t) + \frac{t}{2}|x| \le \overline{P}_{\lambda(x)}(t)$$

which is exactly (6.8). Inequality (6.9) follows in the same way and they both together imply that the pressures are not differentiable at $\eta$. □

*Proof of Theorem 1.5.* We first consider Hausdorff dimension. Let $\underline{h}_\eta > 0$ be the unique zero of $t \mapsto \underline{P}_\eta(t)$ and suppose that the $s_k \in \{-1,1\}$ in Proposition 6.2 are chosen for $t = \underline{h}_\eta$. It follows then from (6.9) in Proposition 6.2 that

$$\underline{P}_{\lambda(x)}(\underline{h}_\eta) \le \underline{P}_\eta(\underline{h}_\eta) - \frac{\underline{h}_\eta}{2}|x| = -\frac{\underline{h}_\eta}{2}|x| < 0.$$

We look for $\underline{h}_x$ zero of $t \mapsto \underline{P}_{\lambda(x)}(t)$ since, by Theorem 5.9, this number equals the Hausdorff dimension of $\mathcal{J}_{\lambda(x)}$. The pressures being stricly decreasing, $\underline{h}_x < \underline{h}_\eta$. Therefore, Proposition 5.8 yields

$$0 = \underline{P}_{\lambda(x)}(\underline{h}_x) \le P_{\lambda(x)}(\underline{h}_\eta) + (\underline{h}_\eta - \underline{h}_x)\log A \le -\frac{\underline{h}_\eta}{2}|x| + (\underline{h}_\eta - \underline{h}_x)\log A$$

from which follows that

$$\underline{h}_x \le \underline{h}_\eta\Big(1 - \frac{|x|}{2\log A}\Big). \tag{6.12}$$

Therefore, $x \mapsto \underline{h}_x = \mathrm{HD}(\mathcal{J}_{\lambda(x)})$ is not differentiable.

Similarly to (6.12) one obtains, with obvious notations,

$$\overline{h}_x \ge \overline{h}_\eta\Big(1 + \frac{|x|}{2\log A}\Big) \tag{6.13}$$

and the non-differentiability of the packing dimension follows. □

*Proof of Theorem 1.6.* In any family $\mathcal{F}$ the set $\Omega = \{\lambda \in \Lambda,\ \mathrm{HD}(\mathcal{J}_\lambda) < \mathrm{PD}(\mathcal{J}_\lambda)\}$ is open in $l^\infty(\Lambda)$ because of Theorem 1.4.

Density of $\Omega$ for the particular quadratic family of this section can be shown as follows. If $\eta \in \Lambda \setminus \Omega$ then it follows immediately from (6.12) and (6.13) together with Bowen's formula (Theorem 5.9) that there are arbitrarily small perturbations of $\eta$ that are in $\Omega$. □

## References

[Bea91] Beardon, A. F.: Iteration of rational functions. - Grad. Texts in Math. 132, Springer-Verlag, New York, 1991.

[Bow79] Bowen, R.: Hausdorff dimension of quasicircles. - Inst. Hautes Études Sci. Publ. Math. 50, 1979, 11–25.

[Brü00] Brück, R.: Connectedness and stability of Julia sets of the composition of polynomials of the form $z^2 + c_n$. - J. London Math. Soc. (2) 61:2, 2000, 462–470.

[Brü01] Brück, R.: Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$. - Pacific J. Math. 198:2, 2001, 347–372.

[Büg97] Büger, M.: Self-similarity of julia sets of the composition of polynomials. - Ergodic Theory Dynam. Systems, 17, 1997, 1298–1297.

[Com03] Comerford, M.: Conjugacy and counterexample in random iteration. - Pacific J. Math. 211:1, 2003, 69–80.

[Com06] Comerford, M.: Hyperbolic non-autonomous Julia sets. - Ergodic Theory Dynam. Systems 26:2, 2006, 353–377.

[Com08] Comerford, M.: Holomorphic motions of hyperbolic nonautonomous Julia sets. - Complex Var. Elliptic Equ. 53:1, 2008, 1–22.

[DU91] Denker, M., and M. Urbański: On the existence of conformal measures. - Trans. Amer. Math. Soc. 328:2, 1991, 563–587.

[FS91] Fornæss, J. E., and N. Sibony: Random iterations of rational functions. - Ergodic Theory Dynam. Systems 11:4, 1991, 687–708.

[Hub76] Hubbard, J. H.: Sur les sections analytiques de la courbe universelle de teichmüller. - Mem. Amer. Math. Soc. 166, 1976, 1–137.

[JM07] Jiang, Y., and S. Mitra: Some applications of universal holomorphic motions. - Kodai Math. J. 30:1, 2007, 85–96.

[Jon99] Jonsson, M.: Dynamics of polynomial skew products on $\mathbf{C}^2$. - Math. Ann. 314:3, 1999, 403–447.

[Jon00] Jonsson, M.: Ergodic properties of fibered rational maps. - Ark. Mat. 38:2, 2000, 281–317.

[Lev81] Levin, G. M.: Irregular values of the parameter of a family of polynomial mappings. - Uspekhi Mat. Nauk 36:6(222), 1981, 219–220.

[Lyu86] Lyubich, M. Yu.: Dynamics of rational transformations: topological picture. - Uspekhi Mat. Nauk 41:4(250), 1986, 35–95, 239.

[Mit00] Mitra, S.: Teichmüller spaces and holomorphic motions. - J. Anal. Math. 81, 2000, 1–33.

[MSS83] Mañé, R., P. Sad, and D. Sullivan: On the dynamics of rational maps. - Ann. Sci. Éc. Norm. Supér. (4) 16:2, 1983, 193–217.

[MUS11] Mayer, V., M. Urbański, and B. Skorułski: Distance expanding random mappings, thermodynamic formalism, Gibbs measures, and fractal geometry. - Lecture Notes in Math. 2036, Springer, New York, 2011.

[PU] Przytycki, F., and M. Urbański: Conformal fractals – Ergodic theory methods. - Cambridge Univ. Press (to appear), available on www.math.unt.edu/∼urbanski.

[Rue82] Ruelle, D.: Repellers for real analytic maps. - Ergodic Theory Dynam. Systems 2:1, 1982, 99–107.

[Rug] Rugh, H. H.: On the dimension of conformal repellors. Randomness and parameter dependency. - Ann. of Math. (2) 168:3, 2008, 695–748.

[Ses99] Sester, O.: Hyperbolicité des polynomes fibrés. - Bull. Soc. Math. France 127:3, 1999, 393–428.

[Slo95] Slodkowski, Z.: Extensions of holomorphic motions. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 22:2, 1995, 185–210.

[Sum97] Sumi, H.: On dynamics of hyperbolic rational semigroups. - J. Math. Kyoto Univ. 37:4, 1997, 717–733.

[Sum01] Sumi, H.: Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products. - Ergodic Theory Dynam. Systems 21:2, 2001, 563–603.

[Sum06] Sumi, H.: Semi-hyperbolic fibered rational maps and rational semigroups. - Ergodic Theory Dynam. Systems 26:3, 2006, 893–922.

[Sum10a] Sumi, H.: Cooperation principle, stability and bifurcation in random complex dynamics. - 2010.

[Sum10b] Sumi, H.: Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles. - Ergodic Theory Dynam. Systems 30:6, 2010, 1869–1902.

[Sum11] Sumi, H.: Random complex dynamics and semigroups of holomorphic maps. - Proc. Lond. Math. Soc. (3) 102:1, 2011, 50–112.

Received 23 March 2012 • Accepted 21 September 2012