

GENERALIZED HAUSDORFF MEASURE FOR GENERIC COMPACT SETS

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Abstract. Let X be a Polish space. We prove that the generic compact set $K \subseteq X$ (in the sense of Baire category) is either finite or there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, where \mathcal{H}^h denotes the h -Hausdorff measure. This answers a question of Cabrelli, Darji, and Molter. Moreover, for every weak contraction $f: K \rightarrow X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$. This is a measure theoretic analogue of a result of Elekes.

1. Introduction

Hausdorff dimension is one of the most important concepts to measure the size of a metric space, but there are some cases when a finer notion of dimension is needed. An important example is the trail of the n -dimensional ($n \geq 2$) Brownian motion defined on $[0, 1]$. It has Hausdorff dimension 2 almost surely, but its \mathcal{H}^2 measure is 0 with probability 1. It is well-known that there is a gauge function h such that the h -Hausdorff measure of the trail is positive and finite almost surely, where $h(x) = x^2 \log \log(1/x)$ if $n \geq 3$ and $h(x) = x^2 \log(1/x) \log \log \log(1/x)$ if $n = 2$. Thus the exact dimension is logarithmically smaller than 2.

Davies [3] constructed a Cantor set $K \subseteq \mathbf{R}$ that is either null or non- σ -finite for every translation invariant Borel measure on \mathbf{R} . This implies that there is no gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, where \mathcal{H}^h denotes the h -Hausdorff measure. Cabrelli, Darji, and Molter [2] dealt with the problem that for ‘how many’ compact sets $K \subseteq \mathbf{R}$ exist a translation invariant Borel measure μ or a gauge function h such that $0 < \mu(K) < \infty$ or $0 < \mathcal{H}^h(K) < \infty$, respectively. They proved that the generic compact set $K \subseteq \mathbf{R}$ (see Definition 4.1) admits a translation invariant Borel measure μ such that $0 < \mu(K) < \infty$. They defined a compact set $K \subseteq \mathbf{R}$ to be \mathcal{H} -visible if there is a gauge function h such that $0 < \mathcal{H}^h(K) < \infty$. They showed that the set of \mathcal{H} -visible compact sets is dense in the space of all non-empty compact subsets of \mathbf{R} endowed with the Hausdorff metric. They posed the problem whether the generic compact set $K \subseteq \mathbf{R}$ is \mathcal{H} -visible. We answer this question affirmatively by the following more general result.

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Theorem 1.1. *Let X be a Polish space. The generic compact set $K \subseteq X$ is either finite or there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$.*

We remark here that for every fixed gauge function h the generic compact set $K \subseteq X$ has zero \mathcal{H}^h measure.

If X is a perfect Polish space then the set of finite compact subsets of X form a meager set in the metric space of all non-empty compact subsets of X endowed with the Hausdorff metric. Therefore Theorem 1.1 implies the following result.

Corollary 1.2. *Let X be a perfect Polish space. For the generic compact set $K \subseteq X$ there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$.*

Elekes [4] studied metric spaces X which are not complete but possess the Banach Fixed Point Theorem, that is, every contraction $f: X \rightarrow X$ has a fixed point. He proved the following theorem which is interesting in its own right.

Theorem 1.3. [Elekes] *For the generic compact set $K \subseteq \mathbf{R}$ for any contraction $f: K \rightarrow \mathbf{R}$ the set $f(K)$ does not contain a non-empty relatively open subset of K .*

The first author of the present paper [1] constructed metric spaces X such that every weak contraction $f: X \rightarrow X$ is constant, where he used measure theoretic methods. Based on [1], we prove the (somewhat stronger) measure theoretic analogue of Theorem 1.3.

