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CHANGE OF VARIABLES FOR A_∞ WEIGHTS BY MEANS OF QUASICONFORMAL MAPPINGS: SHARP RESULTS

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Abstract. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiconformal mapping whose Jacobian is denoted by J_f and let A_∞ be the Muckenhoupt class of weights w satisfying

$$\left(\fint_B w\,dx\right)\left(\exp\fint_B \log\frac{1}{w}\,dx\right) \le A,$$

for every ball $B \subset \mathbf{R}^n$ and for some positive constant $A \ge 1$ independent of B. We consider two characteristic constants $\tilde{A}_\infty(w)$ and $\tilde{G}_1(w)$ which are simultaneously finite for every $w \in A_\infty$. We study the behaviour of the $\tilde{A}_\infty$-constant under the operator already considered by Johnson and Neugebauer [18]

$$w \in A_\infty \mapsto (w \circ f)J_f \in A_\infty,$$

and establish the equivalence of the two constants $\tilde{G}_1(J_f)$ and $\tilde{A}_\infty\left(J_{f^{-1}}\right)$. Our quantitative estimates are sharp.

1. Introduction

Let Ω be an open subset of $\mathbf{R}^n$ with $n \ge 2$. A homeomorphism $f\colon \Omega \to \mathbf{R}^n$ is a K-*quasiconformal mapping* for a constant $K \ge 1$ if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and

$$|Df(x)|^n \le K J_f(x) \quad \text{for a.e. } x \in \Omega. \tag{1.1}$$

Here $Df(x)$ stands for the *differential matrix* of f and $J_f(x) = \det Df(x)$ denotes the *Jacobian determinant* of f. The norm $|Df(x)|$ of $Df(x)$ in (1.1) is defined as $|Df(x)| = \sup\{|Df(x)\xi|\colon \xi \in \mathbf{R}^n,\ |\xi| = 1\}$.

Let $H \ge 1$ be a constant. A homeomorphism $f\colon \Omega \to \mathbf{R}^n$ is called *weakly H-quasisymmetric* if for every $x, y, z \in \Omega$ we have

$$|x-y| \le |x-z| \quad \text{implies} \quad |f(x)-f(y)| \le H|f(x)-f(z)|.$$

As proved in [30] and [33] the notions of weak quasisymmetry and quasiconformality are equivalent in dimension $n \ge 2$ when $\Omega = \mathbf{R}^n$.

Let us recall the definition of the *Muckenhoupt class* A_∞ (see [24]). Here and in the rest of the paper, we say that a measurable function $w\colon \mathbf{R}^n \to \mathbf{R}$ is a *weight* if w

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is positive a.e. and locally integrable in $\mathbf{R}^n$. A weight w belongs to the Muckenhoupt class A_∞ if

$$A_\infty(w) = \sup_B \left(⨍_B w\,dx \right) \left(\exp ⨍_B \log \frac{1}{w}\,dx \right) < \infty. \tag{1.2}$$

The supremum in (1.2) is taken over all balls $B \subset \mathbf{R}^n$. We call $A_\infty(w)$ the A_∞-*constant* of the weight w. The class A_∞ may be characterized in several ways. We mention here (see [5]) that $w \in A_\infty$ if and only if for every ball $B \subset \mathbf{R}^n$ and every measurable set $E \subset B$ it holds

$$\frac{|E|}{|B|} \le M \left(\frac{\displaystyle\int_E w(x)\,dx}{\displaystyle\int_B w(x)\,dx} \right)^\alpha, \tag{1.3}$$

for some $0 < \alpha \le 1 \le M$ independent of E and B.

Another characterization of A_∞ is given in [25] where it is proved that

$$A_\infty = \bigcup_{1<p<\infty} A_p.$$

For the definition of the Muckenhoupt class A_p for $1 \le p < \infty$, see Section 2.2 below.

