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QUASI s-NUMBERS AND MEASURES OF NON-COMPACTNESS OF MULTILINEAR OPERATORS

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Abstract. The main goal of this paper is the study of quasi s-numbers of multilinear operators among Banach spaces. The relationships among multilinear variants of approximation, Kolmogorov and Gelfand numbers of operators and their generalized linear adjoint are shown. In the multilinear case, analogous theorems which are well-known in the linear case, are stated and proved. The estimates of measures of non-compactness of multilinear operators in terms of measures of the adjoint operators are also proved.

1. Introduction

The theory of s-numbers of linear bounded operators among Banach spaces was introduced and studied by Pietsch [10]. It plays a fundamental role in the theory of operators and the local theory of Banach spaces and it is a powerful tool in the study of eigenvalue distribution of Riesz operators in Banach spaces (see, e.g., [9, 13]). In 1983 Pietsch [12] proposed and sketched a theory of ideals and s-numbers of multilinear functionals. While the properties of s-numbers of linear operators have been studied extensively, the theory of s-numbers of multilinear operators has not been studied yet.

In this paper the theory of quasi s-number sequences of bounded multilinear operators among Banach spaces is developed. We investigate the question of how the fundamental properties of important s-numbers of linear operators are inherited to the multilinear case. It should be noted that whereas the work is based on some ideas from the theory of s-numbers of bounded linear operators, some proofs may be extended from the linear case to the multilinear operators and other require new ideas and methods. The difficulty comes from the fact that even in the bilinear case the range or the kernel of a bilinear operator is not necessarily a linear subspace. In

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particular, as a consequence the well-known relations between the dimensions of the kernel and the range in the linear case are not true in general in the multilinear case.

Throughout the paper the standard notation from the Banach space theory is used. If X is a Banach space we denote by X^* its dual Banach space and by U_X, $\overset{\circ}{U}_X$ the closed and open unit balls of X, respectively. As usual κ_X denotes the canonical embedding of X to the bidual X^{**} of X. For each $m \in \mathbf{N}$ the product $X_1 \times \cdots \times X_m$ of Banach spaces is equipped with the norm $\|(x_1, \ldots, x_m)\| = \max_{1\le j\le m} \|x_j\|_{X_j}$. We shall denote by $\mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ the Banach space of all m-linear bounded operators defined on $X_1 \times \cdots \times X_m$ with values in a Banach space Y, equipped with the norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_m)\|_Y;\ (x_1, \ldots, x_m) \in U_{X_1} \times \cdots \times U_{X_m}\}.$$

In the case when $m = 1$, we shall write $\mathcal{L}(X_1, Y)$ instead of $\mathcal{L}_1(X_1, Y)$. In the case when Y is the scalar field $\mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$), we denote the space of all m-linear forms by $\mathcal{L}_m(X_1 \times \cdots \times X_m)$. As usual, $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ will denote the projective tensor product of the Banach spaces $X_1, \ldots, X_m$.

For $m \ge 2$ let $X_1,\ldots,X_m$ be Banach spaces. Following [6] the Banach space $X_1 \times \cdots \times X_m$ is said to have the *multilinear extension property* if, whenever $F_i \subset X_i$ $(1 \le i \le m)$ are closed subspaces and $T \in \mathcal{L}_m(F_1 \times \cdots \times F_m)$ is a continuous multilinear form, then there exists a continuous multilinear form $\widetilde{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ such that $\widetilde{T}|_{F_1\times\cdots\times F_m} = T$. When $m = 2$, we will also say that $X_1 \times X_2$ has the *bilinear extension property*. It was proved in [6, Theorem 2.3] that $X_1 \times \cdots \times X_m$ has the multilinear extension property if, and only if, there exists a constant $M > 0$ such that whenever $F_i \subset X_i$ are closed subspaces, every $\widetilde{T} \in \mathcal{L}_m(F_1 \times \cdots \times F_m)$ has an extension $\widetilde{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ with $\|\widetilde{T}\| \le M\|T\|$. In this case we say that the $X_1 \times \cdots \times X_m$ has the multilinear extension property with constant M, and the least constant is denoted by $M(X_1 \times \cdots \times X_m)$.

Following the theory of s-numbers presented in [11, 13] and [2, 1], we introduce the notion of an m-quasi s-number sequence for m-multilinear bounded operators. For each $m \in \mathbf{N}$, a rule $s = (s_n)\colon \mathcal{L}_m(X_1 \times \cdots \times X_m, Y) \to [0,\infty)^{\mathbf{N}}$ assigning to every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ a non-negative scalar sequence $(s_n(T))$, it is called an m-quasi s-number sequence if the following conditions are satisfied:

(S1) *Monotonicity*: For every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,

$$\|T\| = s_1(T) \ge s_2(T) \ge \cdots \ge 0.$$

(S2) *Additivity*: For every $S, T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,

$$s_{k+n-1}(S + T) \le s_k(S) + s_n(T).$$

(S3) *Ideal-property*: For every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, $S \in \mathcal{L}(Y, Z)$,

$$s_n(ST) \le \|S\|\, s_n(T).$$

(S4) *Rank-property*:

$$\operatorname{rank}(T) < n \;\Rightarrow\; s_n(T) = 0.$$

If (s_n) is an m-quasi s-number sequence for each positive integer m, then (s_n) is called a quasi s-number sequence. A quasi s-number sequence is called an s-number sequence provided it satisfies

(S5) *Norming property*:

$$s_n(I\colon \ell_2^n \to \ell_2^n) = 1, \quad n \in \mathbf{N},$$

where I denotes the identity operator on the n-dimensional Hilbert space ℓ_2^n.

For simplicity of notation and presentation, similarly as in the linear case, we do not indicate the involved Banach spaces and we write for short $s_n(T)$ instead of $s_n(T\colon X_1 \times \cdots \times X_m \to Y)$.

We also need the following notions:

(J) An m-quasi s-number sequence $s = (s_n)$ is called injective if, given any metric injection $j \in \mathcal{L}(Y, Z)$, i.e., $\|j(y)\| = \|y\|$ for all $y \in Y$, $s_n(T) = s_n(jT)$ for all $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and all m-tuples of Banach spaces $(X_1, \ldots, X_m)$.

(S) An m-quasi s-number sequence $s = (s_n)$ is called surjective if, given any metric surjections $Q_j \in \mathcal{L}(Y_j, X_j)$, $1 \le j \le m$, i.e., $Q_j(\mathring{U}_{Y_j}) = \mathring{U}_{X_j}$ for each $1 \le j \le m$, $s_n(T) = s_n(T(Q_1 \times \cdots \times Q_m))$ for all $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and all Banach spaces Y, where $Q_1 \times \cdots \times Q_m$ is the linear operator from $Y_1 \times \cdots \times Y_m$ into $X_1 \times \cdots \times X_m$ defined by

$$Q_1 \times \cdots \times Q_m(y_1, \ldots, y_m) = (Q_1 y_1, \ldots, Q_m y_m), \quad (y_1, \ldots, y_m) \in Y_1 \times \cdots \cdot \times Y_m.$$

(JS) An m-quasi s-number sequence is called injective and surjective, if it satisfies (J) and (S).

(M) A quasi s-number sequence $s = (s_n)$ is called multiplicative if, for $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,

$$s_{k+n-1}(ST) \le s_k(S) s_n(T), \quad k, n \in \mathbf{N}.$$

The paper is organized as follows. In Section 2 we investigate the measure of non-compactness of multilinear operators among Banach spaces. The main results of this section states that, up to universal constants, the measure of non-compactness of every multilinear operator T is equivalent to the measure of non-compactness of its generalized adjoint (adjoint for short) operator $T^{\times}$ of T. As a consequence we obtain a variant of Schauder's theorem for multilinear operators, which was first proved for the bilinear case by Ramanujan and Schock [15].

In Section 3 an s-sequence of approximation numbers of multilinear operators is studied. This sequence is used to provide a multilinear variant of a remarkable Carl's mixing property for any quasi s-number sequence of multilinear operators. The relationships between approximation numbers of multilinear operator and its generalized linear adjoint operator are also obtained. In particular we show that for every compact multilinear operator T, $a_n(T) = a_n(T^{\times})$ for each positive integer n.

