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THE FAST DIFFUSION EQUATION WITH INTEGRABLE DATA WITH RESPECT TO THE DISTANCE TO THE BOUNDARY

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Abstract. In this paper, we study the existence and regularity of very weak solutions to the fast diffusion equations with integrable data with respect to the distance to the boundary.

1. Introduction and statement of the main results

This paper deals with the following problem

$$(P)\qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) = f & \text{in } Q,\\ u = 0 & \text{on } \Sigma,\\ u(x,0) = u_0 & \text{in } \Omega,\end{cases}$$

where Ω is a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $T > 0$, $Q = \Omega \times (0,T)$, Σ denotes the lateral surface of Q, $f \in L^1(Q,\delta)$, $u_0 \in L^1(\Omega,\delta)$, $\delta(x) = \text{distance}(x,\partial\Omega)$, $1 - \frac{2}{N+1} < m < 1$.

If $m < 1$, the above problem is called the fast diffusion problem; if $m > 1$, it is called the porous media problem. There are systematic survey books about the porous media equations written by Vázquez (see [29, 30]). Lukkari has discussed the fast diffusion equation and the porous media equation with measure data (see [22, 23]).

Recently, Díaz and Rakotoson [11] have studied the very weak solutions to linear elliptic equations with right-hand side integrable data with respect to the distance to the boundary and answered the question of the integrablity of the generalized derivative raised in the unpublished manuscript by Brezis (see also [9]). Lately, they have extended these results to semilinear elliptic equations and linear parabolic equations (see [12] and [25]).

My main goal in this paper is to study the existence and regularity of very weak solutions to the fast diffusion equations with integrable data with respect to the distance to the boundary in the framework of weighted spaces by using a different method to that of [11] and [25].

We recall the weighted Lebesgue space and weighted Sobolev space as follows (see [1,7,14,18]): For $1 \leq p < +\infty,\ 1 \leq q < +\infty$,

$$L^p(\Omega,\delta) = \left\{ u\colon \Omega \to R \text{ is Lebesgue measurable}, \int_\Omega |u|^p \delta(x)\,dx < +\infty \right\}$$

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which is equipped with the norm

$$\|u\|_{L^p(\Omega,\delta)} = \left(\int_\Omega |u|^p \delta(x)\, dx\right)^{\frac{1}{p}}, \tag{1.1}$$

$$L^p(Q,\delta) = \{u\colon Q \to R \text{ is Lebesgue measurable}, \int_Q |u|^p \delta(x)\, dx\, dt < +\infty\},$$

which is equipped with the norm

$$\|u\|_{L^p(Q,\delta)} = \left(\int_Q |u|^p \delta(x)\, dx\, dt\right)^{\frac{1}{p}}, \tag{1.2}$$

$$L^q(0,T;L^p(\Omega,\delta)) = \Big\{u\colon Q \to R \text{ is Lebesgue measurable}, \int_0^T \left(\int_\Omega |u|^p \delta(x)\, dx\right)^{\frac{q}{p}} dt < +\infty\Big\},$$

which is equipped with the norm

$$\|u\|_{L^q(0,T;L^p(\Omega,\delta))} = \left(\int_0^T \left(\int_\Omega |u|^p \delta(x)\, dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}, \tag{1.3}$$

$$W^{1,p}(\Omega,\delta) = \{u \in L^p(\Omega,\delta) \mid |Du| \in L^p(\Omega,\delta)\},$$

which is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega,\delta)} = \left(\int_\Omega (|u|^p \delta(x) + |Du|^p \delta(x))\, dx\right)^{\frac{1}{p}} \tag{1.4}$$

$$L^q(0,T;W^{1,p}(\Omega,\delta)) = \{u \in L^q(0,T;L^p(\Omega,\delta)) \mid |Du| \in L^q(0,T;L^p(\Omega,\delta))\},$$

which is equipped with the norm

$$\|u\|_{L^q(0,T;W^{1,p}(\Omega,\delta))} = \left(\int_0^T \left(\int_\Omega (|u|^p \delta(x) + |Du|^p \delta(x))\, dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}. \tag{1.5}$$

We define the space $W_0^{1,p}(\Omega,\delta)$ as the completion of $C_0^\infty(\Omega)$ with respect to the above norm. We can also define similarly the weighted Sobolev space $L^q(0,T;W_0^{1,p}(\Omega,\,\delta))$.

These weighted spaces equipped with the above norms are Banach spaces. Replacing $\delta(x)$ by $\delta(x)^\alpha$, we can define $L^p(\Omega,\delta^\alpha)$, $W^{1,p}(\Omega,\delta^\alpha)$, $W_0^{1,p}(\Omega,\delta^\alpha)$, $L^q(0,T;L^p(\Omega,\delta^\alpha))$, $L^q(0,T;W^{1,p}(\Omega,\delta^\alpha))$, $L^q(0,T;W_0^{1,p}(\Omega,\delta^\alpha))$.

Definition 1.1. A measurable function u will be called a very weak solution to the problem (P) if $u \in L^\infty(0,T;L^1(\Omega,\delta))$, $|u|^m \in L^1(Q)$ and it satisfies

$$\begin{gathered} -\int_Q u\varphi_t\, dx\, dt - \int_Q |u|^{m-1} u \Delta\varphi\, dx\, dt = \int_Q f\varphi\, dx\, dt + \int_\Omega u_0(x)\varphi(x,0)\, dx, \\ \forall \varphi \in C^\infty(\bar{Q}),\ \varphi = 0 \text{ on } \Sigma,\ \varphi(x,T) = 0. \end{gathered} \tag{1.6}$$

Now we state the main results of this paper.

Theorem 1.1. *If $f \in L^1(Q,\delta)$, $u_0 \in L^1(\Omega,\delta)$, then there exists a very weak solution u to problem (P) such that $u \in L^\infty(0,T;L^1(\Omega,\delta))$, $|u|^m \in L^q(0,T;W_0^{1,q}(\Omega,\delta)) \cap$*

$L^{\bar{q}}(Q,\delta)\cap L^{q}(0,T;L^{q_0}(\Omega))$ with

$$1\le q<\frac{m(N+1)+2}{m(N+1)+1},\quad 1\le \bar{q}<\frac{m(N+1)+2}{m(N+1)},\qquad\qquad (1.7)$$
$$1\le q_0<\frac{mN(N+1)+2N}{mN(N+1)+N-1},$$

and it satisfies

$$\|u\|_{L^\infty(0,T;L^1(\Omega,\delta))}\le C[M+\frac{1}{2}\operatorname{meas}_\delta\Omega],\qquad (1.8)$$

$$\||u|^m\|_{L^q(0,T;W_0^{1,q}(\Omega,\delta))\cap L^{\bar{q}}(Q,\delta)\cap L^q(0,T;L^{q_0}(\Omega))}\qquad (1.9)$$
$$\le C\max\{M^{\frac{2m(q_1-1)}{(m+1)q_1-2}},M^{\frac{[(3m+1)q_1-2(m+1)]}{2[(m+1)q_1-2]}}\},$$

where C is a positive constant depending only on q, $\bar{q}$ and q_0,

$$M=\|f\|_{L^1(Q,\delta)}+\|u_0\|_{L^1(\Omega,\delta)},\qquad (1.10)$$

$$2\le q_1<\frac{2(N+1)}{N-1},\qquad (1.11)$$

which only depends on q, $\bar{q}$, m and N.

Remark 1.1. Theorem 1.1 implies that if $f\in L^1(Q,\delta)$ and $u_0\in L^1(\Omega,\delta)$ are replaced by $f\in M(Q,\delta)$ and $u_0\in M(\Omega,\delta)$, respectively, being the weighted Radon measure space (see also [12]) in Theorem 1.1, the same conclusion holds.

Remark 1.2. By a weighted L^1 contraction estimate for the problem (P) in Theorem 6.15 in [30], we can deduce that the very weak solution u to problem (P) is unique in Theorem 1.1, and also get the estimate (1.8) to hold without the measure of Ω on the right hand side.

Remark 1.3. In this paper, the lower bound $1-\frac{2}{N+1}$ for m is due to the fact $|u|^m\in L^{\bar{q}}(Q,\delta)$, $m\bar{q}\ge 1$, in Theorem 1.1.

Theorem 1.2. *If $f\in L^1(Q,\delta^\alpha)$, $u_0\in L^1(\Omega,\delta^\alpha)$ with*

$$0<\alpha<\frac{-(2mN+2-m)+\sqrt{(2mN+2-m)^2+8m(mN+2)}}{4m},$$

then there exists a very weak solution u to the problem (P) such that $u\in L^\infty(0,T;L^1(\Omega,\delta^\alpha))$, $|u|^m\in L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))\cap L^{\bar{q}}(Q,\delta^\alpha)$ with

$$1\le q<\frac{m(N+\alpha)+2}{m(N+\alpha)+1},\quad 1\le\bar{q}<\frac{m(N+\alpha)+2}{m(N+\alpha)}.\qquad (1.12)$$

Furthermore, $|u|^m\in L^{\tilde{q}}(0,T;W_0^{1,\tilde{q}}(\Omega))$ with

$$1\le\tilde{q}<\min\{\frac{2m(N+\alpha)+4}{2m(N+\alpha)+3},\frac{2[m(N+\alpha)(1-\alpha)+2-\alpha]}{m(N+\alpha)+2}\}\qquad (1.13)$$

and u satisfies

$$\|u\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha))} \le C[M_1 + \frac{1}{2}meas_{\delta^\alpha}\Omega], \tag{1.14}$$

$$\||u|^m\|_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))\cap L^{\bar q}(Q,\delta^\alpha)} \le C\max\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\}, \tag{1.15}$$

$$\||u|^m\|_{L^{\tilde q}(0,T;W_0^{1,\tilde q}(\Omega))} \le CM_1^{\frac12}\left(1+\max\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\}\right), \tag{1.16}$$

where C is a positive constant depending only on q, $\bar q$ and q_0, $M_1 = \|f\|_{L^1(Q,\delta^\alpha)} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}$.

Remark 1.4. If $f \in L^1(Q,\delta^\alpha)$ and $u_0 \in L^1(\Omega,\delta^\alpha)$ are also replaced by $f \in M(Q,\delta^\alpha)$ and $u_0 \in M(\Omega,\delta^\alpha)$ in Theorem 1.2, the same conclusion holds.

Remark 1.5. The upper bound for $\tilde q$ in Theorem 1.2 shows that the fact that α must be strictly smaller than

$$\frac{-(2mN+2-m)+\sqrt{(2mN+2-m)^2+8m(mN+2)}}{4m}$$

implies that $\alpha < 1$.

Theorem 1.3. *Let u be the very weak solution of the problem (P) given in Theorem 1.1, $f \in L^1(0,T;L^1\log L^1(\Omega,\delta))$, $u_0 \in L^1\log L^1(\Omega,\delta)$, where $L^1\log L^1(\Omega,\delta)$ is the Orlicz space generated by the function $|s|\log(1+|s|)$ with weighted function $\delta(x)$. Then $|u|^m \in L^q(0,T;W_0^{1,q}(\Omega,\delta)) \cap L^{\bar q}(Q,\delta) \cap L^q(0,T;L^{q_0}(\Omega))$* with

$$q = \frac{m(N+1)+2}{m(N+1)+1}, \quad \bar q = \frac{m(N+1)+2}{m(N+1)}, \quad q_0 = \frac{mN(N+1)+2N}{mN(N+1)+N-1}. \tag{1.17}$$

Remark 1.6. Theorem 1.3 shows that a limit regularity is achieved if one improves the regularity of the right term f and initial value.

Theorem 1.4. *Let u be the very weak solution of the problem (P) given in Theorem 1.1, $f \in L^p(Q,\delta)$ with*

$$1 < p < \frac{2m(N+2)+2}{m(N+3)+2} \tag{1.18}$$

and $u_0 = 0$. Then $|u|^m \in L^q(0,T;W_0^{1,q}(\Omega,\delta)) \cap L^{\bar q}(Q,\delta) \cap L^q(0,T;L^{q_0}(\Omega))$ with

$$\begin{gathered} q = \frac{[m(N+1)+2]p}{m(N+2-p)+1}, \quad \bar q = \frac{[m(N+1)+2]p}{m(N+3-2p)}, \\ q_0 = \frac{[mN(N+1)+2N]p}{m(N+1)(N+2-2p)+(N+1-2p)}. \end{gathered} \tag{1.19}$$

Remark 1.7. The lower bound for p in Theorem 1.4 is due to the fact that q must not be smaller than 1. The upper bound for p implies $q < 2$.

Theorem 1.5. Let u be the very weak solution of the problem (P) given in Theorem 1.1, $f \in L^p(Q,\delta)$ with

$$\frac{2m(N+2)+2}{m(N+3)+2} < p < \frac{N+3}{2} \tag{1.20}$$

and $u_0 \in L^d(\Omega,\delta)$ with

$$d = m+1. \tag{1.21}$$

Then $|u|^m \in L^2(0,T;W_0^{1,2}(\Omega,\delta)) \cap L^{\bar q}(Q,\delta) \cap L^2(0,T;L^{q_0}(\Omega))$ with

$$\bar q < \frac{2m(N+2)+2}{m(N+1)}, \quad q_0 < \frac{2N}{N-1}. \tag{1.22}$$

Theorem 1.6. Let u be the very weak solution of the problem (P) given in Theorem 1.1, $f \in L^p(Q,\delta)$ with

$$p > \frac{N+3}{2} \tag{1.23}$$

and $u_0 \in L^\infty(Q)$. Then $|u|^m \in L^2(0,T;W_0^{1,2}(\Omega,\delta)) \cap L^\infty(Q)$.

