MULTILINEAR LOCAL TB THEOREM
FOR SQUARE FUNCTIONS

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Abstract. In the present work we extend a local Tb theorem for square functions of Christ [3] and Hofmann [17] to the multilinear setting. We also present a new BMO type interpolation result for square functions associated to multilinear operators. These square function bounds are applied to prove a multilinear local Tb theorem for singular integral operators.

1. Introduction

Consider the family of multilinear of operators $\{\Theta_t\}_{t>0}$ given by

$$(1.1) \qquad \Theta_t(f_1, \ldots, f_m)(x) = \int_{\mathbf{R}^{mn}} \theta_t(x, y_1, \ldots, y_m) \prod_{i=1}^{m} f_i(y_i) dy_i$$

where $\theta_t \colon \mathbf{R}^{(m+1)n} \to \mathbf{C}$ and the square functions associated to $\{\Theta_t\}_{t>0}$

$$(1.2) \qquad S(f_1, \ldots, f_m)(x) = \left(\int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

where f_i for $i = 1, \ldots, m$ are initially functions in $C_0^\infty(\mathbf{R}^n)$ (smooth with compact support). The purpose of this work is to find appropriate cancellation conditions on θ_t and indices $p, p_1, \ldots, p_m$ that guarantee L^p boundedness of the square functions S of the form

$$(1.3) \qquad ||S(f_1, \ldots, f_m)||_{L^p} \lesssim \prod_{i=1}^{m} ||f||_{L^{p_i}}$$

given that θ_t satisfies some size and regularity estimates. In particular, we assume that θ_t is a multilinear standard Calderón–Zygmund kernel, i.e. it satisfies for all $x, y_1, \ldots, y_m, x', y_1', \ldots, y_m' \in \mathbf{R}^n$,

$$(1.4) \qquad |\theta_t(x, y_1, \ldots, y_m)| \lesssim \frac{t^{-mn}}{\prod_{i=1}^{m}(1 + t^{-1}|x - y_i|)^{N+\gamma}},$$

(1.5) $|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x, y_1, \ldots, y_i', \ldots, y_m)| \lesssim t^{-mn}(t^{-1}|y_i - y_i'|)^\gamma,$

(1.6) $|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x', y_1, \ldots, y_m)| \lesssim t^{-mn}(t^{-1}|x - x'|)^\gamma,$

for some $N > n$ and $0 < \gamma \leq 1$. It follows from a scaling argument that if (1.3) holds, then the indices $p, p_1, \ldots, p_m$ satisfies the Hölder type relationship

$$(1.7) \qquad\qquad \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}.$$

So throughout this work we assume that all indices $p, p_1, \ldots, p_m$ satisfy (1.7).

There is a rich history of the study of square functions in harmonic analysis. In [24], Semmes studied the linear version ($m = 1$) of the operators (1.1). He proved that if θ_t satisfies (1.4), (1.5), and there exists a para-accretive function b such that $\Theta_t(b) = 0$ for all $t > 0$, then the bound (1.3) is satisfied with $p = p_1 = 2$. (For the definition of para-accretive see e.g. [7], [3], [24] or [15].) In fact the perspective of Semmes was a Besov type square function given in the multilinear setting by

$$(1.8) \qquad\qquad (f_1, \ldots, f_m) \mapsto \left(\int_0^\infty \|\Theta_t(f_1, \ldots, f_m)\|_{L^p}^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

When $m = 1$ and $p = p_1 = 2$ as in (24), the study of this Besov type square function (1.8) coincides with the study of (1.2). The Besov type square function point of view was carried to the multilinear setting by Maldonado in [21] and Maldonado–Naibo in [22], where the authors prove bounds of (1.8) on products of Besov and Lebesgue spaces under kernel conditions essentially equivalent to (1.4) and (1.5), and $\Theta_t(1, f_2, \ldots, f_m) = 0$ for $t > 0$.

In [12], Grafakos–Oliveira proved the bound (1.3) for $p = 2$ and $1 \leq p_i \leq \infty$ for $i = 1, \ldots, m$ assuming (1.4), (1.5) and that there exist para-accretive functions b_i for $i = 1, \ldots, m$ on $\mathbf{R}^n$ such that the cancelation condition

$$(1.9) \qquad\qquad \Theta_t(b_1, \ldots, b_m) = 0$$

holds. In [15], under similar size, regularity and cancellation conditions, Hart showed (in the discrete bilinear setting, but is easily extended to the m-linear setting) that (1.3) holds for $1 < p, p_i < \infty$ for $i = 1, \ldots, m$, and under stronger size and regularity conditions for $1 < p_i < \infty$ and $\frac{1}{2} < p < \infty$. In [15] and [11], Hart and Grafakos–Liu–Maldonado–Yang prove bounds of the square functions (1.2) and (1.8) on products of various spaces of smooth functions assuming (1.4), (1.5) and a variety of cancellation conditions.

The essence of T1 and Tb theorems is to determine the L^2 boundedness of an operator by verifying its behaviour in some particular test functions. In [3], Christ introduced the notion of a *local Tb theorem* in the context of singular integrals, and applied this to estimates for the Cauchy integral on Lipschitz curves. He changed the existence of a (globally defined) para-accretive test function where the operator vanishes, for the existence of a family of (locally defined) test functions when some additional information about the behavior of the operator is known. More recently, in [17] Hofmann gave an analogous result for square functions based on some previous work by Auscher–McIntosh–Hofmann–Lacey–Tchamitchian on the Kato square root problem in [1] (see also related work [19] by Hofmann–McIntosh and [18] by

Hofmann–Lacey–McIntosh). These kind of Tb theorems for square functions have proven to be very useful in many applications in complex function theory and in PDE, as they can be viewed as singular integrals taking values in a Hilbert space.

The principal result in this article is an extension of Hofmann's result to multi-linear square functions, which we state now.

Theorem 1.1. *Let Θ_t and S be defined as in (1.1) and (1.2) where θ_t satisfies (1.4)–(1.6). Suppose there exist $q_i, q > 1$ for $i = 1, \ldots, m$ with $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ and functions b_Q^i indexed by dyadic cubes $Q \subset \mathbf{R}^n$ for $i = 1, \ldots, m$ such that for every dyadic cube Q*

$$(1.10) \qquad \int_{\mathbf{R}^n} |b_Q^i|^{q_i} \leq B_1 |Q|,$$

$$(1.11) \qquad \frac{1}{B_2} \leq \left| \frac{1}{|Q|} \int_Q \prod_{i=1}^{m} b_Q^i(x)\, dx \right|,$$

$$(1.12) \qquad \left| \frac{1}{|R|} \int_R \prod_{i=1}^{m} b_Q^i(x)\, dx \right| \leq B_3 \prod_{i=1}^{m} \left| \frac{1}{|R|} \int_R b_Q^i(x)\, dx \right|$$

for all dyadic subcubes $R \subset Q$, and

$$(1.13) \qquad \int_Q \left(\int_0^{\ell(Q)} |\Theta_t(b_Q^1, \ldots, b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \leq B_4 |Q|.$$

Then S satisfies (1.3) for all $1 < p_i < \infty$ and $2 \leq p < \infty$ satisfying (1.7).

If $\{b_Q\}$ satisfies (1.10) and (1.11), we say that $\{b_Q\}$ is a pseudo-accretive system. This definition of pseudo-accretive system is analogous to the one defined by Christ in [3] in the linear case when restricted to the Euclidean setting. More precisely, Christ defined a pseudo-accretive system to be a collection of functions $\{b_B\}$ indexed by all balls $B = B(x, r) \subset \mathbf{R}^n$ satisfying (1.10) and (1.11) with $m = 1$, $q = q_1 = \infty$ and dyadic cubes Q replaced with balls B. We say that $\{b_Q^i\}$ for $i = 1, \ldots, m$ is an m-compatible, or just compatible, collection of pseudo-accretive systems if they satisfy (1.10)–(1.12). The proof of Theorem 1.1 follows along the lines of the linear version in [17], with modifications to address difficulties that arise in the setting of multilinear operators.

We also prove that if the square function S defined in (1.2) is bounded as in (1.3), then S is also bounded from $L_c^\infty(\mathbf{R}^n) \times \cdots \times L_c^\infty(\mathbf{R}^n)$ into BMO, where L_c^∞ is the set of L^∞ functions with compact support. Note that L_c^∞ is not a Banach space and S is not a linear operator, so this bound does not mean that S is continuous from $L^\infty \times \cdots \times L^\infty$ into BMO. This is simply an estimate for $f_1, \ldots, f_m \in L_c^\infty$

$$||S(f_1, \ldots, f_m)||_{BMO} \lesssim \prod_{i=1}^{m} ||f_i||_{L^\infty},$$

where the constant is independent of $f_1, \ldots, f_m$ (and in particular the support f_i for $i = 1, \ldots, m$). This means that we cannot use this bound to approximate $S(f_1, \ldots, f_m)$ for $f_1, \ldots, f_m \in L^\infty$, but the estimate is still useful for interpolation. This will be discussed more in depth in section 4.

This permits us to prove the following generalization of the multilinear $T(1)$ theorem of Grafakos–Torres [14] as a sort of multilinear version of the local Tb theorem of Christ in [3].

Theorem 1.2. *Let T be a continuous m linear operator from $\mathscr{S} \times \cdots \times \mathscr{S}$ into $\mathscr{S}'$ with standard Calderón–Zygmund kernel K. Suppose that $T \in WBP$ and there exist $2 \leq q < \infty$ and $1 < q_i < \infty$ with $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ and an m-compatible collection of functions $\{b_Q^i\}$ indexed by dyadic cubes Q and $i = 1, \ldots, m$ such that*

$$(1.14) \qquad \int_Q \left(\int_0^{\ell(Q)} |Q_t T(P_t b_Q^1, \ldots, P_t b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \lesssim |Q|$$

$$(1.15) \qquad\qquad T^{*1}(1, \ldots, 1), \ldots, T^{*m}(1, \ldots, 1) \in BMO.$$

Then T is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $1 < p_i < \infty$ such that (1.7) holds. Here P_t is an approximation to the identity and Q_t a Littlewood–Paley–Stein projection operator both with C_0^∞ convolution kernels.

