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THE EXISTENCE OF INFINITELY MANY SOLUTIONS FOR p-LAPLACIAN TYPE EQUATIONS ON $\mathbf{R}^N$ WITH LINKING GEOMETRY

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Abstract. In this paper, we study the existence of infinitely many solutions to the following quasilinear equation of p-Laplacian type in $\mathbf{R}^N$

$$-\triangle_p u + |u|^{p-2}u = \lambda V(x)|u|^{p-2}u + g(x,u), \quad u \in W^{1,p}(\mathbf{R}^N) \tag{0.1}$$

with sign-changing radially symmetric potential $V(x)$, where $1 < p < N$, $\lambda \in \mathbf{R}$ and $\triangle_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian operator, $g(x,u) \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ is subcritical and p-superlinear at 0 as well as at infinity. We prove that under certain assumptions on the potential V and the nonlinearity g, for any $\lambda \in \mathbf{R}$, the problem (0.1) has infinitely many solutions by using a fountain theorem over cones under Cerami condition. A minimax approach, allowing an estimate of the corresponding critical level, is used. New linking structures, associated to certain variational eigenvalues of $-\triangle_p u + |u|^{p-2}u = \lambda V(x)|u|^{p-2}u$ are recognized, even in absence of any direct sum decomposition of $W^{1,p}(\mathbf{R}^N)$ related to the eigenvalue itself.

Our main result can be viewed as an extension to a recent result of Degiovanni and Lancelotti in [10] concerning the existence of nontrivial solutions for the quasilinear elliptic problem:

$$\begin{cases} -\triangle_p u = \lambda V(x)|u|^{p-2}u + g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open domain.

1. Introduction and main result

In this paper, we study the existence of infinitely many solutions to the following nonlinear elliptic equation of p-Laplacian type in the entire space

$$\begin{cases} -\triangle_p u + |u|^{p-2}u = \lambda V(x)|u|^{p-2}u + g(x,u), & x \in \mathbf{R}^N, \\ u \in W^{1,p}(\mathbf{R}^N), \end{cases} \tag{1.1}$$

where $\lambda \in \mathbf{R}$, $1 < p < N$, $\triangle_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian operator. We assume that the potential $V(x)\colon \mathbf{R}^N \to \mathbf{R}$ satisfies the following condition:

(H) $V \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $V^+ \neq 0$ and V is radially symmetric with respect to x.

The nonlinearity $g \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ satisfies the following conditions:

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(g_1) There exists a constant $C > 0$ such that

$$|g(x,t)| \leq C(1+|t|^{q-1}) \text{ for all } (x,t) \in \mathbf{R}^N \times \mathbf{R},$$

where $p < q < p^* = \frac{Np}{N-p}$ and g is radially symmetric with respect to x;

(g_2) $\lim\limits_{|t|\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$ uniformly in $x \in \mathbf{R}^N$;

(g_3) $\lim\limits_{|t|\to +\infty} \frac{G(x,t)}{|t|^p} = +\infty$ uniformly in $x \in \mathbf{R}^N$, where $G(x,t) = \int_0^t g(x,s)ds$;

(g_4) For $\widetilde{G}(x,t) = \frac{1}{p}g(x,t)t - G(x,t)$, we have $\widetilde{G}(x,t) > 0$ for any $x \in \mathbf{R}^N,\ t \neq 0$ and there exist $C > 0$, $M > 0$, $\theta > \frac{N}{p}$ such that for any $x \in \mathbf{R}^N$, $|t| \geq M$,

$$|g(x,t)|^\theta \leq C\widetilde{G}(x,t)|t|^{\theta(p-1)};$$

(g_5) $g(x,-t) = -g(x,t)$ for any $x \in \mathbf{R}^N$, $t \in \mathbf{R}$.

We call $u \in W^{1,p}(\mathbf{R}^N)$ a weak solution to (1.1) if

$$\begin{aligned}&\int_{\mathbf{R}^N} (|Du|^{p-2}Du\, Dv + |u|^{p-2}uv)\, dx - \lambda \int_{\mathbf{R}^N} V(x)|u|^{p-2}uv\, dx \\ &= \int_{\mathbf{R}^N} g(x,u)v\, dx,\ \forall v \in W^{1,p}(\mathbf{R}^N).\end{aligned}$$

By conditions (g_1) and (g_2), we deduce that there exists a constant $C > 0$ such that

$$|g(x,t)| \leq C(|t|^{p-1} + |t|^{q-1}), \tag{1.2}$$

where $p < q < p^*$. Hence $u = 0$ is a trivial solution to (1.1). We are interested in the existence of multiple nontrivial solutions of (1.1).

Since Ambrosetti and Rabinowitz proposed the mountain-pass theorem in 1973 (see [1]), critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. Clearly, weak solutions to (1.1) correspond to critical points of the related variational functional

$$I(u) = \frac{1}{p}\int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx - \frac{\lambda}{p}\int_{\mathbf{R}^N} V(x)|u|^p\, dx - \int_{\mathbf{R}^N} G(x,u)\, dx \tag{1.3}$$

defined on $W^{1,p}(\mathbf{R}^N)$. By (1.2), $I \in C^1(W^{1,p}(\mathbf{R}^N), \mathbf{R})$.

Note that for any bounded open subset $\Omega \subset \mathbf{R}^N$, the following p-Laplacian type problem

$$\begin{cases} -\triangle_p u + a(x)|u|^{p-2}u = g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.4}$$

has been studied extensively in the past decades. The corresponding energy functional to (1.4) is

$$\Phi(u) = \frac{1}{p}\int_\Omega (|Du|^p + a(x)|u|^p)\, dx - \int_\Omega G(x,u)\, dx,$$

where $G(x,t) = \int_0^t g(x,s)\, ds$.

If $a(x) \geq 0$, and $g(x,t)$ is subcritical and p-superlinear at 0 and at infinity, then the functional Φ possesses a mountain-pass geometric structure around $u = 0$. The existence of a nontrivial solution to (1.4) can be obtained by using the mountain-pass theorem and the existence of infinitely many solutions, when $g(x,t)$ is odd in t, can be obtained by the so-called fountain theorem, see e.g. [1, 8, 12, 22, 27, 38].

If $a(x)$ is sign-changing, when $p=2$, the existence result can be obtained if the energy functional possesses a linking geometric structure, see [21, 33, 38]. However, such an existence result relies on a linking theorem on Hilbert spaces, which is based on the fact that each eigenvalue of $-\triangle$ induces a suitable direct sum decomposition of $W_0^{1,2}(\Omega)$. If $p \neq 2$, the p-Laplacian operator $-\Delta_p$ is no longer a linear operator and the properties of the set $\sigma(-\Delta_p)$ of the eigenvalues of $-\Delta_p$ are not clear. The existence of a first eigenvalue $\lambda_1 = \min \sigma(-\Delta_p) > 0$ and a second eigenvalue $\lambda_2 > \lambda_1$ was proved and several equivalent variational characterizations of λ_1 and λ_2 were studied (see [2, 3, 9, 14, 23, 24]). There are at least three different variational ways to define a diverging sequence $\{\lambda_n\} \subset \sigma(-\Delta_p)$ (see [6, 14, 31, 32]). However, one does not know if these definitions are equivalent for $n \geq 3$ or not (see [6, 10]). Also, a direct sum decomposition of the space $W_0^{1,p}(\Omega)$ according to the eigenvalues of $-\Delta_p$ are not expected as one always does when $p=2$. In [37], Szulkin and Willem proved that the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda V(x)|u|^{p-2}u, \quad u \in D_0^{1,p}(\Omega), \tag{1.5}$$

has a sequence of eigenvalues $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$, where $1 < p < N$ and Ω is an open, in general unbounded subset of $\mathbf{R}^N$ and V satisfies the following assumptions:

$$\begin{cases} V \in L^1_{\mathrm{loc}}(\Omega),\ V^+ = V_1 + V_2 \neq 0,\ V_1 \in L^{\frac{N}{p}}(\Omega), \\ \lim\limits_{\substack{x \to y \\ x \in \Omega}} |x-y|^p V_2(x) = 0 \ \text{ for each } y \in \bar{\Omega},\ \lim\limits_{\substack{|x| \to \infty \\ x \in \Omega}} |x|^p V_2(x) = 0. \end{cases}$$

In [18], Frigon introduced a new notion of linking, which includes many notions of linking, such as homotopically linking, homologically linking, etc. In considering continuous functionals, Frigon stated a deformation property in an abstract setting. Then, with the new notion of linking, minimax critical point theorems for continuous functionals on metric spaces were presented (Theorem 3.1 in [18]). After the publication of [18], there are many papers on problem (1.4) for the sign-changing potential case. In [11], Degiovanni and Lancelotti considered problem (1.4) under the case $a(x) = -\lambda$ and $g(x,u) = |u|^{p^*-2}u$, where $1 < p < N$, $p^* = \frac{Np}{N-p}$ and obtained that for every $\lambda \geq \lambda_1$, (1.4) has a nontrivial solution if $\frac{N^2}{N+1} > p^2$ or $\frac{N^3+p^3}{N^2+N} > p^2$ with certain smoothness assumption on the boundary of Ω. In [10], they also studied problem (1.4) with $a(x) = -\lambda V(x)$, where $V \in L^\infty(\Omega)$, $1 < p < N$, $\Omega \subset \mathbf{R}^N$ is a bounded open domain with smooth boundary. They proved that for any $\lambda \in \mathbf{R}$, (1.4) admits a nontrivial weak solution if g is subcritical, p-superlinear at 0 and satisfies the Ambrosetti-Rabinowitz condition ((AR) in short):

(AR) There exist $\mu > p$ and $R > 0$ such that for all $x \in \Omega$,

$$|t| \geq R \implies 0 < \mu G(x,t) \leq t g(x,t).$$

In [39], Yan and Yang established a fountain theorem over cones under Palais–Smale (PS) condition and its dual version (Theorem 1.1 and Theorem 1.2 in [39]).