Theorem 1.7. *Let u be the very weak solution of the problem (P) given in Theorem 1.1, $f \in L^p(Q,\delta)$ with*

$$p = \frac{N+3}{2} \tag{1.24}$$

and $u_0 \in L^\infty(Q)$. Then $|u|^m \in L^2(0,T;W_0^{1,2}(\Omega,\delta)) \cap L^{\bar q}(Q,\delta) \cap L^2(0,T;L^{q_0}(\Omega))$ with

$$1 \le \bar q < +\infty, \quad 1 \le q_0 < +\infty. \tag{1.25}$$

Remark 1.8. Since $L^p(Q,\delta) \subset L^1(Q,\delta^{\frac{1}{p}})$, the conclusion in Theorem 1.2 still holds under the assumpations of Theorems 1.4–1.7, respectively, and $\alpha = \frac{1}{p}$.

Remark 1.9. Theorem 1.7 gives the regularity result in the limit case $p = \frac{N+3}{2}$.

This paper is organized as follows. In Section 2, some preliminary results and the existence of approximate solutions will be given; In Section 3, we will give a priori estimates about the approximate solutions; the proofs of the main results of this paper will be finished in Section 4.

2. Some preliminary results and existence of approximate solutions

Before we prove Theorems 1.1–1.7, we need some preliminary results. Firstly, let us recall the weighted Orlicz spaces (see [1, 18]).

Definition 2.1. Assume that Φ is a N-function and ρ is an integrable and almost everywhere positive function in Ω. The weighted Orlicz class $\mathcal{L}_\Phi(\Omega,\rho)$ (resp. the weighted Orlicz space $L_\Phi(\Omega,\rho)$) is defined as the set of (equivalence class of) measurable functions v on Ω such that $\int_\Omega \Phi(v(x))\rho(x)\,dx < +\infty$ (resp. $\int_\Omega \Phi(\frac{v(x)}{\lambda})\rho(x)\,dx < +\infty$ for some $\lambda > 0$). Weighted Orlicz space $L_\Phi(\Omega,\rho)$ is a Banach space under the norm

$$\|v\|_{L_\Phi(\Omega,\rho)} = \inf\left\{\lambda > 0\colon \int_\Omega \Phi\left(\frac{v(x)}{\lambda}\right)\rho(x)\,dx \le 1\right\}, \tag{2.1}$$

and $\mathcal{L}_\Phi(\Omega,\rho)$ is a convex subset of $L_\Phi(\Omega,\rho)$.

Remark 2.1. In this paper, we take $\Phi(s) = |s|\log(1+|s|)$ and $\rho(x) = \delta(x)$.

We also recall a weighted Sobolev space imbedding theorem.

Lemma 2.1. [17, Theorem 8.7 and Theorem 8.9] *Let $1 \le q \le r < +\infty$, β and γ are two real numbers. If*

$$\frac{1}{N} \ge \frac{1}{q} - \frac{1}{r} \tag{2.2}$$

and

$$\beta \neq q-1, \quad \frac{N+\gamma}{r}+1 \geq \frac{N+\beta}{q}, \tag{2.3}$$

$$\beta = q-1, \quad \frac{N+\gamma}{r}+1 > \frac{N+\beta}{q}, \tag{2.4}$$

then the weighted Sobolev space $W_0^{1,q}(\Omega,\delta^\beta)$ *is a continuous imbedding to the weighted Lebesgue space* $L^r(\Omega,\delta^\gamma)$*, that is,*

$$W_0^{1,q}(\Omega,\delta^\beta) \circlearrowleft L^r(\Omega,\delta^\gamma). \tag{2.5}$$

If the inequalities in (2.2) and (2.3) are strict, then $W_0^{1,q}(\Omega,\delta^\beta)$ *is a compact imbedding to* $L^r(\Omega,\delta^\gamma)$*, that is,*

$$W_0^{1,q}(\Omega,\delta^\beta) \circlearrowleft\circlearrowleft L^r(\Omega,\delta^\gamma). \tag{2.6}$$

To obtain a priori estimates of solutions, we need also the following lemmas. From Lemma A.2. in [4] and Lemma 2.4 in [5], we have the following result.

Lemma 2.2. *Let* $1 \leq q < \hat{q} < +\infty$. *Suppose that there exists a positive constants* M *independent of* k *such that*

$$\text{meas}_{\delta^\alpha}\{|u|>k\} = \mu_{\delta^\alpha}(k) = \int_{\{|u|>k\}} \delta^\alpha \, dx\, dt \leq M k^{-\hat{q}}, \quad \forall k>0. \tag{2.7}$$

Then

$$\int_Q |u|^q \delta^\alpha \, dx\, dt \leq \left(\frac{\hat{q}}{q}\right)^{\frac{q}{\hat{q}}} \frac{\hat{q}}{\hat{q}-q} (\text{meas}_{\delta^\alpha} Q)^{\frac{\hat{q}-q}{\hat{q}}} M^{\frac{q}{\hat{q}}}. \tag{2.8}$$

Proof. Given $\lambda>0$, we have

$$\int_Q |u|^q \delta^\alpha \, dx\, dt \leq \lambda^q \, \text{meas}_{\delta^\alpha} Q + \int_{\{|u|>\lambda\}} |u|^q \delta^\alpha \, dx\, dt.$$

However, by using Hardy–Littlewood inequality, we have

$$\begin{aligned}\int_{\{|u|>\lambda\}} |u|^q \delta^\alpha \, dx\, dt &= -\int_\lambda^{+\infty} k^q \, d\mu_{\delta^\alpha}(k) = \lambda^q \mu_{\delta^\alpha}(\lambda) + q\int_\lambda^{+\infty} k^{q-1}\mu_{\delta^\alpha}(k)\, dk \\ &\leq \lambda^{q-\hat{q}} M + Mq \int_\lambda^{+\infty} k^{q-1-\hat{q}}\, dk \leq \frac{\hat{q}}{\hat{q}-q} M\lambda^{q-\hat{q}}.\end{aligned}$$

Hence

$$\int_Q |u|^q \delta^\alpha \, dx\, dt \leq \lambda^q \, \text{meas}_{\delta^\alpha} Q + \frac{\hat{q}}{\hat{q}-q} M\lambda^{q-\hat{q}}.$$

Minimization of the right-hand side of the above inequality in λ and setting $\lambda = (\frac{\hat{q}}{q})^{\frac{1}{\hat{q}}} (\text{meas}_{\delta^\alpha}\Omega)^{-\frac{1}{\hat{q}}} M^{\frac{1}{\hat{q}}}$, we get (2.8). □

The following lemma is a revised version of Proposition 3.1 in [13].

Lemma 2.3. *Assume that* $v \in L^\infty(0,T; L^r(\Omega,\delta^\alpha)) \cap L^q(0,T; W_0^{1,q}(\Omega,\delta^\alpha))$ *with* $r \geq 1$, $1 \leq q < N+\alpha$, *where* $\alpha > 0$. *Then* $v \in L^s(Q,\delta^\alpha)$ *and there exists a postive constant* C *depending only on* r, q, α *and* $\partial\Omega$ *such that*

(i) if $\alpha \neq q-1$, then $s = \frac{q(N+\alpha+r)}{N+\alpha}$ and

$$\int_Q |v|^s \delta^\alpha \, dx\, dt \le C\|v\|^{\frac{qr}{N+\alpha}}_{L^\infty(0,T;L^r(\Omega,\delta^\alpha))}\|v\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))};$$

(ii) if $\alpha = q-1$, then $s = r+q-\frac{qr}{q_1}$ and

$$\int_Q |v|^s \delta^\alpha \, dx\, dt \le C\|v\|^{(1-\frac{q}{q_1})r}_{L^\infty(0,T;L^r(\Omega,\delta^\alpha))}\|v\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))},$$

where $q \le q_1 < \frac{(N+q-1)q}{N-1}$.

Proof. (i) Let $s = \theta r + (1-\theta)q_1$, where $q_1 = \frac{(N+\alpha)q}{N+\alpha-q}$, $0<\theta<1$. By using (2.2) and (2.3) in Lemma 2.1, for a.e. $t \in (0,T)$ we have

$$\|v(t)\|_{L^{q_1}(\Omega,\delta^\alpha)} \le C\|v(t)\|_{W_0^{1,q}(\Omega,\delta^\alpha)},$$

where C is a postive constant independent of t. Hölder's inequality implies that

$$\begin{aligned}\int_Q |v|^s\delta^\alpha\,dx\,dt &= \int_Q |v|^{\theta r+(1-\theta)q_1}\delta^{\theta\alpha+(1-\theta)\alpha}\,dx\,dt\\ &\le \int_0^T \left(\int_\Omega |v|^r\delta^\alpha\,dx\right)^\theta \left(\int_\Omega |v|^{q_1}\delta^\alpha\,dx\right)^{1-\theta} dt\\ &\le \sup_{0<t<T}\left(\int_\Omega |v(t)|^r\delta^\alpha\,dx\right)^\theta \int_0^T \left(\int_\Omega |v|^{q_1}\delta^\alpha\,dx\right)^{1-\theta} dt\\ &\le C\|v\|^{\theta r}_{L^\infty(0,T;L^r(\Omega,\delta^\alpha))}\int_0^T \|v\|^{q_1(1-\theta)}_{W_0^{1,q}(\Omega,\delta^\alpha)}\,dt.\end{aligned}$$

Let $q_1(1-\theta) = q$. Then $\theta = \frac{q}{N+\alpha}$ and $s = \frac{q(N+\alpha+r)}{N+\alpha}$. Thus we obtain

$$\int_Q |v|^s \delta^\alpha \, dx\, dt \le C\|v\|^{\frac{qr}{N+\alpha}}_{L^\infty(0,T;L^r(\Omega,\delta^\alpha))}\|v\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))}.$$

(ii) As $\alpha = q-1$, by using (2.2) and (2.3) in Lemma 2.1, for a.e. $t \in (0,T)$ we have

$$\|v(t)\|_{L^{q_1}(\Omega,\delta^\alpha)} \le C\|v(t)\|_{W_0^{1,q}(\Omega,\delta^\alpha)},$$

where $q_1 < \frac{(N+q-1)q}{N-1}$. Processing the proof of (i), we only take $\theta = \frac{q_1-q}{q_1}$, then $s = r+q-\frac{qr}{q_1}$ and

$$\int_Q |v|^s \delta^\alpha \, dx\, dt \le C\|v\|^{(1-\frac{q}{q_1})r}_{L^\infty(0,T;L^r(\Omega,\delta^\alpha))}\|v\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))}.$$ □

Lemma 2.4. [11, Lemma 2] *There is a function $\varphi_1 \in W^{2,p}(\Omega)\cap W_0^{1,2}(\Omega)$ and $\lambda_1 > 0$ for all $p \in (1,+\infty)$ satisfying*

$$\begin{cases}-\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega,\\ \varphi_1 = 0 & \text{on } \partial\Omega,\end{cases}$$

and there are two positive constants c_1 and c_2 such that

$$c_1\delta(x) \le \varphi_1(x) \le c_2\delta(x), \quad \forall x \in \Omega. \tag{2.9}$$

For any given $n > 0$, let

$$T_n(s) = \begin{cases} n & \text{if} |s| < n, \\ n\frac{s}{|s|} & \text{if } |s| \geq n. \end{cases} \tag{2.10}$$

In order to discuss problem (P), we need consider the approximate problem

$$(P_n) \qquad \begin{cases} \frac{\partial u_n}{\partial t} - \Delta(|u_n|^{m-1}u_n) = f_n & \text{in } Q, \\ u_n = 0 & \text{on } \Sigma, \\ u_n(x,0) = u_{0n} & \text{in } \Omega, \end{cases}$$

where $f_n = T_n(f),\ u_{0n} = T_n(u_0)$, T_n is defined in (2.10).

Lemma 2.5. *For any given $n > 0$, the approximate problem (P_n) has a unique weak solution $u_n \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(Q)$ such that $u_{nt} \in L^2(0,T; W^{-1,2}(\Omega))$, $|u_n|^{m-1}u_n \in L^2(0,T; W_0^{1,2}(\Omega))$ and satisfies*

$$(P') \quad \begin{cases} \int_\Omega u_{nt}v + D(|u_n|^{m-1}u_n)Dv\,dx = \int_\Omega f_n v\,dx, & \forall v \in W_0^{1,2}(\Omega),\ \text{a.e. } t \in (0,T), \\ u_n(x,0) = u_{0n} & \text{in } \Omega. \end{cases}$$

Proof. To prove the existence and uniqueness of a weak solution to the approximate problem (P_n), we consider first the following approximate problem (P_{nk}):

$$(P_{nk}) \qquad \begin{cases} \frac{\partial u_{nk}}{\partial t} - \operatorname{div}(m|T_k((|u_{nk}| - \frac{1}{k})_+ + \frac{1}{k}) \operatorname{sgn} u_{nk}|^{m-1}\nabla u_{nk}) = f_n & \text{in } Q, \\ u_{nk} = 0 & \text{on } \Sigma, \\ u_{nk}(x,0) = u_{0n} & \text{in } \Omega, \end{cases}$$

where $k > 1$, T_k can be seen in (2.10).

