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ON THE LENGTH SPECTRUM METRIC IN INFINITE DIMENSIONAL TEICHMÜLLER SPACES

Erina Kinjo

Tokyo Institute of Technology, Department of Mathematical and Computing Sciences
Oh-okayama 2-12-1, Meguro-ku, Tokyo 152-8552, Japan; kinjo.e.aa@m.titech.ac.jp

Abstract. We consider the length spectrum metric d_L in infinite dimensional Teichmüller space $T(R_0)$. It is known that d_L defines the same topology as that of the Teichmüller metric d_T on $T(R_0)$ if R_0 is a topologically finite Riemann surface. In 2003, Shiga proved that d_L and d_T define the same topology on $T(R_0)$ if R_0 is a topologically infinite Riemann surface which can be decomposed into pairs of pants such that the lengths of all their boundary components except punctures are uniformly bounded by some positive constants from above and below. In this paper, we extend Shiga's result to Teichmüller spaces of Riemann surfaces satisfying a certain geometric condition.

1. Introduction

Let R_0 be a Riemann surface of infinite topological type. We consider a pair (R, f) of a Riemann surface R and a quasiconformal mapping $f\colon R_0 \to R$. Two such pairs (R_1, f_1) and (R_2, f_2) are called equivalent if $f_2 \circ f_1^{-1}\colon R_1 \to R_2$ is homotopic to some conformal mapping, where the homotopy map does not necessarily keep points of ideal boundary ∂R_0 fixed. We denote the equivalence class of (R, f) by $[R, f]$. The set of all equivalence classes is called *the Teichmüller space* of R_0; we denote it by $T(R_0)$.

The Teichmüller space $T(R_0)$ has a complete metric d_T called *the Teichmüller metric* which is defined by

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal mappings from R_1 to R_2 that is homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of f.

We introduce another metric on $T(R_0)$. Let $\mathcal{C}(R_0)$ be the set of non-trivial and non-peripheral closed curves in R_0. We define *the length spectrum metric* d_L by

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in \mathcal{C}(R_0)} \left| \log \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))} \right|,$$

where $\ell_{R_i}(f_i(\alpha))$ is the hyperbolic length of the closed geodesic on R_i which is freely homotopic to $f_i(\alpha)$.

Proposition 1.1. [15, Proposition 3.5] *Let $\mathcal{S}(R_0)$ be the set of simple closed curves in R_0. Then*

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in \mathcal{S}(R_0)} \left| \log \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))} \right|.$$

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In 1972, Sorvali [14] defined d_L, and showed the following.

Lemma 1.2. [14] *For any* $[R_1, f_1], [R_2, f_2] \in T(R_0)$,

$$d_L([R_1, f_1], [R_2, f_2]) \leq d_T([R_1, f_1], [R_2, f_2])$$

holds.

Sorvali conjectured that d_L defines the same topolpgy as that of d_T on $T(R_0)$ if R_0 is a topologically finite Riemann surface. In 1986, Li [9] proved that the statement holds in the case where R_0 is a compact Riemann surface with genus ≥ 2. In 1999, Liu [10] proved that Sorvali's conjecture is true and he asked whether or not the statement holds for any Riemann surface of infinite type. To this question, Shiga [13] gave a negative answer, that is, he showed that there exists a Riemann surface R_0 of infinite type such that d_L and d_T do not define the same topology on $T(R_0)$. Also, he gave a sufficient condition for these metrics to define the same topology on $T(R_0)$ as follows.

Theorem 1.3. [13] *Let R_0 be a Riemann surface. Assume that there exists a pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$ satisfying the following conditions.*

(1) *Each connected component of ∂P_k ($k = 1, 2, 3 \dots$) is either a puncture or a simple closed geodesic of R_0.*
(2) *There exists a constant $M > 0$ such that if α is a boundary curve of some P_k then*

$$0 < M^{-1} < l_{R_0}(\alpha) < M$$

holds.

Then d_L defines the same topology as that of d_T on $T(R_0)$.

In our previous paper [8], we showed that the converse of Shiga's theorem is not true, that is, there exists a Riemann surface R_0 such that R_0 does not satisfy Shiga's condition, but the two metrics define the same topology on $T(R_0)$. In this paper, we generalize the example and extend Shiga's theorem as follows.

Theorem 1.4. *Let R_0 be a Riemann surface. Assume that there exists a constant $M > 0$ and a decomposition $R_0 = S \cup (R_0 - S)$ such that*

(1) *S is an open subset of R_0 whose relative boundary consists of simple closed geodesics and each connected component of S has a pants decomposition satisfying the same condition as that of Shiga's theorem for M, and*
(2) *$R_0 - S$ is of genus 0 and $d_{R_0}(x, S) < M$ for any $x \in R_0 - S$, where $d_{R_0}(\cdot, \cdot)$ is the hyperbolic distance in R_0.*

Then d_L defines the same topology as that of d_T on $T(R_0)$.

In Section 2, we show that there exists a Riemann surface such that it satisfies the condition of Theorem 1.4 but it does not satisfy that of Theorem 1.3. In Section 3, we introduce lemmas to prove Theorem 1.4. In Section 4, we prove Theorem 1.4.

In Section 5, we consider Riemann surfaces with bounded geometry. Here we say that a Riemann surface R_0 has (M-)bounded geometry if it satisfies the following condition: There exists a constant $M > 0$ such that any closed geodesic has the length greater than $1/M$ and for any $x \in R_0$, there exists a closed curve based on x with the length less than M.