Theorem 4.2. [Main Theorem] *Let X be a Polish space. The generic compact set $K \subseteq X$ is either finite or there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f: K \rightarrow X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.*

In Section 2 we recall some notions from metric spaces which we use in this paper. In Section 3 we introduce the notion of balanced compact sets. It is shown in [1] that for every balanced compact set there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$ and that $\mathcal{H}^h(K \cap f(K)) = 0$ for every weak contraction $f: K \rightarrow X$. In Section 4 we prove that in a perfect Polish space the generic compact set is a balanced compact set, and we conclude the proof of Theorem 4.2 and Theorem 1.1.

2. Preliminaries

Let (X, d) be a metric space, and let $A, B \subseteq X$ be arbitrary sets. We denote by $\text{cl}A$ and $\text{diam}A$ the closure and the diameter of A , respectively. We use the convention $\text{diam} \emptyset = 0$. The *distance* of the sets A and B is defined by $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Let $B(x, r) = \{y \in X : d(x, y) \leq r\}$ and $U(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$. More generally, consider $B(A, r) = \{x \in X : \text{dist}(A, \{x\}) \leq r\}$.

The function $h: [0, \infty) \rightarrow [0, \infty)$ is defined to be a *gauge function* if it is non-decreasing, right-continuous, and $h(x) = 0$ iff $x = 0$. For $A \subseteq X$ and $\delta > 0$ consider

$$\mathcal{H}_\delta^h(A) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam} A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i, \forall i \text{ diam} A_i \leq \delta \right\},$$

$$\mathcal{H}^h(A) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^h(A).$$

We call \mathcal{H}^h the *h -Hausdorff measure*. For more information on these concepts see [6].

Let X be a complete metric space. A set is *somewhere dense* if it is dense in a non-empty open set, otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is *co-meager* if its complement is meager. Baire’s Category Theorem implies that a set is co-meager if and only if it contains a dense G_δ set. We say that the *generic* element $x \in X$ has property \mathcal{P} if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. A metric space X is *perfect* if it has no isolated points. A metric space X is *Polish* if it is complete and separable.

Given two metric spaces (X, d_X) and (Y, d_Y) , a function $f: X \rightarrow Y$ is called a *weak contraction* if $d_Y(f(x_1), f(x_2)) < d_X(x_1, x_2)$ for every $x_1, x_2 \in X, x_1 \neq x_2$.

Let $\mathbf{N}^{<\omega}$ stand for the set of finite sequences of natural numbers. Let us denote the set of positive odd numbers by $2\mathbf{N} + 1$.

3. The definition of balanced compact sets

Following [1] we define balanced compact sets.

Definition 3.1. If a_n ($n \in \mathbf{N}^+$) are positive integers then let us consider, for every $n \in \mathbf{N}^+$,

$$\mathcal{I}_n = \prod_{k=1}^n \{1, 2, \dots, a_k\} \quad \text{and} \quad \mathcal{I} = \bigcup_{n=1}^\infty \mathcal{I}_n.$$

We say that a map $\Phi: 2\mathbf{N} + 1 \rightarrow \mathcal{I}$ is an *index function according to the sequence* $\langle a_n \rangle$ if it is surjective and $\Phi(n) \in \bigcup_{k=1}^n \mathcal{I}_k$ for every odd n .

Definition 3.2. Let X be a Polish space. A compact set $K \subseteq X$ is *balanced* if it is of the form

$$(3.1) \quad K = \bigcap_{n=1}^\infty \left(\bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1 \dots i_n} \right),$$

where the a_n are positive integers and $C_{i_1 \dots i_n} \subseteq X$ are non-empty closed sets with the following properties. There are positive reals b_n and there is an index function $\Phi: 2\mathbf{N} + 1 \rightarrow \mathcal{I}$ according to the sequence $\langle a_n \rangle$ such that for all $n \in \mathbf{N}^+$ and $(i_1, \dots, i_n), (j_1, \dots, j_n) \in \mathcal{I}_n$