Section 4 is devoted to an s-sequence (d_n) of Kolmogorov numbers of multilinear operators. We show the fundamental properties of these numbers. Among others, we show that, as in the linear case, (d_n) is the largest surjective multiplicative quasi s-number sequence.

In Section 5 we define variants of Gelfand's numbers (c_n) of multilinear operators. It is proved that the Gelfand numbers is the largest injective multiplicative quasi s-number sequence and also that, in the multilinear case, variants of important relations between Gelfand and Kolmogorov numbers of an operator and its adjoint are true. Namely, we prove that $c_n(T^{\times}) \le d_n(T)$ and $c_n(T) = d_n(T^{\times})$ are true for every multilinear operator T and each positive integer n. In the last part of Section

5 we investigate the relationships between variants of Gelfand's numbers. The main key here is the multilinear extension property of multilinear forms. Using famous Maurey's extension theorem for the bilinear forms, we show applications to the bilinear operators $T\colon X \times Y \to Z$, where X and Y are finite dimensional Banach spaces of type 2.

2. Measures of non-compactness of multilinear operators

The present section is devoted to the relationships among the corresponding measure of non-compactness of multilinear operator and its adjoint. Let us recall that Schauder's well-known result states that an operator T between Banach spaces is compact if, and only if, its adjoint, T^*, is compact. A slightly more general result says that the Kuratowski–Hausdorff measure of non-compactness of an operator T is equivalent to the measure of non-compactness of its adjoint. Ramanujan and Schock studied in [15] ideals of bilinear operators between Banach spaces, including the ideal of bilinear compact operators, i.e., $T \in \mathcal{L}_2(X \times Y, Z)$ such that $T(U_X \times U_Y)$ is relatively compact in Z. Given $T \in \mathcal{L}_2(X \times Y, Z)$, they defined the adjoint linear map $T^\times\colon Z^* \to \mathcal{L}_2(X \times Y)$ by

$$T^\times z^*(x, y) = z^*(T(x, y)), \qquad (x, y) \in X \times Y.$$

Clearly $T^\times$ is bounded operator, and $\|T\| = \|T^\times\|$. Ramanujan and Schock [15, Theorem 2.6] proved the analogues of Schauder's theorem which states that if $T \in \mathcal{L}_2(X \times Y, Z)$, then T is compact if, and only if $T^\times$ is compact.

The mentioned results give rise to a question: whether the measures of non-compactness of a multilinear operator and its adjoint are equivalent? In the present section, we discuss this problem. We extend the classical well-known results for the linear operators to the case of multilinear operators.

Let us recall that if A is a bounded subset of a metric space $\mathcal{X}$, the *Kuratowski measure of non-compactness* is defined by

$$\alpha(A) = \inf\{\varepsilon > 0;\ A \text{ may be covered by finitely many sets of diameter} \le \varepsilon\};$$

the *Hausdorff ball measure of non-compactness* of A is defined by

$$\beta(A) = \inf\{\varepsilon > 0;\ A \text{ can be covered by finitely many balls of radius} \le \varepsilon\}.$$

It is easy to check that $\beta(A) \le \alpha(A) \le 2\beta(A)$ for every bounded set.

Recall that in a Banach space X, a set S is called an ε-net of A if $A \subset S + \varepsilon U_X$. Thus the definition of β-measure in a Banach space is equivalent to the following:

$$\beta(A) = \inf\{\varepsilon > 0;\ A \text{ has a finite } \varepsilon\text{-net}\}.$$

Let $X_1, \dots, X_m$ and Y be Banach spaces. The Kuratowski, and the Hausdorff measure of non-compactness of $T \in \mathcal{L}_m(X_1 \times \dots \times X_m, Y)$ are defined by

$$\gamma(T) = \alpha(T(U_{X_1 \times \dots \times X_m}))$$

and respectively,

$$\widetilde{\gamma}(T) = \beta(T(U_{X_1 \times \dots \times X_m})).$$

Since $T(U_{X_1 \times \dots \times U_{X_m}}) \subset \|T\| U_Y$, $\gamma(T) \le \|T\|$. It is clear that T is compact if, and only if, $\gamma(T) = 0$.

Throughout the paper the map $J\colon X_1 \times \cdots \times X_m \to \mathcal{L}_m(X_1 \times \cdots \times X_m)^*$ is defined by

$$Jx(B) = Bx, \quad x \in X_1 \times \cdots \times X_m,\ B \in \mathcal{L}_m(X_1 \times \cdots \times X_m).$$

Clearly J is a bounded m-linear operator. Given $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, we define the *generalized adjoint* (adjoint for short) operator $T^\times\colon Y^* \to \mathcal{L}_m(X_1 \times \cdots \times X_m)$ by

$$(T^\times y^*)x = y^*(Tx), \quad y^* \in Y^*,\ x \in X_1 \times \cdots \times X_m.$$

For $B \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ and $y \in Y$, by $B \otimes y$ we denote the m-linear operator

$$B \otimes y = B(x) \cdot y, \quad x \in X_1 \times \cdots \times X_m.$$

Obviously $\operatorname{Im}(B \otimes y)$ is a linear subspace of Y and $\operatorname{rank}(B \otimes y) = 1$ provided $B \neq 0$ and $y \neq 0$. We also have

$$(B \otimes y)^\times = \kappa_Y(y) \otimes B.$$

We will need the following observation.

Lemma 2.1. *Let $m \geq 2$ and let $X_1,\ldots,X_m$, Y be Banach spaces. Then for every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,*

$$(T^\times)^* J = \kappa_Y T.$$

Proof. Let $x \in X_1 \times \cdots \times X_m$ and $y^* \in Y^*$. Then we have

$$\langle y^*, (T^\times)^* J(x)\rangle = \langle T^\times y^*, J(x)\rangle = (y^* \circ T)(x) = \langle Tx, y^*\rangle = \langle y^*, \kappa_Y T(x)\rangle,$$

and this yields the required equality. $\square$

A well-known result about linear operators states that for every operator $T\colon X \to Y$ between Banach spaces X and Y, we have ([4, Theorem 2.9])

$$\gamma(T) \leq \widetilde{\gamma}(T^*) \quad \text{and} \quad \gamma(T^*) \leq \widetilde{\gamma}(T).$$

We will now prove analogous results in the multilinear case.

Theorem 2.1. *Let $m \geq 2$ and let $X_1,\ldots,X_m$, Y be Banach spaces. Then the following estimates hold for every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$:*

$$\gamma(T) \leq \widetilde{\gamma}(T^\times) \quad \textit{and} \quad \gamma(T^\times) \leq \widetilde{\gamma}(T).$$

Proof. We claim that $\gamma(T) \leq \widetilde{\gamma}(T^\times)$. Applying the mentioned above result about linear operators, we have

$$\gamma((T^\times)^*) \leq \widetilde{\gamma}(T^\times).$$

Since κ_Y is an isometry and $\|J\| = 1$, the above inequality in combination with Lemma 2.1 yields

$$\begin{aligned}\gamma(T) &= \alpha(T(U_{X_1\times\cdots\times X_m})) = \alpha(\kappa_Y T(U_{X_1\times\cdots\times X_m}))\\ &= \alpha((T^\times)^* J(U_{X_1\times\cdots\times X_m})) \leq \gamma((T^\times)^*)\,\alpha(J(U_{X_1\times\cdots\times X_m}))\\ &\leq \widetilde{\gamma}(T^\times)\,\alpha(\|J\|\,U_{Z^*}) \leq \widetilde{\gamma}(T^\times),\end{aligned}$$

where $Z := \mathcal{L}_m(X_1 \times \cdots \times X_m)$, and this completes the proof of the claim.

Put $k := \gamma(T)$, and let $S \subset Y^*$ be any set with $\operatorname{diam}(S) \leq d$, $d > 0$. To prove the second inequality we only need to show that $T^\times(S)$ can be covered by finitely many sets with diameter less or equal than kd. To do this fix $\varepsilon > 0$. Then there exist $y_1, \dots, y_n \in Y$, such that

$$T(U_{X_1\times\cdots\times X_m}) \subset \bigcup_{j=1}^{n} (y_j + r\,U_Y),$$

where $r = \gamma(T) + \varepsilon/2d$.