As a corollary of Theorem 1.4, we obtain the following:

Corollary 1.5. *Suppose that R_0 is of finite genus and R_0 has bounded geometry. Then d_L define the same topology as that of d_T on $T(R_0)$.*

Remark 1. We consider surfaces after cutting the flares if such cylindrical ends exist. We assume that the unique complete hyperbolic metric on $R_0 \setminus \partial R_0$ that uniformizes the complex structure on the surface satisfies the following condition: Let $\{P_k\}$ be a pants decomposition of R_0. If we replace each boundary component of P_k $(k = 1, 2, \ldots)$ with the closed geodesic in its homotopy class, then P_k becomes a sphere with three holes, where a hole is either a boundary component which is a closed geodesic or a cusp.

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2. Examples

First, we give examples of Riemann surfaces satisfying the conditions in Theorem 1.4 and Corollary 1.5.

Example 1. Any Riemann surface satisfying Shiga's condition satisfies the condition in Theorem 1.4. Hence, in particular, any Riemann surface of finite topological type satisfies it.

Example 2. The Riemann surface R_0 constructed in our previous paper [8] satisfies conditions in both Theorem 1.4 and Cororally 1.5. For convenience of the reader, we show the construction.

Let Γ be a hyperbolic triangle group of signature (2,4,8) acting on the unit disk $\mathbf{D}$ and let P be a fundamental domain for Γ with angles $(\pi, \pi/4, \pi/4, \pi/4)$. (See the left in Figure 1.) Let O, a, b, c denote the vertices of P, where the angle at O is π. Now, take a sufficiently small number $\varepsilon > 0$. Let b' the point on the segment $[Ob]$ whose hyperbolic distance from b is ε. Similarly, we take a' and c' in P. (See the middle in Figure 1.)

We define a Riemann surface R_0 by removing the Γ-orbits of a', b', c' from the unit disk $\mathbf{D}$. (See the right in Figure 1.)

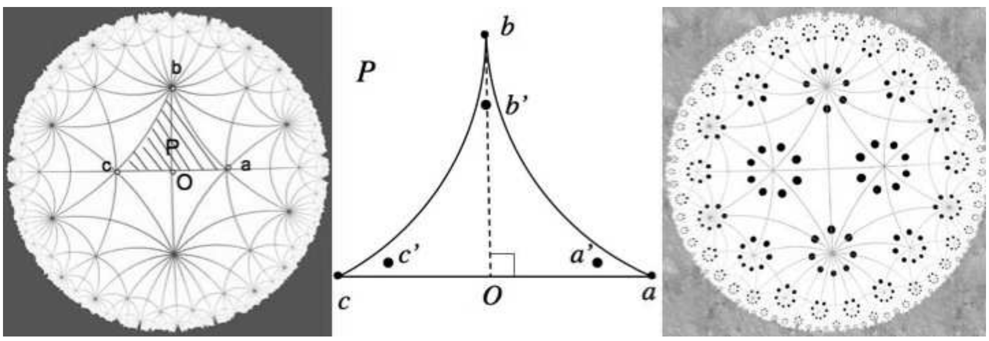

Figure 1. Left: Tessellation by the (2,4,8) group. Middle: Points a', b', c' in P. Right: A Riemann surface $R_0 = \mathbf{D} - \{\gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma\}$.

It is not hard to see that the surface R_0 does not satisfy Shiga's condition (cf. Section 2 in [8]). However, it satisfies the above conditions. Indeed, we decompose R_0 into eight times punctured disks and a multiply-connected domain. (See Figure 2.)

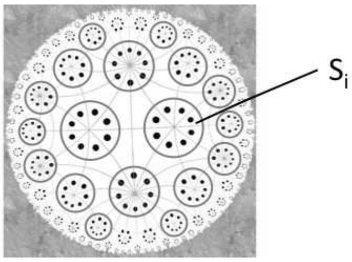

Figure 2. $R_0 = \{$punctured disks$\} \cup \{$a multiply-connected domain$\}$.

Let S_i be a punctured disk and put $S = \cup_{i=1}^{\infty} S_i$. Then we obtain a decomposition in Theorem 1.4: $R_0 = S \cup (R_0 - S)$. On the other hand, R_0 satisfies the condition in Corollary 1.5 obviously.

Also, we can construct a Riemann surfaces satisfying Theorem 1.4 and Corollary 1.5 by replacing a hyperbolic triangle group Γ with an arbitrary Fuchsian group with a compact fundamental region.

Example 3. In Example 2, R_0 is a Riemann surface of genus 0 with ∞ punctures and 1 flare. By tinkering with R_0, we can construct Riemann surfaces of genus ≥ 1 with two or more flares which satisfies the conditions of Theorem 1.4. For example, in Example 2, we replace a punctured disk S_i with a pair of pants. (See the left in Figure 3.) We regard it as a block and make a copy of it and glue them. (See the right in Figure 3.) Then we obtain a Riemann surface X_0 of genus 1 with ∞ punctures and two flares. Obviously X_0 satisfies Theorem 1.4 and Corollary 1.5. Hence, in the similar way, we can construct Riemann surfaces of genus ∞ with ∞ flares which satisfies Theorem 1.4.