To state (1.14) more precisely, we mean the following: For any $\varphi, \psi \in C_0^\infty$ such that $\widehat{\varphi}(0) = 1$ and $\widehat{\psi}(0) = 0$, (1.14) holds for $P_t f = \varphi_t * f$ and $Q_t f = \psi_t * f$ where the constant is independent of the dyadic cube Q, but may depend on φ and ψ where $\varphi_t(x) = \frac{1}{t^n} \varphi(\frac{x}{t})$ and $\psi_t(x) = \frac{1}{t^n} \psi(\frac{x}{t})$. The definition of standard Calderón–Zygmund kernel and the weak boundedness property (WBP) for an m linear operator T are given in section 5.

The article is organized in the following way: In the next section we collect some results that will be useful in the proofs of the results stated above. In section 3, we prove the Theorem 1.1 for $p = 2$. In section 4, we precisely state and prove the BMO endpoint estimate claimed above and complete the proof of Theorem 1.1 for all $2 \leq p < \infty$. In section 5, we prove the Theorem 1.2.

2. Preliminary results

In what follows $A \lesssim B$ means $A \leq CB$ for some positive constant C. From this point on we will always work with smooth and compactly supported functions, since the general result follows from density unless otherwise stated.

Define for $t > 0$ the linear and multilinear dyadic average operators

$$A_t f(x) = \frac{1}{|Q(x,t)|} \int_{Q(x,t)} f(x) \, dx,$$

$$\mathbf{A}_t(f_1, \ldots, f_m)(x) = \prod_{i=1}^{m} A_t f_i(x),$$

where $Q(x,t)$ is the smallest dyadic cube containing x with side length $\ell(Q) > t$. Define the linear and multilinear smooth approximation to the identity operators

$$P_t f(x) = \int \varphi_t(x - y) f(y) \, dy,$$

$$\mathbf{P}_t(f_1, \ldots, f_m)(x) = \prod_{i=1}^{m} P_t f_i(x),$$

where $\varphi \in C_0^\infty(\mathbf{R}^n)$ has integral 1.

Definition 2.1. A positive measure $d\mu(x,t)$ on $\mathbf{R}^{n+1}_+ = \{(x,t)\colon x \in \mathbf{R}^n,\ t > 0\}$ is called a Carleson measure if

$$(2.1) \qquad \|d\mu\|_{\mathcal{C}} = \sup_Q \frac{1}{|Q|}\, d\mu(T(Q)) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$, $|Q|$ denotes the Lebesgue measure of the cube Q, $B(Q) = Q \times (0, \ell(Q)]$ denotes the *box* over Q, and $\ell(Q)$ is the side length of Q.

We now state a multilinear version of the $T(1)$ theorem for square functions due to [15], [11] and [12].

Proposition 2.2. [15, 11, 12] *Suppose that the kernel $\theta_t(x, y_1, \ldots, y_m)$ satisfies* (1.4) *and* (1.5)*. If $\Theta_t(1, \ldots, 1) = 0$ for $t > 0$, then the square function defined in* (1.2) *satisfies the bound* (1.3) *for all $1 < p$, $p_i < \infty$, $i = 1, \ldots, m$ satisfying* (1.7)*.*

Remark 2.3. Under extra size conditions on the kernel $\theta_t(x, y_1, \ldots, y_m)$, i.e. if we require $N > mn$ in (1.4) and (1.6), we can apply the vector-valued Calderón–Zygmund theory developed in [15] to extend the theorem above to the complete quasi-Banach case, that is, with $1/m < p \leq 1$.

The following result relates Carleson measures and a special kind of multilinear operator that will be useful for us. An important tool in the proof of the above theorems is the following multilinear version of a theorem of Christ and Journé [4].

Proposition 2.4. *Assume Θ_t and S are defined as in* (1.1) *and* (1.2) *where θ_t satisfies* (1.4)–(1.6)*. If Θ_t satisfies the Carleson measure estimate*

$$(2.2) \qquad \int_Q \int_0^{\ell(Q)} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt\, dx}{t} \lesssim |Q|$$

for all cubes $Q \subset \mathbf{R}^n$, then (1.3) *holds when $p = 2$ and $1 < p_i < \infty$ for $i = 1, \ldots, m$ satisfying* (1.7)*.*

Proof. We decompose $\Theta_t = \Theta_t - M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t + M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t$, where M_b is the operator defined as pointwise multiplication by b. It is clear that $\Theta_t - M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t$ satisfies (1.4), (1.5) and $\Theta_t(1, \ldots, 1) - M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t(1, \ldots, 1) = 0$. Then by Proposition 2.2, it follows that the square function associated to $\Theta_t - M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t$ is bounded for all $1 < p, p_1, \ldots, p_m < \infty$. Using this bound and that $|\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt\, dx}{t}$ is a Carleson measure (by assumption)

$$\|S(f_1, \ldots, f_m)\|_{L^2} \leq \left\|\left(\int_0^\infty |\Theta_t(f_1, \ldots, f_m) - M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t(f_1, \ldots, f_m)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^2}$$

$$+ \prod_{i=1}^m \left(\int_{\mathbf{R}^{n+1}_+} |\mathbf{P}_t f_i(x)|^{p_i} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dx\, dt}{t}\right)^{\frac{1}{p_i}}$$

$$\lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

The final inequality uses the well-known Carleson measure estimate: If $d\mu(x,t)$ is a Carleson measure, then $P\colon f \mapsto P_t f(x)$ is bounded from $L^q(\mathbf{R}^n)$ into $L^q(\mathbf{R}^{n+1}_+, d\mu)$ for $1 < q < \infty$. $\qquad\square$

The next result allows us to compare the multilinear dyadic averaging operators $\mathbf{A}_t$ and the multilinear smooth approximation to the identity operators $\mathbf{P}_t$. This comparison principle will be important in the proof of Theorem 1.1. This is a particular case of a multilinear version of a result of Duoandikoetxea–Rubio de Francia in [8].

Proposition 2.5. *Let A_t, P_t, $\mathbf{A}_t$ and $\mathbf{P}_t$ be as above. Then for all $1 < p_i < \infty$, $i = 1, \ldots, m$, we have the bound*

$$\left\| \left(\int_0^\infty |\mathbf{A}_t(f_1, \ldots, f_m) - \mathbf{P}_t(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

Note that this even holds for $\frac{1}{m} < p < \infty$ as long as $1 < p_i < \infty$.

Proof. Define for $j = 1, \ldots, m$

$$\mathbf{E}_t^j(f_1, \ldots, f_m) = \left(\prod_{i=1}^{j-1} A_t f_i \right) (A_t f_j - P_t f_j) \left(\prod_{i=j+1}^m P_t f_i \right).$$

Here we use the convection that $\prod_{i=1}^0 A_i = \prod_{i=m+1}^m P_t = 1$. Then we have the following decomposition by successively adding and subtracting the term $A_t f_1 \cdots A_t f_j P_t \cdot f_{j+1} \cdots P_t f_m$

$$\mathbf{A}_t(f_1, \ldots, f_m) - \mathbf{P}_t(f_1, \ldots, f_m) = \mathbf{E}_t^1(f_1, \ldots, f_m) + A_t f_1 \left(\prod_{i=2}^m A_t f_i - \prod_{i=2}^m P_t f_i \right)$$

$$= \sum_{j=1}^2 \mathbf{E}_t^j(f_1, \ldots, f_m) + A_t f_1 A_t f_2 \left(\prod_{i=3}^m A_t f_i - \prod_{i=3}^m P_t f_i \right) = \sum_{j=1}^m \mathbf{E}_t^j(f_1, \ldots, f_m).$$

It is a standard argument to show that $\sup_{t>0} |P_t f(x)| \lesssim Mf(x)$ where M is the Hardy–Littlewood maximal function, and the same inequality holds replacing P_t with A_t. Then we use the linear bound of $A_t - P_t$ which was proved by Duoandikoetxea–Rubio de Francia [8]

$$\left\| \left(\int_0^\infty |\mathbf{E}_t^j(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left(\int_0^\infty |(A_t - P_t)f_j|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \prod_{i \neq j} Mf_i \right\|_{L^p}$$

$$\leq \left\| \left(\int_0^\infty |(A_t - P_t)f_j|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^{p_j}} \prod_{i \neq j} \|Mf_i\|_{L^{p_i}} \lesssim \prod_{i=1}^m \|f\|_{L^{p_i}}.$$

The square function bound for $\mathbf{A}_t - \mathbf{P}_t$ easily follows. $\qquad\qquad\qquad\square$

3. Proof of Theorem 1.1 with $p = 2$

We proceed by reducing our arguments to the dyadic case. Using dyadic covering properties it is easy to see that if (2.2) holds for all dyadic cubes, then (2.2) holds for all cubes Q with a slightly larger constant. In the following, we prove (2.2) for dyadic cubes to conclude (1.3) for $p = 2$, and then proceed with other techniques in the next section.

3.1. Decomposition of dyadic cubes. We start with a proposition similar to one used in [17], applied to m collections of pseudo-accretive systems $\{b_Q^i\}$ for $i = 1, \ldots, m$.