Applying the results in [21], for every k we find that the problem (P_{nk}) has a unique weak solution $u_{nk} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$ such that $u_{nt} \in L^2(0,T; W^{-1,2}(\Omega))$. By the regularity theory in [20], we also deduce that $u_{nk} \in L^\infty(Q)$.

Firstly, we will obtain an estimate to $\|u_{nk}\|_{L^\infty(Q)}$. Let $v_{nk} = e^{-t}u_{nk}$. Then the problem (P_{nk}) can be written as

$$(P'_{nk}) \quad \begin{cases} \frac{\partial v_{nk}}{\partial t} - \operatorname{div}(m|T_k((|e^t v_{nk}| - \frac{1}{k})_+ + \frac{1}{k}) \operatorname{sgn} v_{nk}|^{m-1}\nabla v_{nk}) + v_{nk} = e^{-t}f_n & \text{in } Q, \\ v_{nk} = 0 & \text{on } \Sigma, \\ v_{nk}(x,0) = u_{0n} & \text{in } \Omega, \end{cases}$$

Setting $k_0 = \max(\|u_{0n}\|_{L^\infty(\Omega)}, \|f_n\|_{L^\infty(Q)})$, we can take $(v_{nk} - k_0)_+$ as a test function of the problem (P'_{nk}), and we have

$$\begin{aligned} &\int_Q v_{nkt}(v_{nk} - k_0)_+\,dx\,dt \\ &\quad + m\int_Q |T_k((|e^t v_{nk}| - \frac{1}{k})_+ + \frac{1}{k}) \operatorname{sgn} v_{nk}|^{m-1} Dv_{nk} D(v_{nk} - k_0)_+\,dx\,dt \\ &\quad + \int_Q v_{nk}(v_{nk} - k_0)_+\,dx\,dt \\ &= \int_Q e^{-t} f_n (v_{nk} - k_0)_+\,dx\,dt. \end{aligned} \tag{2.11}$$

By calculating it, we obtain

$$
\begin{aligned}
\int_Q v_{nkt}(v_{nk}-k_0)_+\,dx\,dt &= \frac{1}{2}\int_\Omega (v_{nk}(T)-k_0)_+^2\,dx\,dt - \frac{1}{2}\int_\Omega (u_{0n}-k_0)_+^2\,dx\,dt \\
&= \frac{1}{2}\int_\Omega (v_{nk}(T)-k_0)_+^2\,dx\,dt \geq 0
\end{aligned} \tag{2.12}
$$

and

$$
\begin{aligned}
& m\int_Q |T_k((|e^t v_{nk}|-\frac{1}{k})_+ + \frac{1}{k})\operatorname{sgn} v_{nk}|^{m-1} Dv_{nk} D(v_{nk}-k_0)_+\,dx\,dt \\
&= m\int_Q |T_k((|e^t v_{nk}|-\frac{1}{k})_+ + \frac{1}{k})\operatorname{sgn} v_{nk}|^{m-1} |D(v_{nk}-k_0)_+|^2\,dx\,dt \geq 0.
\end{aligned} \tag{2.13}
$$

Now (2.11), (2.12) and (2.13) imply that

$$
\int_Q v_{nk}(v_{nk}-k_0)_+\,dx\,dt \leq \int_Q f_n(v_{nk}-k_0)_+\,dx\,dt. \tag{2.14}
$$

Adding $-\int_Q k_0(v_{nk}-k_0)_+\,dx\,dt$ to the both sides of (2.13) we get

$$
\int_Q (v_{nk}-k_0)_+^2\,dx\,dt \leq \int_Q (f_n-k_0)(v_{nk}-k_0)_+\,dx\,dt \leq 0. \tag{2.15}
$$

Thus we can deduce

$$
v_{nk} \leq k_0, \quad \text{a.e. in } Q. \tag{2.16}
$$

Replacing v_{nk} by $-v_{nk}$ in the above proof, we can get

$$
-v_{nk} \leq k_0, \quad \text{a.e. in } Q. \tag{2.17}
$$

Hence we have

$$
|v_{nk}| \leq k_0, \quad \text{a.e. in } Q. \tag{2.18}
$$

Thus we get

$$
\|u_{nk}\|_{L^\infty(Q)} \leq e^T k_0. \tag{2.19}
$$

Taking $k > e^T k_0 + 1$ in problem (P_{nk}), then problem (P_{nk}) can be written as

$$
(P''_{nk}) \qquad \begin{cases} \frac{\partial u_{nk}}{\partial t} - \operatorname{div}(m|((|u_{nk}|-\frac{1}{k})_+ + \frac{1}{k})\operatorname{sgn} u_{nk}|^{m-1}\nabla u_{nk}) = f_n & \text{in } Q, \\ u_{nk} = 0 & \text{on } \Sigma, \\ u_{nk}(x,0) = u_{0n} & \text{in } \Omega. \end{cases}
$$

Let $\psi(s) = \int_0^s |((|\xi|-\frac{1}{k})_+ + \frac{1}{k})\operatorname{sgn}\xi|^{m-1}\,d\xi$. Using $\psi(u_{nk})$ as a test function of the problem (P''_{nk}), we have

$$
\begin{aligned}
& \int_Q u_{nkt}\psi(u_{nk})\,dx\,dt + m\int_Q |((|u_{nk}|-\tfrac{1}{k})_+ + \tfrac{1}{k})\operatorname{sgn} u_{nk}|^{m-1} Du_{nk} D\psi(u_{nk})\,dx\,dt \\
&= \int_Q f_n\psi(u_{nk})\,dx\,dt.
\end{aligned} \tag{2.20}
$$

However,

$$
\int_Q u_{nkt}\psi(u_{nk})\,dx\,dt = \int_\Omega\int_0^{u_{nk}(T)} \psi(\xi) - \int_\Omega\int_0^{u_{0n}} \psi(\xi) \geq -\frac{\|u_{0n}\|_{L^\infty(\Omega)}^{m+1}}{m(m+1)}, \tag{2.21}
$$

$$
\begin{aligned}
& m\int_Q |((|u_{nk}|-\tfrac{1}{k})_+ + \tfrac{1}{k})\operatorname{sgn} u_{nk}|^{m-1} Du_{nk} D\psi(u_{nk})\,dx\,dt \\
& = m\int_Q |D\psi(u_{nk})|^2\,dx\,dt \\
& = m\int_Q |((|u_{nk}|-\tfrac{1}{k})_+ + \tfrac{1}{k})\operatorname{sgn} u_{nk}|^{2m-2}|Du_{nk}|^2\,dx\,dt
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
\int_Q f_n\psi(u_{nk})\,dx\,dt &\le \frac{1}{m}\|f_n\|_{L^\infty(Q)}\|u_{nk}\|^m_{L^\infty(Q)} \operatorname{meas} Q \\
&\le \frac{(e^T k_0)^m}{m}\|f_n\|_{L^\infty(Q)} \operatorname{meas} Q.
\end{aligned} \tag{2.23}
$$

Now (2.20)–(2.23) yield

$$
\int_Q |D\psi(u_{nk})|^2\,dx\,dt \le \frac{\|u_{0n}\|^{m+1}_{L^\infty(\Omega)}}{m^2(m+1)} + \frac{(e^T k_0)^m}{m^2}\|f_n\|_{L^\infty(Q)} \operatorname{meas} Q. \tag{2.24}
$$

Hence,

$$
\begin{aligned}
& \int_Q |Du_{nk}|^2\,dx\,dt \\
& = \int_Q |((|u_{nk}|-\tfrac{1}{k})_+ + \tfrac{1}{k})\operatorname{sgn} u_{nk}|^{2m-2}|Du_{nk}|^2 \\
& \quad \cdot |((|u_{nk}|-\tfrac{1}{k})_+ + \tfrac{1}{k})\operatorname{sgn} u_{nk}|^{2-2m}\,dx\,dt \\
& \le (\|u_{nk}\|_{L^\infty(Q)}+1)^{2-2m}\int_Q |D\psi(u_{nk})|^2\,dx\,dt \\
& \le (e^T k_0+1)^{2-2m}\left(\frac{\|u_{0n}\|^{m+1}_{L^\infty(\Omega)}}{m^2(m+1)} + \frac{(e^T k_0)^m}{m^2}\|f_n\|_{L^\infty(Q)} \operatorname{meas} Q\right).
\end{aligned} \tag{2.25}
$$

Inequality (2.24) and the first equation of (P''_{nk}) imply that

$$
\|u_{nkt}\|_{L^2(0,T;W^{-1,2}(\Omega))} \le C, \tag{2.26}
$$

where C is a positive constant depending only on $\|u_{0n}\|_{L^\infty(\Omega)}$, $\|f_n\|_{L^\infty(Q)}$, m, T and $\operatorname{meas} Q$.

Thus there exists a subsequence (still denoted by $\{u_{nk}\}$) and a function $u_n \in L^\infty(Q)\cap L^2(0,T;W_0^{1,2}(\Omega))$ such that as k goes to infinity,

$$
u_{nk} \longrightarrow u_n \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega)), \tag{2.27}
$$

$$
u_{nkt} \longrightarrow u_{nt} \text{ weakly in } L^2(0,T;W^{-1,2}(\Omega)), \tag{2.28}
$$

$$
u_{nk} \longrightarrow u_n \text{ weak* in } L^\infty(Q). \tag{2.29}
$$

Using (2.27), (2.28) and the compactness arguments in [25], we have

$$
u_{nk} \longrightarrow u_n \text{ strongly in } L^2(Q) \tag{2.30}
$$

and

$$
u_{0n} = u_{nk}(0) \longrightarrow u_n(0) \text{ strongly in } L^2(\Omega). \tag{2.31}
$$

The fact (2.29) yields

$$u_{nk} \longrightarrow u_n \text{ a.e. in } Q, \tag{2.32}$$

and (2.31) yields

$$\psi(u_{nk}) \longrightarrow \frac{1}{m}|u_n|^{m-1}u_n \text{ a.e. in } Q. \tag{2.33}$$

Hence we can deduce that

$$D\psi(u_{nk}) \longrightarrow \frac{1}{m}D(|u_n|^{m-1}u_n) \text{ weakly in } L^2(Q). \tag{2.34}$$

Let k go to infinity in the problem (P_{nk}) and by using (2.27)–(2.29), (2.31) and (2.34), we can obtain the existence of a solution to the problem (P_n). The uniqueness of a solution to the problem (P_n) is easily proved. Thus Lemma 2.5 is completed. □

3. A priori estimates about the approximate problem (P_n)

In this section, by using the techniques introduced in [7] and [8] (see also [3]), we obtain a priori estimates on u_n as follows.

Lemma 3.1. *Assume that $f \in L^1(Q,\delta)$, $u_0 \in L^1(\Omega,\delta)$. Then every weak solution u_n of the problem (P_n) satisfies*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta))} \le C[M + \tfrac{1}{2}\operatorname{meas}_\delta \Omega], \tag{3.1}$$

$$\begin{aligned} &\||u_n|^m\|_{L^q(0,T;W_0^{1,q}(\Omega,\delta))\cap L^{\bar q}(Q,\delta)\cap L^q(0,T;L^{q_0}(\Omega))} \\ &\quad \le C\max\left\{M^{\frac{2m(q_1-1)}{(m+1)q_1-2}}, M^{\frac{[(3m+1)q_1-2(m+1)]}{2[(m+1)q_1-2]}}\right\}, \end{aligned} \tag{3.2}$$

$$\||T_k(u_n)|^{m-1}T_k(u_n)\|_{L^2(0,T;W_0^{1,2}(\Omega,\delta))} \le Ck^m, \tag{3.3}$$

where C is a positive constant depending only on q, $\bar q$ and q_0, $M = \|f\|_{L^1(Q,\delta)} + \|u_0\|_{L^1(\Omega,\delta)}$, q, $\bar q$, q_0 and q_1 are seen in (1.7) and (1.11), respectively.