For each $1 \le j \le n$, we define the set

$$\{y^*(y_j);\ y^* \in S\} \subset \mathbf{K}.$$

Since S is bounded, the sets are relatively compact. Without loss of generality, we may assume that $\mathbf{K} = \mathbf{R}$. Thus, for each $1 \le j \le n$, the above set may be covered by closed intervals $I_{j,1}, \dots, I_{j,m(j)}$ with length less or equal than $\varepsilon/2$.

Let $p = (p_1, \dots, p_n)$, where $p_j \in \{1, 2, \dots, m(j)\}$, and let us set

$$E_p := \{y^* \in S;\ \langle y, y^*\rangle \in I_{j,p_j},\ 1 \le j \le n\}.$$

Clearly $T^\times(S) \subset \bigcup_p T^\times(E_p)$ with a finite union. We shall show that $\operatorname{diam}(T^\times(E_p)) < \gamma(T)d + \varepsilon$ for all p. To show this fix p and take $y_1^*,\, y_2^* \in E_p$. Then

$$\begin{aligned} \|T^\times y_1^* - T^\times y_2^*\| &= \sup\{|\langle Tx, y_1^* - y_2^*\rangle|;\ x \in U_{X_1\times\cdots\times X_m}\} \\ &= \sup\{|\langle y, y_1^* - y_2^*\rangle|;\ y \in T(U_{X_1\times\cdots\times X_m})\}. \end{aligned}$$

Now, observe that for every $y \in T(U_{X_1\times\cdots\times X_m})$ there exists $1 \le j \le n$ such that $y \in y_j + rU_Y$. Since $y_1^*,\, y_2^* \in E_p$, $|\langle y_j, y_1^* - y_2^*\rangle| < \varepsilon/2$. This implies, by $\|y_1^* - y_2^*\| \le d$ and $\|y - y_j\| \le \gamma(T) + \varepsilon/2d$,

$$|\langle y - y_j, y_1^* - y_2^*\rangle| \le \|y_1^* - y_2^*\|\,\|y - y_j\| \le \gamma(T)d + \varepsilon/2.$$

Hence

$$|\langle y, y_1^* - y_2^*\rangle| \le |\langle y - y_j, y_1^* - y_2^*\rangle| + |\langle y_j, y_1^* - y_2^*\rangle| < \gamma(T)d + \varepsilon.$$

The combination of the above estimates yields

$$|(T^\times y_1^* - T^\times y_2^*)x| < \gamma(T)d + \varepsilon$$

for all $x \in U_{X_1\times\cdots\times X_m}$. Since $\varepsilon > 0$ was arbitrary,

$$\|T^\times y_1^* - T^\times y_2^*\| \le \gamma(T)d$$

and this completes the proof. □

Corollary 2.1. *Let $m \ge 2$ and let $X_1, \dots, X_m$, Y be Banach spaces. Then the following estimates hold for every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$:*

$$\frac{1}{2}\gamma(T) \le \gamma(T^\times) \le 2\gamma(T) \quad \text{and} \quad \frac{1}{2}\widetilde{\gamma}(T) \le \widetilde{\gamma}(T^\times) \le 2\widetilde{\gamma}(T).$$

As a consequence, we obtain Schauder's theorem for multilinear operators, proved for the bilinear case by Ramanujan and Schock [15, Theorem 2.6].

Corollary 2.2. *Let $m \ge 2$ and let $X_1, \dots, X_m$, Y be Banach spaces. Then $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ is compact if, and only if, $T^\times$ is compact.*

3. Approximation numbers of multilinear operators

One of the most important examples of an s-number sequence is the sequence (a_n) of approximation numbers. Let us recall that for any operator $T \in \mathcal{L}(X, Y)$ and $n \in \mathbf{N}$ the n-th approximation number $a_n(T)$ is given by

$$a_n(T) := \inf\{\|T - A\|;\ A \in \mathcal{L}(X, Y),\ \mathrm{rank}(A) < n\}$$

(cf. [11, 13]).

Let $X_1, \ldots, X_m$ and Y be Banach spaces and let $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$. We denote the image of T by $\mathrm{Im}(T)$. Since it is not generally the case that $\mathrm{Im}(T)$ is a linear subspace of Y, we define $\mathrm{rank}(T)$ as the dimension of $[\mathrm{Im}(T)]$, where $[E]$ denotes the linear span of a subset E in a vector space V.

We define the n-th approximation number $a_n(T)$ of any multilinear operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ by

$$a_n(T) := \inf\{\|T - A\|;\ A \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y),\ \mathrm{rank}(A) < n\}.$$

It is easy to check that (a_n) is an s-number sequence.

We will state below some fundamental properties of approximation numbers of multilinear maps, and for the sake of completeness we include proofs. It should be pointed out that Carl [1] was among the first to discover the mixing multiplicativity property of bounded linear operators and used it to study s-numbers of bounded linear operators among Banach spaces.

Proposition 3.1. *Assume $(s_n)\colon \mathcal{L}_m(X_1 \times \cdots \times X_m, Y) \to [0, \infty)^{\mathbf{N}}$ is a sequence which satisfies the monotonicity $(S1)$, the additivity $(S2)$ and the rank property $(S4)$.*

(i) *An approximation sequence (a_n) is the largest (s_n) sequence which satisfies*

$$s_n(T) \le a_n(T), \quad n \in \mathbf{N}.$$

(ii) *If (s_n) is a quasi s-number sequence, then it has the mixing multiplicativity property, i.e., for all $S \in \mathcal{L}(Y, Z)$, $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and all $k, n \in \mathbf{N}$ we have*

$$s_{k+n-1}(ST) \le s_k(S) a_n(T) \quad \text{and} \quad s_{k+n-1}(ST) \le a_k(S) s_n(T).$$

Proof. (i) Let $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$. Then for any $A \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ with $\mathrm{rank}(A) < n$, we have

$$s_n(T) \le \|T - A\| + s_n(A) = \|T - A\|$$

and this yields $s_n(T) \le a_n(T)$.

(ii) Let $A \in \mathcal{L}(X_1 \times \cdots \times X_m, Y)$ be an operator with $\mathrm{rank}(A) < n$. Since $\mathrm{rank}(SA) < n$, it follows, by properties (S1), (S2) and (S4), that

$$\begin{aligned} s_{k+n-1}(ST) &\le s_{k+n-1}((S(T - A) + SA) \le s_k(S(T - A)) + s_n(SA) \\ &= s_k(S(T - A)) \le s_k(S)\|T - A\| \end{aligned}$$

and this completes the proof of the first inequality. The proof of the second inequality is very similar to the first one and so it will be omitted. □

Below we state and prove certain relationships between approximation numbers of an m-linear bounded operator T and its generalized adjoint operator $T^{\times}$. We need the following.

Lemma 3.1. *Let $m \geq 2$ and $X_1, \dots, X_m, Y$ be Banach spaces. If $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ has finite rank, then*

$$\mathit{rank}(T) = \mathit{rank}(T^\times).$$

Proof. Let $n := \operatorname{rank}(T)$. Then $\dim(V) = n$, where $V = [\operatorname{Im}(T)] \subset Y$. We use Auerbach's lemma which states, that there are unit vectors $v_1, \dots, v_n \in V$ and unit vectors $v_1^*, \dots, v_n^* \in V^*$, such that

$$\langle v_i, v_j^* \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Obviously, $\{v_1, \dots, v_n\}$ forms a basis for V, and $\{v_1^*, \dots, v_n^*\}$ is a dual basis for V^*. By the Hahn–Banach theorem, there exists $y_j^* \in Y^*$, such that y_j^* is v_j^* on V.