Proposition 3.1. *Given a m-compatible system of functions $\{b_Q^i\}$ indexed by dyadic cubes for $i = 1, \ldots, m$ satisfying (1.10)–(1.13), there exists a collection of non-overlapping dyadic subcubes of Q, $\{Q_k\}$, and $\eta \in (0, 1)$*

$$(3.1) \qquad \sum_k |Q_k| < (1 - \eta)|Q|,$$

where η does not depend on Q, and for $t > \tau_Q(x)$ and $x \in Q$

$$(3.2) \qquad \frac{1}{2B_2 B_3} < \prod_{i=1}^m |A_t b_Q^i(x)| \quad (\text{here WLOG we assume that } B_2, B_3 \geq 1),$$

where

$$(3.3) \qquad \tau_Q(x) = \begin{cases} \ell(Q_k), & x \in Q_k, \\ 0, & x \in E, \end{cases}$$

$$E = Q \backslash \bigcup_k Q_k.$$

Proof. Fix a dyadic cube $Q \subset \mathbf{R}^n$ and define

$$a = \frac{1}{|Q|} \int_Q \prod_{i=1}^m b_Q^i(x) \, dx$$

which satisfies $|a| \geq \frac{1}{B_2}$, where B_2 is from (1.11). Now choose from the dyadic children of Q the cubes that are maximal with respect to the property

$$\text{Re}\left[\frac{1}{a|Q_j|} \int_{Q_j} \prod_{i=1}^m b_Q^i(x) \, dx\right] \leq \frac{1}{2},$$

i.e. $Q_j \subset Q$ is the largest dyadic cube such that the above inequality holds. By the properties of dyadic cubes, these maximal cubes are non-overlapping. This stopping time criterion well defines a collection of cubes since

$$\text{Re}\left[\frac{1}{a|Q|} \int_Q \prod_{i=1}^m b_Q^i(x) \, dx\right] = 1.$$

If $x \in Q_k$ for some k and $t > \tau_Q(x)$, then using (1.12)

$$|\mathbf{A}_t(b_Q^1, \ldots, b_Q^m)(x)| = \prod_{i=1}^m \left|\frac{1}{|Q(x,t)|} \int_{Q(x,t)} b_Q^i(y) \, dy\right|$$

$$\geq \frac{1}{B_3} \left|\frac{1}{|Q(x,t)|} \int_{Q(x,t)} \prod_{i=1}^m b_Q^i(y) \, dy\right|$$

$$\geq \frac{|a|}{B_3} \text{Re}\left(\frac{1}{a|Q(x,t)|} \int_{Q(x,t)} \prod_{i=1}^m b_Q^i(y) \, dy\right) \geq \frac{1}{2B_2 B_3}.$$

Also if $x \in E$, then again using (1.12) and by the stopping time criterion it follows that

$$
\begin{aligned}
|\mathbf{A}_t(b_Q^1, \dots, b_Q^m)(x)| &= \prod_{i=1}^m \left| \frac{1}{|Q(x,t)|} \int_{Q(x,t)} b_Q^i(y) \, dy \right| \\
&\geq \frac{1}{B_3} \left| \frac{1}{|Q(x,t)|} \int_{Q(x,t)} \prod_{i=1}^m b_Q^i(y) \, dy \right| \\
&\geq \frac{|a|}{B_3} \operatorname{Re}\left(\frac{1}{a|Q(x,t)|} \int_{Q(x,t)} \prod_{i=1}^m b_Q^i(y) \, dy \right) \geq \frac{1}{2B_2 B_3}.
\end{aligned}
$$

Now we also have for $i = 1, \dots, m$ that

$$
\begin{aligned}
|Q| &= \operatorname{Re}\left[\frac{1}{a} \int_Q \prod_{i=1}^m b_Q^i(x) \, dx \right] \leq \sum_k \operatorname{Re}\left[\frac{1}{a} \int_{Q_k} \prod_{i=1}^m b_Q^i(x) \, dx \right] + \int_E \left| \prod_{i=1}^m b_Q^i(x) \right| dx \\
&\leq \frac{1}{2} \sum_k |Q_k| + |E|^{\frac{1}{q'}} \left(\int_E \left| \prod_{i=1}^m b_Q^i(x) \right|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{1}{2}|Q| + |E|^{\frac{1}{q'}} \prod_{i=1}^m \left(\int_Q |b_Q^i(x)|^{q_i} \, dx \right)^{\frac{1}{q_i}} \leq \frac{1}{2}|Q| + B_1^m |E|^{\frac{1}{q'}} |Q|^{\frac{1}{q}}.
\end{aligned}
$$

It follows that $\eta|Q| < |E|$ where we may take $\eta = \frac{1}{(2B_1^m)^{q'}} \in (0,1)$. $\square$

3.2. Reduction to Carleson estimates. We pause for a moment to discuss the strategy of the remainder of the proof of Theorem 1.1 for $p = 2$. By Proposition 2.4 and the discussion at the beginning of this section, it is sufficient to show that the estimate (2.2) holds for dyadic cubes. In order to show this, we prove an intermediate estimate: For all dyadic cubes $Q \subset \mathbf{R}^n$

$$
(3.4) \qquad \int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \leq C|Q|.
$$

The remainder of this section is dedicated to proving (3.4), and the next section completes the proof of Theorem 1.1 for $p = 2$ by proving (2.2) from the reduction in this section.

Proposition 3.2. *For all dyadic cubes $Q \subset \mathbf{R}^n$, (3.4) holds with τ_Q defined in (3.3).*

Proof. We have from Proposition 3.1 that $|\mathbf{A}_t(b_Q^1, \dots, b_Q^m)(x)| \geq \frac{1}{2B_2^m B_3}$, so it follows that

$$
\int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx
$$

$$
\leq 2B_2^m B_3 \int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |\Theta_t(1, \dots, 1)(x) \mathbf{A}_t(b_Q^1, \dots, b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx.
$$

Now we consider the operator $M_{\Theta_t(1,\ldots,1)} \mathbf{A}_t(f_1, \ldots, f_m)$, which we decompose in the following way

$$M_{\Theta_t(1,\ldots,1)} \mathbf{A}_t = M_{\Theta_t(1,\ldots,1)}(\mathbf{A}_t - \mathbf{P}_t) + (M_{\Theta_t(1,\ldots,1)}\mathbf{P}_t - \Theta_t) + \Theta_t = R_t^{(1)} + R_t^{(2)} + \Theta_t$$

By Proposition 2.5, it follows that

$$\int_{\mathbf{R}^n} \left(\int_0^\infty |R_t^{(1)}(b_Q^1, \ldots, b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \lesssim \prod_{i=1}^m \|b_Q^i\|_{L^{q_1}}^q \lesssim |Q|.$$

Using Proposition 2.2, it follows that the $R_t^{(2)}$ term is controlled as desired

$$\int_{\mathbf{R}^n} \left(\int_0^\infty |R_t^{(2)}(b_Q^1, \ldots, b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \lesssim |Q|,$$

and by hypothesis (1.13),

$$\int_Q \left(\int_0^{\ell(Q)} |\Theta_t(b_Q^1, \ldots, b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \leq B_4 |Q|.$$

Then we may choose C independent of Q such that (3.4) holds. $\qquad\square$

3.3. End of the proof. Finally we use the reduction from the previous section to complete the proof of Theorem 1.1.

Lemma 3.3. *There exist $N > 0$ and $\beta \in (0,1)$ such that for every dyadic cube Q*

$$(3.5) \qquad\qquad |\{x \in Q \colon g_Q(x) > N\}| \leq (1 - \beta)|Q|,$$

where

$$(3.6) \qquad\qquad g_Q(x) = \left(\int_0^{\ell(Q)} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\tau_Q(x)$ is defined as in (3.3).

Proof. Fix a dyadic cube $Q \subset \mathbf{R}^n$, and define for $N > 0$

$$\Omega_N = \{x \in Q \colon g_Q(x) > N\}$$

Let Q_k and E be as in Proposition 3.1, without loss of generality take $N, C > 1$, and using Chebychev's inequality it follows that

$$|\Omega_N| \leq \sum_k |Q_k| + |\{x \in E \colon g_Q(x) > N\}| \leq (1 - \eta)|Q| + \frac{C}{N^q}|Q|$$

where C is chosen from (3.4) in Proposition 3.2 as discussed above. Now fix N large enough so that $\frac{C}{N^q} < \eta/2$. Then (3.5) easily follows

$$|\Omega_N| \leq (1 - \eta)|Q| + \frac{C}{N^q}|Q| < (1 - \beta)|Q|$$

where $\beta = \frac{\eta}{2} > 0$. $\qquad\square$

We can finally prove the main theorem for $p = 2$.

Proof. Fix $\epsilon \in (0, 1)$ and define for dyadic cube $Q \subset \mathbf{R}^n$ with $\ell(Q) > \epsilon$

$$g_{Q,\epsilon}(x) = \left(\int_{\epsilon}^{\min(1/\epsilon, \ell(Q))} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and $g_{Q,\epsilon} = 0$ if $\ell(Q) \le \epsilon$. Also define

$$K(\epsilon) = \sup_Q \frac{1}{|Q|} \int_Q g_{Q,\epsilon}(x)\, dx$$

where the supremum is over all dyadic cubes. Fix a dyadic cube Q and define

$$\Omega_{N,\epsilon} = \{x \in Q \colon g_{Q,\epsilon}(x) > N\}.$$

Note that $g_{Q,\epsilon}$ is defined depending only on the cube Q and ϵ, completely independent of Q_k, $\tau_Q(x)$ and η. It follows from (1.6) that $\Theta_t(1, \dots, 1)(x)$ is γ-Hölder continuous in x and hence so is $g_{Q,\epsilon}$ (with constant depending on ϵ). Fix $0 < \eta < 1$ small enough so that $N(1-\eta) > n$, and we have

$$|g_{Q,\epsilon}(x) - g_{Q,\epsilon}(x')|^2 \lesssim \int_{\epsilon}^{\min(1/\epsilon, \ell(Q))} (t^{-1}|x - x'|)^{2\gamma\eta}$$

$$\cdot \prod_{i=1}^{m} \left(\int_{\mathbf{R}^n} \left(\frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^{N(1-\eta)}} + \frac{t^{-n}}{(1 + t^{-1}|x' - y_i|)^{N(1-\eta)}} \right) dy_i \right)^2 \frac{dt}{t}$$

$$\lesssim \epsilon^{-2-\gamma}|x - x'|^{2\gamma\eta}.$$

Then $g_{Q,\epsilon}$ is continuous, $\Omega_{N,\epsilon}$ is open, and so we may make the Whitney decomposition Q_j of $\Omega_{N,\epsilon}$. That is there exists a collection of cubes $\{Q_j\}$ such that

(3.7)
$$\bigcup_j Q_j = \Omega_{N,\epsilon},$$

(3.8)
$$\sqrt{n}\,\ell(Q_j) \le \text{dist}(Q_j, \Omega^c) \le 4\sqrt{n}\,\ell(Q_j),$$

(3.9)
$$\partial Q_j \cap Q_k \neq \emptyset \implies \frac{1}{4} \le \frac{\ell(Q_j)}{\ell(Q_k)} \le 4,$$

(3.10)
given a cube, there are at most 12^n that touch it.