One of the issues addressed in [18] by Johnson and Neugebauer concerns the composition problem for Muckenhoupt weights. It is proved (see Theorem 3.4 in [18]) that, if $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a quasiconformal mapping, then the condition

$$w \in A_\infty \quad \text{implies} \quad w \circ f \in A_\infty, \tag{1.4}$$

holds if and only if the Jacobian of f satisfies

$$J_f \in \bigcap_{1<p<\infty} A_p. \tag{1.5}$$

It is easily seen by means of examples that not every quasiconformal mapping satisfies (1.5). From a celebrated result of Gehring [13], suitably extended to quasiregular maps in [15, 20, 22], one can only deduce that $J_f \in A_{p_0}$ for some $p_0 > 1$. Therefore, (1.4) does not hold for an arbitrary quasiconformal mapping. In dimension $n \ge 2$, the equivalence of the notions of weak quasisymmetry and quasiconformality implies that each weakly quasisymmetric homeomorphism belongs to $W^{1,s}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ for some $s > n$ (see [13] and [1] for some sharp regularity result in the planar case).

We draw our attention to a similar issue started in [31]. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a given quasiconformal mapping. Then

$$w \in A_\infty \quad \text{implies} \quad (w \circ f)J_f \in A_\infty. \tag{1.6}$$

Actually (1.6) follows from a result in [31] of Uchiyama, where it is proved that if μ is a A_∞-measure then its pull back $f^*\mu$ is A_∞-measure as well. We recall that a positive Borel measure μ on $\mathbf{R}^n$ belongs to A_∞ if $d\mu = w\,dx$ for some $w \in A_\infty$ and the pull back $f^*\mu$ is the measure defined by

$$(f^*\mu)(E) = \mu(f(E)) \quad \text{for every Borel set } E \subset \mathbf{R}^n.$$

Indeed, (1.6) follows from the change of variables formula for quasiconformal mappings (see Section 2.1 below) which gives that $(w \circ f)J_f$ is the Radon–Nikodym derivative of the A_∞-measure $f^*\mu$ with respect to the Lebesgue measure and hence belongs to A_∞ by Uchiyama's result.

Our aim is to give a quantitative version of the statement in (1.6) and hence of Uchiyama's result by means of the auxiliary constant

$$\tilde{A}_\infty(w) = \inf\left\{\frac{M}{\alpha} \colon 0 < \alpha \le 1 \le M \text{ and (1.3) holds}\right\}. \tag{1.7}$$

We briefly refer to $\tilde{A}_\infty(w)$ as the $\tilde{A}_\infty$-constant of w. The interest in studying the behaviour of the constant $\tilde{A}_\infty(w)$ goes back to Gotoh's paper [16], where the composition problem for functions of bounded mean oscillation is taken into account (see also the seminal paper [27] and [7, 8] for sharp estimates involving the distances to L^∞ introduced in [4, 10, 12]).

We are in a position to state our results. The weak quasisymmetry property of a quasiconformal mapping will play a crucial role in the estimates we are going to show, especially for what concerns the optimality of such estimates. For this reason, we introduce the *weakly quasisymmetric constant* H_f of the quasiconformal mapping f, namely

$$H_f = \sup\left\{\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \colon x,y,z\in\Omega,\ x\ne z,\ \frac{|x-y|}{|x-z|}\le 1\right\}. \tag{1.8}$$

Our first result reads as follows.

Theorem 1.1. *Let $f\colon \mathbf{R}^n\to\mathbf{R}^n$ be a quasiconformal mapping with $n\ge 2$. Let $w\in A_\infty$. Then the following estimates hold*

$$\frac{1}{H^n_{f^{-1}}\tilde{A}_\infty(J_{f^{-1}})}\tilde{A}_\infty(w) \le \tilde{A}_\infty[(w\circ f)\,J_f] \le H_f^n \tilde{A}_\infty(J_f)\,\tilde{A}_\infty(w). \tag{1.9}$$

Another important class of weights is furnished by the *Gehring class* G_1. A weight v belongs to the Gehring G_1 class if

$$G_1(v) = \sup_B\left(\exp ⨍_B \frac{v}{v_B}\log\frac{v}{v_B}\,dx\right) < \infty. \tag{1.10}$$

The supremum in (1.10) is taken over all balls $B\subset\mathbf{R}^n$. The link between Muckenhoupt and Gehring classes is given in [9, 23] where it is proved that $A_\infty = G_1$. We mention here (see again [5]) that $v\in G_1$ if and only if for every ball $B\subset\mathbf{R}^n$ and every measurable set $F\subset B$ it holds

$$\frac{\displaystyle\int_F v(x)\,dx}{\displaystyle\int_B v(x)\,dx} \le L\left(\frac{|F|}{|B|}\right)^\beta, \tag{1.11}$$

for some $0<\beta\le 1\le L$ independent of F and B.