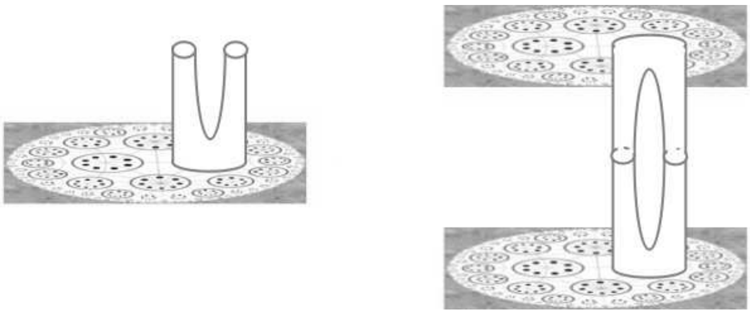

Figure 3. Left: A block obtained by replacing S_i in Example 2 with a pair of pants. Right: A Riemann surface X_0 of genus 1 with ∞ punctures and two flares.

3. Lemmas

In this section, we present some lemmas to prove Theorem 1.4 and Corollary 1.5.

Lemma 3.1. [4, Lemma 3.1] *Let $T_1, T_2 \subset \mathbf{D}$ be two hyperbolic triangles with sides (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Suppose all their angles are bounded below by $\theta > 0$ and*

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$

Then there is a constant $C = C(\theta, A)$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi\colon T_1 \to T_2$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of T_1.

Lemma 3.2. [4, Corollary 3.3] *Let $H, H' \subset \mathbf{D}$ be two hyperbolic hexagons with sides $(a_1, \ldots, a_6)$ and $(b_1, \ldots, b_6)$ respectively. Suppose $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$ are $\leq B$ and are comparable with a constant B. Also assume that three alternating angles of H and the corresponding angles of H' are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. If $\varepsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a constant $C = C(\theta, B)$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi\colon H \to H'$ such that φ maps each vertex to the corresponding vertex and φ is affine on the edge of H.*

Lemma 3.3. [4, Lemma 6.2] *Let $P_1$and P_2 be pants with boundary lengths (a_1, b_1, c_1) and (a_2, b_1, c_1) respectively. Suppose $a_1, a_2, b_1, c_1 \leq L$ (punctures count as length zero). Assume that $\varepsilon := |\log a_1/a_2| \leq 2$, where we define $|\log a_1/a_2| = 0$ if $a_1 = a_2 = 0$ and $|\log a_1/a_2| = +\infty$ if one is zero and the other is not. Then there is a constant $C = C(L)$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi\colon P_1 \to P_2$ such that φ is affine on each of the boundary components.*

Also we note the following lemma.

Lemma 3.4. *Let R_0 be a Riemann surface. Suppose α_1 and α_2 are disjoint simple closed geodesics . Let β_{12} be a simple arc connecting α_1 and α_2. Then there exists a geodesic $\beta_{12}^\star$ connecting α_1 and α_2 such that*

(1) *β_{12} and $\beta_{12}^\star$ are homotopic, where the homotopy map may not keep end points of β_{12} and β_{12}^* fixed;*
(2) *$\beta_{12}^\star$ is orthogonal to α_1 and α_2;*
(3) *the length of $\beta_{12}^\star$ is determined by lengths of three simple closed geodesics which are homotopic to α_1, α_2 and $\alpha_{12} := \alpha_1 \cdot \beta_{12} \cdot \alpha_2 \cdot \beta_{12}^{-1}$.*

Proof. There exists a closed geodesic in R_0 homotopic to α_{12}. We denote it by $[\alpha_{12}]$. Consider a pair of pants P_{12} bounded by α_1, α_2 and $[\alpha_{12}]$. There are three lines which divide P_{12} into two isometric right-hexagons. Let $\beta_{12}^\star$ be a line connecting α_1 and α_2 in those. We denote the length of $\beta_{12}^\star$ by $\ell_{R_0}(\beta_{12}^\star)$. Then, by Theorem 7.19.2 of [3],

$$\cosh \ell_{R_0}(\beta_{12}^\star) = \frac{\cosh(\frac{1}{2}\ell_{R_0}(\alpha_{12})) + \cosh(\frac{1}{2}\ell_{R_0}(\alpha_1))\cosh(\frac{1}{2}\ell_{R_0}(\alpha_2))}{\sinh(\frac{1}{2}\ell_{R_0}(\alpha_1))\sinh(\frac{1}{2}\ell_{R_0}(\alpha_2))}$$

holds. □

In the following lemma, for $R_0 - S$ in Theorem 1.4 we may consider a decomposition by right-hexagons with a bounded condition.

Lemma 3.5. *Let R_0 be a Riemann surface satisfying the condition of Theorem 1.4. Then $R_0 - S$ can be decomposed hyperbolic right-hexagons $\{H_j\}_{j=1}^\infty$ with sides of lengths less than $2M$.*

Proof. By assumption, we can decompose S into domains $\{S_i\}_{i=0}^\infty$ such that $\partial S_i \cap (R_0 - S)$ $(i = 0, 1, 2, \ldots)$ is a closed geodesic with the length less than M. For S_i, we consider the following domain:

$$D_i := \{x \in R_0 - S \mid d_{R_0}(x, S_i) \leq d_{R_0}(x, S_j)(\forall j \neq i)\}.$$

D_i is contained in M-neighborhood of S_i, and $D_i \cup S_i$ is convex. The boundary of D_i consists of two kinds of connected components; the boundary of S_i and the boundary of geodesic polygon with finitely many sides. We denote the polygon with a hole by W_i (i.e. $W_i := \partial D_i$).