Then, if $F_{N,\epsilon} = Q \backslash \Omega_{N,\epsilon}$,

$$\int_Q g_{Q,\epsilon}^2(x)\, dx = \int_{F_{N,\epsilon}} g_{Q,\epsilon}^2(x)\, dx + \sum_j \int_{Q_j} g_{Q,\epsilon}^2(x)\, dx$$

$$\le N^2|Q| + \sum_j \int_{Q_j} \int_{\epsilon}^{\min(1/\epsilon, \ell(Q_j))} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt\, dx}{t}$$

$$+ \sum_j \int_{Q_j} \int_{\max(\epsilon, \ell(Q_j))}^{\min(1/\epsilon, \ell(Q))} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt\, dx}{t}$$

$$\le N^2|Q| + K(\epsilon) \sum_j |Q_j| + \sum_j \int_{Q_j} \int_{\max(\epsilon, \ell(Q_j))}^{\min(1/\epsilon, \ell(Q))} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt\, dx}{t}$$

$$\le N^2|Q| + K(\epsilon)(1 - \beta)|Q| + \sum_j \int_{Q_j} \int_{\max(\epsilon, \ell(Q_j))}^{\min(1/\epsilon, \ell(Q))} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt\, dx}{t}.$$

To control the last term, since Q_j is a Whitney decomposition, there exists $x_j \in F_{N,\epsilon}$ such that

$$\operatorname{dist}(x_j, Q_j) \leq (4\sqrt{n} + 1)\ell(Q_j).$$

We have for $x \in Q_j$

$$|\Theta_t(1, \ldots, 1)(x) - \Theta_t(1, \ldots, 1)(x_j)|$$

$$\leq \int_{\mathbf{R}^{mn}} |\theta_t(x, y_1, \ldots, y_m) - \theta_t(x_j, y_1, \ldots, y_m)| \prod_{i=1}^m dy_i$$

$$\lesssim (t^{-1}|x - x_j|)^{\gamma\eta} \prod_{i=1}^m \int_{\mathbf{R}^n} \left(\frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^{N(1-\eta)}} + \frac{t^{-n}}{(1 + t^{-1}|x_j - y_i|)^{N(1-\eta)}} \right) dy_i$$

$$\lesssim (t^{-1}\ell(Q_j))^{\gamma\eta}.$$

So choose c_1 which depends only on the dimension such that the inequality

$$|\Theta_t(1, \ldots, 1)(x) - \Theta_t(1, \ldots, 1)(x_j)| \leq c_1(t^{-1}\ell(Q_j))^{\gamma\eta}$$

holds for all $x \in Q_j$. Then

$$\sum_j \int_{Q_j} \int_{\max(\epsilon, \ell(Q_j))}^{\min(1/\epsilon, \ell(Q))} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt\, dx}{t}$$

$$\leq \sum_j \int_{Q_j} \int_{\max(\epsilon, \ell(Q_j))}^{c_1\ell(Q_j)} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt\, dx}{t}$$

$$+ \sum_j \int_{Q_j} \int_{c_1\ell(Q)}^{\min(\ell(Q), 1/\epsilon)} |\Theta_t(1, \ldots, 1)(x_j)|^2 \frac{dt\, dx}{t}$$

$$+ \sum_j \int_{Q_j} \int_{c_1\ell(Q)}^{\min(\ell(Q), 1/\epsilon)} |\Theta_t(1, \ldots, 1)(x) - \Theta_t(1, \ldots, 1)(x_j)|^2 \frac{dt\, dx}{t}$$

$$= I + II + III.$$

We have that

$$I \leq \|\Theta_t(1, \ldots, 1)\|_{L^\infty}^2 \sum_j \int_{Q_j} \int_{\ell(Q_j)}^{c_1\ell(Q_j)} \frac{dt\, dx}{t} \lesssim c_1 \sum_j |Q_j| \lesssim |Q|.$$

Since $x_j \in F_{N,\epsilon}$ and $g_{Q,\epsilon}(x_j) \leq N$, it follows that

$$II \leq \sum_j \int_{Q_j} \int_0^{\ell(Q)} |\Theta_t(1, \ldots, 1)(x_j)|^2 \frac{dt\, dx}{t} = \sum_j |Q_j| g_{Q,\epsilon}(x_j)^2 \lesssim N^2 |Q|.$$

For all $x \in Q_j$, $|\Theta_t(1, \ldots, 1)(x) - \Theta_t(1, \ldots, 1)(x_j)| \leq c_1(t^{-1}\ell(Q_j))^{\gamma\eta}$, so

$$III \lesssim \sum_j \int_{Q_j} \int_{c_1\ell(Q_j)}^\infty (t^{-1}\ell(Q_j))^{\gamma\eta} \frac{dt\, dx}{t} \lesssim \sum_j |Q_j| \leq |Q|.$$

Therefore $K(\epsilon) \leq C(1 + N^2) + (1 - \beta)K(\epsilon)$ and hence

$$K(\epsilon) \leq \frac{C(1 + N^2)}{\beta}.$$

Therefore

$$\frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t(1,\dots,1)(x)|^2 \frac{dt\,dx}{t}$$

$$= \sup_{0<\epsilon<1} \sup_{\ell(Q)>\epsilon} \frac{1}{|Q|} \int_Q \int_\epsilon^{\ell(Q)} |\Theta_t(1,\dots,1)(x)|^2 \frac{dt\,dx}{t} = \sup_{0<\epsilon<1} K(\epsilon) \le \frac{C(1+N^2)}{\beta}.$$

Hence $|\Theta_t(1,\dots,1)(x)|^2 \frac{dt\,dx}{t}$ is a Carleson measure and by Proposition 2.2 the square function bound (1.3) holds with constant $C(1+N^2)/\beta$ for $p=2$ and $1<p_1,\dots,p_m<\infty$. $\square$

This proves theorem 1.1 for $p=2$. In the following section we prove (1.3) holds for all $2 \le p < \infty$ and $1 < p_1,\dots,p_m < \infty$, but first we make some remarks on compatible pseudo-accretive systems.

3.4. A comment on compatible pseudo-accretive systems. The purpose of this discussion is to better understand the conditions (1.11) and (1.12) through various examples. In the first example we construct a class of non-trivial classes of compatible pseudo-accretive systems.

Example 3.4.1. Suppose there exists $\epsilon > 0$ such that $\epsilon \le b_Q^i(x) \le \epsilon^{-1}$ for a.e. $x \in Q$, all dyadic cubes $Q \subset \mathbf{R}^n$ and each $i = 1,\dots,m$, then (1.11) and (1.12) hold as well,

$$\epsilon^m \le \left| \frac{1}{|R|} \int_R \prod_{i=1}^m b_Q^i(x)\,dx \right| \le \epsilon^{-m} \le \epsilon^{-2m} \prod_{i=1}^m \left| \frac{1}{|R|} \int_R b_Q^i(x)\,dx \right|.$$

Notice that this is a uniform condition for b_Q^i. That is there is no dependence between the functions, as long as they are each in this class of functions. This class of functions includes many commonly used functions. For example, the following functions defined for each dyadic cube $Q \subset \mathbf{R}^n$ satisfy $\epsilon < b_Q < \epsilon^{-1}$ uniformly on the cube Q for some ϵ.

Characteristic functions: $b_Q(x) = \chi_Q(x),$

Gaussian functions: $b_Q(x) = e^{-\frac{|x-x_Q|^2}{\ell(Q)^2}},$

Poisson kernels: $b_Q(x) = \dfrac{\ell(Q)^{n+1}}{(\ell(Q)^2 + |x-x_Q|^2)^{\frac{n+1}{2}}}.$

Example 3.4.2. Consider the pseudo-accretive systems on $\mathbf{R}$ for dyadic cubes $Q_{j,k} = [j2^{-k}, (j+1)2^{-k})$ defined

$$b_{Q_{j,k}}^1 = b_{j,k}^1 = \chi_{[j2^{-k},(j+3/4)2^{-k})} - \chi_{[(j+3/4)2^{-k},(j+1)2^{-k})},$$
$$b_{Q_{j,k}}^2 = b_{j,k}^2 = \chi_{[(j+1/4)2^{-k},(j+1)2^{-k})} - \chi_{[j2^{-k},(j+1/4)2^{-k})}.$$

It follows that $b_{j,k}^i$ satisfies (1.11) for $i = 1,2$ by a quick computation

$$\frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} b_{j,k}^1(x)\,dx = \frac{1}{|Q_{j,k}|} \int_{Q_{j,k}} b_{j,k}^2(x)\,dx = \frac{1}{2}.$$

It is a bit more complicated to see that $b_{j,k}^1, b_{j,k}^2$ satisfy (1.12), but it does hold: For $R = Q_{j,k}$, it follows that the left hand side of (1.12) is zero so the inequality holds.