Proof. Let $\psi(s) = \min\{|s|, 1\}\operatorname{sgn} s, \forall s \in \mathbf{R}$. Then we get

$$\Psi(s) = \int_0^s \psi(\xi)\,d\xi = \begin{cases} \frac{s^2}{2} & \text{if } |s| \le 1, \\ |s| - \frac{1}{2} & \text{if } |s| > 1, \end{cases}$$

and $|s| - \frac{1}{2} \le \Psi(s) \le |s|$. Taking $v = \psi(u_n)\varphi_1$ in (P') and integrating it over $(0,\tau)$, $\tau \in (0,T)$, where φ_1 denotes the first eigenfunction associated to the Laplacian operator which is defined in Lemma 2.4, we have

$$\int_0^\tau \int_\Omega u_{nt}\psi(u_n)\varphi_1 + D(|u_n|^{m-1}u_n)D(\psi(u_n)\varphi_1)\,dx\,dt = \int_0^\tau \int_\Omega f_n\psi(u_n)\varphi_1\,dx\,dt. \tag{3.4}$$

In the following we will estimate every term in (3.4):

$$\begin{aligned} \int_0^\tau \int_\Omega u_{nt}\psi(u_n)\varphi_1\,dx\,dt &= \int_\Omega \Psi(u_n(\tau))\varphi_1 dx - \int_\Omega \Psi(u_{0n})\varphi_1\,dx \\ &\ge \int_\Omega (|u_n(\tau)| - \frac{1}{2})\varphi_1 dx - \int_\Omega |u_{0n}|\varphi_1\,dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned}(3.6)\qquad &\int_0^\tau\int_\Omega D(|u_n|^{m-1}u_n)D(\psi(u_n))\varphi_1\,dx\,dt\\ &= m\int_0^\tau\int_\Omega|\psi(u_n)|^{m-1}|D(\psi(u_n))|^2\varphi_1\,dx\,dt\geq 0,\end{aligned}$$

$$\begin{aligned}(3.7)\qquad &\int_0^\tau\int_\Omega D(|u_n|^{m-1}u_n)\psi(u_n)D\varphi_1\,dx\,dt\\ &= \int_0^\tau\int_\Omega D\varphi_1 D\int_0^{|u_n|^{m-1}u_n}\psi(|s|^{\frac{1-m}{m}}s)\,ds\,dx\,dt\\ &= -\int_0^\tau\int_\Omega \Delta\varphi_1\int_0^{|u_n|^{m-1}u_n}\psi(|s|^{\frac{1-m}{m}}s)\,ds\,dx\,dt\geq 0.\end{aligned}$$

Now (3.4)–(3.7) yield

$$(3.8)\qquad \int_\Omega(|u_n(\tau)|-\frac{1}{2})\varphi_1\,dx-\int_\Omega|u_{0n}|\varphi_1\,dx\leq\int_0^\tau\int_\Omega|f_n|\varphi_1\,dx\,dt.$$

Thus we get

$$(3.9)\qquad \|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta))}\leq C(\|f_n\|_{L^1(Q,\delta)}+\|u_{0n}\|_{L^1(\Omega,\delta)}+\tfrac{1}{2}\operatorname{meas}_\delta\Omega).$$

Let $|u_n|^{m-1}u_n=w_n$. For a given $k>0$, taking $v=T_k(w_n)\varphi_1$ in (P') and integrating it over $(0,\tau)$, $\tau\in(0,T)$, we have

$$\begin{aligned}(3.10)\qquad &\int_{Q_\tau}u_{nt}T_k(w_n)\varphi_1\,dx\,dt+\int_{Q_\tau}|DT_k(w_n)|^2\varphi_1\,dx\,dt+\int_{Q_\tau}Dw_nT_k(w_n)D\varphi_1\,dx\,dt\\ &=\int_{Q_\tau}f_nT_k(w_n)\varphi_1\,dx\,dt.\end{aligned}$$

By using integration by parts for the third term on the left side of (3.10) and Lemma 2.4, we have

$$\begin{aligned}(3.11)\qquad \int_{Q_\tau}Dw_nT_k(w_n)D\varphi_1\,dx &= -\int_{Q_\tau}\Delta\varphi_1\int_0^{w_n}T_k(s)\,dx\,dt\\ &\geq\frac{\lambda_1}{2}\int_{Q_\tau}\varphi_1|T_k(w_n)|^2\,dx\,dt.\end{aligned}$$

We also get

$$\begin{aligned}(3.12)\qquad &\int_{Q_\tau}u_{nt}T_k(w_n)\varphi_1\,dx\,dt\\ &=\int_\Omega\varphi_1\int_0^{u_n(\tau)}T_k(|s|^{m-1}s)\,ds\,dx-\int_\Omega\varphi_1\int_0^{u_{0n}}T_k(|s|^{m-1}s)\,ds\,dx\\ &\geq\frac{1}{m+1}\int_\Omega|T_k(w_n(\tau))|^{\frac{m+1}{m}}\varphi_1\,dx-k\int_\Omega|u_{0n}|\varphi_1\,dx.\end{aligned}$$

From (3.10)–(3.12) it follows that

$$\begin{aligned}(3.13)\qquad &\operatorname*{ess\,sup}_{\tau\in(0,T)}\int_\Omega|T_k(w_n(\tau))|^{\frac{m+1}{m}}\delta(x)\,dx\\ &\quad+\int_Q(|DT_k(w_n)|^2\delta(x)+|T_k(w_n)|^2\delta(x))\,dx\,dt\\ &\leq Ck(\|f\|_{L^1(Q,\delta)}+\|u_0\|_{L^1(\Omega,\delta)}),\end{aligned}$$

where C is a positive constant independent of k.

By using Lemma 2.3(ii) (here $\alpha = 1,\ v = T_k(w_n),\ r = \frac{m+1}{m},\ q = 2,\ s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1}$), we obtain

$$\int_Q |T_k(w_n)|^s \delta\, dx\, dt \le C(kM)^{2-\frac{2}{q_1}}, \tag{3.14}$$

where

$$2 \le q_1 < \frac{2(N+1)}{N-1}. \tag{3.15}$$

Thus we can deduce that

$$\text{meas}_\delta\{|w_n| > k\} = \int_{\{|w_n|>k\}} \delta\, dx\, dt \le CM^{2-\frac{2}{q_1}} k^{\frac{2}{mq_1} - \frac{m+1}{m}}. \tag{3.16}$$

By using Lemma 2.2 (here $\alpha = 1,\ u = w_n,\ \hat{q} = \frac{m+1}{m} - \frac{2}{mq_1},\ q = \bar{q}$, M is replaced by $CM^{2-\frac{2}{q_1}}$), it follows that

$$\int_Q |w_n|^{\bar{q}} \delta\, dx\, dt \le (\frac{\hat{q}}{\bar{q}})^{\frac{\bar{q}}{\hat{q}}} \frac{\hat{q}}{\hat{q} - \bar{q}} (\text{meas}_\delta\, Q)^{\frac{\hat{q}-\bar{q}}{\hat{q}}} (CM^{2-\frac{2}{q_1}})^{\frac{\bar{q}}{\hat{q}}} = C_1 M^{\frac{2m(q_1-1)\bar{q}}{(m+1)q_1-2}}, \tag{3.17}$$

where $\bar{q} < \hat{q} = \frac{m+1}{m} - \frac{2}{mq_1} < \frac{m(N+1)+2}{m(N+1)},\ C_1 = (\frac{\hat{q}}{\bar{q}})^{\frac{\bar{q}}{\hat{q}}} \frac{\hat{q}}{\hat{q}-\bar{q}} (\text{meas}_\delta\, Q)^{\frac{\hat{q}-\bar{q}}{\hat{q}}} C^{\frac{\bar{q}}{\hat{q}}}$.

For any given $h > 0$, (3.13) yields

$$\text{meas}_\delta\{|DT_k(w_n)| > \frac{h}{2}\} \le CMkh^{-2}. \tag{3.18}$$

From (3.16) and (3.18) it follows that

$$\begin{aligned} &\text{meas}_\delta\{|Dw_n| > h\} \\ &\le \text{meas}_\delta\{|Dw_n - DT_k(w_n)| > \tfrac{h}{2}\} + \text{meas}_\delta\{|DT_k(w_n)| > \tfrac{h}{2}\} \\ &\le \text{meas}_\delta\{|w_n| > k\} + \text{meas}_\delta\{|DT_k(w_n)| > \tfrac{h}{2}\} \\ &\le CM^{2-\frac{2}{q_1}} k^{\frac{2}{mq_1} - \frac{m+1}{m}} + CMkh^{-2}. \end{aligned} \tag{3.19}$$

Minimizing (3.19) in k and setting $k = (\frac{m+1}{m} - \frac{2}{mq_1})^{\frac{mq_1}{(2m+1)q_1-2}} M^{\frac{m(q_1-2)}{(2m+1)q_1-2}} h^{\frac{2mq_1}{(2m+1)q_1-2}}$, we get

$$\text{meas}_\delta\{|Dw_n| > h\} \le CM^{\frac{m(q_1-2)}{(2m+1)q_1-2}+1} h^{-\frac{2[(m+1)q_1-2]}{(2m+1)q_1-2}}. \tag{3.20}$$

By using Lemma 2.2 (here $\alpha = 1,\ u = Dw_n,\ \hat{q} = \frac{2[(m+1)q_1-2]}{(2m+1)q_1-2}$, M is replaced by $CM^{\frac{m(q_1-2)}{(2m+1)q_1-2}+1}$), it follows that

$$\int_Q |Dw_n|^q \delta\, dx\, dt \le CM^{\frac{[(3m+1)q_1-2(m+1)]q}{2[(m+1)q_1-2]}}, \tag{3.21}$$

where $q < \frac{2[(m+1)q_1-2]}{(2m+1)q_1-2} < \frac{m(N+1)+2}{m(N+1)+1}$.

For any given $1 \le q < \frac{m(N+1)+2}{m(N+1)+1}$ and $1 \le \bar{q} < \frac{m(N+1)+2}{m(N+1)}$, (3.15) shows that we can choose q_1, which only depends on $q,\ \bar{q},\ m$ and N, such that $q < \frac{2[(m+1)q_1-2]}{(2m+1)q_1-2}, \bar{q} <$

$\frac{m+1}{m} - \frac{2}{mq_1}$ hold. Furthermore, (3.17) and (3.21) also show that there is a positive constant depending only $q,\ \bar{q}, N$ and $\operatorname{meas}_\delta Q$ such that

$$\left(\int_Q |w_n|^{\bar{q}}\delta\,dx\,dt\right)^{\frac{1}{\bar{q}}} \le CM^{\frac{2m(q_1-1)}{(m+1)q_1-2}} \tag{3.22}$$

and

$$\left(\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\right)^{\frac{1}{q}} \le C\max\left\{M^{\frac{2m(q_1-1)}{(m+1)q_1-2}}, M^{\frac{[(3m+1)q_1-2(m+1)]}{2[(m+1)q_1-2]}}\right\}. \tag{3.23}$$

Taking $r = q_0 = \frac{Nq}{N+1-q},\ q < \frac{2[(m+1)q_1-2]}{(2m+1)q_1-2},\ \gamma = 0,\ \beta = 1$ in Lemma 2.1, and by using (3.23) we have

$$\begin{aligned}\left(\int_0^T\left(\int_\Omega |w_n|^{q_0}\,dx\right)^{\frac{q}{q_0}}dt\right)^{\frac{1}{q}} &\le C\left(\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\,dt\right)\\ &\le C\max\left\{M^{\frac{2m(q_1-1)}{(m+1)q_1-2}}, M^{\frac{[(3m+1)q_1-2(m+1)]}{2[(m+1)q_1-2]}}\right\}.\end{aligned} \tag{3.24}$$

For any given $k > 0$, let $v = |T_k(u_n)|^{m-1}T_k(u_n)\varphi_1$ in (P') and integrating it over $(0,\tau),\ \tau \in (0,T)$, we have

$$\begin{aligned}&\int_0^\tau\int_\Omega u_{nt}|T_k(u_n)|^{m-1}T_k(u_n)\varphi_1\,dx\,dt\\ &\quad+\int_{Q_\tau} D(|u_n|^{m-1}u_n)D(|T_k(u_n)|^{m-1}T_k(u_n))\varphi_1\,dx\,dt\\ &\quad+\int_{Q_\tau} D(|u_n|^{m-1}u_n)|T_k(u_n)|^{m-1}T_k(u_n)D\varphi_1\,dx\,dt\\ &=\int_{Q_\tau} f_n|T_k(u_n)|^{m-1}T_k(u_n)\,dx\,dt.\end{aligned} \tag{3.25}$$

Using the same argument as that of (3.13), we get

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in(0,T)}\int_\Omega |T_k(u_n(\tau))|^{m+1}\delta(x)\,dx\\ &\quad+\int_Q(|D(|T_k(u_n)|^{m-1}T_k(u_n))|^2\delta(x) + ||T_k(u_n)|^{m-1}T_k(u_n)|^2\delta(x))\,dx\,dt\\ &\le Ck^m(\|f\|_{L^1(Q,\delta)} + \|u_0\|_{L^1(\Omega,\delta)}) = CMk^m,\end{aligned} \tag{3.26}$$

where C is a positive constant independent of k and n.

Thus the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Assume that $f \in L^1(Q,\delta^\alpha)$, $u_0 \in L^1(\Omega,\delta^\alpha)$ with $0 < \alpha < \frac{-(2mN+2-m)+\sqrt{(2mN+2-m)^2+8m(mN+2)}}{4m}$. Then every weak solution u_n of the problem (P_n) satisfies*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha))} \le C[M_1 + \tfrac{1}{2}\operatorname{meas}_{\delta^\alpha}\Omega], \tag{3.27}$$

$$\||u_n|^m\|_{L^q(0,T;W_0^{1,q}(\Omega,\delta^\alpha))\cap L^{\bar{q}}(Q,\delta^\alpha)} \le C\max\left\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\right\}, \tag{3.28}$$

$$\| |u_n|^m \|_{L^{\tilde q}(0,T;W_0^{1,\tilde q}(\Omega))} \le C M_1^{\frac{1}{2}} \left(1 + \max\left\{ M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}} \right\} \right), \tag{3.29}$$

where C is a positive constant depending only on q, $\bar q$ and q_0, $M_1 = \|f\|_{L^1(Q,\delta^\alpha)} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}$, q, $\bar q$ and $\tilde q$ are seen in (1.12) and (1.13).