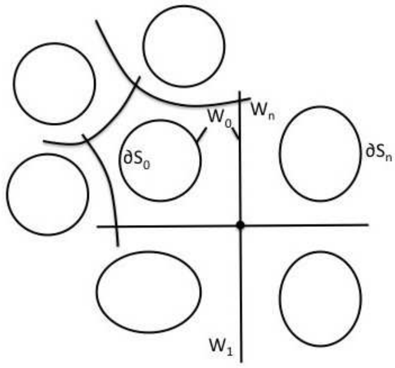

Figure 4. $W_i := \partial D_i$.

W_i is bounded on each side by another W_j. We show that two vertices of W_i coincides with those of W_j if W_i is bounded by W_j. Assume that there exists a side w_i with vertices v_i, v_i' of W_i such that, for a side w_j with vertices v_j, v_j' of W_j, $w_j \subset w_i$ but $0 < d_{R_0}(v_i, v_j) < d_{R_0}(v_i, v_j')$. (See Figure 5.) Then there exists a polygon with a hole W_k which has a side $w_k \cap w_i - w_j \neq \emptyset$. For a domain S_k in $\{S_i\}_{i=0}^{\infty}$ with $\partial S_k \cap W_k \neq \emptyset$, we take a line $b_{i,k} := \{x \in R_0 \mid d_{R_0}(x, S_i) = d_{R_0}(x, S_k)\}$. $b_{i,k}$ is a perpendicular bisector of the shortest geodesic segment $[l_{i,k}]$ connecting ∂S_i and ∂S_k. (Note that $[l_{i,k}]$ is orthogonal to ∂S_i and ∂S_k.) By the definition of domains $\{D_i\}$, $w_k \subset b_{i,k}$. Similarly we take another perpendicular bisector $b_{i,j}$ of the shortest geodesic segment $[l_{i,j}]$ connecting ∂S_i and ∂S_j. Then $b_{i,k} = b_{i,j}$ since they are geodesics and $w_k \subset w_i \subset b_{i,j}$. Take four points $p_{i,k} := \partial S_i \cap [l_{i,k}]$, $m_{i,k} := [l_{i,k}] \cap b_{i,k}$, $p_{i,j} := \partial S_i \cap [l_{i,j}]$, $m_{i,j} := [l_{i,j}] \cap b_{i,j}$ and consider a quadrilateral with vertices $\{p_{i,k}, m_{i,k}, m_{i,j}, p_{i,j}\}$. Then we obtain a right-angled quadrilateral. However there does not exist such a hyperbolic quadrilateral. Hence $d_{R_0}(v_i, v_j) = 0$, i.e. $v_i = v_j$. Similarly $v_i' = v_j'$.

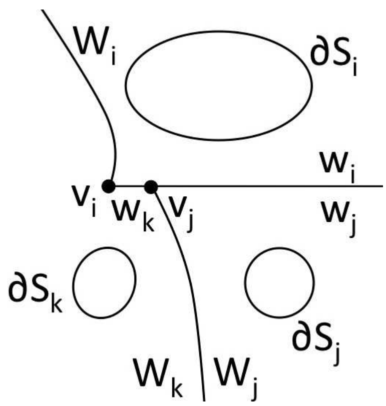

Figure 5. $v_i \neq v_j$.

Now, take an arbitrary vertex v_0 of an arbitrary polygon with a hole W_0 and put W_j ($j = 0, 1, 2, \ldots, n$; in counterclockwise direction) the polygon with a hole which contains v_0. Connect ∂S_j and ∂S_{j+1} by the shortest geodesic segment $[l_{j,j+1}]$ for each $j = 0, 1, 2, \ldots, n$(, where S_j is a domain in $\{S_i\}_{i=0}^{\infty}$ with $\partial S_j \cap W_j \neq \emptyset$ and $S_{n+1} := S_0$). So we obtain a $2n$-sided polygon P which consists of $[l_{j,j+1}]$ and subarcs of ∂S_j ($j = 0, 1, 2, \ldots, n$).

Also we join ∂S_0 and ∂S_j $(j = 2, 3, \ldots, n-1)$ by the shortest geodesic in P which is orthogonal to ∂S_0 and ∂S_j, respectively. (By Lemma 3.4, there exist such geodesics.) Then we obtain $n-1$ right-hexagons. (See Figure 6.) Each right-hexagon has three alternating sides $[l_{0,j}], [l_{j,j+1}], [l_{j+1,0}]$ with the lengths bounded by $2M$ since $[l_{j,j'}] \leq d_{R_0}(v_0, \partial S_j) + d_{R_0}(v_0, \partial S_{j'}) < M + M = 2M$ for $j, j' = 0, 1, 2, \ldots n$, $j \neq j'$. Hence all lengths of sides are bounded by $2M$.