Now if $R \subset Q_{j,k}$ is any dyadic subcube contained in $[j2^{-k}, (j+1/2)2^{-k})$, then $b_{j,k}^1 = 1$ on R and

$$\frac{1}{|R|} \int_R b_{j,k}^1(x) b_{j,k}^2(x) \, dx = \frac{1}{|R|} \int_R b_{j,k}^2(x) \, dx = \prod_{i=1}^2 \frac{1}{|R|} \int_R b_{j,k}^i(x) \, dx.$$

A symmetric argument holds when $R \subset [(j+1/2)2^{-k}, (j+1)2^{-k})$. Therefore $b_{j,k}^1, b_{j,k}^2$ are compatible pseudo-accretive systems. This example is especially interesting because there are subcubes where $b_{j,k}^i$ has mean zero, $b_{j,k}^1 \cdot b_{j,k}^2$ has mean zero, but the particular structure of these functions allow for (1.11) and (1.12) to hold.

Example 3.4.3. There exist pseudo-accretive systems that are not compatible. To construct such a system, we consider the bilinear setting and $\mathbf{R}$. Consider the cube $Q = [0, 2] \subset \mathbf{R}$ and define

$$b_Q = b_Q^1(x) = b_Q^2(x) = \left(x - \frac{1}{2} \right) \chi_{[0,2]}(x).$$

We have that b_Q^i satisfy (1.11) for $i = 1, 2$

$$\left| \int_{[0,2]} b_Q(x) \, dx \right| = 1,$$

but if we consider the dyadic subcube $[0, 1] \subset [0, 2]$, the functions violate (1.12)

$$\left| \int_{[0,1]} b_Q^1(x) b_Q^2(x) \, dx \right| = \int_0^1 \left(x^2 - x + \frac{1}{2} \right) dx = \frac{1}{3},$$

$$\prod_{i=1}^2 \left| \int_{[0,1]} b_Q^i(x) \, dx \right| = \left(\int_0^1 \left(x - \frac{1}{2} \right) dx \right)^2 = 0.$$

Here it is apparent that the failure of condition (1.12) is caused by the cancellation of b_Q^1 and b_Q^2 in the same location.

From Examples 3.4.1 and 3.4.2, we can see that there are non-trivial compatible pseudo-accretive system, even some with cancellation on dyadic subcubes. Example 3.4.3 demonstrates that there are pseudo-accretive systems that aren't compatible, and furthermore the functions in Example 3.4.3 fail to satisfy the compatibility condition (1.12) because they have cancellation behavior in the same location.

4. Extending square function bounds

In this section we prove a multilinear BMO bound and use it as an endpoint for interpolation. More precisely, we prove the following $L_c^\infty \times \cdots \times L_c^\infty \to BMO$ bound.

Theorem 4.1. *Suppose Θ_t satisfies (1.4)-(1.6) and the square function S associated to Θ_t is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some $1 \le p, p_i \le \infty$ that satisfy (1.7). Then for all $f_1, \ldots, f_m \in L_c^\infty$*

$$(4.1) \qquad \qquad \|S(f_1, \ldots, f_m)\|_{BMO} \lesssim \prod_{i=1}^m \|f_i\|_{L^\infty},$$

where the constant is independent of f_i (and in particular the support of f_i) for $i = 1, \ldots, m$.

This is essentially a square function version of a corresponding result for multilinear Calderón–Zygmund operators from Grafakos–Torres [13]: If a multilinear Calderón–Zygmund operator T is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some $1 < p, p_1, \ldots, p_m < \infty$, then T is bounded from $L_c^\infty \times \cdots \times L_c^\infty$ into BMO. In [13], the authors prove this using an inductive argument by reducing the m-linear case to the $m-1$ linear one. Here we present a direct multilinear proof adapted from the classical linear version due to Spanne [25], Peetre [23] and Stein [26], but prior to this proof we briefly discuss why we don't conclude here that S is bounded from $L^\infty \times \cdots \times L^\infty$ into BMO.

In [13], the authors also conclude that if an m-linear Calderón–Zygmund operator T is bounded, then T is bounded from $L^\infty \times \cdots \times L^\infty$ into BMO estimate. One difficultly in this problem is that T is not necessarily even defined for $f_1, \ldots, f_m \in L^\infty$. So one must define T for $f_1, \ldots, f_m \in L^\infty$, and the definition for such functions must be consistent with the given definition of T in the case that $f_i \in L^{p_i} \cap L^\infty$. As it turns out (see [13]), it is reasonable to define for $f_1, \ldots, f_m \in L^\infty$

$$T(f_1, \ldots, f_m) = \lim_{R \to \infty} T(f_1 \chi_{B(0,R)}, \ldots, f_m \chi_{B(0,R)})$$
$$- \int_{|y_i|>1} K(0, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) \chi_{B(0,R)}(y_i) \, dy_i,$$

where the limit is taken in the dual of $C_{c,0}^\infty(\mathbf{R}^n)$. Here $C_{c,0}^\infty(\mathbf{R}^n)$ is the collection of all smooth compactly supported functions with mean zero. As expected, this well defines T on $L^\infty \times \cdots \times L^\infty$ modulo a constant, which is permissible as an element of BMO. The existence of this limit follows from the linearity and kernel estimates of T. Along with the $L_c^\infty \times \cdots \times L_c^\infty \to BMO$ estimate for T, the existence of this limit implies that T is bounded from $L^\infty \times \cdots \times L^\infty$ into BMO.

It is typically reasonable to expect the square function S defined in (1.2) to satisfy the same boundedness properties as m linear Calderón–Zygmund operators, but despite the estimate for S on $L_c^\infty \times \cdots \times L_c^\infty$, we are unable to make the same boundedness conclusion on $L^\infty \times \cdots \times L^\infty$ for S as can be made for T in the last paragraph. The reason for this essentially comes down to the fact that S is not a linear operator. If one tries to mimic the proof from [13] replacing T with S, the above limit does not necessarily exist. So the problem becomes finding a suitable definition for S on $L^\infty \times \cdots \times L^\infty$, as the classical definition does not necessarily exist (at least using the same proof techniques). Another approach to define S on $L^\infty \times \cdots \times L^\infty$ is to view Θ_t as an m-linear operator taking values in $L^2(\mathbf{R}_+, \frac{dt}{t})$. In this case one may be able to define S as a weak limit of an appropriate space of smooth functions taking values in $L^2(\mathbf{R}_+, \frac{dt}{t})$. Since we only need the previous estimate for compactly supported functions to prove our interpolation theorem, we will not pursue this approach here.

Proof. Assume that $f_i \in L_c^\infty$ for $i = 1, \ldots, m$ and $B = B(x_B, R) \subset \mathbf{R}^n$ is a ball for some $R > 0$ and $x_B \in \mathbf{R}^n$. Define

$$c_B = \left(\int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x_B) - \Theta_t(f_1 \chi_{2B}, \ldots, f_m \chi_{2B})(x_B)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

which exists since $f_1, \ldots, f_m \in L^p$ for all $1 \le p \le \infty$ since we have assumed that $f_1, \ldots, f_m$ are compactly supported. Then it follows that

$$
\int_B |S(f_1, \ldots, f_m)(x) - c_B| \, dx \le \int_B S(f_1\chi_{2B}, \ldots, f_m\chi_{2B})(x)| \, dx
$$

$$
+ \sum_{\vec{F} \in \Lambda} \int_B \left(\int_0^R (|\Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x)| + |\Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x_B)|)^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx
$$

$$
+ \sum_{\vec{F} \in \Lambda} \int_B \left(\int_R^\infty |\Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x) - \Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x_B)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx
$$

$$
= I + II + III,
$$

where

$$
\Lambda = \{(F_1, \ldots, F_m) \colon F_i = 2B \text{ or } F_i = (2B)^c\} \setminus \{(2B, \ldots, 2B)\}.
$$

That is Λ is the collection of m vectors of sets with with all combinations of components $2B$ and $(2B)^c$ except for $(2B, \ldots, 2B)$. Note that $|\Lambda| = 2^m - 1$. We can easily estimate I using that S is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p

$$
I \le |2B|^{\frac{1}{p'}} ||S(f_1\chi_{2B}, \ldots, f_m\chi_{2B})||_{L^p} \lesssim |B|^{\frac{1}{p'}} \prod_{i=1}^m ||f_i\chi_{2B}||_{L^{p_i}} \lesssim |B| \prod_{i=1}^m ||f_i||_{L^\infty}.
$$

Then to bound II, take $\vec{F} \in \Lambda$, $x \in B$, and we first look at the integrand for $x \in B$

$$
|\Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x)| \lesssim \int_{\mathbf{R}^n} t^{-mn} \prod_{i=1}^m \frac{f_i(y_i)\chi_{F_i}(y_i)}{(1 + t^{-1}|x - y_i|)^{N+\gamma}} \, dy_i
$$

$$
\le \prod_{j=1}^m ||f_j||_{L^\infty} \left(\prod_{i:F_i=2B} \int_{\mathbf{R}^n} \frac{1}{(1 + |x - y_i|)^{N+\gamma}} \, dy_i \right) \left(\prod_{i:F_i=(2B)^c} \int_{|y_i|>R} \frac{2^{N+\gamma} t^{N+\gamma-n}}{|y_i|^{N+\gamma}} \, dy_i \right)
$$

$$
\lesssim \prod_{j=1}^m ||f_j||_{L^\infty} \left(\prod_{i:F_i=(2B)^c} \frac{t^{N+\gamma-n}}{R^{N+\gamma-n}} \right) \lesssim t^{k_0(N+\gamma-n)} R^{-k_0(N+\gamma-n)} \prod_{j=1}^m ||f_j||_{L^\infty},
$$

where $k_0 \in \mathbf{N}$ is the number of terms in $\vec{F}$ such that $F_i = (2B)^c$. It is important here that $k_0 \ge 1$. Now recall that $|\Lambda| = 2^m - 1$, and it is now trivial to bound I,

$$
II \lesssim \prod_{j=1}^m ||f_j||_{L^\infty} \int_B \left(\int_0^R (t^{k_0(N+\gamma-n)} R^{-k_0(N+\gamma-n)})^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \lesssim |B| \prod_{j=1}^m ||f_j||_{L^\infty}.
$$

