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COMPLEX RICCATI DIFFERENTIAL EQUATIONS REVISITED

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Abstract. We utilise a new approach via the so-called re-scaling method to derive a thorough theory for polynomial Riccati differential equations in the complex domain.

1. Introduction

The basic features concerning the value distribution of the solutions to Riccati differential equations

$$w' = a_0(z) + a_1(z)w + a_2(z)w^2 \tag{1}$$

with polynomial coefficients are well understood due to the pioneering work of Wittich (see his book [15], Chapter V, pp. 73–80). The solutions are meromorphic in the complex plane, and every non-rational solution has order of growth

$$\varrho = \limsup_{r\to\infty} \frac{\log T(r,w)}{\log r} = 1 + n/2 \tag{2}$$

mean type, where the non-negative integer n depends on the coefficients a_ν only. The aim of this paper is to refine the results of Wittich and others (Bank [1], Gundersen [5], Hellerstein and Rossi [7, 8]; see also Laine's book [9], Chapter 5) on equation (1) and the associated linear differential equation (set $a_2w = -u'/u$)

$$u'' - \Big(\frac{a_2'(z)}{a_2(z)} + a_1(z)\Big)u' + a_0(z)a_2(z)u = 0$$

by a new approach which has been developed earlier to investigate the solutions of Painlevé differential equations (see [12]). By a simple change of variables (retaining the original notation z, w) we obtain

$$w' = a(z) - w^2 \tag{R}$$

with

$$a(z) = z^n + O(|z|^{n-1}) \quad (z\to\infty). \tag{3}$$

Up to finitely many, all poles are simple with residue 1; w has counting function

$$n(r,w) = O(r^\varrho). \tag{4}$$

Our proofs are solely based on the estimate (4), a new existence proof for asymptotic expansions, and the method of re-scaling.

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2. Re-scaling and the distribution of poles

Throughout the whole paper w denotes any non-rational solution to the Riccati equation (R). For $h \neq 0$ we set

$$w_h(\mathfrak{z}) = h^{-n/2} w(h + h^{-n/2}\mathfrak{z}),$$

where $h^{-n/2}$ denotes any branch, the same at every occurrence ($h^{-n/2}h^{-n/2} = h^{-n}$).

Theorem 1. *The re-scaled family* $(w_h)_{|h|>1}$ *is normal in the sense of Montel, and every limit function* $\mathfrak{w} = \lim_{h_n\to\infty} w_{h_n}$ *satisfies the differential equation*

$$\mathfrak{w}' = 1 - \mathfrak{w}^2. \tag{5}$$

We note that the solution $\mathfrak{w} = \coth \mathfrak{z}$ with pole at the origin has the poles $k\pi i$, $k \in \mathbf{Z}$, and no others. Any sequence $\sigma = (p_k)$ satisfying the approximate recursion

$$p_{k+1} = p_k + \omega p_k^{-n/2} + o(|p_k|^{-n/2}) \tag{6}$$

with $\omega = \pm i\pi$ fixed is called a *string*.

Theorem 2. *Let w be any solution to* (R). *Then the set of poles on $|z| > r_0$ consists of finitely many* strings of poles. *Each string σ accumulates at some* Stokes ray

$$s_\nu\colon \ \arg z = \theta_\nu = \frac{(2\nu+1)\pi}{n+2} \tag{7}$$

and has counting function

$$n(r,\sigma) = \frac{r^\varrho}{\pi\varrho} + o(r^\varrho).$$

Remark. We note that w has *Nevanlinna characteristic* $T(r,w) = \ell\dfrac{r^\varrho}{\pi\varrho^2} + o(r^\varrho)$, where $\ell = \ell(w)$ denotes the number of strings of poles.

3. Stokes sectors and asymptotic expansions

The open sectors

$$S_\nu\colon \left|\arg z - \frac{2\nu\pi}{n+2}\right| < \frac{\pi}{n+2}$$

are called *Stokes sectors*. They are bounded by the Stokes rays s_ν and $s_{\nu-1}$, and will be enumerated as follows:

(a) $0 \le \nu \le n+1$ if n is even, and

(b) $-m-1 \le \nu \le m+1$ if $n = 2m+1$ is odd.

In the second case $s_{-m-2} = s_{m+1}$ coincides with the negative real axis.

Let f be meromorphic on some sector $S\colon \phi_1 < \arg z < \phi_2$. Then f is said to have the *asymptotic expansion* $f \sim \sum_{k=0}^\infty c_k z^{-k/q}$ for some $q \in \mathbf{N}$, if for every $\delta > 0$ and every $n \in \mathbf{N}_0$

$$f(z) - \sum_{k=0}^n c_k z^{-k/q} = o(|z|^{-n/q}) \quad (z \to \infty)$$

is valid, uniformly on every sub-sector $S(\delta)\colon \phi_1 + \delta < \arg z < \phi_2 - \delta$. Obviously, the sector S is 'pole-free' for f in the following sense: to every $\delta > 0$ there exists $r(\delta) > 0$, such that f has no poles on $S(\delta)$, $|z| > r(\delta)$. It follows from Theorem 2 that the Stokes sectors S_ν are 'pole-free' for every solution to equation (R). By $\sqrt{z}$

we denote the branch of the square root with $\operatorname{Re}\sqrt{z} > 0$ on $|\arg z| < \pi$, and set $z^{n/2} = (\sqrt{z})^n$ if n is odd.