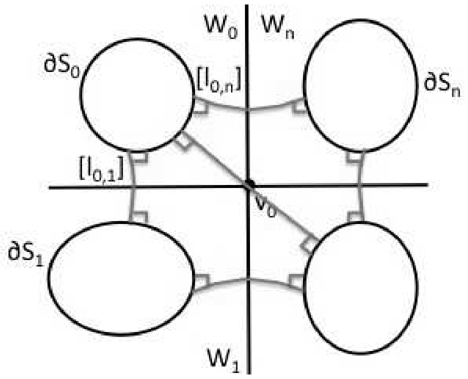

Figure 6. A right-hexagons decomposition around v_0.

Continue the above operation, then $R_0 - S$ is divided into right-hexagons with sides of the lengths bounded by $2M$. □

4. Proof of Theorem 1.4

From Lemma 1.2, it is sufficient to show that for any sequence $\{p_n\}_{n=o}^{\infty} \subset T(R_0)$ with $d_L(p_n, p_0) \to 0$ $(n \to \infty)$, $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$. We assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.

In Section 3, we see that R_0 is decomposed by pairs of pants and hexagons such that lengths of their boundaries are bounded uniformly; $R_0 = \bigcup_{i=1}^{\infty} S_i \cup \bigcup_{j=1}^{\infty} H_j$. We consider a decomposition of R_n for sufficiently large n.

First, for each $j = 1, 2, \ldots$, we replace $f_n(H_j)$ by a right-hexagon in R_n as follows. $H_j \subset R_0$ has edges $a_1, \ldots, a_6$ (in counterclockwise direction). We suppose that a_1, a_3, a_5 are subarcs of $\partial S_1, \partial S_2, \partial S_3$ respectively and a_2 connects ∂S_1 and ∂S_2. Put $\alpha_{12} := \partial S_1 \cdot a_2 \cdot \partial S_2 \cdot a_2^{-1} \in \mathcal{C}(R_0)$. For a closed curve $f_n(\alpha_{12})$ in R_n, we take a closed geodesic $[f_n(\alpha_{12})]$ in R_n. If we consider a pair of pants bounded by $[f_n(\partial S_1)]$, $[f_n(\partial S_2)]$ and $[f_n(\alpha_{12})]$, then there exists a geodesic segment connecting $[f_n(\partial S_1)]$ and $[f_n(\partial S_2)]$ as in Lemma 3.4. We denote it by a_2^n. The length of a_2^n is determined by the lengths of $[f_n(\partial S_1)]$, $[f_n(\partial S_2)]$ and $[f_n(\alpha_{12})]$. The lengths of them are almost the same as that of preimages of closed geodesics in R_0 respectively, so the lengths of a_2 and a_2^n are almost the same. Similarly we take geodesic segments a_4^n and a_6^n in R_n for a_4 and a_6 respectively. Let $H_j^n \subset R_n$ be a right-hexagon bounded by a_2^n, a_4^n, a_6^n and subarcs of $[f_n(\partial S_1)]$, $[f_n(\partial S_2)]$, $[f_n(\partial S_3)]$. (See Figure 7.) Then H_j^n is almost congruous with H_j.

Put $R_n' := \bigcup_{j=1}^{\infty} H_j^n$. By Lemma 3.2, we obtain a quasiconformal mapping g_n from $R_0' (= \bigcup_{j=1}^{\infty} H_j)$ to R_n'. We claim that f_n is homotopic to g_n on R_0', where the homotopy map does not necessarily keep points of ∂R_0 fixed.

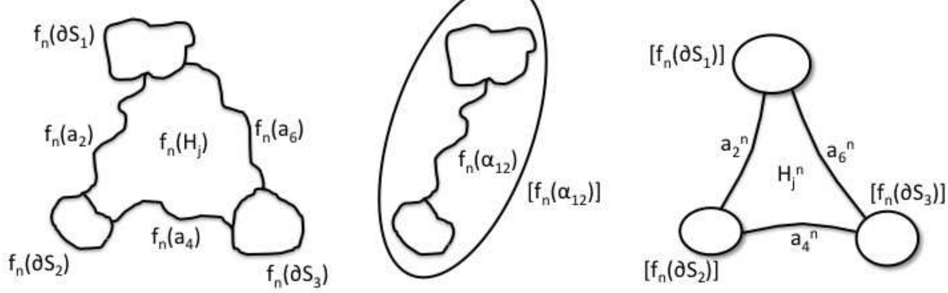

Figure 7. Replacement of $f_n(H_j)$ by a right-hexagon H_j^n in R_n.

It is enough to see that for an arbitrary simple closed geodesic $\alpha \subset R_0'$, $f_n(\alpha)$ and $g_n(\alpha)$ are homotopic (cf. [5, Lemma 4]). Let $\{H_{j(k)}\}_{k \in K} \subset R_0'$ be the set of all the right-hexagons such that $H_{j(k)} \cap \alpha \neq \emptyset$. Since f_n is homeomorphic, $f_n(\alpha) \subset \bigcup_{k \in K} f_n(H_{j(k)})$. Therefore we see that for each $k \in K$, a curve $g_n(\alpha) \cap H_{j(k)}^n$ is homotopic to a curve $[f_n(\alpha)] \cap H_{j(k)}^n$, where the homotopy map does not necessarily fix endpoints. (See Figure 8.) Hence $f_n(\alpha)$ is homotopic to $g_n(\alpha)$, so we verify the claim.