By applying these two theorems to the quasilinear elliptic problem

$$
\begin{cases} -\triangle_p u = \lambda |u|^{q-2}u + \mu |u|^{\gamma-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}
$$

where $1 < p < N$, $1 < q < p < \gamma < p^*$, Ω is a smooth bounded domain and $\lambda, \mu \in \mathbf{R}$, they showed that problem (1.6) possesses a sequence of positive energy solutions for each $\mu > 0$, $\lambda \in \mathbf{R}$ and (1.6) has a sequence of negative energy solutions for each $\lambda > 0$, $\mu \in \mathbf{R}$.

Recently, Liu and Zheng in [25] studied the following p-Laplacian equation in $\mathbf{R}^N$

$$
\begin{cases} -\triangle_p u + b(x)|u|^{p-2}u = \lambda V(x)|u|^{p-2}u + f(x,u), & x \in \mathbf{R}^N, \\ u \in W^{1,p}(\mathbf{R}^N), \end{cases} \tag{1.7}
$$

where $p > 1$, $\lambda \in \mathbf{R}$, $V \in L^\infty(\mathbf{R}^N)$ and $b(x) \in C(\mathbf{R}^N, \mathbf{R})$ satisfies the following condition:

(B) $\inf\limits_{x \in \mathbf{R}^N} b(x) \geq b_0 > 0$, $\operatorname{meas}(\{x \in \mathbf{R}^N |\ b(x) \leq M\}) < \infty\ \forall M > 0$,

and $f(x,t)$ is of subcritical growth, p-superlinear at 0 and at infinity satisfying

(F) $\exists\ \theta \geq 1$ such that $\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st)\ \forall\ (x,t) \in \mathbf{R}^N \times \mathbf{R}$ and $s \in [0,1]$,

where $\mathcal{F}(x,t) = f(x,t)t - pF(x,t)$, $F(x,t) = \int_0^t f(x,s)\,ds$, $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbf{R}^N$. The condition (F) was first introduced in [19] for $p = 2$ and in [26] for gengeral p. By using a linking theorem over cones on a weighted Sobolev space $\mathcal{W} = \{u \in W^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N}(|Du|^p + b(x)|u|^p)\,dx < \infty\}$, in which Sobolev embeddings $\mathcal{W} \hookrightarrow L^q(\mathbf{R}^N)$, $p \leq q < p^*$ are compact since (B) holds, they proved that problem (1.7) has a nontrivial solution for each $\lambda \in \mathbf{R}$.

Motivated by works just described, more precisely by results founded in [10] and [39], a natural question is whether the same phenomenon of existence and multiplicity holds or not when we consider problem (1.1). The purpose of this paper is to study problem (1.1) without assuming the (AR) condition. Our basic assumptions on the nonlinearity are (g_1)–(g_5). Note that the condition (g_4) was introduced in [13] for $p = 2$ and it was used in [28] for Kirchhoff type problems. Here, we employ a fountain theorem over cones under Cerami condition. It is necessary to point out that, if $b(x) \equiv 1$ in (1.7), then the condition (B) does not hold, hence problem (1.1) is not a special case of (1.7).

We state our main result:

Theorem 1.1. *Suppose that (g_1)–(g_5) hold and let V satisfy (H). Then, for each $\lambda \in \mathbf{R}$, the quasilinear elliptic problem* (1.1) *has infinitely many nontrivial solutions in $W^{1,p}(\mathbf{R}^N)$.*

We do not assume that $g(x,t)$ satisfies the (AR) condition. In fact, there are functions which satisfy (g_1)–(g_5) but (AR). For example

$$
g(x,t) = |t|^{p-2}t \log(1 + |t|)
$$

satisfies (g_1)–(g_5) but $g(x,t)$ does not satisfy

$$
G(x,t) \geq C|t|^\mu \ \text{ for all } (x,t) \in \mathbf{R}^N \times \mathbf{R} \tag{1.8}
$$

for any $\mu > p$ and some $C > 0$, which is a direct consequence of (AR).

Remark 1.2. Assume that (g_1) holds. Then (AR) implies (g_3), (g_4).

Remark 1.3. There are indeed functions which satisfy our conditions (g_1)–(g_4) but the assumption (F) above. For example,

$$G(x,t) = |t|^{\mu} + (\mu - p)|t|^{\mu-\varepsilon} \sin^2\left(\frac{|t|^{\varepsilon}}{\varepsilon}\right),$$

where $p < \mu < p^*$, $0 < \varepsilon < \min\left\{\mu - p,\ \mu - \frac{\mu N}{p} + \frac{2N}{p}\right\}$.

We prove Theorem 1.1 by showing that the energy functional I possesses infinitely many critical points in $W^{1,p}(\mathbf{R}^N)$. To do so, we will try to get Cerami sequences for I and to prove that each Cerami sequence is bounded in $W^{1,p}(\mathbf{R}^N)$ and converges to a critical point of I in $W^{1,p}(\mathbf{R}^N)$, which can be distinguished to each other.

There are several difficulties. First, as $\mathbf{R}^N$ is translation invariant, the Sobolev embeddings $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$ for $q \in [p, p^*)$ are not compact, which makes the proof of the convergence of approximate critical points difficult. The assumption that $V(x)$, $g(x,t)$ are radially symmetric in x makes it possible to deal with this difficulty by using the so-called principle of symmetric criticality (see e.g. [29]) to work in the radially symmetric Sobolev space

$$W_r^{1,p}(\mathbf{R}^N) = \{u \in W^{1,p}(\mathbf{R}^N) \mid u(x) = u(|x|)\}.$$

Secondly, one usually gets a Cerami sequence by using a mountain-pass geometric structure or linking geometric structure of I. As the assumption (H) holds, I does not have the mountain-pass geometric structure around $u = 0$. We have to show that I possesses certain linking geometric structure and to establish a suitable fountain theorem over cones. According to what people usually do for problems in bounded domain, one has to deal with the eigenvalues for the p-Laplacian operator. However, for unbounded domain, it is much more complicated. We consider the following eigenvalue problem

$$-\Delta_p u + |u|^{p-2}u = \lambda V(x)|u|^{p-2}u, \quad u \in W_r^{1,p}(\mathbf{R}^N)$$

and get a divergent sequence of eigenvalues

$$\mu_m = \inf_{A \subseteq \mathcal{M}} \left\{ \max_{u \in A} \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \,\middle|\, A \text{ is compact and symmetric, } \mathrm{Index}(A) \geq m \right\},$$

where Index is the $\mathbf{Z}_2$-cohomological index described in [15, 16] and $\mathcal{M} = \{u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} V(x)|u|^p\, dx = 1\}$. Then each $\mu_m < \mu_{m+1}$ induces a generalized linking structure associated with the cones

$$C_-^m = \left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \leq \mu_m \int_{\mathbf{R}^N} V|u|^p\, dx \right\},$$

$$C_+^m = \left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \geq \mu_{m+1} \int_{\mathbf{R}^N} V|u|^p\, dx \right\}.$$

Then we use a fountain theorem under Cerami condition (see Theorem 2.8 in Section 2), a result which we have not found elsewhere, to get infinitely many Cerami sequences. The main difficulty now is to prove that each Cerami sequence $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbf{R}^N)$. To this end, we use the argument in [28] and the assumptions (g_3), (g_4). However, as we deal with problems in $\mathbf{R}^N$, the argument in [28] which deals with problems in bounded domains, needs to be improved. We succeeded in doing so by more careful analysis. Whence the boundedness of $\{u_n\}$ is proved, the result follows by using standard method.