To bound III, for a fixed $\vec{F} \in \Lambda$ and $x \in B$, we look at the integrand

$$
|\Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x) - \Theta_t(f_1\chi_{F_1}, \ldots, f_m\chi_{F_m})(x_B)|
$$

$$
\lesssim \int_{\mathbf{R}^{mn}} t^{-mn} (t^{-1}|x - x_B|)^\gamma \prod_{i=1}^m \left(\frac{f_i(y_i)\chi_{F_i}(y_i)}{(1 + t^{-1}|x - y_i|)^{N+\gamma}} + \frac{f_i(y_i)\chi_{F_i}(y_i)}{(1 + t^{-1}|x_B - y_i|)^{N+\gamma}} \right) dy_i
$$

$$\lesssim t^{-\gamma} R^{\gamma} \prod_{i=1}^{m} ||f_i||_{L^{\infty}} \int_{\mathbf{R}^n} \left(\frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^{N+\gamma}} + \frac{t^{-n}}{(1 + t^{-1}|x_B - y_i|)^{N+\gamma}} \right) dy_i$$

$$\lesssim t^{-\gamma} R^{\gamma} \prod_{i=1}^{m} ||f_i||_{L^{\infty}}.$$

Then once more using that $|\Lambda| = 2^m - 1$, we can bound III

$$III \lesssim |B| \prod_{j=1}^{m} ||f_j||_{L^{\infty}} \left(\int_{R}^{\infty} (t^{-\gamma} R^{\gamma})^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim |B| \prod_{j=1}^{m} ||f_j||_{L^{\infty}}.$$

Then for $f_i \in L_c^{\infty}$, $i = 1, \ldots, m$, (4.1) holds with constant independent of $f_1, \ldots, f_m$.
$\square$

Corollary 4.2. *If θ_t satisfies (1.4)–(1.6) and (1.3) holds for $p = 2$ and $1 < p_1, \ldots, p_m < \infty$, then (1.3) holds for all $2 \le p < \infty$ and $1 < p_1, \ldots, p_m < \infty$.*

Proof. Define the sharp maximal function

$$M^{\#} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

By definition we have that $||f||_{BMO} = ||M^{\#}f||_{L^{\infty}}$. Also it is easy to see that $||M^{\#}f||_{L^p} \lesssim ||Mf||_{L^p}$, where M is the Hardy–Littlewood maximal operator. Then using the L^2 bound of M and the hypothesis on S, it follows that for all $f_1, \ldots, f_m \in L_c^{\infty}$

$$||M^{\#} S(f_1, \ldots, f_m)||_{L^2} \lesssim ||MS(f_1, \ldots, f_m)||_{L^2} \lesssim \prod_{i=1}^{m} ||f_i||_{L^{p_i}}$$

and by theorem 4.1

$$||M^{\#} S(f_1, \ldots, f_m)||_{L^{\infty}} = ||S(f_1, \ldots, f_m)||_{BMO} \lesssim \prod_{i=1}^{m} ||f_i||_{L^{\infty}}.$$

Then by multilinear Marcinkiewicz interpolation, it follows that

$$||M^{\#} S||_{L^p} \lesssim \prod_{i=1}^{m} ||f_i||_{L^{p_i}}$$

for all $f_i \in L_c^{\infty}$ where $2 \le p < \infty$ and $1 < p_1, \ldots, p_m < \infty$ satisfying (1.7) with constant independent of $f_1, \ldots, f_m$. Since L_c^{∞} is dense in L^q for all $1 \le q < \infty$, it follows that $M^{\#}S$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $2 \le p < \infty$ and $1 < p_1, \ldots, p_m < \infty$. We have also from a result of Fefferman–Stein [9] that $||f||_{L^q} \lesssim ||M^{\#}f||_{L^q}$ when $1 \le q < \infty$ and f satisfies $M^d f \in L^q$ where M^d is the dyadic maximal function (in particular, when $f \in L^q$ for $1 < q < \infty$). Therefore

$$||S(f_1, \ldots, f_m)||_{L^p} \lesssim ||M^{\#} S(f_1, \ldots, f_m)||_{L^p} \lesssim \prod_{i=1}^{m} ||f_i||_{L^{p_i}},$$

which completes the proof. $\square$

5. Proof of Theorem 1.2

The way we will prove theorem 1.2 is to first assume that T satisfies

$$(5.1) \qquad T^{*1}(1,\ldots,1) = \cdots = T^{*m}(1,\ldots,1) = 0$$

in place of (1.15), and prove that T is bounded. Then we proceed by using a multilinear version of the T1 paraproduct used in the original T1 theorem by David–Journé [6]. The bilinear version of this paraproduct was constructed in [16].

Lemma 5.1. *Given $\beta \in BMO$, there exists a multilinear Calderón–Zygmund operator L bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $1 < p_i < \infty$ satisfying (1.7) such that*

$$(5.2) \qquad L(1,\ldots,1) = \beta \quad and \quad L^{*i}(1,\ldots,1) = 0 \ \ for \ i = 1,\ldots,m.$$

We will give a proof of this lemma at the end of this section. First we state the definition of standard Calderón–Zygmund kernel, and we prove the theorem 1.2 assuming lemma 5.1.

Definition 5.2. A function $K \colon \mathbf{R}^{n+1} \backslash \{(x,\ldots,x) \in \mathbf{R}^{n+1} \colon x \in \mathbf{R}^n\} \to \mathbf{C}$ is a standard m-linear Calderón–Zygmund kernel if for all $x, x', y_1, \ldots, y_m, y'_1, \ldots, y'_m \in \mathbf{R}^n$,

$$|K(x,y_1,\ldots,y_m)| \lesssim \frac{1}{(|x-y_1| + \cdots + |x-y_m|)^{mn}},$$

$$|K(x,y_1,\ldots,y_m) - K(x,y_1,\ldots,y'_i,\ldots,y_m)| \lesssim \frac{|y_i - y'_i|^\gamma}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\gamma}}$$

whenever $|y_i - y'_i| \leq \max(|x-y_1|,\ldots,|x-y_m|)/2$,

$$|K(x,y_1,\ldots,y_m) - K(x',y_1,\ldots,y_m)| \lesssim \frac{|x-x'|^\gamma}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\gamma}}$$

whenever $|x - x'| \leq \max(|x-y_1|,\ldots,|x-y_m|)/2$ for some $\gamma > 0$. Given a m-linear operator T that is continuous from $\mathscr{S} \times \cdots \times \mathscr{S}$ into $\mathscr{S}'$, T has standard Calderón–Zygmund kernel K if for all $f_0,\ldots,f_m \in C_0^\infty$ such that

$$\bigcap_{i=0}^m \operatorname{supp}(f_i) = \emptyset,$$

T can be written as an absolutely convergent integral

$$\langle T(f_1,\ldots,f_m), f_0 \rangle = \int_{\mathbf{R}^{(m+1)n}} K(y_0,y_1,\ldots,y_m) \prod_{i=0}^m f_i(y_i) \, dy_i.$$

We continue now to prove theorem 1.2 assuming lemma 5.1.

Proof. Denote by P_t be a smooth approximation to identity operators with smooth compactly supported kernels that satisfy

$$f = \lim_{t \to 0} P_t f \quad and \quad 0 = \lim_{t \to \infty} P_t f$$

in $\mathscr{S}$ for $f \in \mathscr{S}_0$. There exist Littlewood–Paley–Stein projection operators $Q_t^{(i)}$ for $i = 1, 2$ with smooth compactly supported kernels such that $t\frac{d}{dt}P_t^2 = Q_t^{(2)}Q_t^{(1)}$. Using

these operators, we decompose T for $f_i \in \mathscr{S}_0$, $i = 0, \ldots, m$,

$$|\langle T(f_1, \ldots, f_m), f_0 \rangle| = \left| \int_0^\infty t \frac{d}{dt} \langle T(P_t^2 f_1, \ldots, P_t^2 f_m), P_t^2 f_0 \rangle \frac{dt}{t} \right|$$

$$\leq \sum_{i=0}^m \int_0^\infty \left| \langle \Theta_t^{(i)}(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots f_m), Q_t^{(1)} f_i \rangle \right| \frac{dt}{t}$$

$$\leq \sum_{i=0}^m \left\| \left(\int_0^\infty |\Theta_t^{(i)}(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^{p_i'}} \left\| \left(\int_0^\infty |Q_t^{(1)} f_i|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^{p_i}},$$

where we define $p_0 = p'$ and

$$\Theta_t^{(i)}(f_1, \ldots, f_m) = Q_t^{(2)*} T^{*i}(P_t^2 f_1, \ldots, P_t^2 f_m)$$

and T^{*i} is the ith formal transpose of T defined by the pairing for $f_0, \ldots, f_m \in \mathscr{S}$

$$\langle T^{*i}(f_1, \ldots, f_m), f_0 \rangle = \langle T(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots, f_m), f_i \rangle .$$

This type of decomposition was originally done by Coifman–Meyer in [5], and then in the bilinear setting in [16]. Since $1 < p_1, \ldots, p_m < \infty$, the second term in above can be bounded by $||f_i||_{L^{p_i}}$ using a Littlewood–Paley–Stein estimate for $Q_t^{(1)}$. We have also assumed that $T \in WBP$ which we define now.

Definition 5.3. For $M \in \mathbf{N}$, a function $\phi \in C_0^\infty(\mathbf{R}^n)$ is a normalized bump of order M if $\operatorname{supp}(\phi) \subset B(0,1)$ and for all multi-indices $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leq M$,

$$||\partial^\alpha \phi||_{L^\infty} \leq 1.$$

An m-linear operator $T \colon \mathscr{S} \times \cdots \times \mathscr{S}^m \to \mathscr{S}'$ satisfies the weak boundedness property, written $T \in WBP$, if there exists $M \in \mathbf{N}$ such that for all normalized bumps $\phi_0, \ldots, \phi_m \in C_0^\infty$ of order M

$$\left| \langle T(\phi_1^{x,R}, \ldots, \phi_m^{x,R}), \phi_0^{x,R} \rangle \right| \lesssim R^n,$$

where $\phi^{x,R}(y) = \phi\left(\frac{y-x}{R}\right)$.