As was done above related to Muckenhoupt classes, we define an auxiliary constant for the Gehring classes

$$\tilde{G}_1(v) = \inf\left\{\frac{L}{\beta}\colon 0<\beta\le 1\le L \text{ and (1.11) holds}\right\}.$$

Let us recall here some results which are valid in dimension $n=1$. Let $h\colon\mathbf{R}\to\mathbf{R}$ be an increasing homeomorphism which is locally absolutely continuous with its inverse. It is well known (see e.g. [5]) that the derivative h' belongs to A_∞ if and only if $(h^{-1})'$ belongs to A_∞ and hence to G_1. Quantitative versions of this result may be found in [18] and [26], where the two identities

$$A_\infty\left((h^{-1})'\right) = G_1(h'), \tag{1.12}$$

and

$$\tilde{A}_\infty\left((h^{-1})'\right) = \tilde{G}_1(h'), \tag{1.13}$$

are respectively proved. Note that in general one has

$$A_p\left((h^{-1})'\right) = G_q(h'), \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.14}$$

as proved in [18, Lemma 2.5]. Thus the identity (1.12) follows taking the limit as $p \to \infty$ and using the relations

$$A_\infty(w) = \lim_{p\to\infty} A_p(w), \tag{1.15}$$

$$G_1(v) = \lim_{q\to 1^+} G_q(v) \tag{1.16}$$

proved in [29] and [23] respectively. Identities like (1.12), (1.13) and (1.14) are related to the study of the one-dimensional Dirichlet energy

$$\mathcal{D}_p\colon u \in W^{1,p}(a,b) \mapsto \int_a^b |u'|^p\, dt, \quad p > 1.$$

In [21] it is proved that the inverse of a quasiminimizer of $\mathcal{D}_p$ is a quasiminimizer of $\mathcal{D}_s$ for suitable values of s and the optimal range of such exponents s is explicitely computed using (1.14) among other facts.

Inspired by these one-dimensional results, our next goal is to establish the equivalence of the two constants $\tilde{A}_\infty\left(J_{f^{-1}}\right)$ and $\tilde{G}_1\left(J_f\right)$ whenever $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a quasiconformal mapping in higher dimension $n \geq 2$.

Our second result reads as follows.

Theorem 1.2. *Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiconformal mapping with $n \geq 2$. Then*

$$\frac{1}{H^n_{f^{-1}}}\,\tilde{A}_\infty(J_{f^{-1}}) \leq \tilde{G}_1(J_f) \leq H^n_f \tilde{A}_\infty(J_{f^{-1}}). \tag{1.17}$$

We point out that the estimates above are sharp. Indeed, equalities hold in (1.9) and in (1.17) if we let f be the identity map $\mathrm{Id}(x) = x$; this follows by observing that $\tilde{A}_\infty(u) = 1$ if and only if u is a constant weight (see Proposition 2.1 in [26]) and that $H_{\mathrm{Id}} = 1$.

It is worth pointing out that condition (1.5) is also equivalent to requiring that if $1 < p_0 < \infty$ then $w \in A_{p_0}$ implies $(w \circ f)J_f^\lambda \in A_{p_0}$ for each $\lambda \in [0,1]$ (see Theorem 2.10 in [18]). One may wonder if the condition

$$w \in A_\infty \quad \text{implies} \quad (w\circ f)J_f^\lambda \in A_\infty \quad \text{for each } \lambda \in [0,1),$$

holds without the further assumption (1.5). In Section 4 we will prove that this is not the case, by means of some counterexample (see Proposition 1 below).

2. Preliminaries

2.1. Quasiconformal and quasisymmetric mappings. We need to recall here some well known facts about quasiconformal mappings and quasisymmetric mappings. Our main sources here will be [2, 32].