Our paper is organized as follows. In Section 2 we give some basic definitions and some preliminary results, including a fountain theorem over cones under Cerami condition and an existence result of an eigenvalue problem for p-Laplacian type operator in $\mathbf{R}^N$. In Section 3 we prove our main result Theorem 1.1.

We use standard notations. For example, for $1 \le p < +\infty$, $\|u\|_{L^p} = (\int_{\mathbf{R}^N} |u|^p\,dx)^{\frac{1}{p}}$ denotes the usual L^p-norm. The N-dimensional Lebesgue measure of a set $E \subset \mathbf{R}^N$ is denoted by $|E|$. We use "$\rightarrow$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. C will denote a positive constant unless specified. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

2. Preliminary results

In this section, we give some preliminary results which will be used to prove our main result.

We first give some definitions and results about the linking. Let (X, d) be a metric space and H^* be Alexander–Spanier cohomology (see [34]).

Definition 2.1. Let $A \subset B \subset X$, $P \subset Q \subset X$. We say that (B, A) links (Q, P), if $A \cap Q = B \cap P = \varnothing$, and for every deformation $\eta\colon B \times [0,1] \to X \backslash P$ with $\eta(A \times [0,1]) \cap P = \varnothing$, then $\eta(B \times \{1\}) \cap Q \neq \varnothing$.

Definition 2.2. Suppose $A \subset B \subset X$, $P \subset Q \subset X$. Let m be a nonnegative integer and $\mathbf{K}$ be a field. We say that (B, A) links (Q, P) cohomologically in dimension m over $\mathbf{K}$, if $A \cap Q = B \cap P = \varnothing$, and the restriction homomorphism $H^m(X\backslash P, X\backslash Q; \mathbf{K}) \to H^m(B, A; \mathbf{K})$ is not identically zero.

In the setting of Definition 2.1 and 2.2, if $P = \varnothing$ (resp. $A = \varnothing$), we simply write Q instead of $(Q, \varnothing)$ (resp. B instead of $(B, \varnothing)$).

Proposition 2.3. (Proposition 2.4 in [10]) *If (B, A) links (Q, P) cohomologically, then (B, A) links (Q, P).*

Definition 2.4. Let A be a subset of a real normed space X. A is said to be symmetric, if $-u \in A$ whenever $u \in A$. A is said to be a cone, if $tu \in A$ whenever $u \in A$ and $t > 0$.

Proposition 2.5. (Proposition 2.1 in [39]) *Let X ba a real normed space and C_-, C_+ be two cones in X such that C_+ is closed in X, $C_- \cap C_+ = \{0\}$ and $(X, C_-\backslash\{0\})$ links C_+ cohomologically in dimension m. For $r_-, r_+ > 0$, set*

$$D_- = \{u \in C_- \mid \|u\| \le r_-\}, \quad S_- = \{u \in C_- \mid \|u\| = r_-\},$$
$$D_+ = \{u \in C_+ \mid \|u\| \le r_+\}, \quad S_+ = \{u \in C_+ \mid \|u\| = r_+\}.$$

Denote

$$\Gamma = \{\gamma \in C(D_-, X) \mid \gamma|_{S_-} = \mathrm{id}\}.$$

Then, for every $\gamma \in \Gamma$, $(\gamma(D_-), S_-)$ links S_+ cohomologically in dimension $m+1$, i.e.,

$$H^{m+1}(X, X\backslash S_+; \mathbf{K}) \to H^{m+1}(\gamma(D_-), S_-; \mathbf{K})$$

is not identically zero.

For readers' convenience, let us recall the definition and some properties of $\mathbf{Z}_2$-cohomological index (see [15, 16, 32] for details). For simplicity, we only consider the usual $\mathbf{Z}_2$-action on a linear space, i.e., $Z_2 = \{-1, 1\}$ and the action is the usual multiplication. In this case, a $\mathbf{Z}_2$-set A is just a symmetric set, which means that $-A = A$.

Let W be a normed linear space and $\mathcal{S}(W)$ denote the class of symmetric subsets of $W\backslash\{0\}$. If $2 \le \dim W \le \infty$, the index is defined as follows. Let $\sim$ be the equivalence relation in $W\backslash\{0\}$ which identifies u with $-u$. It is well known that $H^1((W\backslash\{0\})/\sim; \mathbf{Z}_2) \approx \mathbf{Z}_2$. Let α be the generator of $H^1((W\backslash\{0\})/\sim; \mathbf{Z}_2)$. For any $A \in \mathcal{S}(W)$, Index(A) is defined as the smallest nonnegative integer k such that $\alpha^k|_{A/\sim} = 0$. If no such integer exists, then Index$(A) = \infty$. We list some properties of the cohomological index here for further use. Let $A, B \in \mathcal{S}(W)$.

(i) (Definiteness) Index$(A) = 0 \iff A = \varnothing$.
(ii) (Monotonicity) If there is an odd map $A \to B$, then Index$(A) \le$ Index(B). In particular, equality holds if A and B are homeomorphic.
(iii) (Continuity) If A is closed, then there is a closed neighborhood $N \in \mathcal{S}(W)$ of A such that Index$(N) =$ Index(A). (iv)] (Neighborhood of zero) If U is a bounded closed symmetric neighborhood of 0 in W, then Index $(\partial U) = \dim W$.
(v) For every symmetric, open subset A of W,

$$\text{Index}(A) = \sup\{\text{Index}(K) \mid K \text{ is compact and symmetric with } K \subset A\}.$$

Note that Index$(A) \le \gamma(A)$, where the Krasnoselskii genus $\gamma(A)$ is by definition the smallest nonnegative integer k for which there exists an odd mapping $A \to \mathbf{R}^k\backslash\{0\}$. If there is no such mapping for any k, then $\gamma(A) = +\infty$.

Let us also recall that Index$(Y\backslash\{0\}) = \dim(Y)$, whenever Y is a linear subspace of W. Moreover, $\gamma^+(A) \le$ Index(A), where, as defined in [5], $\gamma^+(A) = \sup\{m \in \mathbf{N} \mid$ there exists an odd continuous map $\psi\colon \mathbf{R}^m\backslash\{0\} \to A\}$.

Lemma 2.6. (Theorem 2.7 in [10]) *Let X be a real normed space and let S, A be two symmetric subsets of X such that $S \cap A = \varnothing$, $0 \in A$ and* Index$(S) =$ Index$(X\backslash A) < \infty$. *Then (X, S) links A cohomologically in dimension* Index(S) *over* $\mathbf{Z}_2$.

In particular, Proposition 2.5 holds if C_-, C_+ are symmetric cones such that $C_- \cap C_+ = \{0\}$ and Index$(C_-\backslash\{0\}) =$ Index$(X\backslash C_+) < \infty$.

Let $(X, \|\cdot\|)$ be a real Banach space with its dual space $(X^*, \|\cdot\|_*)$, $c \in \mathbf{R}$, $\varphi \in C^1(X, \mathbf{R})$. Recall that $\{u_n\} \subset X$ is called a Palais–Smale sequence of φ at level c ($(PS)_c$ sequence in short) if $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$. If $\varphi(u_n) \to c$ and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, then $\{u_n\}$ will be called a Cerami sequence at level c ($(C)_c$ sequence in short).

Let $c \in \mathbf{R}$, we say that φ satisfies $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_n\} \subset X$ has a strongly convergent subsequence in X. If any $(C)_c$ sequence $\{u_n\} \subset X$ has a strongly convergent subsequence in X, then we say that φ satisfies $(C)_c$ condition.

The following lemma, which is a special case of a deformation lemma on a Banach space (Theorem 2.6 in [21]), will be useful in this paper.

Lemma 2.7. *Let X be a Banach space. Assume that $\varphi \in C^1(X, \mathbf{R})$ is an even functional and $S \subset X$ is symmetric. Let $c \in \mathbf{R}$, $\varepsilon, \delta > 0$ such that*

$$\forall u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta} \Longrightarrow (1+\|u\|)\|\varphi'(u)\|_* \geq \frac{8\varepsilon}{\delta}.$$

Then there exists $\eta \in C([0,1]\times X, X)$ such that

(i) $\eta(t,u)=u$ *if* $t=0$ *or if* $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$,

(ii) $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$,

(iii) $\eta(t,\cdot)$ *is a homeomorphism of* X *for each* $t \in [0,1]$,

(iv) $\varphi(\eta(t,u))$ *is nonincreasing for each* $u \in X$, *and* $\eta(t,\cdot)$ *is odd for every* $t \in [0,1]$.

The following fountain theorem over cones under $(C)_c$ condition extends Theorem 1.1 of [39], which is under the (PS) condition.