It follows that $\theta_t^{(i)}$ satisfy (1.4)–(1.6) for $i = 0, 1, \ldots, m$ when $|x - y| \lesssim t$ since $T \in WBP$ and for $|x - y| \gtrsim t$ using the kernel representation of T (for details see [16]). It follows from Theorem 1.1 and (1.14) that (1.3) holds for all $2 \leq p < \infty$ and $1 < p_i < \infty$ where S is the square function associated to $\Theta_t^{(0)}$ defined by (1.2). Also it follows from [15] or [11] that (1.3) holds for all $1 < p, p_i < \infty$ where S is the square function associated to $\Theta_t^{(i)}$ defined by (1.2) for $i = 1, \ldots, m$. Now fix $2 \leq p < \infty$ and $1 < p_i < \infty$ such that (1.7) holds. For example take $p_i = 2m$ and $p = 2$. Then $p_i' = \frac{2m}{2m-1} > 1$ for $i = 1, \ldots, m$. Using this choice of indices, it follows from (1.3) that T is bounded from $L^{2m} \times \cdots \times L^{2m}$ into L^2, and hence is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $1 < p_1, \ldots, p_m < \infty$ such that (1.7) holds (see for example [13]). Here we have used that

$$\left\| \left(\int_0^\infty |\Theta_t^{(0)}(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \prod_{i=1}^m ||f_i||_{L^{p_i}}$$

and that for $j = 1, \ldots, m$,

$$\left\lVert \left(\int_0^\infty |\Theta_t^{(j)}(f_1, \ldots, f_{j-1}, f_0, f_{j+1}, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\rVert_{L_i^{p'_i}} \lesssim \lVert f_0 \rVert_{L^{p'}} \prod_{i \neq j} \lVert f_i \rVert_{L^{p_i}}.$$

This proves the reduction case of theorem 1.2 where we assumed (5.1) in place of (1.15). Now assuming that lemma 5.1 holds, we prove the full theorem 1.2 where T satisfies (1.15). Given T satisfying the hypotheses of theorem 1.2, by lemma 5.1 there exist operators bounded m-linear Calderón–Zygmund operators $L_1, \ldots, L_m$ such that

$$L_i^{*i}(1, \ldots, 1) = T^{*i}(1, \ldots, 1) \quad \text{and} \quad L_i^{*j}(1, \ldots, 1) = 0 \quad \text{for} \ i \neq j.$$

Define

$$\widetilde{T}(f_1, \ldots, f_m) = T(f_1, \ldots, f_m) - \sum_{i=1}^m L_i(f_1, \ldots, f_m).$$

Then $\widetilde{T}$ satisfies for $i = 1, \ldots, m$

$$\widetilde{T}^{*i}(1, \ldots, 1) = T^{*i}(1, \ldots, 1) - \sum_{i=1}^m L_i^{*i}(1, \ldots, m) = 0.$$

Now for any dyadic cube $Q \subset \mathbf{R}^n$ we bound $\widetilde{T}$ as in (1.14),

$$\int_Q \left(\int_0^{\ell(Q)} |Q_t \widetilde{T}(P_t b_Q^1, \ldots, P_t b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx$$

$$\leq \sum_{i=1}^m \int_Q \left(\int_0^{\ell(Q)} |Q_t L_i^{*i}(P_t b_Q^1, \ldots, P_t b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx$$

$$+ \int_Q \left(\int_0^{\ell(Q)} |Q_t T(P_t b_Q^1, \ldots, P_t b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx.$$

The second term is bounded by $|Q|$ by hypothesis. If we prove that the square function associated to each term $Q_t L_i^{*i}(P_t f_1, \ldots, P_t f_m)$ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ into L^q, then we bound the first term as well and we can apply the reduced version to complete the proof. So we have reduced the proof to showing that (1.3) holds for $2 \leq p < \infty$ and $1 < p_1, \ldots, p_m < \infty$ for $\Theta_t(f_1, \ldots, f_m) = Q_t L_i^{*i}(P_t f_1, \ldots, P_t f_m)$ with its associated kernel $\theta_t(x, y_1, \ldots, y_m)$ and square function S as in (1.2). Since L_i is bounded, it follows that

$$|\theta_t(x, y_1, \ldots, y_m)| = |\langle L_i(\varphi_t^{y_1}, \ldots, \varphi_t^{y_m}), \psi_t^x \rangle| \lesssim \lVert \psi_t \rVert_{L^2} \prod_{i=1}^m \lVert \varphi_t \rVert_{L^{2m}} \lesssim t^{-mn}.$$

Also if $|x - y_{i_0}| > 4t$, then it follows that

$$|\theta_t(x, y_1, \ldots, y_m)| = \left| \int_{\mathbf{R}^{(m+1)n}} \ell(u, v_1, \ldots, v_m) \psi_t(x - u) \left(\prod_{i=1}^m \varphi_t(y_i - v_i) \, dv_i \right) du \right|$$

$$= \left| \int_{\mathbf{R}^{(m+1)n}} (\ell(u, v_1, \ldots, v_m) - \ell(x, v_1, \ldots, v_m)) \psi_t(x-u) \left(\prod_{i=1}^{m} \varphi_t(y_i - v_i) \, dv_i \right) du \right|$$

$$\lesssim \int_{\mathbf{R}^{(m+1)n}} \frac{|x-u|^\gamma}{(\sum_{i=1}^{m} |x-v_i|)^{mn+\gamma}} |\psi_t(x-u)| \left(\prod_{i=1}^{m} |\varphi_t(y_i - v_i)| dv_i \right) du$$

$$\lesssim \int_{|v_{i_0} - y_{i_0}| < t} \int_{|x-u| < t} \frac{t^\gamma}{|x - v_{i_0}|^{mn+\gamma}} t^{-(m+1)n} du \, dv_1 \cdots dv_m$$

$$(5.3) \quad \lesssim \frac{t^{-mn}}{(1 + t^{-1}|x - y_{i_0}|)^{mn+\gamma}}.$$

In this computation we use that $|x - y_{i_0}| > 4t$ to replace $|x - v_{i_0}|$ with $|x - y_{i_0}| + t$:
For v_{i_0} such that $|v_{i_0} - y_{i_0}| < t$, we have

$$|x - v_{i_0}| \geq |x - y_{i_0}| - |y_{i_0} - v_{i_0}| > \frac{1}{2}|x - y_{i_0}| + t.$$

Since $|\theta_t(x, y_1, \ldots, y_m)| \lesssim t^{-mn}$ as well, it follows that θ_t satisfies (5.3) for all $x, y_{i_0} \in \mathbf{R}^n$ and $i_0 = 1, \ldots, m$ (not just for $|x - y_{i_0}| > 4t$). Then it follows that θ_t satisfies (1.4),

$$|\theta_t(x, y_1, \ldots, y_m)| \lesssim \prod_{i=1}^{m} \left(\frac{t^{-mn}}{(1 + t^{-1}|x - y_i|)^{mn+\gamma}} \right)^{1/m} = \prod_{i=1}^{m} \frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^{n+\gamma/m}}.$$

It follows as well that θ_t satisfies (1.5) and (1.6). Consider

$$|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x', y_1, \ldots, y_m)| = \left| \left\langle L_i(\varphi_t^{y_1}, \ldots, \varphi_t^{y_m}), \psi_t^x - \psi_t^{x'} \right\rangle \right|$$
$$\lesssim t^{-mn}(t^{-1}|x - x'|).$$

This estimate is sufficient for (1.6). Then by symmetric arguments for $y_1, \ldots, y_m \in \mathbf{R}^n$, θ_t satisfies (1.4)–(1.6). Moreover, since L_i is bounded it follows that $L_i(1, \ldots, 1) \in BMO$ and so

$$|\Theta_t(1, \ldots, 1)(x)|^2 \, dx \frac{dt}{t} = |Q_t L_i(1, \ldots, 1)|^2 \, dx \frac{dt}{t}$$

is a Carleson measure. Therefore by proposition 2.4 and corollary 4.2, it follows that (1.3) holds for the square function S associated to $\Theta_t = Q_t L_i(P_t \otimes \cdots \otimes P_t)$ for any $2 \leq p < \infty$, $1 < p_1, \ldots, p_m < \infty$ satisfying (1.7) and for each $i = 1, \ldots, m$. The second term above can be bounded since $q \geq 2$

$$\int_Q \left(\int_0^{\ell(Q)} |Q_t T(P_t b_Q^1, \ldots, P_t b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \leq |Q|^{\frac{q}{q'}} \left\| \left(\int_0^{\ell(Q)} |\Theta_t(b_Q^1, \ldots, b_Q^m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^q}^q$$

$$\lesssim |Q|^{\frac{q}{q'}} \prod_{i=1}^{m} \|b_Q^1\|_{L^{q_i}}^q \lesssim |Q|.$$

Therefore $\widetilde{T}$ satisfies (1.14) as well and hence is bounded for from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $1 < p_1, \ldots, p_m < \infty$ satisfying (1.7). It follows easily that T is bounded on the same spaces since $\widetilde{T}$ and L_i for each $i = 1, \ldots, m$ are. $\square$

Finally we prove the paraproduct construction in lemma 5.1.