- (i) $a_1 \geq 2$ and $a_{n+1} \geq na_1 \cdots a_n$,
- (ii) $C_{i_1 \dots i_{n+1}} \subseteq C_{i_1 \dots i_n}$,
- (iii) $\text{diam } C_{i_1 \dots i_n} \leq b_n$,
- (iv) $\text{dist}(C_{i_1 \dots i_n}, C_{j_1 \dots j_n}) > 2b_n$ if $(i_1, \dots, i_n) \neq (j_1, \dots, j_n)$.
- (v) If n is odd, $C_{i_1 \dots i_n} \subseteq C_{\Phi(n)}$ and $C_{j_1 \dots j_n} \not\subseteq C_{\Phi(n)}$, then for all $s, t \in \{1, \dots, a_{n+1}\}, s \neq t$, we have

$$\text{dist}(C_{i_1 \dots i_n s}, C_{i_1 \dots i_n t}) > \text{diam} \left(\bigcup_{j_{n+1}=1}^{a_{n+1}} C_{j_1 \dots j_n j_{n+1}} \right).$$

Remark 3.3. Property (v) and the notion of an index function Φ are not needed for the proof of Theorem 1.1, only for Theorem 4.2.

Note that we cannot require property (v) for every positive integer. The proof of Lemma 4.11 only works if we restrict this property to odd numbers.

Remark 3.4. In a countable Polish space X there is no balanced compact set $K \subseteq X$, since every balanced compact set has cardinality 2^{\aleph_0} .

4. The Main Theorem

Definition 4.1. If X is a Polish space then let $(\mathcal{K}(X), d_H)$ be the set of non-empty compact subsets of X endowed with the *Hausdorff metric*; that is, for each $K_1, K_2 \in \mathcal{K}(X)$,

$$d_H(K_1, K_2) = \min \{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\}.$$

It is well-known that $(\mathcal{K}(X), d_H)$ is a Polish space, see e.g. [5], hence we can use Baire category arguments. Let $B_H(K, r) \subseteq \mathcal{K}(X)$ denote the closed ball around K with radius r .

The main goal of this paper is to prove the following theorem.

Theorem 4.2. [Main Theorem] *Let X be a Polish space. The generic compact set $K \subseteq X$ is either finite or there is a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f: K \rightarrow X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.*

Remark 4.3. If X is a Polish space and h is a fixed gauge function then it is easy to see that for the generic compact set $K \subseteq X$ we have $\mathcal{H}^h(K) = 0$. If X is uncountable then infinite compact sets form a second category subset in $\mathcal{K}(X)$, therefore the gauge function h must depend on K in the Main Theorem.

The first author of the paper proved the following theorem [1, Thm. 5.1].

Theorem 4.4. *Let X be a Polish space, and let $K \subseteq X$ be a balanced compact set. Then there exists a continuous gauge function h such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f: K \rightarrow X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.*

If h is a gauge function then finite sets have zero \mathcal{H}^h measure, so Theorem 4.4 also holds for compact sets $K \subseteq X$ that can be written as a union of a balanced compact set and a finite set. Therefore the following theorem implies our Main Theorem.

Theorem 4.5. *If X is a Polish space then the generic compact set $K \subseteq X$ is either finite or it can be written as the union of a balanced compact set and a finite set.*

To prove Theorem 4.5 first we give definitions and prove two key lemmas.

Definition 4.6. Let us fix an onto map $\Psi: 2\mathbf{N} + 1 \rightarrow \mathbf{N}^{<\omega}$ such that $\Psi(n)$ has at most n coordinates for every odd n .

For $n \in \mathbf{N}^+$ and sequence $(a_1, a_2, \dots, a_{2n-1})$, we define the function

$$\Phi = \Phi_{a_1 a_2 \dots a_{2n-1}} : \{2k - 1 : 1 \leq k \leq n\} \rightarrow \bigcup_{m=1}^{2n-1} \mathcal{I}_m$$

by setting

$$\Phi(2k - 1) = \begin{cases} \Psi(2k - 1) & \text{if } \Psi(2k - 1) \in \bigcup_{m=1}^{2k-1} \mathcal{I}_m \\ 1 \in \mathcal{I}_1 & \text{otherwise.} \end{cases}$$

Remark 4.7. If $\langle a_n \rangle_{n \in \mathbf{N}^+}$ is a sequence of positive integers then the above definition implies that the functions $\Phi_{a_1 \dots a_{2n-1}}$ have a common extension $\Phi: 2\mathbf{N} + 1 \rightarrow \mathcal{I}$, and Φ is an index function according to the sequence $\langle a_n \rangle$.