Let X be a Banach space with $\dim X \geq 2$ and for each $m \in \mathbf{N}$, $\{C_-^m\}, \{C_+^m\}$ be two sequences of symmetric cones in X. For $r_-^m, r_+^m > 0$, we define

$$D_-^m = \{u \in C_-^m \mid \|u\| \leq r_-^m\}, \quad S_-^m = \{u \in C_-^m \mid \|u\| = r_-^m\},$$
$$D_+^m = \{u \in C_+^m \mid \|u\| \leq r_+^m\}, \quad S_+^m = \{u \in C_+^m \mid \|u\| = r_+^m\}.$$

Theorem 2.8. *Let $I \in C^1(X, \mathbf{R})$ be an even functional. Suppose that for every $m \in \mathbf{N}$, C_+^m is closed in X with $C_-^m \cap C_+^m = \{0\}$ and*

$$\operatorname{Index}(C_-^m \backslash \{0\}) = \operatorname{Index}(X \backslash C_+^m) < \infty \tag{2.1}$$

holds. Let

$$c_m = \inf_{\gamma \in \Gamma_m} \max_{u \in D_-^m} I(\gamma(u)),$$

where $\Gamma_m = \{\gamma \in C(D_-^m, X) \mid \gamma$ is even and $\gamma|_{S_-^m} = \mathrm{id}\}$. If there exist $r_-^m > r_+^m > 0$ such that

(1) $b_m := \inf_{S_+^m} I(u) \to \infty$ *as* $m \to \infty$,

(2) $a_m := \max_{S_-^m} I(u) \leq 0$,

then $c_m \geq b_m$ and there is a $(C)_{c_m}$ sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c_m, \quad \|I'(u_n)\|_*(1+\|u_n\|) \to 0.$$

Moreover, if I satisfies the $(C)_c$ condition for any $c > 0$, then I has an unbounded sequence of critical values $\{c_m\}$.

Proof. By Lemma 2.6 and Proposition 2.5, (2.1) implies that $(\gamma(D_-^m), S_-^m)$ links S_+^m cohomologically for each $\gamma \in \Gamma_m$, then $\gamma(D_-^m) \cap S_+^m \neq \varnothing$ for every $\gamma \in \Gamma_m$. So $c_m \geq b_m$.

By contradiction, if there is no $(C)_{c_m}$ sequence $\{u_n\} \subset X$ as stated, let $S = X$, then there exist $\varepsilon, \delta > 0$ such that

$$\forall\, u \in I^{-1}([c_m - 2\varepsilon, c_m + 2\varepsilon]) \ \Rightarrow \ (1+\|u\|)\|I'(u)\|_* \geq \frac{8\varepsilon}{\delta}.$$

We can further require that $\varepsilon > 0$ such that

$$c_m - 2\varepsilon > a_m. \tag{2.2}$$

By the definition of c_m, there exists $\gamma \in \Gamma_m$ such that

$$\max_{u \in D_-^m} I(\gamma(u)) \leq c_m + \varepsilon.$$

Then by Lemma 2.7, there exists an odd decent flow $\eta \in C([0,1] \times X, X)$ such that

$$\max_{u \in D_-^m} I(\eta(1, \gamma(u))) \leq c_m - \varepsilon.$$

Denote $\beta(u) = \eta(1, \gamma(u))$. For $u \in S_-^m$, by (2.2), $\beta(u) = \eta(1, \gamma(u)) = \eta(1, u) = u$. Then it follows that $\beta \in \Gamma_m$. Hence

$$c_m \leq \max_{u \in D_-^m} I(\beta(u)) = \max_{u \in D_-^m} I(\eta(1, \gamma(u))) \leq c_m - \varepsilon,$$

which is a contradiction. □

Since V and g are radially symmetric in x, we can study problem (1.1) in the radially symmetric Sobolev space

$$W_r^{1,p}(\mathbf{R}^N) := \{u \in W^{1,p}(\mathbf{R}^N) \mid u(x) = u(|x|)\}$$

with the norm defined by $\|u\| = \left(\int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \right)^{\frac{1}{p}}$ for all $u \in W_r^{1,p}(\mathbf{R}^N)$. It is well known that $W_r^{1,p}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$ for $p \leq q \leq p^*$. Moreover, the embeddings $W_r^{1,p}(\mathbf{R}^N) \hookrightarrow L^s(\mathbf{R}^N)$ are compact for $p < s < p^*$.

Consider the following nonlinear eigenvalue problem

$$-\Delta_p u + |u|^{p-2}u = \lambda V(x)|u|^{p-2}u, \quad u \in W_r^{1,p}(\mathbf{R}^N), \tag{2.3}$$

where $\Delta_p = div(|Du|^{p-2}Du)$ is the p-Laplacian operator with $1 < p < N$ and V satisfies (H). Denote

$$\varphi(u) = \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx$$

and

$$\psi(u) = \int_{\mathbf{R}^N} V(x)|u|^p\, dx.$$

Note that $V \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, then $V \in L^{\frac{N}{p}}(\mathbf{R}^N)$, hence $\varphi, \psi \in C^1(W_r^{1,p}(\mathbf{R}^N), \mathbf{R})$. Set

$$\mathcal{M} = \{u \in W_r^{1,p}(\mathbf{R}^N) \mid \psi(u) = 1\}. \tag{2.4}$$

Recall that $c \in \mathbf{R}$ is a regular value of a C^1 function f if and only if $f'(x) \neq 0$ for all $x \in f^{-1}(c)$. Since $\psi(u) = \frac{1}{p}\langle \psi'(u), u\rangle$, 1 is a regular value of the functional ψ in $\mathcal{M}$. Hence the implicit function theorem implies that $\mathcal{M}$ is a C^1-Finsler manifold with the natural Finsler structure induced by ψ. Since ψ is continuous and even, $\mathcal{M}$ is complete, symmetric and 0 is not contained in $\mathcal{M}$.

By Proposition 5.12 in [38], the norm of the derivative of the restriction of φ to $\mathcal{M}$ at u is given by $\|(\varphi|_{\mathcal{M}})'(u)\|_* = \min_{\mu \in \mathbf{R}} \|\varphi'(u) - \mu\psi'(u)\|$ (here the norm $\|\cdot\|_*$ is the norm in $T_u\mathcal{M}$, which is the tangent space of $\mathcal{M}$ at u).

Similar to the proof of Lemma 2.13 in [38], it is easy to prove the next lemma:

Lemma 2.9. *If V satisfies (H), then*

(i) *ψ' is a compact operator from $W_r^{1,p}(\mathbf{R}^N)$ to $(W_r^{1,p}(\mathbf{R}^N))^*$,*
(ii) *$\psi(u)$ is weakly continuous in $W_r^{1,p}(\mathbf{R}^N)$.*

The following Lemma is a direct consequence of Lemma 2.7 in [20].

Lemma 2.10. *Let $u_n, u \in W_r^{1,p}(\mathbf{R}^N)$. If*

$$\lim_{n\to+\infty}\int_{\mathbf{R}^N}(|Du_n|^{p-2}Du_n-|Du|^{p-2}Du)(Du_n-Du)\,dx=0$$

and

$$\lim_{n\to+\infty}\int_{\mathbf{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,dx=0,$$

then $u_n \to u$ in $W_r^{1,p}(\mathbf{R}^N)$.

Lemma 2.11. *$\varphi|_{\mathcal{M}}$ satisfies the $(PS)_c$ condition for any $c>0$.*

Proof. For $c>0$, suppose that $\{u_n\}$ is a $(PS)_c$ sequence for $\varphi|_{\mathcal{M}}$, i.e.,

$$(\varphi|_{\mathcal{M}})(u_n)\to c \quad \text{and} \quad (\varphi|_{\mathcal{M}})'(u_n)\to 0 \ \text{ in } (W_r^{1,p}(\mathbf{R}^N))^*$$

as $n\to+\infty$. Then there exists a sequence $\{\mu_n\}\subset\mathbf{R}$ such that

$$\varphi'(u_n)-\mu_n\psi'(u_n)\to 0 \ \text{ in } (W_r^{1,p}(\mathbf{R}^N))^*. \tag{2.5}$$

Since $\varphi(u_n)\to c$, $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbf{R}^N)$, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W_r^{1,p}(\mathbf{R}^N)$ for some $u\in W_r^{1,p}(\mathbf{R}^N)$. Hence by Lemma 2.9,

$$\psi'(u_n)\to\psi'(u) \ \text{ in } (W_r^{1,p}(\mathbf{R}^N))^* \quad \text{and} \quad \psi(u)=\lim_{n\to\infty}\psi(u_n)=1,$$

then we have that $u\neq 0$. By (2.5),

$$p(\varphi(u_n)-\mu_n)=\langle\varphi'(u_n),u_n\rangle-\mu_n\langle\psi'(u_n),u_n\rangle\to 0.$$