Let $\eta\colon [0,\infty) \to [0,\infty)$ be an increasing homeomorphism. A homeomorphism $f\colon \Omega \to \mathbf{R}^n$ is called *η-quasisymmetric* if for every $x, y, z \in \Omega$ we have

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right).$$

The notions of quasiconformality, quasisymmetry and weak quasisymmetry coincide for mappings in dimension $n \ge 2$ (see e.g. [30] and [33]).

We recall that the change of variables formula holds for a quasiconformal mapping $f\colon \Omega \to \Omega'$. More precisely, if $\varphi \in L^1_{\mathrm{loc}}(\Omega')$ then $(\varphi \circ f)\, J_f \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_E \varphi(f(x)) J_f(x)\, dx = \int_{f(E)} \varphi(y)\, dy,$$

for every $E \subset\subset \Omega$.

2.2. A_p and G_q classes. We recall here the definition of the *Muckenhoupt class* A_p (see [24]) for $1 \le p < \infty$. A weight w belongs to the Muckenhoupt class A_p for $1 < p < \infty$ if

$$A_p(w) = \sup_B \left(\fint_B w\, dx\right)\left(\fint_B w^{-\frac{1}{p-1}}\, dx\right)^{p-1} < \infty. \tag{2.1}$$

As a natural extention of the above definition, one can consider the Muckenhoupt classes A_1 which cover the limit case $p = 1$. A weight w belongs to the Muckenhoupt class A_1 if

$$A_1(w) = \sup_B \frac{\fint_B w\, dx}{\operatorname*{ess\,inf}_{x\in B} w(x)} < \infty. \tag{2.2}$$

The suprema in (2.1) and (2.2) are taken over all balls $B \subset \mathbf{R}^n$. For each $1 \le p < \infty$ we call $A_p(w)$ the *A_p-constant* of the weight w.

We recall here the definition of the *Gehring class* G_q for $1 < q \le \infty$. A weight v belongs to the Gehring class G_q for $1 < q < \infty$ if

$$G_q(v) = \sup_B \left[\frac{\left(\fint_B v^q\, dx\right)^{\frac{1}{q}}}{\fint_B v\, dx}\right]^{\frac{q}{q-1}} < \infty. \tag{2.3}$$

As a natural extention of the above definition, one can consider the G_∞ which cover the limit case $q = \infty$. A weight v belongs to the Gehring class G_∞ if

$$G_\infty(v) = \sup_B \frac{\operatorname*{ess\,sup}_{x\in B} v(x)}{\fint_B v\, dx} < \infty. \tag{2.4}$$

The suprema in (2.3) and (2.4) are taken over all balls $B \subset \mathbf{R}^n$. For each $1 < q \leq \infty$ we call $G_q(v)$ the G_q*-constant* of the weight v.

Each weight in the G_q class satisfies a reverse Hölder inequality. This is a key fact in order to study the regularity of the Jacobian of quasiconformal mappings (see [13]). More generally, we refer for instance to [14, 17] for the study of the self-improving property and the regularity of the Jacobian of a mapping of finite distortion.

For more details related to the Muckenhoupt and Gehring classes we refer to [3, 6, 11, 19, 23, 28, 29].

3. Proofs

Proof of Theorem 1.1. Let $H = H_f$ where H_f is given by (1.8). We fix some $\varepsilon > 0$. We appeal to the definition (1.7) of the $\tilde{A}_\infty$-constant of w and we find some constants M, α with

$$0 < \alpha \leq 1 \leq M,$$

and

$$\frac{M}{\alpha} < \tilde{A}_\infty(w) + \varepsilon, \tag{3.1}$$

such that, for every ball $B' \subset \mathbf{R}^n$ and for every measurable $E' \subset B'$ we have

$$\frac{|E'|}{|B'|} \leq M \left(\frac{\displaystyle\int_{E'} w(y)\, dy}{\displaystyle\int_{B'} w(y)\, dy} \right)^{\alpha}. \tag{3.2}$$

We recall that $J_f \in A_\infty$. Therefore, appealing to the definition of the $\tilde{A}_\infty$–constant of J_f, we find some constants M', γ with

$$0 < \gamma \leq 1 \leq M',$$

and

$$\frac{M'}{\gamma} < \tilde{A}_\infty(J_f) + \varepsilon, \tag{3.3}$$

such that, for every ball $B \subset \mathbf{R}^n$ and for every measurable set $E \subset B$ we have

$$\frac{|E|}{|B|} \leq M' \left(\frac{|f(E)|}{|f(B)|} \right)^{\gamma}. \tag{3.4}$$

