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ON WEIGHTED POINCARÉ INEQUALITIES

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Abstract. The aim of this note is to show that Poincaré inequalities imply corresponding weighted versions in a quite general setting. Fractional Poincaré inequalities are considered, too. The proof is short and does not involve covering arguments.

1. Introduction

Let (X,ρ) be a metric space with a positive σ-finite Borel measure dx, we will write $|E| = \int_E dx$ for the measure of a Borel set $E \subset X$. We fix some point $x_0 \in X$ and set $B_r = \{x \in X\colon \rho(x,x_0) < r\}$, $\overline{B}_r = \{x \in X\colon \rho(x,x_0) \le r\}$. We call a function $\phi\colon B_1 \to [0,\infty)$ a *radially decreasing weight*, if ϕ is a radial function, i.e. $\phi = \Phi(\rho(\cdot,x_0))$ and its profile Φ is nonincreasing and right-continuous with left-limits. We assume that ϕ is not identically zero on $B_1 \setminus \overline{B}_{1/2}$. For any such weight ϕ there exists a positive, non-zero σ-finite Borel measure ν on $(\frac{1}{2},1]$, such that

$$\phi(x) = \int_{\rho(x,x_0)\vee 1/2}^1 \nu(dt) = \int_{1/2}^1 \chi_{B_t}(x)\,\nu(dt), \quad x \in B_1 \setminus \overline{B}_{1/2}. \tag{1}$$

Note that we put $\int_a^b f(t)\,\nu(dt) = \int_{(a,b]} f(t)\,\nu(dt)$. For a function u we denote by

$$u_E = \frac{1}{|E|}\int_E u(x)\,dx$$

the mean of u over the set E, and by

$$u_E^\phi = \frac{\int_E u(x)\phi(x)\,dx}{\int_E \phi(x)\,dx}$$

the mean of u over the set $E \subset B_1$ with respect to the weight function ϕ.

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Our main result is the following:

Theorem 1. *Let* $1 \le p < \infty$ *and let* ϕ *be a radially decreasing weight with* $\phi = \Phi(\rho(\cdot, x_0))$*. Let* $F\colon L^p(X) \times (\frac{1}{2}, 1] \to [0, \infty]$ *be a functional satisfying*

$$F(u + a, r) = F(u, r), \quad a \in \mathbf{R}, \tag{2}$$

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \le F(u, r), \tag{3}$$

for every $r \in (\frac{1}{2}, 1]$ *and every* $u \in L^p(X)$*. Then for* $M = \frac{8^p |B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(1/2)}$

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x) \, dx \le M \int_{1/2}^1 F(u, t) \, \nu(dt) \tag{4}$$

for every $u \in L^p(B_1)$*, where* ν *is as in* (1).

By choosing the functional F appropriately, (4) becomes a Poincaré inequality with weight ϕ, see Section 3. Such inequalities have been studied extensively because of their importance for the regularity theory of partial differential equations, see the exposition in [5].

2. Proof

Lemma 2. *Let* Ω *be a finite measure space and* $p \ge 1$*. Assume* $f \in L^p(\Omega)$ *with* $\int_\Omega f = 0$*. Then*

$$\|f + a\|_{L^p(\Omega)} \ge \tfrac{1}{2} \|f\|_{L^p(\Omega)}$$

for every $a \in \mathbf{R}$.

Proof. We may assume $a > 0$. Then

$$\int_{\Omega \cap \{f > 0\}} |f + a|^p \ge \int_{\Omega \cap \{f > 0\}} |f|^p \quad \text{and} \quad \int_{\Omega \cap \{f < -2a\}} |f + a|^p \ge 2^{-p} \int_{\Omega \cap \{f < -2a\}} |f|^p.$$

Furthermore, since $\int_{\Omega \cap \{f \le 0\}} |f| = \int_{\Omega \cap \{f > 0\}} |f|$, we obtain

$$\begin{aligned} \int_{\Omega \cap \{-2a \le f \le 0\}} |f|^p &\le (2a)^{p-1} \int_{\Omega \cap \{-2a \le f \le 0\}} |f| \\ &\le (2a)^{p-1} \int_{\Omega \cap \{f > 0\}} |f| \le 2^{p-1} \int_{\Omega \cap \{f > 0\}} |f + a|^p, \end{aligned}$$

where we use $a^{p-1} b \le (b + a)^{p-1} (b + a)$ for positive a, b. Combining these observations we obtain the result. □