So $\{\mu_n\}$ is bounded and we may assume that $\mu_n\to\mu$ for some $\mu\in\mathbf{R}$. Moreover, $0<\varphi(u)\le\mu$. It follows from $\mu_n\to\mu$ and $\psi'(u_n)\to\psi'(u)$ in $(W_r^{1,p}(\mathbf{R}^N))^*$ that

$$\begin{aligned}\langle\varphi'(u_n),u_n-u\rangle &= \langle\varphi'(u_n)-\mu_n\psi'(u_n),u_n-u\rangle+(\mu_n-\mu)\langle\psi'(u_n),u_n-u\rangle\\ &\quad+\mu\langle\psi'(u_n)-\psi'(u),u_n-u\rangle+\mu\langle\psi'(u),u_n-u\rangle\to 0,\end{aligned}$$

and then $\langle\varphi'(u_n)-\varphi'(u),u_n-u\rangle\to 0$, i.e.,

$$\begin{aligned}&\int_{\mathbf{R}^N}(|Du_n|^{p-2}Du_n-|Du|^{p-2}Du)(Du_n-Du)\,dx\\ &+\int_{\mathbf{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,dx\to 0,\end{aligned}$$

hence

$$\int_{\mathbf{R}^N}(|Du_n|^{p-2}Du_n-|Du|^{p-2}Du)(Du_n-Du)\,dx\to 0$$

and

$$\int_{\mathbf{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,dx\to 0.$$

Then by Lemma 2.10, $u_n\to u$ in $W_r^{1,p}(\mathbf{R}^N)$. □

Denote by $\mathcal{A}$ the class of compact symmetric subsets of $\mathcal{M}$. For $m\in\mathbf{N}$, let

$$\mathcal{F}_m:=\{A\in\mathcal{A}\mid \operatorname{Index}(A)\ge m\}$$

and set

$$\mu_m=\inf_{A\in\mathcal{F}_m}\max_{u\in A}\varphi(u). \tag{2.6}$$

Since $\mathcal{M} \neq \varnothing$, for each m there is a set $A \subset \mathcal{M}$ which is homeomorphic to the unit sphere $S^{m-1} \subset \mathbf{R}^m$ by an odd homeomorphism. Since $\gamma^+(S^{m-1}) = m$, all μ_m are well defined. Moreover, $\mu_1 = \inf_{u \in \mathcal{M}} \varphi(u) > 0$.

Following [4], to prove that $\{\mu_m\}$ is a sequence of eigenvalues of problem (2.3), we define a family of manifolds on $\mathcal{M}$. For fixed $\alpha > 0$, we denote

$$\mathcal{F}_\alpha = \{\mathcal{M}_\beta \mid \beta \in [0, \alpha]\} \quad \text{and} \quad \mathcal{F}^*_\alpha = \{\mathcal{M}_\beta \mid \beta \in [-\alpha, 0]\},$$

where $\mathcal{M}_\beta = \{u \in W^{1,p}_r(\mathbf{R}^N) \mid \psi(u) = 1 + \beta\}$.

Definition 2.12. Let $c \in \mathbf{R}$ be a regular value of $\varphi|_{\mathcal{M}}$. The family $\mathcal{F}_\alpha$ is said to be admissible for φ at c if φ is defined on $\mathcal{M}_\beta$ for all $\beta \in [0, \alpha]$ and there exist constants τ, ρ, $\varepsilon_1 > 0$ such that

$$\text{(2.7)} \quad \forall\, \beta \in [0, \alpha],\ u \in \mathcal{M}_\beta,\ |\varphi(u) - c| \le \varepsilon_1 \Longrightarrow \|(\varphi|_{\mathcal{M}_\beta})'(u)\|_* > \rho \text{ and } \|\psi'(u)\| > \tau.$$

Lemma 2.13. (A deformation lemma on a C^1-manifold, Theorem 2.5 in [4]) *Let $\mathcal{M} = \{u \in X \mid \psi(u) = 1\}$ be a C^1 submanifold of a Banach space X, where ψ is a C^1 functional and 1 is not a critical value of ψ. Let φ be a C^1 functional on a neighborhood of $\mathcal{M}$ and c be a noncritical value of $\varphi|_{\mathcal{M}}$ and $\alpha > 0$ be such that $\mathcal{F}_\alpha$ or $\mathcal{F}^*_\alpha$ is admissible for φ at c. Then there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ there exists an homeomorphism η of $\mathcal{M}$ onto $\mathcal{M}$ such that*

(1) *$\eta(u) = u$ if $\varphi(u) \notin [c - \hat{\varepsilon}, c + \hat{\varepsilon}]$;*
(2) *$\varphi(\eta(u)) \le \varphi(u)$ for all $u \in \mathcal{M}$;*
(3) *$\varphi(\eta(u)) \le c - \varepsilon$ for all $u \in \mathcal{M}$ such that $\varphi(u) \le c + \varepsilon$;*
(4) *if $\mathcal{M}$ is symmetric ($\mathcal{M} = -\mathcal{M}$) and φ is even, then η is odd.*

Proposition 2.14. *Under the assumption* (H), *$\{\mu_m\}$ is a sequence of eigenvalues of problem* (2.3) *and $\mu_m \to \infty$ as $m \to \infty$.*

Proof. 1. Note first that critical values of $\varphi|_{\mathcal{M}}$ coincide with eigenvalues of problem (2.3).

2. Claim: For each m, if μ_m is not a critical value of $\varphi|_{\mathcal{M}}$, then there exists $\alpha > 0$ such that $\mathcal{F}_\alpha$ is admissible for φ at μ_m, i.e. there exist constants τ, ρ, $\varepsilon_1 > 0$ satisfying (2.7).

Proof of the Claim. In fact, if there are sequences $\varepsilon_n \to 0$, $\{u_n\} \subset \mathcal{M}_{\varepsilon_n}$ with $|\varphi(u_n) - \mu_m| \to 0$ such that

$$\|(\varphi|_{\mathcal{M}_{\varepsilon_n}})'(u_n)\|_* \to 0 \quad \text{or} \quad \|\psi'(u_n)\|_* \to 0.$$

If $\|(\varphi|_{\mathcal{M}_{\varepsilon_n}})'(u_n)\|_* \to 0$, then $\{u_n\}$ is a $(PS)_{\mu_m}$ sequence of $\varphi|_{\mathcal{M}_{\varepsilon_n}}$. By Lemma 2.11, $\{u_n\}$ possesses a convergent subsequence, thus μ_m is a critical value of $\varphi|_{\mathcal{M}}$, which is a contradiction.

Therefore we must have $\|\psi'(u_n)\|_* \to 0$.

Since $\varphi(u_n)$ is bounded, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}_r(\mathbf{R}^N)$. Then by Lemma 2.9, we have

$$\psi(u) = \lim_{n\to\infty} \psi(u_n) = 1 \quad \text{and} \quad \langle \psi'(u), v\rangle = \lim_{n\to\infty} \langle \psi'(u_n), v\rangle = 0 \ \ \forall\, v \in W^{1,p}_r(\mathbf{R}^N),$$

i.e., 1 is a critical value of ψ, which is also a contradiction. Hence the Claim follows.

3. For each m, if μ_m is not a critical value of φ, then Lemma 2.13 implies that there exist $\hat{\varepsilon} > 0$ and an odd homeomorphism η of $\mathcal{M}$ such that $\eta(\varphi^{\mu_m+\varepsilon}) \subset \varphi^{\mu_m-\varepsilon}$ for

all $\varepsilon \in (0, \hat{\varepsilon})$. Taking $A \in \mathcal{F}_m$ with $\max \varphi(A) \leq \mu_m + \varepsilon$, then we have $\widetilde{A} = \eta(A) \in \mathcal{F}_m$, hence $\max \varphi(\widetilde{A}) \leq \mu_m - \varepsilon$, which contradicts to the definition of μ_m.

4. Obviously, $\mu_m \leq \mu_{m+1}$. Recall that the Ljusternik–Schnirelmann eigenvalues $\lambda_m = \inf_{\gamma(A) \geq m} \sup_{u \in A} \varphi(u) \to \infty$, which are defined by the genus γ (see Theorem 4.4 in [37]). But $\mathrm{Index}(A) \leq \gamma(A)$, so $\mu_m \geq \lambda_m$. Then $\mu_m \to \infty$ as $m \to +\infty$. □

Let $\{\mu_m\}$ be the sequence defined in (2.6). For each μ_m with $\mu_m < \mu_{m+1}$, define

$$C_-^m = \left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \leq \mu_m \int_{\mathbf{R}^N} V(x)|u|^p\, dx \right\}, \tag{2.8}$$

$$C_+^m = \left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \geq \mu_{m+1} \int_{\mathbf{R}^N} V(x)|u|^p\, dx \right\}, \tag{2.9}$$

which are two sequences of symmetric cones in $W_r^{1,p}(\mathbf{R}^N)$.