Proof. The proof of this lemma is similar to that of Lemma 3.1, here we only simply revise the proof of Lemma 3.1. In the process of the proof of Lemma 3.1, we only need to replace φ_1 by φ_1^α, δ by δ^α. Since

$$\Delta\varphi_1^\alpha = \alpha(\alpha-1)\varphi_1^{\alpha-2}|D\varphi_1|^2 + \alpha\varphi_1^{\alpha-1}\Delta\varphi_1, \tag{3.30}$$

then as $\alpha < 1$ (3.7) and (3.11) are replaced by the following inequalities:

$$\begin{aligned}
&\int_0^\tau \int_\Omega D(|u_n|^{m-1}u_n)\psi(u_n) D\varphi_1^\alpha \, dx\, dt \\
&= \int_0^\tau \int_\Omega D\varphi_1^\alpha D \int_0^{|u_n|^{m-1}u_n} \psi(|s|^{\frac{1-m}{m}} s)\, ds\, dx\, dt \\
&= -\int_0^\tau \int_\Omega \Delta\varphi_1^\alpha \int_0^{|u_n|^{m-1}u_n} \psi(|s|^{\frac{1-m}{m}} s)\, ds\, dxdt \\
&= \alpha(1-\alpha)\int_0^\tau \int_\Omega \varphi_1^{\alpha-2}|D\varphi_1|^2 \int_0^{|u_n|^{m-1}u_n} \psi(|s|^{\frac{1-m}{m}} s)\, ds\, dx\, dt \\
&\quad + \alpha\lambda_1 \int_0^\tau \int_\Omega \varphi_1^\alpha \int_0^{|u_n|^{m-1}u_n} \psi(|s|^{\frac{1-m}{m}} s)\, ds\, dx\, dt \ge 0,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau \int_\Omega Dw_n T_k(w_n) D\varphi_1^\alpha \, dx\, dt = -\int_0^\tau \int_\Omega \Delta\varphi_1^\alpha \int_0^{w_n} T_k(s)\, ds\, dx\, dt \\
&\ge \alpha(1-\alpha)\int_0^\tau \int_\Omega \varphi_1^{\alpha-2}|D\varphi_1|^2 \int_0^{w_n} T_k(s)\, ds\, dx\, dt + \frac{\alpha\lambda_1}{2}\int_0^\tau \int_\Omega \varphi_1^\alpha |T_k(w_n)|^2\, dx\, dt \\
&\ge \frac{\alpha\lambda_1}{2}\int_0^\tau \int_\Omega \varphi_1^\alpha |T_k(w_n)|^2\, dx\, dt.
\end{aligned}$$

Now (3.9), (3.13), (3.22) and (2.23) are changed into

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega,\delta^\alpha))} \le C(\|f_n\|_{L^1(Q,\delta^\alpha)} + \|u_{0n}\|_{L^1(\Omega,\delta^\alpha)} + \tfrac{1}{2}\operatorname{meas}_{\delta^\alpha}\Omega), \tag{3.31}$$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |T_k(w_n(\tau))|^{\frac{m+1}{m}}\delta^\alpha\, dx + \int_Q (|DT_k(w_n)|^2\delta^\alpha + |T_k(w_n)|^2\delta^\alpha)\, dx\, dt \\
&\le Ck(\|f\|_{L^1(Q,\delta^\alpha)} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}),
\end{aligned} \tag{3.32}$$

$$\left(\int_Q |w_n|^{\bar q}\delta^\alpha\, dx\, dt\right)^{\frac{1}{\bar q}} \le C M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}} \tag{3.33}$$

and

$$\left(\int_Q |Dw_n|^q\delta^\alpha + |w_n|^q\delta^\alpha\, dx\, dt\right)^{\frac{1}{q}} \le C\max\left\{ M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}} \right\}, \tag{3.34}$$

where $1 \le q < \frac{m(N+\alpha)+2}{m(N+\alpha)+1}$, $1 \le \bar q < \frac{m(N+\alpha)+2}{m(N+\alpha)}$ and $M_1 = \|f\|_{L^1(Q,\delta^\alpha)} + \|u_0\|_{L^1(\Omega,\delta^\alpha)}$.

For a given $\lambda > 0$, set

$$\psi(s) = \int_0^s \frac{dt}{(1+|t|)^\lambda}, \ \forall s \in R. \tag{3.35}$$

If $\lambda > 1$, then

$$\psi(s) = \frac{1}{\lambda - 1}[1 - \frac{1}{(1+|s|)^{\lambda-1}}]sgn(s), \ \forall s \in \mathbf{R}. \tag{3.36}$$

Let $v = \psi(w_n)\varphi_1^\alpha$ in (P') and integrating it over $(0,T)$, we obtain

$$\int_Q \frac{|Dw_n|^2}{(1+|w_n|)^\lambda}\delta^\alpha(x)\,dx\,dt \le CM_1. \tag{3.37}$$

For all $1 < \tilde{q} < 2$, using Hölder's inequality we obtain

$$\begin{aligned} \int_Q |Dw_n|^{\tilde{q}}\,dx\,dt &= \int_Q \frac{|Dw_n|^{\tilde{q}}}{(1+|w_n|)^{\frac{\tilde{q}\lambda}{2}}}\delta^{\frac{\tilde{q}\alpha}{2}}(1+|w_n|)^{\frac{\tilde{q}\lambda}{2}}\delta^{-\frac{\tilde{q}\alpha}{2}}\,dx\,dt \\ &\le \left(\int_Q \frac{|Dw_n|^2}{(1+|w_n|)^\lambda}\delta^\alpha\,dx\,dt\right)^{\frac{\tilde{q}}{2}}\left(\int_\Omega (1+|w_n|)^{\frac{\tilde{q}\lambda}{2-\tilde{q}}}\delta^{-\frac{\tilde{q}\alpha}{2-\tilde{q}}}\,dx\,dt\right)^{\frac{2-\tilde{q}}{2}} \\ &\le CM_1^{\frac{\tilde{q}}{2}}\left(1 + \int_Q |w_n|^{\frac{\tilde{q}\lambda}{2-\tilde{q}}}\delta^{-\frac{\tilde{q}\alpha}{2-\tilde{q}}}\,dx\,dt\right)^{\frac{2-\tilde{q}}{2}}. \end{aligned} \tag{3.38}$$

Taking $r = \frac{\tilde{q}\lambda}{2-\tilde{q}} = q$, $\gamma = -\frac{\tilde{q}}{2-\tilde{q}}$, $\beta = \alpha$ in Lemma 2.1, (2.2), (2.3) and (2.4) yield

$$\frac{N - \frac{\tilde{q}}{2-\tilde{q}}}{\frac{\tilde{q}\lambda}{2-\tilde{q}}} + 1 > \frac{N+\alpha}{q}. \tag{3.39}$$

Now $\lambda > 1$, $\frac{\tilde{q}\lambda}{2-\tilde{q}} = q$ and (3.39) imply that

$$\tilde{q} < \frac{2q}{q+1} \quad \text{and} \quad \tilde{q} < 2\left(1 - \frac{\alpha}{q}\right). \tag{3.40}$$

From (3.34) and (3.38) it follows that

$$\begin{aligned} \left(\int_Q |Dw_n|^{\tilde{q}}\,dx\,dt\right)^{\frac{1}{\tilde{q}}} &\le CM_1^{\frac{1}{2}}\left(1 + \max\left\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\right\}^q\right)^{\frac{2-\tilde{q}}{2\tilde{q}}} \\ &\le CM_1^{\frac{1}{2}}\left(1 + \max\left\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\right\}\right)^{\frac{(2-\tilde{q})q}{2\tilde{q}}} \\ &\le CM_1^{\frac{1}{2}}\left(1 + \max\left\{M_1^{\frac{m(N+\alpha+2)+1}{m(N+\alpha)+2}}, M_1^{\frac{m(N+\alpha+2)}{m(N+\alpha)+2}}\right\}\right). \end{aligned} \tag{3.41}$$

Now $q < \frac{m(N+\alpha)+2}{m(N+\alpha)+1}$ and (3.40) yield

$$1 < \tilde{q} < \min\left\{\frac{2m(N+\alpha)+4}{2m(N+\alpha)+3}, \frac{2[m(N+\alpha)(1-\alpha)+2-\alpha]}{m(N+\alpha)+2}\right\}. \tag{3.42}$$

To ensure $1 \le \tilde{q}$, this needs $0 < \alpha < \frac{-(2mN+2-m)+\sqrt{(2mN+2-m)^2+8m(mN+2)}}{4m}$.

Thus the proof Lemma 3.2 is completed. □

Lemma 3.3. *Assume that* $f \in L^1(0,T; L^1 \log L^1(\Omega,\delta))$, $u_0 \in L^1 \log L^1(\Omega,\delta)$. *Then for the unique weak solution* u_n *of the problem* (P_n), *there exists a positive constant* C *independent of* n *such that*

$$\||u_n|^m\|_{L^q(0,T;W_0^{1,q}(\Omega,\delta))\cap L^{\bar q}(Q,\delta)\cap L^q(0,T;L^{q_0}(\Omega))} \le C, \tag{3.43}$$

where $q, \bar q$ *and* q_0 *can be seen in (1.17).*

Proof. Let $\lambda = 1$ in (3.35). Then

$$\psi(s) = \ln(1+|s|)\,\mathrm{sgn}(s),\ \forall s \in \mathbf{R}. \tag{3.44}$$

Taking $v = \psi(w_n)\varphi_1$ in (P') and integrating it over $(0,T)$, similarly to (3.37), we obtain

$$\begin{aligned}&\int_Q \frac{|Dw_n|^2}{1+|w_n|}\delta(x)\,dx\,dt\\ &\le C\left[\int_Q |f_n|\ln(1+|w_n|)\delta\,dx\,dt + \int_\Omega |u_0|(1+\ln(1+|u_0|))\delta\,dx\right].\end{aligned} \tag{3.45}$$

By using the inequality $ab \le a\ln(1+a) + e^b$, $\forall a,b>0$, we get

$$\begin{aligned}&\int_Q |f_n|\ln(1+|w_n|)\delta\,dx\,dt\\ &\le \int_Q |f_n|\ln(1+|f_n|)\delta\,dx\,dt + \int_Q (1+|w_n|)\delta\,dx\,dt\\ &\le \int_Q |f|\ln(1+|f|)\delta\,dx\,dt + \int_Q (1+|w_n|)\delta\,dx\,dt\\ &\le \int_Q |f|\ln(1+|f|)\delta\,dx\,dt + \left(\int_Q (1+|w_n|)^{\frac{1}{m}}\delta\,dx\,dt\right)^m (\mathrm{meas}_\delta\, Q)^{1-m}.\end{aligned} \tag{3.46}$$

By virtue of $f \in L^1(Q,\delta)$, $u_0 \in L^1(\Omega,\delta)$, by using of the estimates (3.1) and (3.2) in Lemma 3.1, we have

$$\int_Q (1+|w_n|)^{\frac{1}{m}}\delta\,dx\,dt \le 2^{\frac{1-m}{m}} \int_Q 1+|u_n|\delta\,dx\,dt \le C, \tag{3.47}$$

$$\int_Q |w_n|^r\delta\,dx\,dt \le C, \tag{3.48}$$

where $1 \le r < \frac{m(N+1)+2}{m(N+1)}$, the constant C depending only on $\|f\|_{L^1(Q,\delta)}, \|u_0\|_{L^1(\Omega,\delta)}$ and $\mathrm{meas}_\delta\,\Omega$, $\mathrm{meas}_\delta\,Q$.