Proof of Theorem 1. First we observe that it is enough to prove that

$$\int_{B_1} |u(x) - u^{\tilde{\phi}}_{B_1}|^p \tilde{\phi}(x) \, dx \le \frac{2^{2p} |B_1|}{|B_{1/2}|} \int_{1/2}^1 F(u, t) \, \nu(dt), \tag{5}$$

where $\tilde{\phi}(x) = \phi(x) \wedge \Phi(\frac{1}{2})$. Indeed, we have

$$\frac{\Phi(\frac{1}{2})}{\Phi(0)} \phi(x) \le \phi(x) \wedge \Phi(\tfrac{1}{2}) \le \phi(x).$$

Hence if (5) holds, then

$$\int_{B_1} |u(x) - u^{\tilde\phi}_{B_1}|^p \tilde\phi(x)\,dx \geq \frac{\Phi(\frac12)}{\Phi(0)} \int_{B_1} |u(x) - u^{\tilde\phi}_{B_1}|^p \phi(x)\,dx$$
$$\geq \frac{\Phi(\frac12)}{\Phi(0)} 2^{-p} \int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\,dx,$$

where in the last line we have used Lemma 2. Now we prove (5). To simplify the notation, we assume that $\phi(x) = \Phi(\frac12)$ for $x \in B_{1/2}$, so that $\tilde\phi = \phi$. Because of (2), by subtracting a constant from u, we may and do assume that $u^{\phi}_{B_1} = 0$, which means that

$$0 = \int_{B_1} u(x)\phi(x)\,dx = \int_{1/2}^{1} \int_{B_t} u(x)\,dx\,\nu(dt) = \int_{1/2}^{1} u_{B_t}|B_t|\,\nu(dt). \tag{6}$$

We start from the integral on the right hand side of (4) and use (3)

$$\begin{aligned} R &:= \int_{1/2}^{1} F(u,t)\,\nu(dt) \geq \int_{1/2}^{1} \int_{B_t} |u(x) - u_{B_t}|^p\,dx\,\nu(dt) \\ &= \frac12 \int_{1/2}^{1} \int_{B_t} |u(x) - u_{B_t}|^p\,dx\,\nu(dt) + \frac12 \int_{B_1} \int_{1/2}^{1} |u(x) - u_{B_t}|^p \chi_{B_t}(x)\,\nu(dt)\,dx \\ &=: I_1 + I_2 \end{aligned}$$

(In fact $I_1 = I_2$, but we treat them differently.) We now deal with the inner integral in I_2. For $x \in B_{1/2}$ we have

$$\int_{1/2}^{1} |u(x) - u_{B_t}|^p \chi_{B_t}(x)\,\nu(dt) \geq \frac{1}{|B_1|} \int_{1/2}^{1} |u(x) - u_{B_t}|^p |B_t|\,\nu(dt).$$

Since $\int_{1/2}^{1} u_{B_t}|B_t|\,\nu(dt) = 0$, by Lemma 2 we obtain

$$\int_{1/2}^{1} |u(x) - u_{B_t}|^p |B_t|\,\nu(dt) \geq 2^{-p} \int_{1/2}^{1} |u_{B_t}|^p |B_t|\,\nu(dt).$$

Therefore

$$I_2 \geq \frac{2^{-p}}{2|B_1|} \int_{B_{1/2}} \int_{1/2}^{1} |u_{B_t}|^p |B_t|\,\nu(dt)\,dx = \frac{2^{-p}|B_{1/2}|}{2|B_1|} \int_{1/2}^{1} |u_{B_t}|^p |B_t|\,\nu(dt).$$