Lemma 2.15. *If $m \geq 1$ is such that $\mu_m < \mu_{m+1}$, then*

$$\mathrm{Index}(C_-^m \backslash \{0\}) = \mathrm{Index}\left(W_r^{1,p}(\mathbf{R}^N) \backslash C_+^m \right) = m.$$

Moreover, $(W_r^{1,p}(\mathbf{R}^N), C_-^m \backslash \{0\})$ links C_+^m cohomologically in dimension m over $\mathbf{Z}_2$.

Proof. The proof is similar to that of Theorem 3.2 in [10], where the cones were similarly defined on $W_0^{1,p}(\Omega)$, where Ω is a bounded domain. We give a detailed proof here for the readers' convenience.

Set

$$C = \{u \in \mathcal{M} \mid \varphi(u) \leq \mu_m\}, \quad U = \{u \in \mathcal{M} \mid \varphi(u) < \mu_{m+1}\},$$

we have $\mathrm{Index}(C) \leq m \leq \mathrm{Index}(U)$. By contradiction, if $\mathrm{Index}(C) \leq m-1$, by the continuity of the index, there exists a closed symmetric neighborhood $N \in \mathcal{A}$ of C such that $\mathrm{Index}(N) = \mathrm{Index}(C)$. By Lemma 2.13 above, there exist $\varepsilon > 0$ and an odd continuous map η satisfying $\eta(\{u \in \mathcal{M} \mid \varphi(u) \leq \mu_m + \varepsilon\}) \subset \{u \in \mathcal{M} \mid \varphi(u) \leq \mu_m - \varepsilon\} \cup N = N$. It follows that $\mathrm{Index}(\{u \in \mathcal{M} \mid \varphi(u) \leq \mu_m + \varepsilon\}) \leq m-1$. By the definition of μ_m, there exists $M \in \mathcal{F}_m$ such that $\max \varphi(M) < \mu_m + \varepsilon$, then $M \subseteq \{u \in \mathcal{M} \mid \varphi(u) \leq \mu_m + \varepsilon\}$ and thus $\mathrm{Index}(M) \leq m-1$, which contradicts to the choice of M.

Since the index is invariant by odd deformation maps, it follows that

$$\mathrm{Index}\left(\left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx \leq \mu_m \int_{\mathbf{R}^N} V|u|^p\, dx \right\} \right) = \mathrm{Index}(C) = m.$$

Assume, by contradiction, that $\mathrm{Index}(U) \geq m+1$, then there is a symmetric, compact subset K of U with $\mathrm{Index}(K) \geq m+1$, which contradicts the definition of μ_{m+1} since $\max\{\varphi(u) | \ u \in K\} < \mu_{m+1}$. Again, since the index is invariant by odd deformations, we have that

$$\mathrm{Index}\left(\left\{ u \in W_r^{1,p}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} (|Du|^p + |u|^p)\, dx < \mu_{m+1} \int_{\mathbf{R}^N} V|u|^p\, dx \right\} \right) = \mathrm{Index}(U) = m.$$

Then $\mathrm{Index}(C_-^m \backslash \{0\}) = \mathrm{Index}(W_r^{1,p}(\mathbf{R}^N) \backslash C_+^m) = m$. From Lemma 2.6, we conclude that $(W_r^{1,p}(\mathbf{R}^N), C_-^m \backslash \{0\})$ links C_+^m cohomologically in dimension m over $\mathbf{Z}_2$. □

Let $r_-^m, r_+^m > 0$, denote

$$B_-^m = \{u \in C_-^m \mid \|u\| \leq r_-^m\}; \quad S_-^m = \{u \in C_-^m \mid \|u\| = r_-^m\}; \tag{2.10}$$

$$B_+^m = \{u \in C_+^m \mid \|u\| \le r_+^m\}; \quad S_+^m = \{u \in C_+^m \mid \|u\| = r_+^m\}. \tag{2.11}$$

Let us define

$$b_m = \inf_{S_+^m} I(u), \quad a_m = \sup_{S_-^m} I(u).$$

Lemma 2.16. *For any* $\lambda \in \mathbf{R}$*, there exist* $r_-^m > r_+^m > 0$ *such that* $b_m \to \infty$ *as* $m \to \infty$ *and* $a_m \le 0$.

Proof. By (g_1), (g_2), for $\forall\, \varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|G(x,u)| \le \varepsilon |u|^p + C_\varepsilon |u|^q, \quad p < q < p^*.$$

For $u \in S_+^m$, by the inequality $|t|^p \le C(1+|t|^q)$ for all $t \in \mathbf{R}$ and $V \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, then

$$\begin{aligned}\left|\int_{\mathbf{R}^N} V(x)|u|^p\,dx\right| &\le C\int_{\mathbf{R}^N} |V(x)|(1+|u|^q)\,dx \le C\|V\|_{L^1} + C\|V\|_{L^\infty}\int_{\mathbf{R}^N}|u|^q\,dx \\ &\le C\left(1+\int_{\mathbf{R}^N}|u|^q\,dx\right),\end{aligned}$$

where $\|\cdot\|_{L^\infty}$ denotes the L^∞ norm in $\mathbf{R}^N$. Hence, choosing ε small satisfying $\varepsilon \le \frac{1}{2Cp}$, we have that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_{\mathbf{R}^N} V(x)|u|^p\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{|\lambda|C}{p}\left(1+\int_{\mathbf{R}^N}|u|^q\,dx\right) - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &\ge \frac{1}{p}\|u\|^p - \varepsilon\int_{\mathbf{R}^N}|u|^p\,dx - C_\varepsilon\int_{\mathbf{R}^N}|u|^q\,dx - \frac{|\lambda|C}{p}\int_{\mathbf{R}^N}|u|^q\,dx - \frac{|\lambda|C}{p} \\ &\ge \left(\frac{1}{p} - \varepsilon C\right)\|u\|^p - C_\varepsilon\int_{\mathbf{R}^N}|u|^q\,dx - \frac{|\lambda|C}{p}\int_{\mathbf{R}^N}|u|^q\,dx - \frac{|\lambda|C}{p} \\ &\ge \frac{1}{2p}\|u\|^p - \frac{|\lambda|C}{p}\beta_m^q\|u\|^q - \frac{|\lambda|C}{p}, \end{aligned} \tag{2.12}$$

where $\beta_m = \sup\limits_{u\in C_+^m, \|u\|=1} \|u\|_{L^q}$.

We choose $r_+^m = \left(\frac{1}{2Cq\beta_m^q}\right)^{\frac{1}{q-p}} > 0$, then

$$I(u) \ge \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)(r_+^m)^p - C.$$

We claim that $\beta_m \to 0$. Indeed, since $\beta_m \ge \beta_{m+1}$, we assume that $\beta_m \to \beta \ge 0$. Let $\{u_m\} \subset C_+^m$ be a sequence satisfying $\|u_m\| = 1$ and $\beta_m \ge \|u_m\|_{L^q} \ge \frac{1}{2}\beta_m$. We may assume that $u_m \rightharpoonup u$ in $W_r^{1,p}(\mathbf{R}^N)$, then $\psi(u_m) \to \psi(u)$. On the other hand, since $u_m \in C_+^m$, then

$$1 = \|u_m\|^p \ge \mu_{m+1}\int_{\mathbf{R}^N} V(x)|u_m|^p dx,$$

which and $\mu_{m+1} \to \infty$ imply that $\psi(u_m) \to 0$, then $u = 0$. The Sobolev embedding implies that $\|u_m\|_{L^q} \to 0$. The claim follows. As a result, $r_+^m \to \infty$ and $b_m \to \infty$ as $m \to \infty$.