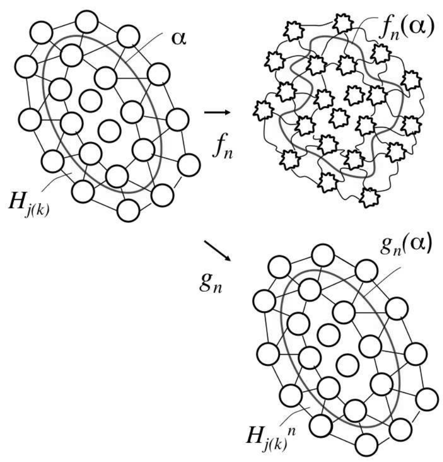

Figure 8. $f_n(\alpha)$ and $g_n(\alpha)$.

Next, we consider a quasiconformal mapping of S_i for each $i = 1, 2, \ldots$ We decompose S_i into pairs of pants satisfying Shiga's condition; $S_i = \bigcup_{k=1}^{\infty} P_k$.

Let G_P be the set of closed geodesics which are boundaries of some P_k in S_i. For each $\alpha \in G_P$, there exists a closed geodesic $[f_n(\alpha)]$ in R_n homotopic to $f_n(\alpha)$. The set $\{[f_n(\alpha)]\}_{\alpha \in G_P}$ gives a pants decomposition of $f_n(S_i)$. (Indeed, $f_n(S_i)$ is Nielsen-convex; cf. [1, Theorem 4.5.])

Now, we put α_1, α_2, α_3 three closed geodesics of ∂P_1 and assume that $\alpha_1 \subset \overline{R_0'}$.

By lemmas of Bishop, we obtain a quasiconformal mapping on $\overline{S_i - P_1}$. However, g_n on R_0' is locally affine on α_1, so we construct a quasiconformal mapping on P_1.

Let $x_1, \ldots, x_m \in \alpha_1$ be vertices of right-hexagons $\{H_j\}$, and let $y_1, \ldots, y_6 \in \partial P_1$ be the vertices of two symmetric right-hexagons constructing P_1 (See Figure 9). Suppose that y_1 is on the segment $[x_1 x_2]$, and y_6 is on the segment $[x_r x_{r+1}]$ $(1 \leq r \leq m)$. Let d_1 be the length of $[x_1 x_2]$ and let d_1' be the length of the $[x_1 y_1]$. Then there is a number $t \in [0, 1]$ such that $d_1 = t d_1'$. Similarly take d_r, d_r' for $[x_r x_{r+1}]$, $[x_r y_6]$, then there is a number $s \in [0, 1]$ such that $d_r = s d_r'$.

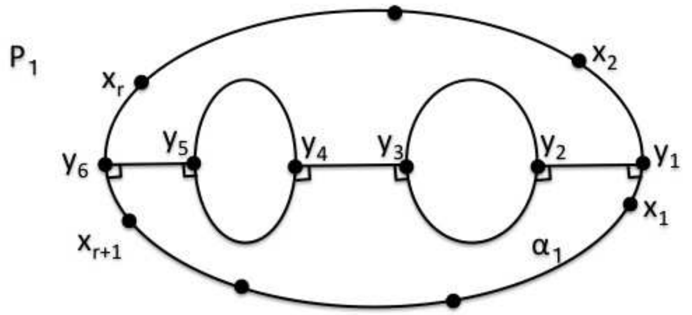

Figure 9. Points on ∂P_1.

On the other hand, let $x_1^n, \ldots, x_m^n \in [f_n(\alpha_1)]$ in R_n be vertices of right-hexagons H_j^n. We take the points $g_n(y_1), \ldots, g_n(y_6)$ on ∂P_1^n, where P_1^n is a pair of pants corresponding to $f_n(P_1)$

We consider a hyperbolic hexagon with vertices $g_n(y_1), \ldots, g_n(y_6)$. We claim that the angle formed by $[g_n(y_2)g_n(y_3)]$ and $[g_n(y_3)g_n(y_4)]$ is almost $\pi/2$. Indeed, for $S_i \subset R_0$, let $\hat{S}_i$ be the Nielsen extension of S_i. We consider the Fenchel–Nielsen coordinates of the Teichmüller space $T(\hat{S}_i)$. Then the twist parameter along $[f_n(\alpha_2)]$ is almost the same as that along α_2 (cf. [13, Lemma 4.1]). Hence we verify the claim. The remaining angles are almost $\pi/2$, similarly.

Let d_i^n be the hyperbolic length of the segment $[x_i^n x_{i+1}^n]$ $(1 \leq i \leq m)$, and let $d_i^{n\prime}$ be the hyperbolic length of the segment $[x_i^n g_n(*)]$ $(i = 1, r, * = y_1, y_6)$. Then, for $t \in [0,1]$ and $s \in [0,1]$ we took above, $d_1^{n\prime} = t d_1^n$ and $d_r^{n\prime} = s d_r^n$ hold, because g_n of R_0' is locally affine on α_1. Moreover, since the quasiconformal mapping g_n of $\overline{S_i - P_1}$ is affine on α_2 and α_3, the lengths of sides $[y_1 y_2], \ldots, [y_6 y_1]$ and the lengths of sides $[g_n(y_1)g_n(y_2)], \ldots, [g_n(y_6)g_n(y_1)]$ are almost the same respectively. Hence the right-hexagon with vertices $(y_1, \cdots, y_6)$ and the hexagon with vertices $(g_n(y_1), \cdots, g_n(y_6))$ are almost congruous.