Proof. Let P_t be a smooth approximation to the identity with convolution kernel supported in $B(0,1)$. Also fix $\psi \in C_0^\infty$ radial, real-valued with mean zero such that

$$\int_0^\infty \widehat{\psi}(te_1)^3 \frac{dt}{t} = 1,$$

where $e_1 = (1,0,\ldots,0) \in \mathbf{R}^n$, and define $Q_t f = \psi_t * f$. It follows that

$$\int_0^\infty Q_t^3 f \frac{dt}{t} = f$$

in L^p for all $1 < p < \infty$ and in H^1, where Q_t^3 is the composition of Q_t with itself three times. Now define L with kernel $\ell(x, y_1, \ldots, y_m)$ by the following

$$L(f_1, \ldots, f_m) = \int_0^\infty L_t(f_1, \ldots, f_m) \frac{dt}{t} = \int_0^\infty Q_t \left((Q_t^2 \beta) \prod_{i=1}^m P_t f_i \right) \frac{dt}{t},$$

$$\ell(x, y_1, \ldots, y_m) = \int_0^\infty \ell_t(x, y_1, \ldots, y_m) \frac{dt}{t}$$

$$= \int_0^\infty \int_{\mathbf{R}^n} \psi_t(x - u) Q_t^2 \beta(u) \prod_{i=1}^m \varphi_t(u - y_i) \, du \frac{dt}{t}.$$

We start by analyzing L_t. Define the non-negative measure $d\mu$ on $\mathbf{R}_+^{n+1}$ by

$$d\mu(x, t) = |\widetilde{L}_t(1, \ldots, 1)(x)|^2 \, dx \frac{dt}{t} = |Q_t^2 \beta(x)|^2 \, dx \frac{dt}{t},$$

where

$$\widetilde{L}_t = M_{Q_t^2 \beta} \prod_{i=1}^m P_t f_i.$$

It follows then that $d\mu(x, t)$ is a Carleson measure. It is straightforward to show that the kernels of $\widetilde{L}_t$ satisfy (1.4)–(1.6) as well (in fact we can take $N > mn + 1$ since $\varphi, \psi \in C_0^\infty$, which we will use later). The smoothness in x is easy to show since we have that $\widetilde{L}_t$ is multiplied by

$$Q_t^2 \beta(x) = \int \psi_t(x - u) Q_t \beta(u) \, du$$

and ψ_t is smooth. So by proposition 2.4 and corollary 4.2, we have

$$\left\| \left(\int_0^\infty |\widetilde{L}_t(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}}$$

for all $2 \leq p < \infty$ and $1 < p_1, \ldots, p_m < \infty$. Then for any $f_0, f_1, \ldots, f_m \in \mathscr{S}$ with $\|f_0\|_{L^{p'}} \leq 1$,

$$|\langle L(f_1, \ldots, f_m), f_0 \rangle| \leq \int_0^\infty \left| \int_{\mathbf{R}^n} \left(Q_t^2 \beta(x) \prod_{i=1}^m P_t f_i \right)(x) Q_t f_0(x) \, dx \right| \frac{dt}{t}$$

$$\leq \left\| \left(\int_0^\infty |\widetilde{L}_t(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p} \left\| \left(\int_0^\infty |Q_t f_0|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^{p'}}$$

$$\lesssim \|f_0\|_{L^{p'}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

Therefore L is bounded for appropriate indices $p, p_1, \ldots, p_m$. It also follows that ℓ is a Calderón–Zygmund kernel. To see this, take $d = \sum_{i=1}^{m} |x - y_i|$ and use (1.4) to compute

$$|\ell(x, y_1, \ldots, y_m)| \lesssim d^{-(N+\gamma)} \int_0^d t^{N+\gamma-mn} \frac{dt}{t} + \int_d^\infty t^{-mn} \frac{dt}{t} \lesssim d^{-mn}.$$

Similarly when $|x - x'| < \max(|x - y_1|, \ldots, |x - y_m|)/2$, we have

$$|\ell(x, y, z) - \ell(x', y, z)| \lesssim |x - x'|^\gamma d^{-(N+\gamma)} \int_0^d t^{N+\gamma-mn} \frac{dt}{t} + |x - x'|^\gamma \int_d^\infty t^{-mn-\gamma} \frac{dt}{t}$$

$$\lesssim |x - x'|^\gamma d^{-(mn+\gamma)}.$$

With symmetric arguments for the regularity in $y_1, \ldots, y_m$, it follows that the kernel ℓ is an m-linear Calderón–Zygmund kernel. So L is an m-linear Calderón–Zygmund operator, and is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for all $1 < p_i < \infty$ when (1.7) holds.

Now we show (5.2). Let $\eta \in C_0^\infty$ with $\eta \equiv 1$ on $B(0, 1)$, $\text{supp}(\eta) \subset B(0, 2)$, and $\eta_R(x) = \eta(x/R)$. Let $\phi \in C_0^\infty$ with mean zero and N such that $\text{supp}(\phi) \subset B(0, N)$. Then to compute $L(1, \ldots, 1)$,

$$\langle L(1, \ldots, 1), \phi \rangle = \lim_{R \to \infty} \int_{R/4}^\infty \int_{\mathbf{R}^n} Q_t \phi(x) \left[P_t \eta_R(x)\right]^m Q_t^2 \beta(x) \, dx \frac{dt}{t}$$

$$(5.4) \qquad\qquad + \lim_{R \to \infty} \int_0^{R/4} \int_{\mathbf{R}^n} Q_t \phi(x) \left[P_t \eta_R(x)\right]^m Q_t^2 \beta(x) \, dx \frac{dt}{t}.$$

We may write this only if the two limits on the right hand side of the equation exist. As we are taking $R \to \infty$ and N is a fixed quantity determined by ϕ, without loss of generality assume that $R > 2N$. Note that for $t \leq R/4$ and $|x| < N + t$,

$$\text{supp}(\varphi_t(x - \cdot)) \subset B(x, t) \subset B(0, N + 2t) \subset B(0, R).$$

Since $\eta_R \equiv 1$ on $B(0, R)$, it follows that $P_t \eta_R(x) = 1$ for all $|x| < N + t$ when $t \leq R/4$. Therefore

$$\lim_{R \to \infty} \int_{R/4}^\infty \int_{\mathbf{R}^n} Q_t \phi(x) \left[P_t \eta_R(x)\right]^m Q_t^2 \beta(x) \, dx \frac{dt}{t} = \int_{\mathbf{R}^n} \int_0^\infty Q_t^3 \phi(x) \frac{dt}{t} \beta(x) \, dx = \langle \beta, \phi \rangle,$$

where we have used that Calderón's reproducing formula holds in H^1. This fact is due originally due to Folland–Stein [10] in the discrete setting and by Wilson in [27] in the continuous setting as used here. For any $t > 0$,

$$(5.5) \qquad\qquad ||P_t \eta_R||_{L^1} \lesssim ||\varphi_t||_{L^1} ||\eta_R||_{L^1} \lesssim R^n,$$

$$(5.6) \qquad\qquad ||P_t \eta_R||_{L^\infty} \leq ||\varphi_t||_{L^1} ||\eta_R||_{L^\infty} = 1,$$

and for any $x \in \mathbf{R}^n$,

$$|Q_t \phi(x)| = \left| \int_{\mathbf{R}^n} (\psi_t(x - y) - \psi_t(x)) \phi(y) \, dy \right|$$

$$(5.7) \qquad\qquad \lesssim \int_{\mathbf{R}^n} t^{-n} (t^{-1}|y|) |\phi(y)| \, dy \lesssim t^{-(n+1)}.$$

Therefore

$$\int_{R/4}^{\infty} \int_{\mathbf{R}^n} |Q_t\phi(x)\,[P_t\eta_R(x)]^m\,Q_t^2\beta(x)|\,dx\frac{dt}{t}$$

$$\leq \int_{R/4}^{\infty} ||P_t\eta_R||_{L^1}||P_t\eta_R||_{L^\infty}^{m-1}||Q_t^2\beta||_{L^\infty}||Q_t\phi||_{L^\infty}\frac{dt}{t}$$

(5.8) $$\lesssim R^n \int_{R/4}^{\infty} t^{-(n+1)}\frac{dt}{t} \lesssim R^{-1}.$$

Hence the second limit in (5.4) exists and tends to 0 as $R \to \infty$. Then $\langle L(1,\ldots,1),\phi\rangle = \langle\beta,\phi\rangle$ for all $\phi \in C_0^\infty$ with mean zero and hence $L(1,\ldots,1) = \beta$ as an element of BMO. Again for any $\phi \in C_0^\infty$ with mean zero and $\mathrm{supp}(\phi) \subset B(0,N)$, we have for $i = 1,\ldots,m$,

$$\langle L^{i*}(1,\ldots,1),\phi\rangle = \lim_{R\to\infty} \int_0^{R/4} \int_{|x|<N+t} Q_t^2\beta(x)P_t\phi(x)[P_t\eta_R(x)]^{m-1}Q_t\eta_R(x)\,dx\frac{dt}{t}$$

(5.9) $$+ \lim_{R\to\infty} \int_{R/4}^{\infty} \int_{|x|<N+t} Q_t^2\beta(x)P_t\phi(x)[P_t\eta_R(x)]^{m-1}Q_t\eta_R(x)\,dx\frac{dt}{t}.$$

Once more without loss of generality take $R > 2N$. When $|x| < N + t$ and $t \leq R/4$

$$\mathrm{supp}(\psi_t(x - \cdot)) \subset B(x,t) \subset B(0,N+2t) \subset B(0,R)$$

and hence $Q_t\eta_R(x) = Q_t1(x) = 0$. With this it is apparent that the first limit in (5.9) is 0. Similar to (5.5)–(5.7), for the terms of (5.9) we have $||P_t\eta_R||_{L^1} \lesssim R^n$, $||Q_t\eta_R||_{L^\infty} \lesssim 1$, and $||P_t\phi||_{L^\infty} \lesssim t^{-(n+1)}$. So the second term of (5.9) tends to 0 as $R \to \infty$ just like the second term in computing $L(1,\ldots,1)$ from (5.8). Then $L^{*1}(1,\ldots,1) = 0$, which concludes the proof of Lemma 5.2. $\square$

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