Since $Tx \in V$ for all $x \in X_1 \times \cdots \times X_m$,

$$Tx = \sum_{j=1}^n v_j^*(Tx) v_j = \sum_{j=1}^n (T^\times y_j^*)(x) v_j = \sum_{j=1}^n (T^\times y_j^* \otimes v_j)(x);$$

that is, $T = \sum_{j=1}^n T^\times y_j^* \otimes v_j$. Consequently,

$$T^\times = \sum_{j=1}^n \kappa_Y(v_j) \otimes B_j,$$

where $B_j := T^\times y_j^* \otimes v_j$ for each $1 \leq j \leq n$. Since $\kappa_Y(v_j) \in Y^{**} \neq 0$ and $B_j \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ with $\operatorname{rank}(B_j) = 1$, $\operatorname{rank}(T^\times) \leq n$. We claim that $\operatorname{rank}(T^\times) = n$. To see this we only need to show that $\{B_1, \dots, B_n\}$ is a linearly independent set in $\mathcal{L}_m(X_1 \times \cdots \times X_m)$. Let $\lambda_1, \dots, \lambda_n \in \mathbf{K}$ be such that

$$\sum_{j=1}^n \lambda_j B_j = 0;$$

that is, $\sum_{j=1}^n \lambda_j (T^\times y_j^* \otimes v_j)(x) = \sum_{j=1}^n \lambda_j v_j^*(Tx) v_j = 0$, for all $x \in X_1 \times \cdots \times X_m$. This implies that for all $v \in V$ we get

$$\sum_{j=1}^n \lambda_j v_j^*(v) v_j = 0.$$

Since $v_i^*(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$, it follows that $\lambda_j = 0$ for each $1 \leq j \leq n$. This completes the proof of the claim. □

Below we will state and prove certain relationships between approximation numbers of an m-linear bounded operator T and its generalized adjoint operator $T^\times$.

Proposition 3.2. *For every operator $\mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ we have*

$$a_n(T^\times) \leq a_n(T), \quad n \in \mathbf{N}.$$

Proof. Given $\varepsilon > 0$, there exists $A \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ with $\operatorname{rank}(A) < n$, such that

$$\|T - A\| \leq (1 + \varepsilon)\, a_n(T).$$

An application of Lemma 3.1 ensures that $\operatorname{rank}(A^\times) < n$. Thus combining with

$$\|T^\times - A^\times\| = \|T - A\| \leq (1 + \varepsilon)\, a_n(T),$$

we conclude that

$$a_n(T^\times) \leq (1 + \varepsilon)\, a_n(T).$$

Since $\varepsilon > 0$ is arbitrary, the result follows. $\square$

Proposition 3.3. *Let Y be a Banach space such that there exists a linear projection P of unit norm from Y^{**} onto $\kappa_Y(Y)$. Then for every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$*

$$a_n(T^{\times}) = a_n(T), \qquad n \in \mathbf{N}.$$

Proof. Fix $\varepsilon > 0$. Then, there exists a linear operator $S\colon \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)^* \to Y^{**}$ with $\operatorname{rank}(S) < n$, such that

$$\|(T^{\times})^* - S\| < a_n((T^{\times})^*) + \varepsilon.$$

Let $A = PSJ$, where $J\colon X_1 \times \cdots \times X_m \to \mathcal{L}_m(X_1 \times \cdots \times X_m)^*$ is given by $Jx(B) = B(x)$ for all $x \in X_1 \times \cdots \times X_m$ and $B \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$. Then $A \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ with $\operatorname{rank}(A) < n$. Our hypothesis $\|P\| = 1$ in combination with $\|J\| = 1$ yields

$$\|T - S\| = \|P(T^{\times})^*J - PSJ\| \leq \|(T^{\times})^* - S\| < a_n((T^{\times})^*) + \varepsilon.$$

Since ε is arbitrary, we conclude that

$$a_n(T) \leq a_n((T^{\times})^*) \leq a_n(T^{\times}).$$

This completes the proof by Proposition 3.2. $\square$

Applications of above results will be shown. Note that Edmunds and Tylli [5] proved that, for any operator $T \in \mathcal{L}(E, F)$ between Banach spaces E and F, the following estimate holds

$$a_n(T) \leq a_n(T^{**}) + 2\gamma(T), \quad n \in \mathbf{N}.$$

In the multilinear case, we have the following result.

Theorem 3.1. *For every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ we have*

$$a_n(T) \leq a_n((T^{\times})^*) + 2\,\widetilde{\gamma}(T), \quad n \in \mathbf{N}.$$

Proof. Let $\varepsilon > 0$ and $\lambda > \widetilde{\gamma}(T)$. Then there exists a linear operator $A\colon \mathcal{L}_m(X_1 \times \cdots \times X_m)^* \to Y^{**}$ with $\operatorname{rank}(A) < n$ such that

$$\|(T^{\times})^* - A\| < a_n((T^{\times})^*) + \varepsilon.$$

Let $y_1, \ldots, y_k \in Y$ with $T(U_{X_1 \times \cdots \times X_m}) \subset \{y_1, \ldots, y_k\} + \lambda U_Y$. Let M be the linear span of $\operatorname{Im}(A) \cup \{\kappa_Y(z_j);\ 1 \leq j \leq k\}$. By the principle of local reflexivity, there exists $R\colon M \to Y$ such that $\|R\| \leq 1 + \varepsilon$ and $R\kappa_Y(y_j) = y_j$, for each $1 \leq j \leq k$.

Define $S := RAJ \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$. Then $\operatorname{rank}(S) \leq n$. For every $x \in U_{X_1 \times \cdots \times X_m}$, we choose y_j with $\|Tx - y_j\| \leq \lambda$ for some $1 \leq j \leq k$. Since $(T^{\times})^*J = \kappa_Y T$,

$$\begin{aligned}
\|Tx - Sx\| &\leq \|Tx - y_j\| + \|y_j - Sx\| \leq \lambda + \|R\kappa_Y y_j - RAJx\| \\
&\leq \lambda + (1+\varepsilon)\big(\|\kappa_Y y_j - \kappa_Y Tx\| + \|(T^{\times})^*Jx - AJx\|\big) \\
&\leq \lambda + (1+\varepsilon)\big(\lambda + a_n((T^{\times})^*) + \varepsilon\big).
\end{aligned}$$

Since $\varepsilon > 0$ and $\lambda > \widetilde{\gamma}(T)$ are arbitrary, we obtain the required estimate. $\square$

As an application of the above results, we obtain the following multilinear variants of the well-known results in the linear case (see [5]).

Corollary 3.1. *If $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ is a compact operator, then*

$$a_n(T) = a_n(T^{\times}), \quad n \in \mathbf{N}.$$

Proof. Since $T^\times$ is a linear operator among Banach spaces, $a_n((T^\times)^*) \le a_n(T^\times)$. If T is compact, then $\widetilde{\gamma}(T) = 0$ and so

$$a_n(T) \le a_n((T^\times)^*) \le a_n(T^\times) \le a_n(T),$$

by Theorem 3.1 and Proposition 3.2. □

Corollary 3.2. *For every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ we have,*

$$a_n(T) \le 5\, a_n(T^\times), \quad n \in \mathbf{N}.$$

Proof. Fix $n \in \mathbf{N}$. Let $S_n \colon Y^* \to \mathcal{L}_m(X_1 \times \cdots \times X_m)$ be an arbitrary linear operator with $\mathrm{rank}(S_n) < n$. Then

$$\widetilde{\gamma}(T^\times) = \widetilde{\gamma}(T^\times - S_n) \le \|T^\times - S_n\|.$$

This shows that $\widetilde{\gamma}(T^\times) \le a_n(T^\times)$ and so

$$a_n(T) \le a_n((T^\times)^*) + 2\,\widetilde{\gamma}(T) \le a_n(T^\times) + 4\,\widetilde{\gamma}(T^\times) \le 5\, a_n(T^\times),$$

and this completes the proof. □

4. Kolmogorov numbers of multilinear operators

Since for every multilinear operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and any m-quasi s-sequence (s_n), the sequence $(s_n(T))$ is non-increasing and bounded below by 0, so it has a limit. In particular the measure of non-approximability, $a(T) := \lim_{n\to\infty} a_n(T)$ exists. If $a(T) = 0$, then clearly T is approximable (i.e., there exists a sequence (A_n) of finite dimensional operators $A_n \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, such that $\lim_{n\to\infty} \|T - A_n\| = 0$) and so it is compact. It is well-known that in general the converse is false. Roughly speaking, the quantity $a(T)$ is not a useful measure of the deviation of an operator T from compactness.