Let $B = B_r(x_0)$ and let $E \subset B$ be measurable. Define

$$R = \max\{|f(x') - f(x_0)| \colon |x' - x_0| = r\}. \tag{3.5}$$

The following inclusions hold

$$B_{\frac{R}{H}}(f(x_0)) \subset f(B) \subset B_R(f(x_0)). \tag{3.6}$$

Indeed, the second inclusion in (3.6) follows directly from the definition of R in (3.5); on the other hand, the quasisymmetry of f shows that

$$H|f(x) - f(x_0)| < |f(x') - f(x_0)| \quad \text{implies} \quad |x - x_0| < |x' - x_0|,$$

and this proves the first inclusion in (3.6). We deduce from (3.4) and (3.6) that

$$\frac{|E|}{|B|} \leq M' \left(\frac{|f(E)|}{\left| B_{\frac{R}{H}}\left(f(x_0)\right) \right|} \right)^{\gamma} = H^{n\gamma} M' \left(\frac{|f(E)|}{|B_R(f(x_0))|} \right)^{\gamma}.$$

Let us remark that $H \geq 1$ and $0 < \gamma \leq 1$ implies

$$H^{n\gamma} \leq H^n.$$

Therefore

$$\frac{|E|}{|B|} \leq H^n M' \left(\frac{|f(E)|}{|B_R(f(x_0))|} \right)^{\gamma}. \tag{3.7}$$

It follows from (3.6) that $f(E) \subset B_R(f(x_0))$. Hence, in (3.7) we apply (3.2) with $E' = f(E)$ and $B' = B_R(f(x_0))$ and from (3.6) we deduce

$$\begin{aligned} \frac{|E|}{|B|} &\leq H^n M' \left[M \left(\frac{\int_{f(E)} w(y)\, dy}{\int_{B_R(f(x_0))} w(y)\, dy} \right)^{\alpha} \right]^{\gamma} \\ &= H^n M' M^{\gamma} \left(\frac{\int_{f(E)} w(y)\, dy}{\int_{B_R(f(x_0))} w(y)\, dy} \right)^{\gamma\alpha} \\ &\leq H^n M' M^{\gamma} \left(\frac{\int_{f(E)} w(y)\, dy}{\int_{f(B)} w(y)\, dy} \right)^{\gamma\alpha}. \end{aligned} \tag{3.8}$$

Let us remark that $M \geq 1$ and $0 < \gamma \leq 1$ implies

$$M^{\gamma} \leq M. \tag{3.9}$$

Hence, (3.8), (3.9) and the change of variable formula imply

$$\frac{|E|}{|B|} \leq H^n M' M \left(\frac{\int_E (w \circ f) J_f\, dx}{\int_B (w \circ f) J_f\, dx} \right)^{\gamma\alpha}.$$

It follows that

$$\tilde{A}_\infty \left[(w \circ f) J_f \right] \leq H^n \frac{M'}{\gamma} \frac{M}{\alpha}.$$

We use (3.1) and (3.3) and we have

$$\tilde{A}_\infty \left[(w \circ f) J_f \right] \leq H^n \left[\tilde{A}_\infty \left(J_f \right) + \varepsilon \right] \left[\tilde{A}_\infty \left(w \right) + \varepsilon \right].$$

Therefore, taking the limit as $\varepsilon \to 0$ we obtain

$$\tilde{A}_\infty \left[(w \circ f) J_f \right] \leq H_f^n \tilde{A}_\infty \left(J_f \right) \tilde{A}_\infty \left(w \right). \tag{3.10}$$