Thus it follows from (3.45)–(3.47) that

$$\int_Q \frac{|Dw_n|^2}{1+|w_n|}\delta(x)\,dx\,dt \le C. \tag{3.49}$$

For all $1 < q < \frac{m(N+1)+2}{m(N+1)}$, Hölder's inequality and (3.49) imply that

$$\begin{aligned}
&\int_Q |Dw_n|^q \delta(x)\,dx\,dt = \int_Q \frac{|Dw_n|^q}{(1+|w_n|)^{\frac{q}{2}}} \delta^{\frac{q}{2}} (1+|w_n|)^{\frac{q}{2}} \delta^{\frac{2-q}{2}}\,dx\,dt \\
&\le \left(\int_Q \frac{|Dw_n|^2}{1+|w_n|}\delta(x)\,dx\,dt\right)^{\frac{q}{2}} \left(\int_Q (1+|w_n|)^{\frac{q}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}} \qquad (3.50)\\
&\le C\left(1+\int_\Omega |w_n|^{\frac{q}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}}.
\end{aligned}$$

By using Lemma 2.3(i) (here $\alpha = 1,\ v = w_n,\ r = \frac{1}{m},\ s = \frac{[m(N+1)+1]q}{m(N+1)} = \bar{q}$) and (3.1), (3.48) and (3.50), we obtain

$$\begin{aligned}
\int_Q |w_n|^{\bar{q}}\delta\,dx\,dt &\le C\|w_n\|^{\frac{q}{m(N+1)}}_{L^\infty(0,T;L^{\frac{1}{m}}(\Omega,\delta))}\|w_n\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta))} \qquad (3.51)\\
&\le C\left(1+\left(\int_\Omega |w_n|^{\frac{q}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}}\right).
\end{aligned}$$

Let

$$\frac{[m(N+1)+1]q}{m(N+1)} = \bar{q} = \frac{q}{2-q}. \qquad (3.52)$$

Then we get

$$q = \frac{m(N+1)+2}{m(N+1)+1}, \quad \bar{q} = \frac{m(N+1)+2}{m(N+1)}. \qquad (3.53)$$

From (3.50)–(3.52) and Young's inequality, it follows

$$\int_Q |w_n|^{\bar{q}}\delta dxdt \le C, \qquad (3.54)$$

$$\int_Q |Dw_n|^q\delta\,dx\,dt \le C. \qquad (3.55)$$

Combining it to (3.48) ($r = q$), we obtain

$$\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\,dt \le C. \qquad (3.56)$$

Taking $r = q_0,\ \gamma = 0,\ \beta = 1$ in Lemma 2.1, (2.2) and (2.3) yield

$$\frac{N}{q_0} + 1 \ge \frac{N+1}{q}. \qquad (3.57)$$

This implies that q_0 admits the maximum and

$$q_0 = \frac{mN(N+1)+2N}{mN(N+1)+N-1}. \qquad (3.58)$$

By using (2.5) in Lemma 2.1, (3.56) implies

$$\int_0^T \left(\int_\Omega |w_n|^{q_0}\,dx\right)^{\frac{q}{q_0}} dt \le C\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\,dt \le C. \qquad (3.59)$$

Thus Lemma 3.3 is proved. □

Lemma 3.4. *Assume that $f \in L^p(Q,\delta)$ with $1 < p < \frac{2m(N+2)+2}{m(N+3)+2}$ and $u_0 = 0$. Then for the unique weak solution u_n of the problem (P_n), there exists a positive constant C independent of n such that*

$$\| |u_n|^m \|_{L^q(0,T;W_0^{1,q}(\Omega,\delta)) \cap L^{\bar q}(Q,\delta) \cap L^q(0,T;L^{q_0}(\Omega))} \le C, \tag{3.60}$$

where $q, \bar q$ and q_0 can be seen in (1.19).

Proof. Similarly to the proof of (3.13) and (3.14), we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |T_k(w_n(\tau))|^{\frac{m+1}{m}} \delta(x)\,dx + \int_Q (|DT_k(w_n)|^2\delta(x) \\ &\quad + |T_k(w_n)|^2\delta(x))\,dx\,dt \le C\|f\|_{L^p(Q,\delta)} (\int_\Omega |T_k(w_n)|^{p'}\delta\,dx\,dt)^{\frac{1}{p'}}, \end{aligned} \tag{3.61}$$

$$\int_Q |T_k(w_n)|^s \delta\,dx\,dt \le C\left(\|f\|_{L^p(Q,\delta)} \left(\int_Q |T_k(w_n)|^{p'}\delta\,dx\,dt \right)^{\frac{1}{p'}} \right)^{2-\frac{2}{q_1}}, \tag{3.62}$$

where $s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1}$, $2 \le q_1 < \frac{2(N+1)}{N-1}$. Due to $p < \frac{2m(N+2)+2}{m(N+3)+2}$ and $q_1 < \frac{2(N+1)}{N-1}$, then $p' > \frac{2m(N+2)+2}{m(N+1)} > s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1}$. Thus

$$\left(\int_Q |T_k(w_n)|^{p'}\delta\,dx\,dt \right)^{\frac{1}{p'}} \le k^{\frac{p'-s}{p'}} \left(\int_Q |T_k(w_n)|^s\delta\,dx\,dt \right)^{\frac{1}{p'}}. \tag{3.63}$$

Now (3.62) and (3.63) yield

$$\int_Q |T_k(w_n)|^s\delta(x)\,dx \le C\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}} k^{\frac{2(p'-s)(q_1-1)}{p'q_1}} \left(\int_Q |T_k(w_n)|^s\delta\,dx\,dt \right)^{\frac{2(q_1-1)}{p'q_1}}, \tag{3.64}$$

Young's inequality implies that

$$\int_Q |T_k(w_n)|^s\delta\,dx\,dt \le C\|f\|_{L^p(Q,\delta)}^{\frac{2(q_1-1)p'}{(p'-2)q_1+2}} k^{\frac{2(p'-s)(q_1-1)}{(p'-2)q_1+2}}. \tag{3.65}$$

By using the same proceeding as (3.16) and (3.17), we get

$$\int_Q |w_n|^{\bar q_1}\delta\,dx\,dt \le C, \tag{3.66}$$

where $\bar q_1 < \frac{[(m+1)q_1-2]p}{[(2-p)q_1+2(p-1)]m} < \frac{[m(N+1)+2]p}{m(N+3-2p)}$.

Let $0 < \lambda < 1$ in (3.35), then

$$\psi(s) = \frac{1}{1-\lambda}[(1+|s|)^{1-\lambda} - 1]\operatorname{sgn}(s), \ \forall s \in \mathbf{R}. \tag{3.67}$$

Let $v = \psi(w_n)\varphi_1$ in (P') and integrating it over $(0,T)$, by the same process as that of (3.37) and using Hölder's inequality, we obtain

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |w_n(\tau)|^{1-\lambda+\frac{1}{m}}\delta\,dx + \int_Q \frac{|Dw_n|^2}{(1+|w_n|)^\lambda}\delta\,dx\,dt \\ &\le C\left(\int_Q (1+|w_n|)^{\frac{(1-\lambda)p}{p-1}}\delta\,dx\,dt \right)^{1-\frac{1}{p}}. \end{aligned} \tag{3.68}$$

For all $1 < q < \min\{2, \frac{[m(N+1)+2]p}{m(N+3-2p)}\}$, Hölder's inequality and (3.68) imply that

$$\begin{aligned}
&\int_Q |Dw_n|^q \delta(x)\,dx\,dt = \int_Q \frac{|Dw_n|^q}{(1+|w_n|)^{\frac{q\lambda}{2}}} \delta^{\frac{q}{2}} (1+|w_n|)^{\frac{q\lambda}{2}} \delta^{\frac{2-q}{2}}\,dx\,dt \\
&\le \left(\int_Q \frac{|Dw_n|^2}{(1+|w_n|)^\lambda}\delta(x)\,dx\,dt\right)^{\frac{q}{2}} \left(\int_Q (1+|w_n|)^{\frac{q\lambda}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}} \\
&\le C\left(\int_\Omega (1+|w_n|)^{\frac{(1-\lambda)p}{p-1}}\delta(x)\,dx\,dt\right)^{\frac{(p-1)q}{2p}} \left(\int_Q (1+|w_n|)^{\frac{q\lambda}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}}.
\end{aligned} \tag{3.69}$$

By using Lemma 2.3(i) (here $\alpha = 1,\ v = w_n,\ r = 1-\lambda+\frac{1}{m},\ s = \frac{(N+2-\lambda+\frac{1}{m})q}{N+1} = \bar{q}$), we obtain

$$\begin{aligned}
&\int_Q |w_n|^{\bar{q}}\delta\,dx\,dt \le C\|w_n\|^{\frac{q(1-\lambda+\frac{1}{m})}{N+1}}_{L^\infty(0,T;L^{1-\lambda+\frac{1}{m}}(\Omega,\delta))} \|w_n\|^q_{L^q(0,T;W_0^{1,q}(\Omega,\delta))} \\
&\le C\left(\int_Q (1+|w_n|)^{\frac{(1-\lambda)p}{p-1}}\delta\,dx\,dt\right)^{\frac{(p-1)q}{(N+1)p}} \left(\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\,dt\right).
\end{aligned} \tag{3.70}$$

From (3.66) (here let $\bar{q}_1 = q$) and (3.68) it follows that

$$\begin{aligned}
&\int_Q |w_n|^{\bar{q}}\delta\,dx\,dt \\
&\le C\left(\int_Q (1+|w_n|)^{\frac{(1-\lambda)p}{p-1}}\delta\,dx\,dt\right)^{\frac{(p-1)q}{(N+1)p}} \Bigg[\left(\int_\Omega (1+|w_n|)^{\frac{(1-\lambda)p}{p-1}}\delta(x)\,dx\,dt\right)^{\frac{(p-1)q}{2p}} \\
&\quad\cdot\left(\int_Q (1+|w_n|)^{\frac{q\lambda}{2-q}}\delta\,dx\,dt\right)^{\frac{2-q}{2}} + 1\Bigg] \\
&\le C\left(1 + (\int_Q |w_n|^{\frac{(1-\lambda)p}{p-1}}\delta\,dx\,dt)^{\frac{(p-1)q}{(N+1)p}+\frac{(p-1)q}{2p}} (\int_Q |w_n|^{\frac{q\lambda}{2-q}}\delta\,dx\,dt)^{\frac{2-q}{2}}\right).
\end{aligned} \tag{3.71}$$

Let $\bar{q} = \frac{(1-\lambda)p}{p-1} = \frac{q\lambda}{2-q} = \frac{(N+2-\lambda+\frac{1}{m})q}{N+1}$. We can deduce

$$\lambda = \frac{(2-q)p}{2p-q} < 1, \quad q = \frac{[m(N+1)+2]p}{m(N+2-p)+1}, \quad \bar{q} = \frac{[m(N+1)+2]p}{m(N+3-2p)}. \tag{3.72}$$

Thanks to $p < \frac{2m(N+2)+2}{m(N+3)+2}$, we have $\frac{(p-1)q}{(N+1)p} + \frac{(p-1)q}{2p} + \frac{2-q}{2} < 1$. Hence, by using Young's inequality, we obtain

$$\int_Q |w_n|^{\bar{q}}\delta\,dx\,dt \le C, \quad \int_Q |Dw_n|^q\delta\,dx\,dt \le C. \tag{3.73}$$

The above estimate yields

$$\int_Q |w_n|^q\delta + |Dw_n|^q\delta\,dx\,dt \le C. \tag{3.74}$$

Taking $r = q_0,\ \gamma = 0,\ \beta = 1$ in Lemma 2.1, (2.2) and (2.3) yield

$$\frac{N}{q_0} + 1 \ge \frac{N+1}{q}. \tag{3.75}$$

From this, it follows

$$q_0 = \frac{[mN(N+1)+2N]p}{m(N+1)(N+2-2p)+(N+1-2p)}. \tag{3.76}$$

Now (2.5) in Lemma 2.1 and (3.74) imply that

$$\int_0^T \left(\int_\Omega |w_n|^{q_0}\,dx\right)^{\frac{q}{q_0}} dt \le C\int_Q |Dw_n|^q\delta + |w_n|^q\delta\,dx\,dt \le C. \tag{3.77}$$

Thus we can get (3.60) by using (3.73), (3.74) and (3.77). □

Lemma 3.5. *Assume that $f \in L^p(Q,\delta)$ with $\frac{2m(N+2)+2}{m(N+3)+2} < p < \frac{N+3}{2}$ and $u_0 \in L^d(\Omega,\delta)$ with $d = m+1$. Then for the unique weak solution u_n of the problem (P_n), there exists a positive constant C independent of n such that*

$$\||u_n|^m\|_{L^2(0,T;W_0^{1,2}(\Omega,\delta))\cap L^{\bar q}(Q,\delta)\cap L^2(0,T;L^{q_0}(\Omega))} \le C, \tag{3.78}$$

where $\bar q$ and q_0 are defined in (1.22).