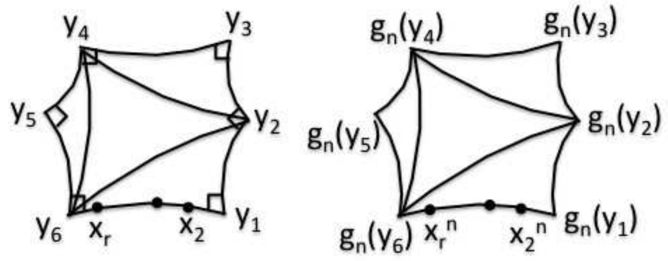

Figure 10. Triangulation.

From the First Cosine Rule for hyperbolic geometry (cf. [3]), the length of the new sides are determined by the sides and angles of the hexagon. Quasiconformal mappings of the triangles with vertices (y_2, y_3, y_4), (y_4, y_5, y_6) and (y_2, y_4, y_6) are obtained from Lemma 3.1.

We consider a quasiconformal mapping of the triangle T with vertices (y_1, y_2, y_6). In T, connect the points $x_2, \ldots, x_r$ by geodesics segments to y_2. Similarly, in the triangle T_n with vertices $(g_n(y_1), g_n(y_2), g_n(y_6))$, connect the points $x_2^n, \ldots, x_r^n$ by geodesics segments to $g_n(y_2)$. Then we obtain a quasiconformal mapping of the triangle T from Lemma 3.1.

Hence we obtain a quasiconformal mapping g_n of the whole of R_0 such that g_n is homotopic to f_n and $K(g_n) \to 1$ $(n \to \infty)$. Thus $d_T(p_n, p_0) \to 0$ $(n \to \infty)$.

In the case where $p_0 \neq [R_0, id]$, we can show that $d_T(p_n, p_0) \to 0$ $(n \to \infty)$ similarly. Indeed, any Riemann surface which is quasiconformally equivalent to R_0 satisfies the condition of Theorem 1.4 for some constant. □

5. Corollary of Theorem 1.4

In this section, we consider Riemann surfaces with bounded geometry.

Proposition 5.1. *Let R_0 be a Riemann surface of finite genus with M-bounded geometry. Then R_0 satisfies the assumption of Theorem 1.4.*

Proof. On R_0, we construct S in the condition of Theorem 1.4 as a union of pairs of pants. Note that we may construct a pair of pants from two disjoint simple closed geodesics and an simple arc connecting the two geodesics.

At first, we take a constant $d = d(M) > 0$ as the following:

For any $x \in R_0$, there exists a closed curve c_x passing through x with $1/M <$ the length of $c_x < M$. We take a geodesic α_x in the homotopy class of c_x. We put

$$d_x := \max_{\alpha_x}\{M - \ell_{R_0}(\alpha_x)\} > 0,$$

where $\alpha_x \in \{\alpha_x :$ a geodesic $|$ In the homotopy class of α_x, there exists a closed curve c_x passing through x with $1/M <$ the length of $c_x < M.\}$. Moreover we put

$$d := \sup_{x \in R_0} d_x$$

Then d satisfies the following property: For any $x \in R_0$ and any closed curve c passing through x with $1/M <$ the length of $c < M$, the geodesic α in the homotopy class of c satisfies $d_{R_0}(x, \alpha) < d$.

Indeed, if $d_{R_0}(x, \alpha) \geq d$ holds, then $d_{R_0}(x, \alpha) \geq d_x \geq M - \ell_{R_0}(\alpha)$ i.e. $d_{R_0}(x, \alpha) + \ell_{R_0}(\alpha) \geq M$. Hence the length of $c(\geq d_{R_0}(x, \alpha) + \ell_{R_0}(\alpha))$ is larger than M. This contradicts. Therefore $d_{R_0}(x, \alpha) < d$.

(Note that, in other words, d is a constant such that if x is a point in R_0 and α is a simple closed geodesic with $1/M < \ell_{R_0}(\alpha) < M$ and $d_{R_0}(x, \alpha) \geq d$, then the length of any closed curve c passing through x which is homotopic to α is larger than M.)

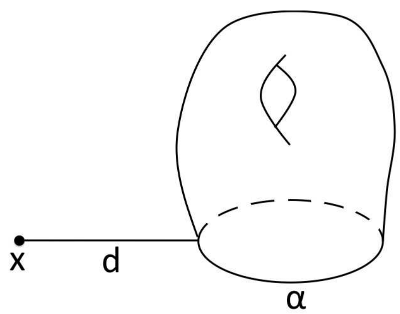

Figure 11. A closed curve c such that the length $= d + \ell_{R_0}(\alpha)$.

Now let us start to construct pairs of pants. Let x_0 be an arbitrary point in R_0. Then there exists a closed curve c_0 passing through x_0 with the length less than M.

We take a geodesic α_0 in the homotopy class of c_0. Then $0 < 1/M < \ell(\alpha_0) < M$ and $d_{R_0}(x_0, \alpha_0) < d$.

Put $D := \max\{d, M\}$. Next, take $y_0 \in R_0$ with $d_{R_0}(y_0, \alpha_0) = D + 1$. Also, take a geodesic β_0 for y_0 in the above way. Then $\beta_0 \neq \alpha_0$ and $d_{R_0}(y_0, \beta_0) < d$. Hence $d_{R_0}(\alpha_0, \beta_0) < D + 1 + d < 2D + 1$. Thus there exists a simple arc $\hat{\gamma_0}$ connecting α_0 and β_0 such that the length of $\hat{\gamma_0} \leq 2D + 1$.