We next show that there is $r_-^m > 0$ such that $a_m \le 0$. For $u_k \in S_-^m$, since $V \in L^\infty(\mathbf{R}^N)$, we have that

$$\|u_k\|^p \le \mu_m \int_{\mathbf{R}^N} V(x)|u_k|^p dx \le \mu_m \|V\|_{L^\infty} \int_{\mathbf{R}^N} |u_k|^p dx.$$

Then by Sobolev embedding inequality and (g_3), we have that

$$\begin{aligned} I(u_k) &= \frac{1}{p}\|u_k\|^p - \frac{\lambda}{p}\int_{\mathbf{R}^N} V(x)|u_k|^p\,dx - \int_{\mathbf{R}^N} G(x,u_k)\,dx \\ &\le \frac{1}{p}\|u_k\|^p + \frac{|\lambda|}{p}\left|\int_{\mathbf{R}^N} V(x)|u_k|^p\,dx\right| - \int_{\mathbf{R}^N} G(x,u_k)\,dx \\ &\le \frac{1+C\|V\|_{L^\infty}|\lambda|}{p}\|u_k\|^p - \int_{\mathbf{R}^N} G(x,u_k)\,dx \\ &= \|u_k\|^p\left(\frac{1+C\|V\|_{L^\infty}|\lambda|}{p} - \int_{\mathbf{R}^N}\frac{G(x,u_k)}{\|u_k\|^p}\,dx\right) \\ &\to -\infty \quad \text{as } \|u_k\| \to +\infty. \end{aligned}$$

Hence there exist $r_-^m > r_+^m > 0$ such that $I(u) \le 0$ for $u \in S_-^m$, then $a_m \le 0$. □

Lemma 2.17. *I satisfies $(C)_c$ condition for any $c > 0$.*

Proof. Suppose that $\{u_n\} \subset W_r^{1,p}(\mathbf{R}^N)$ is a $(C)_c$ sequence for I, i.e., $I(u_n) \to c$ and $(1+\|u_n\|)\|I'(u_n)\|_* \to 0$. Then

$$I(u_n) \to c, \quad \langle I'(u_n), u_n\rangle \to 0 \tag{2.13}$$

as $n \to +\infty$. Hence for large n, there exists a positive constant C such that

$$C \ge I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle = \int_{\mathbf{R}^N} \widetilde{G}(x,u_n)\,dx. \tag{2.14}$$

Arguing by contradiction, we may assume that $\|u_n\| \to +\infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$, up to a subsequence, we have for some $v \in W_r^{1,p}(\mathbf{R}^N)$ that

$$v_n \rightharpoonup v \text{ in } W_r^{1,p}(\mathbf{R}^N), \quad v_n \to v \text{ in } L^s(\mathbf{R}^N) \text{ for } s \in (p,p^*)$$

and

$$v_n(x) \to v(x) \ \text{ a.e. in } \mathbf{R}^N$$

as $n \to +\infty$.

We first consider the case where $v \ne 0$ in $W_r^{1,p}(\mathbf{R}^N)$. Observe that

$$0 \leftarrow \frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^p} = 1 - \lambda\int_{\mathbf{R}^N}\frac{V(x)|u_n|^p}{\|u_n\|^p}\,dx - \int_{\mathbf{R}^N}\frac{g(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx. \tag{2.15}$$

From $V \in L^\infty(\mathbf{R}^N)$, we have that $\left|\int_{\mathbf{R}^N} V(x)|u_n|^p\,dx\right| \le C\|u_n\|^p$ for some $C > 0$. It follows from (2.15) that

$$\left|\int_{\mathbf{R}^N}\frac{g(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx\right| \le C. \tag{2.16}$$

Since $0 \neq v(x) = \lim_{n\to+\infty} v_n(x) = \lim_{n\to+\infty} \frac{u_n(x)}{\|u_n\|}$ a.e. in $\mathbf{R}^N$, $|u_n(x)| \to +\infty$ a.e. in $\mathbf{R}^N$. By Fatou's Lemma and (g_3), we have

$$\int_{\mathbf{R}^N} \frac{g(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx = \int_{\mathbf{R}^N} \frac{g(x,u_n)}{|u_n|^{p-1}}|v_n|^{p-1}v_n\,dx \to +\infty$$

as $n \to +\infty$, which contradicts to (2.16).

If $v = 0$ in $W_r^{1,p}(\mathbf{R}^N)$, then by Lemma 2.9, we have that

$$\int_{\mathbf{R}^N} \frac{V(x)|u_n|^p}{\|u_n\|^p}\,dx = \int_{\mathbf{R}^N} V(x)|v_n|^p\,dx \to 0.$$

By (2.15) again, we have

$$\lim_{n\to+\infty} \int_{\mathbf{R}^N} \frac{g(x,u_n)v_n}{\|u_n\|^{p-1}} = 1. \tag{2.17}$$

Inspired by [28], for $r \geq 0$, we set

$$h(r) = \inf\{\widetilde{G}(x,u) \mid x \in \mathbf{R}^N \text{ and } u \in \mathbf{R} \text{ with } |u| \geq r\}. \tag{2.18}$$

Then (g_4) implies that $h(r) > 0$ for all $r > 0$, and $(g_3)(g_4)$ imply that $h(r) \to +\infty$ as $r \to +\infty$. For $0 \leq \alpha < \beta \leq +\infty$, set

$$A_n(\alpha,\beta) = \{x \in \mathbf{R}^N \mid \alpha \leq |u_n(x)| < \beta\}.$$

and

$$S_\alpha^\beta = \inf\left\{\frac{\widetilde{G}(x,u)}{|u|^p} \mid x \in \mathbf{R}^N \text{ and } u \in \mathbf{R} \text{ with } \alpha \leq |u(x)| < \beta\right\}.$$

Since $\widetilde{G}(x,u) > 0$ for any $|u| > 0$, we have that for each $\alpha > 0$, $S_\alpha^\beta > 0$ and $\widetilde{G}(x,u_n) \geq S_\alpha^\beta|u_n|^p$ for all $x \in A_n(\alpha,\beta)$. It follows from (2.14) that

$$\begin{aligned}
C &\geq \int_{A_n(0,\alpha)} \widetilde{G}(x,u_n)\,dx + \int_{A_n(\alpha,\beta)} \widetilde{G}(x,u_n)\,dx + \int_{A_n(\beta,+\infty)} \widetilde{G}(x,u_n)\,dx \\
&\geq S_\alpha^\beta \int_{A_n(\alpha,\beta)} |u_n|^p\,dx + h(\beta)|A_n(\beta,+\infty)| \geq h(\beta)|A_n(\beta,+\infty)|.
\end{aligned}$$

Since $h(r) \to +\infty$ as $r \to +\infty$, $|A_n(\beta,+\infty)| \to 0$ as $\beta \to +\infty$ uniformly in n. By Hölder's inequality, for any $s \in [1,p^*)$,

$$\begin{aligned}
\int_{A_n(\beta,+\infty)} |v_n|^s\,dx &\leq \left(\int_{A_n(\beta,+\infty)} |v_n|^{p^*}\right)^{\frac{s}{p^*}} |A_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}} \\
&\leq C|A_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}} \to 0
\end{aligned} \tag{2.19}$$

as $\beta \to +\infty$ uniformly in n.

Since for any $\alpha > 0$, it may occur that $|A_n(0,\alpha)| = +\infty$, the methods used in [28] to prove that

$$\left|\int_{A_n(0,\alpha)} \frac{g(x,u_n)u_n}{\|u_n\|^p}\,dx\right| \to 0 \quad \text{as } n \to +\infty$$

does not work now. More careful analysis is needed as follows.

By (g_1) and (g_2), for any $\sigma > 0$, there exists some $C_\sigma > 0$ independent of n such that

$$|g(x,u_n)| \le \sigma |u_n|^{p-1} + C_\sigma |u_n|^{q-1} \quad \text{for all } (x,u_n) \in \mathbf{R}^N \times \mathbf{R}, \tag{2.20}$$

where $q \in (p, p^*)$. Then

$$\begin{aligned}\Big| \int_{A_n(0,\alpha)} \frac{g(x,u_n)u_n}{\|u_n\|^p}\,dx \Big| &\le \frac{\sigma \int_{A_n(0,\alpha)} |u_n|^p\,dx}{\|u_n\|^p} + \frac{C_\sigma \int_{A_n(0,\alpha)} |u_n|^q\,dx}{\|u_n\|^p} \\ &\le \sigma \int_{A_n(0,\alpha)} |v_n|^p\,dx + C_\sigma \frac{\int_{A_n(0,\alpha)} |u_n|^p\,dx \cdot \alpha^{q-p}}{\|u_n\|^p} \\ &\le \sigma \int_{A_n(0,\alpha)} |v_n|^p\,dx + C_\sigma \int_{A_n(0,\alpha)} |v_n|^p\,dx \cdot \alpha^{q-p} \\ &\le \sigma + C_\sigma\, \alpha^{q-p}.\end{aligned}$$

Set $0 < \varepsilon < \frac{1}{3}$. We can first choose $\sigma > 0$ small such that $\sigma < \frac{\varepsilon}{2}$. Then we take $\alpha > 0$ small enough such that $C_\sigma \alpha^{q-p} < \frac{\varepsilon}{2}$. Hence

$$\Big| \int_{A_n(0,\alpha)} \frac{g(x,u_n)u_n}{\|u_n\|^p} dx \Big| < \varepsilon \tag{2.21}$$

for all n.