Similarly to the linear case, if the target space Y has the *approximation property*, that is, given any compact subset K of Y and every $\varepsilon > 0$, there is *a* finite dimensional linear map $S \colon Y \to Y$, such that $\|Sy - y\| < \varepsilon$ for all $y \in K$, then, it is easy to see that any compact m-linear operator $T \colon X_1 \times \cdots \times X_m \to Y$ can be approximated arbitrarily and closely by finite-dimensional m-linear operators. In consequence, $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ is compact if, and only if, $a(T) = 0$. We will use below this simple fact without any further references.

The quantity which is a useful measure of the deviation of change linear operator from compactness is connected with the Kolmogorov numbers.

Following the linear case, we define the *n-th Kolgomorov number* $d_n(T)$ of an operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ by

$$d_n(T) = \inf\{\varepsilon > 0;\ T(U_{X_1 \times \cdots \times X_m}) \subset N_\varepsilon + \varepsilon U_Y,\ N_\varepsilon \subset Y,\ \dim(N_\varepsilon) < n\}.$$

Clearly that $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ is a compact operator if and only if

$$d(T) := \lim_{n\to\infty} d_n(T) = 0.$$

It is also obvious that $d_n(T) = 0$ provided $\mathrm{rank}(T) < n$.

Since in the above definition of $d_n(T)$ we may replace closed unit balls $U_{X_1 \times \cdots \times X_m}$ or U_Y by the corresponding open unit balls $\mathring{U}_{X_1 \times \cdots \times X_m}$ or $\mathring{U}_Y$, it may be shown, similarly as in the linear case, that the following multilinear variant of Pietsch's formula holds (see [12, 1])

$$d_n(T) = \inf\{\|Q_N^Y T\|;\ N \subset Y,\ \dim(N) < n\}.$$

This formula easily gives (d_n) is an s-number sequence, and so Proposition 3.1 implies that

$$d_n(T) \le a_n(T), \quad n \in \mathbf{N}.$$

Note also that (d_n) is a surjective quasi s-number sequence. We next present properties of Kolmogorov numbers similar to the linear case. To do this we need some further definition.

For a given $m \in \mathbf{N}$ and $i = 1, \ldots, m$, let X_i be a Banach space. We say that the Banach space $X_1 \times \cdots \times X_m$ has the *multilinear metric lifting property* if, for every $\varepsilon > 0$ and every bounded m-linear operator T from $X_1 \times \cdots \times X_m$ to any quotient space Y/N, there is $\widetilde{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, such that

$$T = Q_N^Y \widetilde{T} \quad \text{and} \quad \|\widetilde{T}\| \le (1+\varepsilon)\|T\|.$$

Proposition 4.1. *Let $X_1, \ldots, X_m$ and Y be Banach spaces. If $X_1 \times \cdots \times X_m$ has the multilinear metric lifting property, then for every $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,*

$$d_n(T) = a_n(T), \quad n \in \mathbf{N}.$$

Proof. Since $d_n(T) \le a_n(T)$ for each n, we need to show the reverse inequality. Fix $\varepsilon > 0$. Then there exists a subspace $N \subset Y$, such that $\dim(N) < n$ and $\|Q_N^Y T\| < d_n(T) + \varepsilon$. Our hypothesis yields that there exists $\widetilde{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, such that

$$Q_N^Y T = Q_N^Y \widetilde{T} \quad \text{and} \quad \|\widetilde{T}\| \le (1+\varepsilon)\|Q_N^Y T\|.$$

For $A := T - \widetilde{T}$, we have $Q_N^Y A = 0$. This implies $[\mathrm{Im}(T)] \subset N$, and so $\mathrm{rank}(S) \le \dim(N) < n$. In consequence

$$a_n(T) \le \|T - A\| = \|\widetilde{T}\| \le (1+\varepsilon)\|Q_N^Y T\| < (1+\varepsilon)(d_n(T) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, $a_n(T) \le d_n(T)$ and so the proof is complete. □

By the similar proof as in the linear case, one can see that $\ell_1(\Gamma_1) \times \cdots \times \ell_1(\Gamma_m)$ has the multilinear metric lifting property for each $m \ge 2$. Here, as usual, $\ell_1(\Gamma)$ is the Banach space of summable number families $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ over an arbitrary index set.

It is well-known (see [11]) that in the linear case the Kolmogorov numbers $(d_n(T))$ of every operator $T\colon X \to Y$ between Banach spaces X and Y may be characterized by the approximation numbers as follows

$$d_n(T) = a_n(TQ_1)$$

for each $n \in \mathbf{N}$, where Q_1 is the canonical metric surjection from $\ell_1(U_X)$ onto X, defined by

$$Q_1(\{\lambda_x\}) = \sum_{x \in U_X} \lambda_x x, \quad \{\lambda_x\} \in \ell_1(U_X).$$

In the multilinear case, we have the following analogous result.

Theorem 4.1. *Let $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ be an m-linear operator between Banach spaces, and let $Q := Q_1 \times \cdots \times Q_j$ where Q_j is the canonical metric surjection from $\ell_1(U_{X_j})$ onto X_j for each $1 \le j \le m$. Then*

$$d_n(T) = a_n(TQ), \quad n \in \mathbf{N}.$$

Proof. Since $\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m})$ has the multilinear metric lifting property, it follows from Proposition 4.1 that for each $n \in \mathbf{N}$,

$$d_n(TQ) = a_n(TQ).$$

As noted, (d_n) is a surjective s-number sequence, and this gives the required equality. $\square$

Note that for any surjective quasi s-number sequence the following estimate holds for all $S \in \mathcal{L}(Y,Z)$ and all $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,

$$s_{k+n-1}(ST) \leq s_k(S) d_n(T), \quad k, n \in \mathbf{N}.$$

The proof is similar to the linear case proof (see [1]). In fact, let Q_j be the metric surjection from $\ell_1(U_{X_j})$ onto X_j for each $1 \leq j \leq m$. Then combining Theorem 4.1 with the multilinear multiplicativity property (M), yields

$$s_{k+n-1}(ST) = s_{k+n-1}(STQ) \leq s_k(S)s_n(TQ) \leq s_k(S)a_n(TQ) = s_k(S)d_n(T).$$

An immediate consequence of the above fact is a multilinear variant of the well-known result for the linear case, that the sequence (d_n) of the Kolmogorov numbers is the largest surjective quasi s-number sequence which satisfies the multiplicativity property (M):

$$d_{k+n-1}(ST) \leq d_k(S)d_n(T), \quad S \in \mathcal{L}(Y,Z), \ \ T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y).$$

5. Gelfand numbers of mulitilinear operators

In the theory of s-numbers of linear operators, the Gelfand numbers play an important role. There are many equivalent definitions of Gelfand numbers; recall that the usual n-th Gelfand number $c_n(T)$ of an operator $T \in \mathcal{L}(X,Y)$ acting between arbitrary Banach spaces X and Y is defined to be the infimum of all $\varepsilon > 0$, such that there are functionals $x_i^* \in X^*$, $1 \leq i \leq k < n$, which admit an estimate

$$\|Tx\| \leq \sup_{1 \leq i \leq k} |\langle x, x_i^* \rangle| + \varepsilon \, \|x\|, \quad x \in X.$$

It is well-known that in the linear case (c_n) is an s-number sequence. For the basic facts about these numbers, which are given below, we refer to the books of Pietsch [11, 13].

In the multilinear case we make the following definition of Gelfand numbers; the n-th Gelfand number $c_n(T)$ of an operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ is defined by

$$c_n(T) = a_n(\kappa_Y T).$$

This definition is motivated by the fact that the above formula holds in the case of linear operators, and many interesting applications may be found. Obviously (c_n) is an s-number sequence by the fact that (a_n) is an s-number sequence. Clearly we have $c_n(T) \leq a_n(T)$ for each $n \in \mathbf{N}$. In the case when Y is a Banach space with the metric extension property (i.e., every operator $S \in \mathcal{L}(X,Y)$ from every Banach space X to Y can be extended to any Banach space $\widetilde{X}$ containing X as a subspace, where the extension $\widetilde{T} \in \mathcal{L}(\widetilde{X}, Y)$ with $\|\widetilde{T}\| = \|T\|$), then $c_n(T) = a_n(T)$.