It remains to prove the validity of the first inequality in (1.9). In (3.10) we may always replace f by f^{-1} and w by $(w \circ f) J_f$. We let

$$v = (w \circ f)\, J_f.$$

It is clear from the first part of our proof that $v \in A_\infty$. We recall (see e.g. [32]) that the Jacobians J_f and $J_{f^{-1}}$ are both positive a.e. and they are related by

$$J_{f^{-1}}(y) = \frac{1}{J_f(f^{-1}(y))},$$

so we have

$$\left(v \circ f^{-1}\right) J_{f^{-1}} = w(J_f \circ f^{-1}) J_{f^{-1}} = w.$$

Therefore

$$\begin{aligned}\tilde{A}_\infty(w) &= \tilde{A}\left[\left(v \circ f^{-1}\right) J_{f^{-1}}\right] \leq H_{f^{-1}}^n \tilde{A}_\infty(J_{f^{-1}}) \tilde{A}_\infty(v) \\ &= H_{f^{-1}}^n \tilde{A}_\infty(J_{f^{-1}}) \tilde{A}\left[(w \circ f) J_f\right].\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2. Let $H = H_f$ where H_f is given by (1.8). We start by observing that, since both f and f^{-1} are quasiconformal, the following identities follow directly by the change of variable formula

$$|f(F)| = \int_F J_f(x)\,dx \quad \text{for every measurable set } F \subset \mathbf{R}^n,$$
$$|f^{-1}(E)| = \int_E J_{f^{-1}}(y)\,dy \quad \text{for every measurable set } E \subset \mathbf{R}^n.$$

Hence, the constant $\tilde{A}_\infty(J_{f^{-1}})$ is the infimum of all quotients M/α where $0 < \alpha \leq 1 \leq M$ and the following estimate holds

$$\frac{|E|}{|B|} \leq M\left(\frac{|f^{-1}(E)|}{|f^{-1}(B)|}\right)^\alpha, \tag{3.11}$$

for every ball $B \subset \mathbf{R}^n$ and for every measurable $E \subset B$. Similarly, the constant $\tilde{G}_1(J_f)$ is the infimum of all quotients L/β where $0 < \beta \leq 1 \leq L$ and the following estimate holds

$$\frac{|f(F)|}{|f(B)|} \leq L\left(\frac{|F|}{|B|}\right)^\beta, \tag{3.12}$$

for every ball $B \subset \mathbf{R}^n$ and for every measurable $F \subset B$.

Our aim is to prove that

$$\tilde{G}_1(J_f) \leq H^n \tilde{A}_\infty(J_{f^{-1}}). \tag{3.13}$$

Let $B = B_r(x_0)$ be a ball of $\mathbf{R}^n$ and let $F \subset B$ be a measurable set. We fix $\varepsilon > 0$ and we find some constants M, α with $0 < \alpha \leq 1 \leq M$ for which (3.11) holds and

$$\frac{M}{\alpha} < \tilde{A}_\infty(J_{f^{-1}}) + \varepsilon. \tag{3.14}$$

Arguing as in the proof of Theorem 1.1 we find a radius $R > 0$ for which the following inclusions holds

$$B_{\frac{R}{H}}\left(f(x_0)\right) \subset f(B) \subset B_R\left(f(x_0)\right). \tag{3.15}$$

In particular, we see that

$$B \subset f^{-1}\left(B_R\left(f(x_0)\right)\right). \tag{3.16}$$

We set

$$E := f(F).$$

From (3.15) we deduce that

$$\frac{|f(F)|}{|f(B)|} \leq \frac{|E|}{\left|B_{\frac{R}{H}}(f(x_0))\right|} = H^n \frac{|E|}{|B_R(f(x_0))|}.$$

Appealing to (3.11) and (3.16)

$$\frac{|f(F)|}{|f(B)|} \le H^n M\left(\frac{|f^{-1}(E)|}{|f^{-1}(B_R(f(x_0)))|}\right)^\alpha \le H^n M\left(\frac{|F|}{|B|}\right)^\alpha .$$

It follows directly from the definition of $\tilde{G}_1(J_f)$ and from (3.14) that

$$\tilde{G}_1(J_f) \le H^n \frac{M}{\alpha} < H^n \left(\tilde{A}_\infty(J_{f^{-1}}) + \varepsilon\right).$$

We take the limit as $\varepsilon \to 0$ and we obtain (3.13).