Let X be a Polish space.

Definition 4.8. Let $n \in \mathbf{N}^+$. We call the pair of (a_1, \dots, a_{2n}) and

$$\left\{ ((i_1, \dots, i_k), U_{i_1 \dots i_k}) : (i_1, \dots, i_k) \in \mathcal{I}_k, 1 \leq k \leq 2n \right\}$$

a *balanced scheme* of size n if the numbers a_k are positive integers, the sets $U_{i_1 \dots i_k}$ are non-empty open subsets of X , and there exist positive reals b_k for which

- (1) $a_1 \geq 2$ and $a_k \geq (k - 1)a_1 \cdots a_{k-1}$ for all $2 \leq k \leq 2n$,
- (2) $\text{cl} U_{i_1 \dots i_k} \subseteq U_{i_1 \dots i_{k-1}}$ for all $(i_1, \dots, i_k) \in \mathcal{I}_k$ and $2 \leq k \leq 2n$,
- (3) $\text{diam} U_{i_1 \dots i_k} \leq b_k$ for all $(i_1, \dots, i_k) \in \mathcal{I}_k$ and $1 \leq k \leq 2n$,
- (4) $\text{dist}(U_{i_1 \dots i_k}, U_{j_1 \dots j_k}) > 2b_k$ if $(i_1, \dots, i_k) \neq (j_1, \dots, j_k) \in \mathcal{I}_k$ and $1 \leq k \leq 2n$.
- (5) Let $\Phi = \Phi_{a_1 \dots a_{2n-1}}$. If $k < 2n$ is odd, $U_{i_1 \dots i_k} \subseteq U_{\Phi(k)}$ and $U_{j_1 \dots j_k} \not\subseteq U_{\Phi(k)}$, then for all $s, t \in \{1, \dots, a_{k+1}\}$, $s \neq t$, we have

$$\text{dist}(U_{i_1 \dots i_k s}, U_{i_1 \dots i_k t}) > \text{diam} \left(\bigcup_{j_{k+1}=1}^{a_{k+1}} U_{j_1 \dots j_k j_{k+1}} \right).$$

Let (\emptyset, \emptyset) be the *balanced scheme* of size 0.

Definition 4.9. If $n \in \mathbf{N}^+$ and π is a balanced scheme of size n as in Definition 4.8, then we define a non-empty open subset of $\mathcal{K}(X)$,

$$\mathcal{U}(\pi) = \left\{ K \in \mathcal{K}(X) : K \subseteq \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_{2n}=1}^{a_{2n}} U_{i_1 \dots i_{2n}}, \forall (i_1, \dots, i_{2n}) \in \mathcal{I}_{2n} K \cap U_{i_1 \dots i_{2n}} \neq \emptyset \right\}.$$

For $\pi = (\emptyset, \emptyset)$ we define $\mathcal{U}(\pi) = \mathcal{K}(X)$.

Assume $n \in \mathbf{N}$, and let π and π' be balanced schemes of size n and $n + 1$, respectively. We say that π' is *consistent* with π if $a_k(\pi') = a_k(\pi)$ and $U_{i_1 \dots i_k}(\pi') = U_{i_1 \dots i_k}(\pi)$ for all $k \in \{1, \dots, 2n\}$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$.

Remark 4.10. Let π and π' be balanced schemes of size n and $n+1$, respectively. If π' is consistent with π then $\mathcal{U}(\pi') \subseteq \mathcal{U}(\pi)$, and we may assume $b_k(\pi') = b_k(\pi)$ for every $k \in \{1, \dots, 2n\}$.