Using the inequality $|a|^p + |b|^p \geq 2^{1-p}|a+b|^p$ we obtain

$$\begin{aligned} I_1 + I_2 &\geq \frac12 \int_{1/2}^{1} \int_{B_t} \left(|u(x) - u_{B_t}|^p + \frac{2^{-p}|B_{1/2}|}{|B_1|} |u_{B_t}|^p \right) dx\,\nu(dt) \\ &\geq \frac{2^{-p}|B_{1/2}|}{2|B_1|} 2^{1-p} \int_{1/2}^{1} \int_{B_t} |u(x)|^p\,dx\,\nu(dt) \\ &= \frac{|B_{1/2}|}{|B_1|} 2^{-2p} \int_{B_1} |u(x)|^p \phi(x)\,dx \end{aligned}$$

and the proof is finished. □

3. Applications

Let us discuss some corollaries. Corollary 3 is well-known [5]. However, our approach allows for very general weights. Proposition 4 allows to deduce a weighted Poincaré inequality for fractional Sobolev norms from an unweighted version. Corollaries 5 and 6 give a more concrete result for fractional Sobolev norms. The first allows for more general kernels and exponents p. Corollary 6 improves [2, Theorem 5.1] because the result is robust for $s \to 1-$ and allows for general weights and exponents p.

Corollary 3. *Let $p \geq 1$ and ϕ be a radially decreasing weight. Consider $X = \mathbf{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. There exists a positive constant C depending on p, d and ϕ such that*

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq C \int_{B_1} |\nabla u(x)|^p \phi(x)\, dx, \tag{7}$$

for every $u \in W^{1,p}(B_1)$.

Proposition 4. *Let $p \geq 1$ and let ϕ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$. Assume that for some kernel $k\colon B_1 \times B_1 \to [0, \infty)$ and some positive constant C the following inequality holds*

$$\int_{B_r} |u(x) - u_{B_r}|^p\, dx \leq C \int_{B_r}\int_{B_r} |u(x) - u(y)|^p k(x, y)\, dy\, dx, \tag{8}$$

whenever $r \in (\frac{1}{2}, 1]$ and $u \in L^p(X)$. Then with $M = \frac{8^p |B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(1/2)}$

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq CM \int_{B_1}\int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x))\, dy\, dx \tag{9}$$

for $u \in L^p(X)$.

Corollary 5. *Let ϕ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$ and $p \geq 1$. Let $k\colon B_1 \times B_1 \to [0, \infty)$ be a kernel satisfying $k \geq c$ for some constant $c > 0$. There is a positive constant M depending on d, p and Φ such that for $u \in L^p(X)$*

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \leq \frac{M}{c} \int_{B_1}\int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x))\, dy\, dx \tag{10}$$

for $u \in L^p(X)$.

Corollary 6. *Let $p \geq 1$, $R \geq 1$ and $0 < s_0 \leq s < 1$. Consider $X = \mathbf{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. Let ϕ be a radially decreasing weight of the form $\phi = \Phi(|\cdot|)$. Then there exists a positive constant C depending on p, d, s_0 and Φ such that*

$$\begin{aligned} &\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x)\, dx \\ &\leq C(1-s)\, R^{p(1-s)} \int_{B_1}\int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{R}\}} (\phi(y) \wedge \phi(x))\, dy\, dx \end{aligned} \tag{11}$$

for all $u \in L^p(B_1)$.

Proof of Corollary 3. It is well-known that the following Poincaré inequality holds

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \le c \, r^p \int_{B_r} |\nabla u(x)|^p \, dx \tag{12}$$

for every $u \in W^{1,p}(B_r)$ and $r > 0$ where $c > 0$ depends on p and d. Set

$$F(u, r) = c \, r^p \int_{B_r} |\nabla u(x)|^p \, dx,$$

for $u \in W^{1,p}(B_1)$ and $F(u, r) = \infty$ otherwise. Then for $u \in W^{1,p}(B_1)$

$$\begin{aligned} \int_{1/2}^1 F(u, t) \, \nu(dt) &= c \int_{1/2}^1 t^p \int_{B_1} |\nabla u(x)|^p \chi_{B_t}(x) \, dx \, \nu(dt) \\ &\le c \int_{B_1} |\nabla u(x)|^p \int_{1/2}^1 \chi_{B_t}(x) \, \nu(dt) \, dx = c \int_{B_1} |\nabla u(x)|^p \phi(x) \, dx. \end{aligned}$$