We collect some properties of the Gelfand numbers (c_n) for the multilinear case. The following result may be easily verified: If Y is a Banach space with the metric extension property, $X_1, \dots, X_m$ arbitrary Banach spaces and $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, then

$$c_n(T) = a_n(T), \quad n \in \mathbf{N}.$$

Since any $\ell_\infty(\Gamma)$-space has the metric extension property, the above result immediately implies that (c_n) is an injective s-number sequence.

The sequence (c_n) of Gelfand numbers is the largest injective quasi s-number sequence, which satisfies the multiplicativity property (M):

$$c_{k+n-1}(ST) \leq c_k(S)c_n(T), \quad S \in \mathcal{L}(Y, Z), \ T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y).$$

The proof of this property is similar to the linear case proof. One should first observe that a multilinear variant of Carl's mixing multiplicativity of an injective s-number sequence (s_n) states that, for all $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,

$$s_{k+n-1}(ST) \leq c_k(T)s_n(S), \quad k, n \in \mathbf{N}.$$

In fact, the mixing multiplicativity property (MI) implies

$$s_{k+n-1}(ST) = s_{k+n-1}(\kappa_Z TS) \leq a_k(\kappa_Z T)s_n(S) = c_k(T)s_n(S), \quad k, n \in \mathbf{N}.$$

For each $m \in \mathbf{N}$, we define the function $c\colon \mathcal{L}_m(X_1 \times \cdots \times X_m) \to [0, \infty)$ by

$$c(T) = \lim_{n\to\infty} c_n(T).$$

The following proposition gives characterization of compactness of multilinear operators in terms of the quantity c.

Proposition 5.1. *Let $m \geq 2$ and let $X_1,\dots,X_m$, Y be Banach spaces. Then the following statements about an operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ are equivalent*:

(i) *T is compact.*
(ii) $c(T) = 0$.
(iii) $c(T^\times) = 0$.

Proof. Obviously T is compact if, and only if, $\kappa_Y T\colon X_1 \times \cdots \times X_m \to \ell_\infty(U_{Y^*})$ is compact. Since $\ell_\infty(U_{Y^*})$ has an approximation property, $\kappa_Y T$ is an approximable operator and, we may conclude that T is compact if, and only if,

$$c(T) = \lim_{n\to\infty} a_n(\kappa_Y T) = 0.$$

This shows that (i) and (iii) are equivalent. To complete the proof, it is enough to recall that T is compact if, and only if, $T^\times$ is compact. □

Our next result shows the relation between Gelfand and Kolmogorov numbers of a multilinear operator T and its adjoint $T^\times$.

Theorem 5.1. *Let $m \geq 2$ and let $X_1,\dots,X_m$, Y be Banach spaces. Then, for every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and for each $n \in \mathbf{N}$, we have*

(i) $c_n(T^\times) \leq d_n(T)$,
(ii) $c_n(T) = d_n(T^\times)$,
(iii) $c_n(T) \leq 2\sqrt{n}\, c_n(T^\times)$.

Proof. (i) For each $1 \leq j \leq m$ let Q_j be the canonical metric surjection from $\ell_1(U_{X_j})$ onto X_j. For abbreviation, let Q stand for the operator

$$Q_1 \times \cdots \times Q_m\colon \ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m}) \to X_1 \times \cdots \times X_m$$

defined by for every $(\lambda_1, \dots, \lambda_m) \in \ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m})$ by

$$Q(\lambda_1, \dots, \lambda_m) = (Q_1\lambda_1, \dots, Q_m\lambda_m).$$

It is easy to verify that $(TQ)^\times \colon Y^* \to \mathcal{L}_m(\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m}))$ factorizes through $\mathcal{L}_m(X_1 \times \cdots \times X_m)$ as follows

$$(TQ)^\times \colon Y^* \xrightarrow{T^\times} \mathcal{L}_m(X_1 \times \cdots \times X_m) \xrightarrow{\Phi} \mathcal{L}_m(\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m})),$$

where Φ is given by

$$\Phi(S) = SQ, \quad S \in \mathcal{L}_m(X_1 \times \cdots \times X_m).$$

Since Q_j is a metric surjection for each $1 \leq j \leq m$, Φ is a metric injection from $\mathcal{L}_m(X_1 \times \cdots \times X_m)$ into $\mathcal{L}_m(\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_m}))$. Combining these with the injectivity of the Gelfand numbers, Proposition 3.2 and Theorem 4.1, one has

$$c_n(T^\times) = c_n(\Phi T^\times) = c_n((TQ)^\times) \leq a_n((TQ)^\times) \leq a_n(TQ) = d_n(T).$$

(ii) Since $\kappa_Y \colon Y \to \ell_\infty(U_{Y^*})$ is a metric injection, $(\kappa_Y)^* \colon \ell_\infty(U_{Y^*})^* \to Y^*$ is a metric surjection. Thus the surjectivity of the Kolgomorov numbers in combination with Proposition 3.2 yields (by $(\kappa_Y T)^\times = T^\times(\kappa_Y)^*$)

$$d_n(T^\times) = d_n(T^\times(\kappa_Y)^*) \leq a_n(T^\times(\kappa_Y)^*) = a_n((\kappa_Y T)^\times) \leq a_n(\kappa_Y T) = c_n(T).$$

To prove the reverse inequality, we use the well-known fact that, for every operator $S \in \mathcal{L}(E, F)$ between Banach spaces E and F (see [11, Proposition 11.7.6]),

$$c_n(S^*) \leq d_n(S), \quad n \in \mathbf{N}.$$

To estimate $c_n(T)$ from the above, we apply the equality $\kappa_Y T = T^\times J$ from Lemma 2.1 and the injectivity of the Gelfand numbers, namely,

$$c_n(T) = c_n(\kappa_Y T) = c_n(T^\times J) \leq c_n((T^\times)^*) \leq d_n(T^\times)$$

and so this gives the required estimate.

(iii) It is well-known (see, e.g., [4, Proposition 3.8, p. 75]) that for any operator $S \colon E \to F$ between Banach spaces and all $n \in \mathbf{N}$,

$$a_n(S) \leq 2\sqrt{n}\, c_n(S).$$

Since $d_n(T^\times) \leq a_n(T^\times)$, the required estimate follows by (ii). $\square$

In the case of any m-linear operator $T \colon X_1 \times \cdots \times X_m \to Y$ acting between Banach spaces, we also define sequences $(\widehat{c}_n(T))$ and $(\widetilde{c}_n(T))$ as follows: $\widehat{c}_n(T)$ is to be the infimum of all $\varepsilon > 0$, such that there are functionals $B_i \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$, $1 \leq i \leq k < n$, which admit an estimate for all $(x_1, \dots, x_m) \in (X_1, \dots, X_m)$,

$$\|T(x_1, \dots, x_m)\| \leq \sup_{1 \leq i \leq k} |B_i(x_1, \dots, x_m)| + \varepsilon \|x_1\| \cdots \|x_m\|,$$

and, respectively, $\widetilde{c}_n(T)$ is defined to be the infimum of all $\|S\|$, with $S \in \mathcal{L}_m(X_1 \times \cdot\cdot \cdot \times X_m, Z)$ for some Banach space Z, such that, there are $B_i \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$, $1 \leq i \leq k < n$, which satisfy

$$\|Tx\|_Y \leq \sup_{1 \leq i \leq k} |B_i x| + \|Sx\|_Z, \quad x \in X_1 \times \cdots \times X_m.$$

To show some properties of the above introduced sequences, we need a characterization of multilinear compact operators in terms of factorization. We will use a characterization of compact linear operators due to Terzioglou [17] (see also [16];

in a more precise form, see [8]), which states: If X, Y are Banach spaces, $\varepsilon > 0$ is given and $T \in \mathcal{L}(X, Y)$ is compact, then there exists a closed subspace Z of c_0, such that, T admits a factorization through Z:

$$T\colon X \xrightarrow{B} Z \xrightarrow{A} Y,$$

where $B\colon X \to Z$ and $A\colon Z \to Y$ are compact operators with $\|A\| \le 1$ and $\|B\| \le (1+\varepsilon)\,\|T\|$.