Furthermore, for any fixed $0 < \alpha < \beta$,

$$\begin{aligned}\int_{A_n(\alpha,\beta)} |v_n|^p\,dx &= \frac{1}{\|u_n\|^p} \int_{A_n(\alpha,\beta)} |u_n|^p\,dx = \frac{1}{\|u_n\|^p} \int_{A_n(\alpha,\beta)} \frac{S_\alpha^\beta |u_n|^p}{S_\alpha^\beta}\,dx \\ &\le \frac{1}{\|u_n\|^p\ S_\alpha^\beta} \int_{A_n(\alpha,\beta)} \widetilde{G}(x,u_n)\,dx \le \frac{C}{\|u_n\|^p\ S_\alpha^\beta} \to 0\end{aligned} \tag{2.22}$$

as $n \to +\infty$. Note that there exists $C = C(\alpha,\beta) > 0$ independent of n such that

$$|g(x,u_n)| \le C|u_n|^{p-1} \quad \text{for all } x \in A_n(\alpha,\beta).$$

Then by (2.22), $\exists N_0$, $n > N_0$ implies that

$$\begin{aligned}\Big| \int_{A_n(\alpha,\beta)} \frac{g(x,u_n)u_n}{\|u_n\|^p}\,dx \Big| = \Big| \int_{A_n(\alpha,\beta)} \frac{g(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx \Big| &\le \int_{A_n(\alpha,\beta)} \frac{C|u_n|^{p-1}v_n}{\|u_n\|^{p-1}}\,dx \\ &= C \int_{A_n(\alpha,\beta)} |v_n|^p\,dx < \varepsilon.\end{aligned} \tag{2.23}$$

Let $\theta' = \frac{\theta}{\theta-1}$. Since $\theta > \frac{N}{p}$, we have $\theta' \in (1, \frac{N}{N-p})$, then $p\theta' \in (p, p^*)$. By (2.19) we can take β large enough such that

$$\begin{aligned}&\Big| \int_{A_n(\beta,+\infty)} \frac{g(x,u_n)v_n}{\|u_n\|^{p-1}}\,dx \Big| \le \int_{A_n(\beta,+\infty)} \frac{|g(x,u_n)|}{|u_n|^{p-1}} |v_n|^p\,dx \\ &\le \left(\int_{A_n(\beta,+\infty)} \left| \frac{g(x,u_n)}{|u_n|^{p-1}} \right|^\theta dx \right)^{\frac{1}{\theta}} \left(\int_{A_n(\beta,+\infty)} |v_n|^{p\theta'}\,dx \right)^{\frac{1}{\theta'}} \\ &\le \left(\int_{A_n(\beta,+\infty)} C\widetilde{G}(x,u_n)\,dx \right)^{\frac{1}{\theta}} \left(\int_{A_n(\beta,+\infty)} |v_n|^{p\theta'}\,dx \right)^{\frac{1}{\theta'}} < \varepsilon\end{aligned} \tag{2.24}$$

for all $n > N_0$.

Therefore, (2.21), (2.23), (2.24) imply that for $n > N_0$,

$$\int_{R^N} \frac{g(x,u_n)u_n}{\|u_n\|^p}\,dx < 3\varepsilon < 1,$$

which contradicts (2.17). Then $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbf{R}^N)$. So we may assume, up to a subsequence,

$$u_n \rightharpoonup u \ \text{ in } W_r^{1,p}(\mathbf{R}^N)$$

for some $u \in W_r^{1,p}(\mathbf{R}^N)$, and

$$u_n \to u \text{ in } L^s(\mathbf{R}^N), \quad s \in (p, p^*),$$

as $n \to +\infty$.

Set $J(u) = \int_{\mathbf{R}^N} G(x,u)\,dx$ for all $u \in W_r^{1,p}(\mathbf{R}^N)$, then $J \in C^1(W_r^{1,p}(\mathbf{R}^N), \mathbf{R})$ and $\langle J'(u), v\rangle = \int_{\mathbf{R}^N} g(x,u)v\,dx$ for any $u, v \in W_r^{1,p}(\mathbf{R}^N)$.

From (2.20), we have

$$\begin{aligned}
&\Big| \int_{\mathbf{R}^N} \Big(g(x,u_n) - g(x,u)\Big)(u_n - u)\,dx \Big| \\
&\le \int_{\mathbf{R}^N} \sigma\Big(|u_n|^{p-1} + |u|^{p-1}\Big)(u_n - u)\,dx + \int_{\mathbf{R}^N} C_\sigma\Big(|u_n|^{q-1} + |u|^{q-1}\Big)(u_n - u)\,dx \\
&\le \sigma C\Big(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1}\Big)\|u_n - u\|_{L^p} + C_\sigma\Big(\|u_n\|_{L^q}^{q-1} + \|u\|_{L^q}^{q-1}\Big)\|u_n - u\|_{L^q} \\
&\le C(\sigma + C_\sigma\|u_n - u\|_{L^q}),
\end{aligned}$$

where C is independent of σ and of n. Then we have

$$\int_{\mathbf{R}^N} \Big(g(x,u_n) - g(x,u)\Big)(u_n - u)\,dx \to 0$$

as $n \to +\infty$, i.e.

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0 \tag{2.25}$$

as $n \to +\infty$.

Since $I'(u_n) = \varphi'(u_n) - \lambda\psi'(u_n) - J'(u_n) \to 0$ and $\psi'(u_n) \to \psi'(u)$, we have that $\varphi'(u_n) + J'(u_n) \to \lambda\psi'(u)$ in $W_r^{1,p}(\mathbf{R}^N)$, then it follows from (2.25) and $u_n \rightharpoonup u$ in $W_r^{1,p}(\mathbf{R}^N)$ that

$$\begin{aligned}
\langle \varphi'(u_n), u_n - u\rangle &= \langle \varphi'(u_n) + J'(u_n) - \lambda\psi'(u), u_n - u\rangle + \langle \lambda\psi'(u), u_n - u\rangle \\
&\quad - \langle J'(u_n) - J'(u), u_n - u\rangle + \langle J'(u), u_n - u\rangle \to 0.
\end{aligned}$$

Then $\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \to 0$, hence $u_n \to u$ in $W_r^{1,p}(\mathbf{R}^N)$ by Lemma 2.10. □

3. The proof of the main result

Proof of Theorem 1.1. By Lemma 2.14, $\mu_m \to +\infty$ as $m \to +\infty$ where $\{\mu_m\}$ is given in (2.6). For each m with $\mu_m < \mu_{m+1}$. Define C_-^m, C_+^m as in (2.8) (2.9), then C_-^m, C_+^m are two symmetric closed cones in $W_r^{1,p}(\mathbf{R}^N)$ with $C_-^m \cap C_+^m = \{0\}$.

For each m, by Lemma 2.15, $\operatorname{Index}(C_-^m\backslash\{0\}) = \operatorname{Index}\Big(W_r^{1,p}(\mathbf{R}^N)\backslash C_+^m\Big) = m$. Let B_+^m, B_-^m, S_+^m, S_-^m be defined as in (2.10)(2.11) and

$$c_m = \inf_{\gamma\in\Gamma_m} \sup_{u\in B_-^m} I(\gamma(u))),$$

where $\Gamma_m = \{\gamma \in C(B^m_-, W^{1,p}_r(\mathbf{R}^N)) \colon \gamma$ is even and $\gamma|_{S^m_-} = \mathrm{id}\}$. By Lemma 2.16, Lemma 2.17 and Theorem 2.8, $\{c_m\}$ is a sequence of critical values of I and $c_m \geq b_m$. Since $b_m \to \infty$ as $m \to \infty$, I has infinitely many unbounded positive critical values. Then by the principle of symmetric criticality the proof is completed. □

Remark. If we want to get the existence of at least one nontrivial weak solution instead of infinitely many solutions to problem (1.1), then the assumption on $V(x)$ could be weaken as

(H_p) $V \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $V^+ = V_1 + V_2 \neq 0$, $V_1, V^- \in L^{\frac{N}{p}}(\mathbf{R}^N)$, $\lim_{x\to y} |x-y|^p V_2(x) = 0$ for each $y \in \mathbf{R}^N$, $\lim_{|x|\to\infty} |x|^p V_2(x) = 0$.

Under the assumption (H_p) on V, following the arguments in [37], we can also get a divergent sequence $\{\mu_m\}$ of eigenvalues defined in (2.6). By exchanging (λ, V) with $(-\lambda, -V)$, we may assume that $\lambda \geq 0$. For $\lambda \geq \mu_1$, there exists some $m \geq 1$ such that $\mu_m \leq \lambda < \mu_{m+1}$, then we could prove that I possesses a linking structure over cones under the condition (H_p). By the linking theorem over cones under $(C)_c$ condition (see, e.g., Lemma 2.2 in [25]), we can get the existence of one nontrivial solution for problem (1.1). For $0 \leq \lambda < \mu_1$, we could obtain the existence result by mountain pass theorem.

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