It remains to prove the validity of the first inequality in (1.17). We set

$$H' = H_{f^{-1}}. \tag{3.17}$$

If we replace f by f^{-1} in the argument which proves the validity of (3.15) we see that, if $B \subset \mathbf{R}^n$ is a ball, then

$$B_{\frac{R'}{H'}}\left(f^{-1}(y_0)\right) \subset f^{-1}(B) \subset B_{R'}\left(f^{-1}(y_0)\right), \tag{3.18}$$

where

$$R' = \max\{|f^{-1}(y) - f^{-1}(y_0)| \colon |y - y_0| = r'\}.$$

We fix $\theta > 0$ and we find some constants L, β with $0 < \beta \le 1 \le L$ for which (3.12) holds and

$$\frac{L}{\beta} < \tilde{G}_1(J_f) + \theta. \tag{3.19}$$

We fix $E \subset B$ and we set

$$F := f^{-1}(E).$$

From (3.12) and (3.18) we deduce that

$$\frac{|E|}{|B|} = \frac{|f(F)|}{|f(f^{-1}(B)|} \le L\left(\frac{|F|}{|f^{-1}(B)|}\right)^\beta \le L\left(\frac{|F|}{\left|B_{\frac{R'}{H'}}(f^{-1}(y_0))\right|}\right)^\beta . \tag{3.20}$$

Therefore we have

$$\frac{|E|}{|B|} \le (H')^{n\beta} L\left(\frac{|F|}{|B_{R'}(f^{-1}(y_0))|}\right)^\beta .$$

Since $H' \ge 1$, from $0 < \beta \le 1$ immediately follows $(H')^{n\beta} \le (H')^n$; moreover, again from (3.18), we get

$$\frac{|E|}{|B|} \le (H')^n L\left(\frac{|f^{-1}(E)|}{|f^{-1}(B)|}\right)^\beta .$$

It follows directly from the definition of $\tilde{A}_\infty(J_{f^{-1}})$ and from (3.19) that

$$\tilde{A}_\infty(J_{f^{-1}}) \le (H')^n \frac{L}{\beta} < (H')^n \left(\tilde{G}_1(J_f) + \theta\right).$$

Recalling the definition of H' as in (3.17), we take the limit as $\theta \to 0$ and we obtain

$$\tilde{A}_\infty(J_{f^{-1}}) \le H^n_{f^{-1}} \tilde{G}_1(J_f).$$

This completes the proof. □

4. Final remarks

In this section we prove a result announced in the Introduction. We recall that the Jacobian of the radial stretching

$$f(x) = \rho(|x|)\frac{x}{|x|},$$

satisfies

$$J_f(x) \sim \dot{\rho}(|x|)\left(\frac{\rho(|x|)}{|x|}\right)^{n-1}.$$

Here $\rho(\cdot)$ is a smooth increasing function such that $\rho(0) = 0$ and $\dot{\rho}(\cdot)$ is its derivative. Moreover, we use the notation

$$\varphi(x) \sim \psi(x),$$

to mean that the couple of weights φ and ψ satisfies

$$\varphi(x) = c\psi(x),$$

for some constant $c > 0$.

Proposition 1. *For each $\lambda \in [0, 1)$ there exists a weight $w \in A_\infty$ and a quasiconformal mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ such that*

$$(w \circ f)J_f^\lambda \notin A_\infty.$$

Proof. Before we start the proof of the claimed result, we recall that

$$|x|^\theta \in A_\infty \quad \text{if and only if} \quad -n < \theta < \infty. \tag{4.1}$$

We consider quasiconformal mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$f(x) = |x|^\gamma \frac{x}{|x|},$$

and the weight

$$w(x) = |x|^\theta,$$

with the special choices

$$-n < \theta < -n\lambda,$$

$$\frac{n(\lambda - 1)}{\theta + n\lambda} \le \gamma < \infty.$$

Thus $w \in A_\infty$ (observe that $-n < \theta \le 0$ in this case) and $\gamma > 1$. We compute the Jacobian of f and we get

$$J_f(x) \sim |x|^{n(\gamma-1)}.$$

The function

$$u(x) = w(f(x))J_f(x)^\lambda,$$

satisfies the property

$$u(x) \sim |x|^{\theta\gamma + n\lambda(\gamma-1)}.$$

Observing that

$$\theta\gamma + n\lambda(\gamma - 1) \le -n,$$

from (4.1) we conclude that $u \notin A_\infty$ as desired. □

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