In the multilinear case, we have the following variant.

Theorem 5.2. *Let $m \in \mathbf{N}$ and let $X_1, \dots, X_m$, Y be Banach spaces. Given $\varepsilon > 0$ and an operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$, then there exists a closed subspace Z of c_0, such that, T admits a factorization through Z:*

$$T\colon X_1 \times \cdots \times X_m \xrightarrow{B} Z \xrightarrow{A} Y,$$

where $B \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Z)$ and $A \in \mathcal{L}(Z, Y)$ are compact operators with $\|A\| \le 1$ and $\|B\| \le (1+\varepsilon)\,\|T\|$.

Proof. It follows from the theory of the projective tensor product analysis that there exist bounded linear operators $\bigotimes\colon X_1 \times \cdots \times X_m \to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ and $\widetilde{T}\colon X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \to Y$, such that,

$$\bigotimes(x_1, \dots, x_m) = x_1 \otimes \cdots \otimes x_m, \quad (x_1, \dots, x_m) \in X_1 \times \cdots \times X_m,$$

and

$$T = \widetilde{T} \circ \bigotimes \quad \text{with } \|\bigotimes\| \le 1,\ \|\widetilde{T}\| \le \|T\|.$$

Using the representation of the projective tensor norm, we deduce that

$$\widetilde{T}(U_{X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m}) \subset \overline{\operatorname{conv}}\,(T(U_{X_1 \times \cdots \times X_m})).$$

Our hypothesis that T is compact in combination with the well-known Mazur theorem yields that $\widetilde{T}$ is a compact operator. To conclude, it is enough to apply the above shown factorization result to $\widetilde{T}$. $\square$

Lemma 5.1. *Both $(\widehat{c}_n)$ and $(\widetilde{c}_n)$ are injective s-number sequences which satisfy the following estimates*:

$$\widehat{c}_n(T) \le \widetilde{c}_n(T) \le c_n(T), \quad n \in \mathbf{N}.$$

Proof. It is easy to check that the properties (S1)–(S3) are satisfied for both sequences $(\widehat{c}_n)$ and $(\widetilde{c}_n)$. We claim that the rank property (S3) holds for $(\widetilde{c}_n)$. To see this, fix $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ with $\operatorname{rank}(T) < n$. Since

$$T\colon X_1 \times \cdots \times X_m \to [\operatorname{Im}(T)]$$

is compact, it follows by Theorem 5.2 that there exists a closed subspace Z of c_0 such that T admits a factorization through Z,

$$T\colon X_1 \times \cdots \times X_m \xrightarrow{B} Z \xrightarrow{A} [\operatorname{Im}(T)]$$

with $\|A\| \le 1$ and $\|B\| \le (1+\varepsilon)\,\|T\|$.

Since $\operatorname{rank}(T) = \dim([\operatorname{Im}(T)]) < n$, $\operatorname{rank}(A) < n$ and so $c_n(A) = 0$. Thus for every $\varepsilon > 0$ there are functionals $z_i^* \in Z^*$, $1 \le k < n$, which admit an estimate

$$\|Az\|_Y \le \sup_{1 \le i \le k} |\langle z, z_i^*\rangle| + \varepsilon \|z\|_Z, \quad z \in Z.$$

This implies that for $B_i := z_i^* \circ B \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$, $1 \leq i \leq k < n$, and $S := \varepsilon\, B \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Z)$ we have

$$\|Tx\|_Y = \|A(Bx)\|_Y \leq \sup_{1\leq i\leq k} |B_i x| + \|Sx\|_Z, \quad x \in X_1 \times \cdots \times X_m.$$

Consequently we obtain

$$\widetilde{c}_n(T) \leq \|S\| = \varepsilon\|B\| \leq \varepsilon\,(1+\varepsilon)\|T\|.$$

Since $\varepsilon > 0$ is arbitrary, $\widetilde{c}_n(T) = 0$ and so the claim is proved.

Obviously both $(\widehat{c}_n)$ and $(\widetilde{c}_n)$ are injective s-number sequences, which satisfy $\widehat{c}_n(T) \leq \widetilde{c}_n(T)$ for all $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$ and $n \in \mathbf{N}$. Since the sequence of Gelfand numbers is the largest injective s-number sequence, the proof is complete. $\square$

Pietsch in [11, p. 149] proved that the n-th Gelfand number $c_n(T)$ of an operator between Banach spaces X and Y allows the representation

$$c_n(T) = \inf\{\|TI_F\|;\ F \subset X,\ \operatorname{codim}(F) < n\},$$

where I_F is the inclusion map from F into X.

Fix a positive integer $m \geq 2$. Let $X_1,\ldots,X_m$ and Y be arbitrary Banach spaces. Motivated by Pietsch's result, for any bounded m-linear operator $T\colon X_1 \times \cdots \times X_m \to Y$ and $(n_1, \ldots, n_m) \in \mathbf{N}^m$, we define a sequence $(\overline{c}_{(n_1,\ldots,n_m)}(T))$ by

$$\overline{c}_{(n_1,\ldots,n_m)}(T) = \inf\{\|TI_{F_1\times\cdots\times F_m}\|;\ F_i \subset X_i,\ 1 \leq i \leq m,\ \operatorname{codim}(F_i) < n_i\},$$

where F_i is closed subspaces of X_i for each $1 \leq j \leq m$.

Unfortunately in the multilinear case, the relations between the sequence $(c_n(T))$ of Gelfand numbers and the sequence $(\overline{c}_{(n_1,\ldots,n_m)}(T))$ seems to be generally complicated.

We show some relationships between mentioned sequences for m-linear operators defined on the product of finite dimensional spaces. We need some definitions and preliminary results.

Suppose $m \geq 2$ and $F_1,\ldots,F_m$ are closed subspaces, respectively, of the Banach spaces $X_1,\ldots,X_m$. Throughout the rest of the paper we put

$$(F_1 \times \cdots \times F_m)^\circ := \{T \in \mathcal{L}_m(X_1 \times \cdots \times X_m);\ Tx = 0 \text{ for all } x \in F_1 \times \cdots \times F_m\}.$$

Obviously $(F_1 \times \cdots \times F_m)^\circ$ is a closed subspace of $\mathcal{L}_m(X_1 \times \cdots \times X_m)$. For every operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$, $\overline{T} := T + (F_1 \times \cdots \times F_m)^\circ$ denotes an element of the quotient space $\mathcal{L}_m(X_1 \times \cdots \times X_m)/(F_1 \times \cdots \times F_m)^\circ$. The quotient map from $\mathcal{L}_m(X_1 \times \cdots \times X_m)$ onto $\mathcal{L}_m(X_1 \times \cdots \times X_m)/(F_1 \times \cdots \times F_m)^\circ$ will be denoted by $Q_{(F_1\times\cdots\times F_m)^\circ}$.

Lemma 5.2. *Let $m \geq 2$ and let $F_1,\ldots,F_m$ be closed subspaces respectively of the Banach spaces $X_1,\ldots,X_m$, such that for every $T \in \mathcal{L}_m(F_1 \times \cdots \times F_m)$ there exists $\widetilde{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ with $\widetilde{T}|_{F_1\times\cdots\times F_m} = T$ and $\|\widetilde{T}\| \leq M\|T\|$ for some numerical constant $M \geq 1$. Then the operator $\Psi\colon \mathcal{L}_m(X_1 \times \cdots \times X_m)/(F_1 \times \cdots \times F_m)^\circ \to \mathcal{L}_m(F_1 \times \cdots \times F_m)$ given by*

$$\Psi(\overline{T}) := T|_{F_1\times\cdots\times F_m}, \quad \overline{T} \in \mathcal{L}_m(X_1 \times \cdots \times X_m)/(F_1 \times \cdots \times F_m)^\circ$$

is a linear isomorphism such that $M^{-1} \leq \|\Psi\| \leq 1$.