By Theorem 1 the assertion follows with $C = 2^{3p+d} \frac{\Phi(0)}{\Phi(1/2)} c$. □

Proof of Proposition 4. Let

$$F(u, r) = C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx.$$

Then

$$\begin{aligned} \int_{1/2}^1 F(u, t) \, \nu(dt) &= C \int_{1/2}^1 \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \chi_{B_t}(y) \chi_{B_t}(x) \, dy \, dx \, \nu(dt) \\ &= C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \int_{1/2}^1 \chi_{B_t}(y) \chi_{B_t}(x) \, \nu(dt) \, dy \, dx \\ &= C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) (\phi(y) \wedge \phi(x)) \, dy \, dx. \end{aligned}$$

The assertion now follows from Theorem 1. □

Proof of Corollary 5. First we use a well-known argument to obtain a non-weighted Poincaré inequality. By calculus and convexity of the function $x \mapsto |x|^p$ we conclude that $|a + b|^p \ge |a|^p + bp|a|^{p-1} \operatorname{sgn}(a)$. Thus

$$\begin{aligned} \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx &\ge c \int_{B_r} \int_{B_r} |(u(x) - u_{B_r}) + (u_{B_r} - u(y))|^p \, dy \, dx \\ &\ge c|B_r| \int_{B_r} |u(x) - u_{B_r}|^p \, dx \\ &\ge c|B_{1/2}| \int_{B_r} |u(x) - u_{B_r}|^p \, dx, \end{aligned}$$

whenever $u \in L^p(B_r)$ and $\frac{1}{2} < r \le 1$. The assertion follows now from Proposition 4. □

In the proof of Corollary 6 we use the following auxiliary result.

Lemma 7. *Let $R \ge 1$, $p \ge 1$ and $0 < s < 1$. Then*

$$\int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \, dy \, dx \le (3R)^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \chi_{\{|x-y| \le \frac{1}{R}\}} \, dy \, dx \tag{13}$$

for all $u \in L^p(B_1)$.

Proof. Let n be a natural number such that $n \geq 2R > n-1$. We introduce

$$A_k = A_k(x,y) = \tfrac{k}{n} y + \tfrac{n-k}{n} x, \quad k = 0, 1, \ldots n.$$

Then

$$I = \int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\, dy\, dx = \int_{B_1}\int_{B_1} \frac{|\sum_{k=1}^n (u(A_{k-1}) - u(A_k))|^p}{|x-y|^{d+ps}}\, dy\, dx$$
$$\leq n^{p-1} \sum_{k=1}^n \int_{B_1}\int_{B_1} \frac{|u(A_{k-1}) - u(A_k)|^p}{|x-y|^{d+ps}}\, dy\, dx.$$

Note that $|A_{k-1} - A_k| = \frac{1}{n}|x-y|$. If we substitute $\tilde{x} = A_{k-1}$, $\tilde{y} = A_k$, then $d\tilde{y}\, d\tilde{x} = n^{-d}\, dy\, dx$ (which follows by an elementary calculation, see also [3, p. 570]). Moreover, $\tilde{x}$, $\tilde{y} \in B_1$ with $|\tilde{x} - \tilde{y}| \leq \frac{2}{n} \leq \frac{1}{R}$. Hence

$$I \leq n^{p-ps} \int_{B_1}\int_{B_1} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{|\tilde{x}-\tilde{y}|^{d+ps}} \chi_{\{|\tilde{x}-\tilde{y}| \leq \frac{1}{R}\}}\, d\tilde{y}\, d\tilde{x}.$$

Since $n < 2R + 1 \leq 3R$, the assertion follows. □

Proof of Corollary 6. From [4] and [1, p. 80] we know that there exists a constant $C = C(p, d, s_0)$, such that for $s_0 \leq s < 1$

$$\int_{B_r} |u(x) - u_{B_r}|^p\, dx \leq C(1-s) r^{ps} \int_{B_r}\int_{B_r} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}\, dy\, dx, \tag{14}$$

for all $u \in L^p(B_1)$. The assertion now follows from (14), Proposition 4 and Lemma 7. □

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