Lemma 4.11. Assume $n \in \mathbf{N}$. Let X be a non-empty perfect Polish space, let π be a balanced scheme of size n , and let $\mathcal{V} \subseteq \mathcal{U}(\pi)$ be a non-empty open subset of $\mathcal{K}(X)$. There exists a balanced scheme π' of size $n + 1$ such that π' is consistent with π and $\mathcal{U}(\pi') \subseteq \mathcal{V}$.

Proof. Let $a_k(\pi') = a_k(\pi) = a_k$, $b_k(\pi') = b_k(\pi) = b_k$, $U_{i_1 \dots i_k}(\pi') = U_{i_1 \dots i_k}(\pi) = U_{i_1 \dots i_k}$ for every $k \leq 2n$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$. Then π' will satisfy properties (1)-(5) for all $k \leq 2n$, since the map $\Phi_{a_1 \dots a_{2n+1}}$ extends $\Phi_{a_1 \dots a_{2n-1}}$ by Definition 4.6. Therefore it is enough to construct $a_k(\pi') = a_k$, $b_k(\pi') = b_k$, and $U_{i_1 \dots i_k}(\pi') = U_{i_1 \dots i_k}$ for $k \in \{2n + 1, 2n + 2\}$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$.

As finite compact sets form a dense subset in $\mathcal{K}(X)$ and X is perfect, it is easy to see that there is a finite set $K_0 \in \mathcal{V}$ with the following property. There is an integer $N \geq 2$ such that $N \geq 2n(a_1 \cdots a_{2n})$ and $\#(K_0 \cap U_{i_1 \dots i_{2n}}) = N$ for every $(i_1, \dots, i_{2n}) \in \mathcal{I}_{2n}$. Set $a_{2n+1} = N$, then (1) holds for $k = 2n + 1$. For $(i_1, \dots, i_{2n}) \in \mathcal{I}_{2n}$ let

$$K_0 \cap U_{i_1 \dots i_{2n}} = \{x_{i_1 \dots i_{2n+1}} : 1 \leq i_{2n+1} \leq a_{2n+1}\}.$$

For $(i_1, \dots, i_{2n+1}) \in \mathcal{I}_{2n+1}$ consider the non-empty open sets

$$U_{i_1 \dots i_{2n+1}} = U(x_{i_1 \dots i_{2n+1}}, b_{2n+1}/2),$$

where $b_{2n+1} > 0$ is sufficiently small. Then the sets $U_{i_1 \dots i_{2n+1}}$ satisfy properties (2)–(4), and $B_H(K_0, b_{2n+1}) \subseteq \mathcal{V}$. (Notice that we did not require property (5) to hold for even numbers, and indeed, we could not satisfy it here for an arbitrary \mathcal{V} .)

Let $a_{2n+2} = (2n + 1)(a_1 \cdots a_{2n+1})$, so (1) holds for $k = 2n + 2$. First consider those (i_1, \dots, i_{2n+1}) for which $U_{i_1 \dots i_{2n+1}} \subseteq U_{\Phi(2n+1)}$, where $\Phi = \Phi_{a_1 \dots a_{2n+1}}$. Then by the perfectness of X we can fix distinct points $x_{i_1 \dots i_{2n+2}} \in U_{i_1 \dots i_{2n+1}}$ ($i_{2n+2} \in \{1, \dots, a_{2n+2}\}$).

Let δ be the minimum distance between the points $x_{i_1 \dots i_{2n+2}}$ we have defined so far. Now consider those (i_1, \dots, i_{2n+1}) for which $U_{i_1 \dots i_{2n+1}} \not\subseteq U_{\Phi(2n+1)}$. For each of them, fix distinct points $x_{i_1 \dots i_{2n+2}} \in U_{i_1 \dots i_{2n+1}}$ ($i_{2n+2} \in \{1, \dots, a_{2n+2}\}$) such that

$$\text{diam} \left(\bigcup_{i_{2n+2}=1}^{a_{2n+2}} \{x_{i_1 \dots i_{2n+2}}\} \right) \leq \frac{\delta}{2}.$$

For $(i_1, \dots, i_{2n+2}) \in \mathcal{I}_{2n+2}$ consider the non-empty open sets

$$U_{i_1 \dots i_{2n+2}} = U(x_{i_1 \dots i_{2n+2}}, b_{2n+2}/2),$$

where $b_{2n+2} > 0$ is sufficiently small. Then the sets $U_{i_1 \dots i_{2n+2}}$ satisfy properties (2)–(5). Therefore π' is a balanced scheme of size $n + 1$, and π' is consistent with π .