Proof. Let $v = w_n\varphi_1$ in (P'). Integrating it over $(0,T)$, similarly to (3.13), we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |w_n(\tau)|^{\frac{m+1}{m}}\delta\,dx + \int_Q |Dw_n|^2\delta + |w_n|^2\delta\,dx\,dt\\
&\le C\left[\|f\|_{L^p(Q,\delta)}\left(\int_\Omega |w_n|^{p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + \int_\Omega |u_0|^{m+1}\delta\right]\\
&\le C\left[\left(\int_\Omega |w_n|^{p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + 1\right].
\end{aligned} \tag{3.79}$$

Taking $\alpha = 1,\ v = w_n,\ r = \frac{m+1}{m},\ q = 2,\ s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1} = \bar q$ in Lemma 2.3(ii), we have

$$\begin{aligned}
&\int_Q |w_n|^s\delta\,dx\,dt \le C\|w_n\|_{L^\infty(0,T;L^{1+\frac{1}{m}}(\Omega,\delta))}^{(1-\frac{2}{q_1})(1+\frac{1}{m})}\|w_n\|^2_{L^2(0,T;W_0^{1,2}(\Omega,\delta))}\\
&\le C\left[\left(\int_\Omega |w_n|^{p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + 1\right]^{2-\frac{2}{q_1}} \le C\left[\left(\int_\Omega |w_n|^{p'}\delta\,dx\,dt\right)^{\frac{2(q_1-1)}{q_1p'}} + 1\right],
\end{aligned} \tag{3.80}$$

where $2 \le q_1 < \frac{2(N+1)}{N-1}$. Let $s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1} = \bar q$. Then we can get

$$\bar q < \frac{2m(N+2)+2}{m(N+1)}. \tag{3.81}$$

Now $\frac{2m(N+2)+2}{m(N+3)+2} < p < \frac{N+3}{2}$ impies $\frac{N+3}{N+1} < p' < \frac{2m(N+2)+2}{m(N+1)}$, thus we can deduce that $\frac{2(q_1-1)}{q_1p'} < 1$ and choose $\bar q$ such that $\bar q \ge p'$. By using (3.79), (3.80) and Young's inequality, we obtain

$$\int_Q |w_n|^{\bar q}\delta\,dx\,dt \le C, \tag{3.82}$$

$$\int_Q |Dw_n|^2\delta + |w_n|^2\delta\,dx\,dt \le C. \tag{3.83}$$

Using Lemma 2.1 (here $r = q_0,\ q = 2,\ \gamma = 0,\ \beta = 1$) again, (2.2) and (2.4) yield

$$\frac{N}{q_0} + 1 > \frac{N+1}{2}. \tag{3.84}$$

From this, it follows

$$q_0 < \frac{2N}{N-1}. \tag{3.85}$$

By using (2.6) and (3.83), we get

$$\int_0^T \left(\int_\Omega |w_n|^{q_0}\, dx \right)^{\frac{2}{q_0}} dt \le C \int_Q |Dw_n|^2 \delta + |w_n|^2 \delta \, dx\, dt \le C. \tag{3.86}$$

From (3.82), (3.83) and (3.86), it is easy to get (3.78). □

Lemma 3.6. *Assume that $f \in L^p(Q, \delta)$ with $p > \frac{N+3}{2}$ and $u_0 \in L^\infty(\Omega)$. Then for the unique weak solution u_n of the problem (P_n), there exists a positive constant C independent of n such that*

$$\| |u_n|^m \|_{L^2(0,T;W_0^{1,2}(\Omega,\delta)) \cap L^\infty(Q)} \le C. \tag{3.87}$$

Proof. By Lemma 3.5, we obtain a priori estimate about $\| |u_n|^m \|_{L^2(0,T;W_0^{1,2}(\Omega,\delta))}$. Here we need to estimate $\| |u_n|^m \|_{L^\infty(Q)}$. That is $\|w_n\|_{L^\infty(Q)}$. For any given $k \ge k_0 = \|u_0\|_\Omega$, let $v = \operatorname{sgn} w_n (|w_n| - k)_+ \varphi_1$ in (P') and integrating it over $(0, \tau)$, $\tau \in (0, T)$, we have

$$\begin{aligned} &\int_0^\tau \int_\Omega u_{nt} \operatorname{sgn} w_n (|w_n| - k)_+ \varphi_1 \, dx\, dt \\ &\quad + \int_{Q_\tau} Dw_n D(\operatorname{sgn} w_n (|w_n| - k)_+) \varphi_1 \, dx\, dt \\ &\quad + \int_{Q_\tau} Dw_n \operatorname{sgn} w_n (|w_n| - k)_+ D\varphi_1 \, dx\, dt \\ &= \int_{Q_\tau} f_n \operatorname{sgn} w_n (|w_n| - k)_+ \varphi_1 \, dx\, dt. \end{aligned} \tag{3.88}$$

By calculating, we obtain

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau \in (0,T)} \int_\Omega (|\operatorname{sgn} w_n (|w_n(\tau)| - k)_+|)^{\frac{m+1}{m}} \delta \, dx \\ &\quad + \int_Q |D(\operatorname{sgn} w_n (|w_n| - k)_+)|^2 \delta \, dx\, dt + \int_Q |\operatorname{sgn} w_n (|w_n| - k)_+|^2 \delta \, dx\, dt \\ &\le C \int_Q |f_n| |\operatorname{sgn} w_n (|w_n| - k)_+| \delta \, dx\, dt. \end{aligned} \tag{3.89}$$

Let $\alpha = 1,\ v = \operatorname{sgn} w_n(|w_n| - k)_+,\ r = \frac{m+1}{m},\ q = 2,\ s = \frac{m+1}{m} + 2 - \frac{2(m+1)}{mq_1}$ in Lemma 2.3(ii). We have

$$\begin{aligned}
&\int_Q |\operatorname{sgn} w_n(|w_n| - k)_+|^s \delta\, dx\, dt \\
&\le C\|\operatorname{sgn} w_n(|w_n| - k)_+\|_{L^\infty(0,T;L^{1+\frac{1}{m}}(\Omega,\delta))}^{(1-\frac{2}{q_1})(1+\frac{1}{m})} \|\operatorname{sgn} w_n(|w_n| - k)_+\|^2_{L^2(0,T;W_0^{1,2}(\Omega,\delta))} \\
&\le C\left[\int_Q |f_n||\operatorname{sgn} w_n(|w_n| - k)_+|\delta\, dx\, dt\right]^{2-\frac{2}{q_1}} \\
&\le C\left[\int_Q |f||\operatorname{sgn} w_n(|w_n| - k)_+|\delta\, dx\, dt\right]^{2-\frac{2}{q_1}},
\end{aligned} \tag{3.90}$$

where $2 \le q_1 < \frac{2(N+1)}{N-1}$. Taking $l = \frac{s}{2-\frac{2}{q_1}}$, due to $q_1 \ge 2$, then we have $l > 1$. Furthermore, $p > \frac{N+3}{2}$ implies that we can choose $2 \le q_1 < \frac{2(N+1)}{N-1}$ such that $2 - \frac{2}{q_1} > p'$.

Let $\varphi(k) = meas_\delta\{|w_n| > k\} = \int_{\{|w_n|>k\}} \delta\, dx\, dt$. Hölder's inequality, Young's inequality and the term on the right-hand side of (3.89) imply that

$$\begin{aligned}
&C\Big(\int_Q |f||(|w_n| - k)_+|\delta\, dx\, dt\Big)^{2-\frac{2}{q_1}} \\
&\le C\left(\varepsilon\int_Q |f||(|w_n| - k)_+|^l\delta\, dx\, dt + \varepsilon^{-\frac{1}{l-1}}\int_{\{|w_n|>k\}} |f|\delta\, dx\, dt\right)^{2-\frac{2}{q_1}} \\
&\le C\varepsilon^{2-\frac{2}{q_1}}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\left(\int_Q |(|w_n| - k)_+|^{lp'}\delta\, dx\, dt\right)^{\frac{1}{p'}(2-\frac{2}{q_1})} \\
&\quad + C\varepsilon^{-\frac{1}{l-1}(2-\frac{2}{q_1})}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\varphi(k)^{\frac{1}{p'}(2-\frac{2}{q_1})} \\
&\le C\varepsilon^{2-\frac{2}{q_1}}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\int_Q |(|w_n| - k)_+|^s\delta\, dx\, dt \\
&\quad + C\varepsilon^{-\frac{1}{l-1}(2-\frac{2}{q_1})}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\varphi(k)^{\frac{2-\frac{2}{q_1}}{p'}}.
\end{aligned} \tag{3.91}$$

Let $\varepsilon^{2-\frac{2}{q_1}} = \frac{1}{2(C\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}+1)}$. Then we have

$$\int_Q (|w_n| - k)_+|^s\delta\, dx\, dt \le 2^{\frac{l}{l-1}}C\left(C\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}} + 1\right)^{\frac{1}{l-1}}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\varphi(k)^{\frac{2-\frac{2}{q_1}}{p'}}. \tag{3.92}$$

Thus, for every $h > k > 0$, we can deduce that

$$\varphi(h) \le \frac{2^{\frac{l}{l-1}}C\left(C\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}} + 1\right)^{\frac{1}{l-1}}\|f\|_{L^p(Q,\delta)}^{2-\frac{2}{q_1}}\varphi(k)^{\frac{2-\frac{2}{q_1}}{p'}}}{(h-k)^s}. \tag{3.93}$$

By using Lemma 4.1 in [28], there exists a positive constant h_0 depending only on $\|f\|_{L^p(Q,\delta)},\ \|u_0\|_\Omega$ and $\operatorname{meas}_\delta \Omega$ such that

$$\varphi(h_0) = 0. \tag{3.94}$$

Hence

$$\|w_n\|_{L^\infty(Q)} \le h_0. \tag{3.95}$$

Thus we finish the proof of Lemma 3.6. □

Lemma 3.7. *Assume that $f \in L^p(Q,\delta)$ with $p = \frac{N+3}{2}$ and $u_0 \in L^\infty(Q)$. Then for the unique weak solution u_n of the problem (P_n), there exists a positive constant C independent of n such that*

$$\||u_n|^m\|_{L^2(0,T;W_0^{1,2}(\Omega,\delta))\cap L^{\bar q}(Q,\delta)\cap L^2(0,T;L^{q_0}(\Omega))} \le C, \tag{3.96}$$

where $1 \le \bar q < +\infty,\ 1 \le q_0 < +\infty$.

Proof. Firstly, we obtain a priori estimate about $\||u_n|^m\|_{L^2(0,T;W_0^{1,2}(\Omega,\delta))}$ by Lemma 3.5. In the following we will obtain a priori estimate about $\||u_n|^m\|_{L^{\bar q}(Q,\delta)\cap L^2(0,T;L^{q_0}(\Omega))}$.

For any given $\theta > 0$, let $v = |w_n|^{2\theta} w_n \varphi_1$ in (P'). Integrating it over $(0,\tau),\ \tau \in (0,T)$, we have

$$\begin{aligned} &\int_0^\tau \int_\Omega u_{nt}|w_n|^{2\theta} w_n \varphi_1\,dx\,dt + \int_{Q_\tau} Dw_n D(|w_n|^{2\theta} w_n)\varphi_1\,dx\,dt \\ &\quad + \int_{Q_\tau} Dw_n |w_n|^{2\theta} w_n D\varphi_1\,dx\,dt = \int_{Q_\tau} f_n |w_n|^{2\theta} w_n \varphi_1\,dx\,dt. \end{aligned} \tag{3.97}$$

Similarly to (3.13), we can deduce

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |w_n(\tau)|^{2\theta+1+\frac{1}{m}}\delta\,dx + \int_Q |D(|w_n|^\theta w_n)|^2\delta + \||w_n|^\theta w_n|^2 \delta\,dx\,dt \\ &\le C\left(\|f\|_{L^p(Q,\delta)}\left(\int_\Omega |w_n|^{(2\theta+1)p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + \|u_0\|_{L^\infty(Q)}\right) \\ &\le C\left(\left(\int_\Omega |w_n|^{(2\theta+1)p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + 1\right). \end{aligned} \tag{3.98}$$

Taking $\alpha = 1,\ v = |w_n|^\theta w_n,\ r = \frac{2\theta+1+\frac{1}{m}}{\theta+1},\ q = 2,\ s = \frac{2\theta+1+\frac{1}{m}}{\theta+1} + 2 - \frac{2(2\theta+1+\frac{1}{m})}{q_1(\theta+1)}$ in Lemma 2.3(ii), we have

$$\begin{aligned} &\int_Q ||w_n|^\theta w_n|^s \delta\,dx\,dt \\ &\le C\||w_n|^\theta w_n\|_{L^\infty(0,T;L^{\frac{2\theta+1+\frac{1}{m}}{\theta+1}}(\Omega,\delta))}^{(1-\frac{2}{q_1})\frac{(2\theta+1+\frac{1}{m})}{\theta+1}} \||w_n|^\theta w_n\|^2_{L^2(0,T;W_0^{1,2}(\Omega,\delta))} \\ &\le C\left(\left(\int_Q |w_n|^{(2\theta+1)p'}\delta\,dx\,dt\right)^{\frac{1}{p'}} + 1\right)^{2-\frac{2}{q_1}} \\ &\le C\left(\left(\int_Q ||w_n|^\theta w_n|^{\frac{(2\theta+1)p'}{\theta+1}}\delta\,dx\,dt\right)^{\frac{2(q_1-1)}{q_1 p'}} + 1\right), \end{aligned} \tag{3.99}$$

where $2 \le q_1 < \frac{2(N+1)}{N-1}$. Since $p = \frac{N+3}{2}$, we can choose q_1 such that $s > \frac{(2\theta+1)p'}{\theta+1}$. Hölder's inequality and (3.99) yield

$$\int_Q |w_n|^{(\theta+1)s}\delta\,dx\,dt \le C\left(\left(\int_Q |w_n|^{(\theta+1)s}\delta\,dx\,dt\right)^{\frac{2(q_1-1)}{q_1p'}} + 1\right). \tag{3.100}$$

By virtue of $\frac{2(q_1-1)}{q_1p'} < 1$, then by using Young's inequality, we get

$$\int_Q |w_n)|^{(\theta+1)s}\delta(x)\,dx \le C. \tag{3.101}$$

Thus from (3.98) and (3.101) it follows that

$$\int_Q |D(|w_n|^\theta w_n)|^2\delta + ||w_n|^\theta w_n|^2\delta\,dx\,dt \le C. \tag{3.102}$$

Doing the same work as that of (3.86) we obtain

$$\begin{aligned}\int_0^T\left(\int_\Omega ||w_n|^\theta w_n|^{q_3}\,dx\right)^{\frac{2}{q_3}} dt &\le C\int_Q |D(|w_n|^\theta w_n)|^2\delta + |(|w_n|^\theta w_n)|^2\delta\,dx\,dt \\ &\le C,\end{aligned} \tag{3.103}$$

where $q_3 < \frac{2N}{N-1}$.