If we construct a pair of pants P_0 by α_0, β_0 and $\hat{\gamma_0}$, then the length of each boundary component is bounded by some constant $L = L(M)$ from above and below.

Next, we take a point $x_1 \in R_0$ such that $d_{R_0}(P_0, x_1) = 3D + 2$. Also, take a geodesic α_1 for x_1 in the above way. (Note that $d_{R_0}(\alpha_1, x_1) < D$.) Then we can take a point y_1 in $R_0 - P_0$ such that $d_{R_0}(y_1, \alpha_1) = D + 1$. Indeed, since

$$d_{R_0}(\alpha_1, P_0) \geq d_{R_0}(x_1, P_0) - d_{R_0}(\alpha_1, x_1) \geq 2D + 2,$$

$\{y \in R_0 \mid d_{R_0}(y, \alpha_1) = D + 1\} \cap P_0 = \emptyset$.

Now, we take a geodesic β_1 for y_1 similarly. $\beta_1 \neq \alpha_1$ and $d_{R_0}(y_1, \beta_1) < d < D$ hold. Hence $d_{R_0}(\alpha_1, \beta_1) < 2D + 1$. Thus $\beta_1 \cap P_0 = \emptyset$ since

$$d_{R_0}(\beta_1, P_0) \geq d_{R_0}(P_0, \alpha_1) - d_{R_0}(\alpha_1, \beta_1) \geq 2D + 2 - (2D + 1) = 1.$$

Also, there exists a simple arc $\hat{\gamma_1}$ connecting α_1 and β_1 with the length of $\hat{\gamma_1} < 2D+1$. Then we see that $\hat{\gamma_1} \cap P_0 = \emptyset$ since $d_{R_0}(\alpha_1, P_0) \geq 2D + 2$.

We construct a pair of pants P_1 by α_1, β_1 and $\hat{\gamma_1}$. Then $P_0 \cap P_1 = \emptyset$. Indeed, if we take a geodesic γ_1 which is homotopic to a closed curve $\alpha_1 \cdot \hat{\gamma_1} \cdot \beta_1 \cdot \hat{\gamma_1}^{-1}$ then $\gamma_1 \cap P_0 = \emptyset$ by property of geodesics.

Similarly, we take a point $x_2 \in R_0$ such that $d_{R_0}(x_2, P_0 \cup P_1) = 3D + 2$. Let α_2 be the geodesic in homotopy class of a simple closed curve c_2 passing through x_2 with $M^{-1} <$ the length of $c_2 < M$. $d_{R_0}(\alpha_2, x_2) < d < D$. We can take a point $y_2 \in R_0 - (P_0 \cup P_1)$ such that $d_{R_0}(\alpha_2, y_2) = D+1$ since $d_{R_0}(\alpha_2, P_0 \cup P_1) \geq 2D+2$. Let β_2 be the geodesic in homotopy class of a simple closed curve c_2' passing through y_2 with $M^{-1} <$ the length of $c_2' < M$. $d_{R_0}(\beta_2, y_2) < d < D$. Hence $d_{R_0}(\alpha_2, \beta_2) < 2D+1$. Thus $\beta_2 \cap (P_0 \cup P_1) = \emptyset$. Also, there exists a simple arc $\hat{\gamma_2}$ connecting α_2 and β_2 with $\ell(\hat{\gamma_2}) < 2D + 1$. $\hat{\gamma_2} \cap (P_0 \cup P_1) = \emptyset$ since $d_{R_0}(\alpha_2, P_0 \cup P_1) \geq 2D + 2$. If we construct a pair of pants P_2 by α_2, β_2 and $\hat{\gamma_2}$ then $P_2 \cap (P_0 \cup P_1) = \emptyset$.

Inductively, we construct a sequence of pairs of pants $\{P_i\}$. Then it satisfies the following.

(i) $P_i \cap P_j = \emptyset$ if $i \neq j$.

(ii) For any i, the length of each connected component of ∂P_i is bounded by L from above and below.

Finally we consider the set

$$X_n := \{x \in R_0 \mid d_{R_0}(x_0, x) < n\}$$

for a sufficient large number $n > 0$. Since the closure $\overline{X_n}$ of X_n is relatively compact in R_0, there exists a finite sequence of pairs of pants $\{P_i\}_{i=0}^k$ such that $X_n \cap P_i \neq \emptyset$ $(i = 0, 1, \ldots, k)$ and it satisfies (i) and (ii); moreover

$$d_{R_0}(x, \bigcup_{i=0}^{k} P_i) < 3D + 2$$

for any $x \in X_n - \bigcup_{i=0}^k P_i$.

Construct $\{P_i\}_{i=0}^{\infty}$ as $n \to \infty$. Since $\bigcup_{n=1}^{\infty} X_n = R_0$, we obtain a subset $S = \bigcup_{i=0}^{\infty} P_i \subset R_0$ in the condition of Theorem 1.4.

(We note that $R_0 - S$ is of genus 0. Indeed, the lengths of geodesics cutting genus of R_0 are bounded by M since R_0 is of finite genus. Hence we can choose them as the curves of ∂P_i.) □

Hence we obtain Corollary 1.5.

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