Finally, we need to prove that $\mathcal{U}(\pi') \subseteq \mathcal{V}$. We show that for every $K \in \mathcal{U}(\pi')$,

$$(4.1) \quad d_H(K, K_0) \leq b_{2n+1}.$$

Let $K \in \mathcal{U}(\pi')$. By the definition of $\mathcal{U}(\pi')$ we have $K \subseteq \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_{2n+1}=1}^{a_{2n+1}} U_{i_1 \dots i_{2n+1}}$ and $K \cap U_{i_1 \dots i_{2n+1}} \neq \emptyset$ for all $(i_1, \dots, i_{2n+1}) \in \mathcal{I}_{2n+1}$. The set K_0 has the above properties by its definition, too. As $\text{diam} U_{i_1 \dots i_{2n+1}} \leq b_{2n+1}$ for all $(i_1, \dots, i_{2n+1}) \in \mathcal{I}_{2n+1}$, (4.1) follows. Equation (4.1) implies $\mathcal{U}(\pi') \subseteq B_H(K_0, b_{2n+1})$, therefore $B_H(K_0, b_{2n+1}) \subseteq \mathcal{V}$ yields $\mathcal{U}(\pi') \subseteq \mathcal{V}$. \square

Lemma 4.12. *Assume $n \in \mathbf{N}$. Let X be a non-empty perfect Polish space, and let π be a balanced scheme of size n . Then there are balanced schemes π_j ($j \in \mathbf{N}$) of size $n + 1$ such that each π_j is consistent with π , the sets $\mathcal{U}(\pi_j)$ ($j \in \mathbf{N}$) are pairwise disjoint, and $\bigcup_{j=0}^{\infty} \mathcal{U}(\pi_j)$ is dense in $\mathcal{U}(\pi)$.*

Proof. Let $\mathcal{U}_i \subseteq \mathcal{U}(\pi)$ ($i \in \mathbf{N}$) be non-empty disjoint open sets such that $\bigcup_{i=0}^{\infty} \mathcal{U}_i$ is dense in $\mathcal{U}(\pi)$. For all $i \in \mathbf{N}$ let \mathcal{B}_i be a countable basis of \mathcal{U}_i , and let $\mathcal{B} = \bigcup_{i=0}^{\infty} \mathcal{B}_i$. We may assume $\emptyset \notin \mathcal{B}$ and let us consider an enumeration $\mathcal{B} = \{\mathcal{V}_n : n \in \mathbf{N}\}$. Let $j \in \mathbf{N}$ and assume that π_k and $n(k) \in \mathbf{N}$ ($k < j$) are already defined such that $\mathcal{U}(\pi_k) \subseteq \mathcal{V}_{n(k)}$ for $k < j$. Consider

$$n(j) = \min \{n \in \mathbf{N} : \mathcal{V}_n \cap (\bigcup_{k < j} \mathcal{U}(\pi_k)) = \emptyset\}.$$

The definition of \mathcal{B} and the induction hypothesis easily imply that $\bigcup_{k < j} \mathcal{U}(\pi_k)$ can intersect at most j open sets \mathcal{U}_i , so $n(j) < \infty$ exists. Lemma 4.11 implies that there is a balanced scheme π_j of size $n + 1$ such that π_j is consistent with π and $\mathcal{U}(\pi_j) \subseteq \mathcal{V}_{n(j)}$.