Proof. It is clear that Ψ is linear and one-to-one. Let $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$. Then for all $S \in (F_1 \times \cdots \times F_m)^\circ$, we have

$$\big\|\Psi(\overline{T})\big\| = \big\|T|_{F_1\times\cdots\times F_m}\big\| = \big\|(T+S)|_{F_1\times\cdots\times F_m}\big\| \le \|T+S\|.$$

In consequence $\|\Psi\| \le 1$.

To complete the proof, for a given $B \in \mathcal{L}_m(F_1 \times \cdots \times F_m)$, our hypothesis implies that there exists $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$ with $\|T\| \le M\|B\|$, such that, T is B on $F_1 \times \cdots \times F_m$. In particular this implies that

$$\Psi(\overline{T}) = T|_{F_1\times\cdots\times F_m} = B$$

and so Ψ is onto. We also have

$$M^{-1}\big\|\overline{T}\big\| \le M^{-1}\|T\| \le \|B\| = \big\|\Psi(\overline{T})\big\|,$$

which gives $M^{-1} \le \|\Psi\|$. $\square$

Theorem 5.3. *Let $m \ge 2$, $X_1, \ldots, X_m$, Y be Banach spaces, and let $F_1,\ldots,F_m$ be closed subspaces of the Banach spaces $X_1,\ldots,X_m$, respectively, which satisfy hypotheses of Lemma 5.2. Then for any operator $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,*

$$\big\|Q_{(F_1\times\cdots\times F_m)^\circ}T^\times\big\| \le M\big\|TI_{F_1\times\cdots\times F_m}\big\|,$$

where $I_{F_1\times\cdots\times F_m}$ is the inclusion map from $F_1 \times \cdots \times F_m$ into $X_1 \times \cdots \times X_m$.

Proof. Let $R\colon \mathcal{L}_m(X_1 \times \cdots \times X_m) \to \mathcal{L}_m(F_1 \times \cdots \times F_m)$ be the restriction map defined by

$$R(S) := S|_{F_1\times\cdots\times F_m}, \quad S \in \mathcal{L}_m(X_1 \times \cdots \times X_m).$$

For any $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m)$, we have

$$RT^\times : Y^* \to \mathcal{L}_m(F_1 \times \cdots \times F_m).$$

Since for all $y^* \in Y^*$,

$$RT^\times(y^*) = R(T^\times y^*) = R(y^* \circ T) = (y^* \circ T)|_{F_1\times\cdots\times F_m} = y^* \circ (TI_{F_1\times\cdots\times F_m}),$$

it follows that

$$\|RT^\times\| = \big\|TI_{F_1\times\cdots\times F_m}\big\|.$$

To conclude observe that $R = \Psi Q_{(F_1\times\cdots\times F_m)^\circ}$ where

$$\Psi\colon \mathcal{L}_m(X_1 \times \cdots \times X_m)/(F_1 \times \cdots \times F_m)^\circ \to \mathcal{L}_m(F_1 \times \cdots \times F_m)$$

is an operator defined as in Lemma 5.2 and apply 5.2. $\square$

Theorem 5.4. *Let $m \ge 2$ and let $X_1,\ldots,X_m$, Y be Banach spaces such that $N_i = \dim(X_i) < \infty$ for each $1 \le i \le m$. Assume $X_1 \times \cdots \times X_m$ has the multilinear extension property with M. Then for all $T \in \mathcal{L}_m(X_1 \times \cdots \times X_m, Y)$,*

$$c_n(T) \le M\,\overline{c}_{(k_1,\ldots,k_m)}(T), \quad 1 \le k_i < N_i,\ 1 \le i \le m,$$

where $n \ge N_1 \cdots N_m - (N_1 - k_1) \cdots (N_m - k_m)$.

Proof. Fix $\varepsilon > 0$. Then for each $1 \le i \le m$ there exists a closed subspace F_i of X_i, such that $\operatorname{codim}(F_i) < k_i$ and

$$\|TI_{F_1\times\cdots\times F_m}\| \le \overline{c}_{(k_1,\ldots,k_m)}(T) + \varepsilon/M. \qquad (*)$$

It follows from Lemma 5.2 that $\operatorname{codim}((F_1\times\cdots\times F_m)^\circ) = \dim(\mathcal{L}_m(F_1\times\cdots\times F_m)) < \infty$. Since $\dim(F_i) > N_i - k_i$ for each $1 \le i \le m$,

$$\begin{aligned} N := \dim((F_1 \times \cdots \times F_m)^\circ) &= \dim(\mathcal{L}_m(X_1 \times \cdots \times X_m)) - \dim(\mathcal{L}_m(F_1 \times \cdots \times F_m)) \\ &= \dim(X_1)\cdots\dim(X_m) - \dim(F_1)\cdots\dim(F_m) \\ &< N_1 \cdots N_m - (N_1 - k_1)\cdots(N_m - k_m). \end{aligned}$$

Combining Theorems 5.1 and 5.3 and with $(*)$ yields

$$\begin{aligned} c_n(T) = d_n(T^\times) \le d_N(T^\times) &\le \big\|Q_{(F_1\times\cdots\times F_m)^\circ} T^\times\big\| \\ &\le M\big\|TI_{F_1\times\cdots\times F_m}\big\| \le M\overline{c}_{(k_1,\dots,k_m)}(T) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary the proof, the required estimate follows. □

It is clear that if $H_1,\dots,H_m$ are Hilbert spaces, then $H_1 \times \cdots \times H_m$ has the multilinear extension property. Let us remark that Hayden's extension theorem [7, Theorem 7] gives more precise result that $M(H_1 \times \cdots \times H_m) = 1$. In consequence, we obtain the following.

Corollary 5.1. *Let $m \ge 2$ and $H_1,\dots,H_m$ be Hilbert spaces and let Y be a Banach space such that $N_i = \dim(H_i) < \infty$ for each $1 \le i \le m$. Then for all $T \in \mathcal{L}_m(H_1 \times \cdots \times H_m, Y)$,*

$$c_n(T) \le \overline{c}_{(k_1,\dots,k_m)}(T), \quad 1 \le k_i < N_i,\ \ 1 \le i \le m,$$

where $n \ge N_1 \cdots N_m - (N_1 - k_1)\cdots(N_m - k_m)$.

To show applications to bilinear operators we recall that a Banach X has *type* p, $1 < p \le 2$, provided there exists a constant $C > 0$, such that, for every choice of finitely many elements $x_1,\dots,x_n \in X$

$$\bigg(\int_0^1 \Big\|\sum_{k=1}^n r_k(t)x_k\Big\|^2\,dt\bigg)^{1/2} \le C\bigg(\sum_{k=1}^n \|x_k\|^p\bigg)^{1/p},$$

where (r_n) is the sequence of the Rademacher functions. The least constant C in the above inequality is called the *type constant* of X and is denoted by $T_p(X)$.

Maurey's celebrated extension theorem (see [3, pp. 246–248]) implies that $X_1\times X_2$ has the bilinear extension property with M for arbitrary Banach spaces X_1 and X_2 of type 2, where M depends type constants $T_2(X_1)$ and $T_2(X_2)$ of X_1 and X_2, respectively. Examples of Banach spaces with type 2 are L_p-spaces for $2 \le p < \infty$.

We conclude with the following corollary.

Corollary 5.2. *Let X_1, X_2 and Y be Banach spaces. If X_1 and X_2 are finite dimensional spaces of type 2 with $\dim(X_1) = N_1$ and $\dim(X_2) = N_2$, then there exists a constant $C > 0$, such that, for every bounded bilinear operator $T\colon X_1 \times X_2 \to Y$,*

$$c_n(T) \le C\,\overline{c}_{(k_1,k_2)}(T), \quad 1 \le k_i < N_i,\ \ 1 \le i \le 2,$$

where $n \ge N_1N_2 - (N_1 - k_1)(N_2 - k_2)$.

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