Set $\bar{q} = (\theta+1)s,\ q_0 = (\theta+1)q_3$. Due to θ is an arbitrary nonegative real number, then $\bar{q}$ and q_0 are two arbitrary nonegative finite real numbers. Thus Lemma 3.7 is proved. □

4. Proofs of the main results

In this section, we will finish the proofs of Theorems 1.1–1.7. Because the proofs of Theorems 1.2–1.7 are similar to that of Theorem 1.1, here we only give the proof of Theorem 1.1.

Proof of Theorem 1.1. To establish the compactness in the weighted L^1 space, we need the following truncated function

$$h_k(s) = \begin{cases} 1 & \text{if } |s| \le k, \\ 1 - s + k & \text{if } k < s \le k+1, \\ 1 + s + k & \text{if } -k-1 \le s < -k, \\ 0 & \text{if } |s| > k+1. \end{cases} \tag{4.1}$$

Let

$$H_k(s) = \int_0^s h_k(\tau)\,d\tau,\ \forall s \in \mathbf{R},\ \forall k > 0. \tag{4.2}$$

If we multiply the approximate equation of the problem (P_n) by $h_k(u_n)$, we get in the sense of distributions

$$(H_k(u_n))_t = \operatorname{div}(mh_k(u_n)|u_n|^{m-1}Du_n) - m|u_n|^{m-1}|Du_n|^2h_k'(u_n) + f_nh_k(u_n). \tag{4.3}$$

Note that $\text{supp}(h_k) \subseteq [-k-1, k+1]$, $0 \le h_k \le 1$, $|h_k'| \le 1$. If $n > k+1$,

$$\begin{aligned} mh_k(u_n)|u_n|^{m-1}Du_n &= mh_k(u_n)|T_{k+1}(u_n)|^{m-1}DT_{k+1}(u_n) \\ &= h_k(u_n)D(|T_{k+1}(u_n)|^{m-1}T_{k+1}(u_n)), \end{aligned} \tag{4.4}$$

$$\begin{aligned} m|u_n|^{m-1}|Du_n|^2h_k'(u_n) &= m|T_{k+1}(u_n)|^{m-1}|DT_{k+1}(u_n)|^2h_k'(u_n) \\ &= \frac{1}{m}|T_{k+1}(u_n)|^{1-m}|D(|T_{k+1}(u_n)|^{m-1}T_{k+1}(u_n))|^2h_k'(u_n) \\ &\le \frac{1}{m}(k+1)^{1-m}|D(|T_{k+1}(u_n)|^{m-1}T_{k+1}(u_n))|^2. \end{aligned} \tag{4.5}$$

By Lemma 3.1 and (4.1)–(4.5), for fixed $k > 0$, we deduce $mh_k(u_n)|u_n|^{m-1}Du_n$ is bounded in $L^2(Q,\delta)$, and $m|u_n|^{m-1}|Du_n|^2h_k'(u_n)$ is bounded in $L^1(Q,\delta)$. Hence $(H_k(u_n))_t$ is bounded in $L^2(0,T;(W_0^{1,2}(\Omega,\delta))^*)) + L^1(Q,\delta)$. By virtue of $DH_k(u_n) = h_k(u_n)Du_n = h_k(u_n)DT_{k+1}(u_n) = \frac{1}{m}h_k(u_n)D(|T_{k+1}(u_n)|^{m-1}T_{k+1}(u_n))|T_{k+1}(u_n)|^{1-m}$, (3.3) implies that $H_k(u_n)$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega,\delta))$. Hence a compactness result (see Corollary 4 in [26]) allows to conclude that $H_k(u_n)$ is compact in $L^1(Q,\delta)$. Thus there exists a subsequence of $\{H_k(u_n)\}$ (still be denoted by $\{H_k(u_n)\}$) such that it also converges in measure and almost everywhere in Q.

For all $\sigma > 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} \text{meas}_\delta\{|u_n - u_m| > \sigma\} \le{}& \text{meas}_\delta\{|u_n| > k\} + \text{meas}_\delta\{|u_m| > k\} \\ &+ \text{meas}_\delta\{|H_k(u_n) - H_k(u_m)| > \sigma\}. \end{aligned} \tag{4.6}$$

By (3.1) in Lemma 3.1, we can choose k large enough to have

$$\text{meas}_\delta\{|u_n| > k\} + \text{meas}_\delta\{|u_m| > k\} < \frac{\varepsilon}{2}, \quad \forall n, m. \tag{4.7}$$

Furthermore, for the above fixed k, we can choose a large $\bar{N}$ such that

$$\text{meas}_\delta\{|H_k(u_n) - H_k(u_m)| > \sigma\} < \frac{\varepsilon}{2}, \quad \forall n, m > \bar{N}. \tag{4.8}$$

Now (4.6), (4.7) and (4.8) yield

$$\text{meas}_\delta\{|u_n - u_m| > \sigma\} < \varepsilon, \quad \forall n, m > \bar{N}, \tag{4.9}$$

and (4.9) implies that $\{u_n\}$ is a Cauchy sequence in measure in Q. Hence there exists a measurable function u such that

$$u_n \longrightarrow u \ \text{ a.e. in Q.} \tag{4.10}$$

Now (3.1) in Lemma 3.1, (4.10) and Fatou's lemma yield $u \in L^\infty(0,T;L^1(\Omega,\delta))$. By (4.10) and (3.2)–(3.3) in Lemma 3.1 and Vitali's theorem, as $n \to \infty$ we have

$$\begin{aligned} |u_n|^{m-1}u_n \longrightarrow |u|^{m-1}u \ &\text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega,\delta)) \\ &\forall 1 \le q < \frac{m(N+1)+2}{m(N+1)+1}, \end{aligned} \tag{4.11}$$

$$|u_n|^{m-1}u_n \longrightarrow |u|^{m-1}u \ \text{ strongly in } L^{\bar{q}}(\Omega,\delta)\ \forall 1 \le \bar{q} < \frac{m(N+1)+2}{m(N+1)}, \tag{4.12}$$

$$\begin{aligned} |u_n|^{m-1}u_n \longrightarrow |u|^{m-1}u \ &\text{ weakly in } L^q(0,T;L^{q_0}(\Omega)) \\ &\forall 1 \le q_0 < \frac{mN(N+1)+2N}{mN(N+1)+N-1}. \end{aligned} \tag{4.13}$$

Due to $1-\frac{2}{N+1}<m<1$ and $\bar{q}<\frac{m(N+1)+2}{m(N+1)}$, we can choose $\bar{q}$ such that $m\bar{q}>1$. Thus from (4.10), (4.12) and Vitali's theorem, we can obtain

$$u_n \longrightarrow u \text{ strongly in } L^{m\bar{q}}(\Omega,\delta). \tag{4.14}$$

From (4.11)–(4.13), it follows that (1.9) holds. For any given $\varphi\in C^\infty(\bar{Q})$, $\varphi=0$ on Σ, $\varphi(x,T)=0$ and taking $v=\varphi$ in (P') and integrating it over $(0,T)$, we have

$$\int_Q u_{nt}\varphi\,dx\,dt+\int_Q D(|u_n|^{m-1}u_n)D\varphi\,dx\,dt=\int_Q f_n\varphi\,dx\,dt. \tag{4.15}$$

By using integration by parts for the left-hand side of (4.15), we get

$$-\int_Q u_n\varphi_t\,dx\,dt-\int_Q |u_n|^{m-1}u_n\Delta\varphi\,dx\,dt=\int_Q f_n\varphi\,dx\,dt+\int_\Omega u_{0n}(x)\varphi(x,0)\,dx. \tag{4.16}$$

Let $n\to\infty$ in (4.16), (4.13) and (4.14) yield

$$-\int_Q u\varphi_t\,dx\,dt-\int_Q |u|^{m-1}u\Delta\varphi\,dx\,dt=\int_Q f\varphi\,dx\,dt+\int_\Omega u_0(x)\varphi(x,0)\,dx. \tag{4.17}$$

Thus we obtain u is a very weak solution to the problem (P) in the sense of Definition 1.1. So the proof of Theorem 1.1 is finished. □

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References

[1] Adams, R. A., and J. J. F. Fournier: Sobolev spaces. Second edition. - Pure and Applied Mathematics (Amsterdam) 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] Alt, H. W., and S. Luckhaus: Quasilinear elliptic-parabolic differential equations. - Math. Z. 183:3, 1983, 311–341.

[3] Bénilan, P., L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez: An L^1-theory of existence and uniqueness of solutions of non-linear elliptic equations. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 22:2, 1995, 241–273.

[4] Bénilan, P., H. Brezis, and M. G. Crandall: A semilinear equation in $L^1(R^N)$. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 2:4, 1975, 523–555.

[5] Bidaut-Véron, M. F., and L. Vivier: An elliptic semilinear equation with source term involving boundary measures: the subcritical case. - Rev. Mat. Iberoam. 16:3, 2000, 477–513.

[6] Blanchard, D., and G. A. Francfort: A few results on a class of degenerate parabolic equations. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 18:2, 1991, 213–249.

[7] Boccardo, L., and T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data. - J. Funct. Anal. 87:1, 1989, 149–169.

[8] Boccardo, L., and T. Gallouët: Nonlinear elliptic equations with right-hand side measures. - Comm. Partial Differential Equations 17:3-4, 1992, 641–655.

[9] Brezis, H., T. Cazenave, Y. Martel, and A. Raminandrisoa: Blow up for $u_t-\Delta u=g(u)$ revisited. - Adv. Differential Equations 1:1, 1996, 73–90.

[10] Dall'aglio, A., D. Giachetti, C. Leone, and S. Segura De León: Quasi-linear parabolic equations with degenerate coercivity having a quadratic gradient term. - Ann. Inst. H. Poincaré Anal. Non Linéaire 23:1, 2006, 97–126.

[11] Díaz, J. I., and J. M. Rakotoson: On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary. - J. Funct. Anal. 257:3, 2009, 807–831.

[12] Díaz, J. I., and J. M. Rakotoson: On very weak solutions of semi-linear elliptic equations in the framework of weighted spaces with respect to the distance to the boundary. - Discrete Contin. Dyn. Syst. 27:3, 2010, 1037–1058.

[13] Di Benedetto, E.: Degenerate parabolic equations. - Springer-Verlag, New York, 1993.

[14] Filo, J., and J. Kačur: Local existence of general nonlinear parabolic systems. - Nonlinear Anal. 24:11, 1995, 1597–1618.

[15] Giachetti, D., and G. Maroscia: Porous medium type equations with a quadratic gradient term. - Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 10:3, 2007, 753–759.

[16] Giachetti, D., and G. Maroscia: Existence results for a class of porous medium type equations with a quadratic gradient term. - J. Evol. Equ. 8:1, 2008, 155–188.

[17] Gurka, P., and B. Opic: Continuous and compact imbeddings of weighted Sobolev spaces. I, II. - Czechoslovak Math. J. 38:4, 1988, 730–744; 39:1, 1989, 78–94.

[18] Krasnosel'skii, M., and Y. Rutickii: Convex functions and Orlicz spaces. - Noordhoff, Groningen, 1969.

[19] Kufner, A.: Weighted Sobolev spaces. - John Wiley Sons, Chichester, 1985.

[20] Ladyzhenskaja, O. A., V. A. Solonikov, and N. N. Ural'ceva: Linear and quasilinear equations of parabolic type. - Transl. Math. Monogr., Amer. Math. Soc., Providence. RI, 1968.

[21] Lions, J. L.: Quelques méthodes de résolution des problémes aux limites nonlinéaires. - Dunod, Paris, 1969.

[22] Lukkari, T.: The porous medium equation with measure data. - J. Evol. Equ. 10:3, 2010, 711–729.

[23] Lukkari, T.: The fast diffusion equation with measure data. - NoDEA Nonlinear Differential Equations Appl. 19, 2012, 329–343.

[24] Porretta, A.: Existence results for nonlinear parabolic equations via strong convergence of truncations. - Ann. Mat. Pura Appl. 177, 1999, 143–172.

[25] Rakotoson, J. M.: Regularity of a very weak solution for parabolic equations and applications. - Adv. Differential Equations 16:9-10, 2011, 867–894.

[26] Simon, J.: Compact sets in the space $L^p(0,T;B)$. - Ann. Mat. Pura Appl. 146, 1987, 65–96.

[27] Souplet, P., and P. Quittner: Superlinear parabolic problems. Blow-up, global existence and steady states. - Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser, Basel, 2007.

[28] Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. - Ann. Inst. Fourier (Grenoble) 15:1, 1965, 189–258.

[29] Vázquez, J. L.: Smoothing and decay estimates for nonlinear parabolic equations of porous medium type. - Oxford Univ. Press, Oxford, 2006.

[30] Vázquez, J. L.: The porous medium equation. Mathematical theory. - Oxford Math. Monogr., The Clarendon Press, Oxford Univ. Press, Oxford, 2007.

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