The construction yields that $\bigcup_{j=0}^{\infty} \mathcal{U}(\pi_j)$ intersects each \mathcal{V}_i , thus it is dense in each \mathcal{U}_i , therefore it is dense in $\mathcal{U}(\pi)$, and the union is clearly a disjoint union. \square

Now we are ready to prove Theorem 4.5 that implies our Main Theorem.

Proof of Theorem 4.5. First assume that X is perfect, we prove that the generic compact set $K \subseteq X$ is balanced. We may assume that $X \neq \emptyset$. Let $\mathcal{G}_0 = \mathcal{K}(X)$.

Lemma 4.12 implies that there are balanced schemes π_j ($j \in \mathbf{N}$) of size 1 such that the disjoint union

$$\mathcal{G}_1 = \bigcup_{j_1=0}^{\infty} \mathcal{U}(\pi_{j_1})$$

is a dense open set in $\mathcal{K}(X)$. Assume by induction that the balanced schemes $\pi_{j_1 \dots j_n}$ of size n and the dense open set \mathcal{G}_n are already defined. Lemma 4.12 implies that for every $j_1, \dots, j_n \in \mathbf{N}$ there exist balanced schemes $\pi_{j_1 \dots j_{n+1}}$ ($j_{n+1} \in \mathbf{N}$) of size $n + 1$ such that $\pi_{j_1 \dots j_{n+1}}$ is consistent with $\pi_{j_1 \dots j_n}$ and the disjoint union $\bigcup_{j_{n+1}=0}^{\infty} \mathcal{U}(\pi_{j_1 \dots j_{n+1}})$ is dense in $\mathcal{U}(\pi_{j_1 \dots j_n})$. Then the disjoint union

$$\mathcal{G}_{n+1} = \bigcup_{j_1=0}^{\infty} \dots \bigcup_{j_{n+1}=0}^{\infty} \mathcal{U}(\pi_{j_1 \dots j_{n+1}})$$

is dense in \mathcal{G}_n , and the induction hypothesis yields that \mathcal{G}_{n+1} is a dense open set in $\mathcal{K}(X)$. Consider

$$\mathcal{G} = \bigcap_{n=0}^{\infty} \mathcal{G}_n.$$

As a countable intersection of dense open sets \mathcal{G} is co-meager in $\mathcal{K}(X)$. Let $K \in \mathcal{G}$ be arbitrary fixed, it is enough to prove that K is balanced. Since the n th level open sets $\mathcal{U}(\pi_{j_1 \dots j_n})$ are pairwise disjoint, there is a (unique) sequence $\langle j_n \rangle_{n \in \mathbf{N}^+}$ such that $K \in \mathcal{U}(\pi_{j_1 \dots j_n})$ for all $n \in \mathbf{N}^+$. As the balanced scheme $\pi_{j_1 \dots j_{n+1}}$ is consistent with $\pi_{j_1 \dots j_n}$ for every $n \in \mathbf{N}^+$, there are positive integers a_n and non-empty open sets $U_{i_1 \dots i_n}$ witnessing this fact. By Remark 4.7, the functions $\Phi_{a_1 a_2 \dots a_{2n-1}}$ have a common extension $\Phi: 2\mathbf{N} + 1 \rightarrow \mathcal{I}$, and Φ is an index function according to the sequence $\langle a_n \rangle$. For $n \in \mathbf{N}^+$ and $(i_1, \dots, i_n) \in \mathcal{I}_n$ let us define

$$C_{i_1 \dots i_n} = \text{cl} U_{i_1 \dots i_n}.$$

Since $K \in \mathcal{U}(\pi_{j_1 \dots j_n})$ for every n , Definition 4.9 implies that

$$K = \bigcap_{n=1}^{\infty} \left(\bigcup_{i_1=1}^{a_1} \dots \bigcup_{i_n=1}^{a_n} C_{i_1 \dots i_n} \right).$$

From Definition 4.8 it follows that the positive integers a_n and the non-empty closed sets $C_{i_1 \dots i_n}$ satisfy properties (i)–(v) of Definition 3.2. Therefore K is balanced.

Now let X be an arbitrary non-empty Polish space. Then there is a perfect set $X^* \subseteq X$ such that $U = X \setminus X^*$ is countable open, see [5, (6.4) Thm.]. Let S be the set of isolated points of X . Then S is open, and $S \subseteq U$. We claim that S is dense in U , thus $U \subseteq \text{cl} S$. Indeed, assume to the contrary that there is a non-empty open set $V \subseteq U$ such that $V \cap S = \emptyset$. By shrinking V , we may suppose that $\text{cl} V \subseteq U$. Then $\text{cl} V \subseteq U$ is a non-empty perfect set, so it has cardinality 2^{\aleph_0} by [5, (6.3) Cor.], which is a contradiction.

For a set $A \subseteq X$ let us denote by $\mathcal{K}(A)$ the metric space of non-empty compact subsets of A , similarly as in Definition 4.1.

Since S is open, compact non-empty subsets of S form a dense open subset of $\mathcal{K}(\text{cl} S)$. As S is the set of isolated points, every compact subset of S is finite.

The first part of the proof implies that there is a dense G_δ set $\mathcal{F}^* \subseteq \mathcal{K}(X^*)$ such that every $K^* \in \mathcal{F}^*$ is balanced.

Let $\mathcal{F} \subseteq \mathcal{K}(X)$ be the set of those non-empty compact subsets $K \subseteq X$ for which $K \cap \text{cl} S \subseteq S$ and $K \cap X^* \in \mathcal{F}^* \cup \{\emptyset\}$. Clearly, every $K \in \mathcal{F}$ is a union of \emptyset or a balanced compact set in X^* and finitely many points in S . We claim that \mathcal{F} is a dense G_δ subset of $\mathcal{K}(X)$. Let us define the continuous map

$$R: \mathcal{K}(X) \rightarrow \mathcal{K}(X^*) \cup \{\emptyset\}, \quad R(K) = K \cap X^*,$$

where the distance of \emptyset to points of $\mathcal{K}(X^*)$ is defined to be 1.

We show that the map R is open. Let $K \in \mathcal{K}(X)$ and $C^* \in \mathcal{K}(X^*) \cup \{\emptyset\}$ be arbitrary, and set $K^* = K \cap X^*$. It is enough to construct $C \in \mathcal{K}(X)$ such that $C \cap X^* = C^*$ and $d_H(K, C) \leq d_H(K^*, C^*)$. If $K \subseteq X^*$ or $K^* = C^*$, then $C = C^*$ or $C = K$ works, respectively. Thus we may assume that $K \setminus X^* \neq \emptyset$ and $d_H(K^*, C^*) > 0$. The compactness of K implies that there are finitely many open sets V_i such that $K \setminus X^* \subseteq \bigcup_{i=1}^m V_i$, $V_i \cap (K \setminus X^*) \neq \emptyset$, and $\text{diam } V_i \leq d_H(K^*, C^*)$ for all $i \in \{1, \dots, m\}$. Let us choose $x_i \in V_i \setminus X^*$ for all $i \in \{1, \dots, m\}$ arbitrarily, and consider $C = C^* \cup \bigcup_{i=1}^m \{x_i\}$. It is easy to see that $C \in \mathcal{K}(X)$ fulfills the required properties.

Since R is open, $R^{-1}(\mathcal{F}^* \cup \{\emptyset\})$ is dense G_δ in $\mathcal{K}(X)$. We clearly have

$$\mathcal{F} = R^{-1}(\mathcal{F}^* \cup \{\emptyset\}) \cap \mathcal{K}((X \setminus \text{cl} S) \cup S).$$

As $(X \setminus \text{cl} S) \cup S$ is dense open in X , $\mathcal{K}((X \setminus \text{cl} S) \cup S)$ is dense open in $\mathcal{K}(X)$. Thus \mathcal{F} is dense G_δ in $\mathcal{K}(X)$, which concludes the proof. \square

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