Theorem 3. *The function $z^{-n/2}w(z)$ has an asymptotic expansion*

(a) $\varepsilon + \sum_{k=1}^{\infty} c_k z^{-k}$ *if n is even, and*

(b) $\varepsilon + \sum_{k=1}^{\infty} c_k z^{-k/2}$ *if n is odd*

on every 'pole-free' sector S, with $\varepsilon = \varepsilon(w) \in \{-1, 1\}$ and coefficients c_k only depending on ε, but neither on w nor the sector S. The solution w is uniquely determined by its asymptotic expansion if S contains some sub-sector S' such that

$$\varepsilon \operatorname{Re} z^{\varrho} < 0 \quad \text{on } S'. \tag{8}$$

Remark. In particular, Theorem 3 holds on Stokes sectors S_ν with $\varepsilon = \varepsilon_\nu = \varepsilon_\nu(w)$. If (8) is valid on S_ν, then the corresponding solution is uniquely determined and is denoted by w_ν. With every solution w we associate its *symbol*

(a) $\Sigma = \Sigma(w) = [\varepsilon_0, \ldots, \varepsilon_{n+1}]$ if n is even, and

(b) $\Sigma = \Sigma(w) = [\varepsilon_{-m-1}, \ldots, \varepsilon_{m+1}]$ if $n = 2m + 1$ is odd.

Solutions having the symbol $\Sigma(w)$ with entries $\varepsilon_\nu = (-1)^\nu$ are called *generic*. Noting that $(-1)^\nu \operatorname{Re} z^{\varrho} > 0$ holds on S_ν, we obtain from Theorem 3:

Theorem 4. *Any generic solution w has counting function of poles*

$$n(r, w) = \frac{2r^{\varrho}}{\pi} + o(r^{\varrho}).$$

Theorem 5. *Suppose w has symbol Σ. Then w has*

(a) *no string of poles asymptotic to the Stokes ray s_ν if $\varepsilon_\nu = \varepsilon_{\nu+1}$,*

(b) *exactly one such string if $(-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = 2$, while*

(c) *$(-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = -2$ is impossible.*

If $n = 2m+1$ is odd and $\nu = m+1$, the term $\varepsilon_{\nu+1}$ has to be replaced by $-\varepsilon_{-m-1}$. In case (a)*, w has an asymptotic expansion on $\theta_{\nu-1} < \arg z < \theta_{\nu+1}$. Generic solutions have exactly one string of poles along* every *Stokes ray, and in any case we have*

$$n(r, w) = \frac{r^{\varrho}}{\pi\varrho} \sum_{\nu} (-1)^\nu \epsilon_\nu + o(r^{\varrho}).$$

4. Exceptional solutions

The non-generic solutions are called *exceptional*. Exceptional solutions w_ν have the 'false' asymptotics

$$w_\nu \approx -(-1)^\nu z^{n/2} \quad \text{on } S_\nu \tag{9}$$

and are uniquely determined by that condition.

Example 1. The Riccati equation $w' = z^2 + a_0 - w^2$ is closely related to the *Weber–Hermite* equation

$$y' = y^2 + 2zy - 2 - 2\alpha \quad (w = -y - z,\ a_0 = 1 + 2\alpha).$$

There are four exceptional solutions which may be described by their respective symbols $[-\mathbf{1}, -1, 1, -1]$, $[1, \mathbf{1}, 1, -1]$, $[1, -1, -\mathbf{1}, -1]$, and $[1, -1, 1, \mathbf{1}]$. The poles are distributed along two rays: $|\arg z - \pi| = \frac{\pi}{4}$, $|\arg z + \frac{\pi}{2}| = \frac{\pi}{4}$, $|\arg z| = \frac{\pi}{4}$, and $|\arg z - \frac{\pi}{2}| = \frac{\pi}{4}$, respectively.

Example 2. The Riccati equation $w' = z + a_0 - w^2$ is closely related to the *Airy* equation $y' = z/2 + y^2$. It has three exceptional solutions with symbols $[-1, -\mathbf{1}, -1]$, $[\mathbf{1}, 1, -1]$, and $[-1, 1, \mathbf{1}]$, and strings of poles asymptotic to (actually: *on*) $\arg z = \pi$, $\arg z = \pi/3$, and $\arg z = -\pi/3$, respectively.

Theorem 6. *To every Stokes sector S_ν there exists a unique exceptional solution w_ν. It has the asymptotic expansion* (9) *also on the Stokes sectors adjacent to S_ν, and no strings of poles along the Stokes rays that form the boundary of S_ν. The number $d_\nu = n - \ell_\nu$, where ℓ_ν denotes the number of strings of poles of w_ν, is even.*

Remark. The exceptional solutions w_ν correspond to those solutions to the linear differential equation $y'' = a(z)y$ that are *sub-dominant* on S_ν; $y_\nu = \exp \int w(z)\,dz$ is called sub-dominant on S_ν, if y_ν tends to zero exponentially as $z \to \infty$ on S_ν.

Example 3. Gundersen and Steinbart [6] considered the linear differential equation $f'' - z^n f = 0$. They proved among others that certain contour integrals

$$f_\nu(z) = \frac{1}{2\pi i} \int_{C_\nu} e^{P(z,w)}\,dw$$

represent solutions having no zeros along given Stokes rays $s_{\nu-1}$ and s_ν. These solutions give rise to exceptional solutions $w_\nu = f_\nu'/f_\nu$ to the special Riccati equation $w' = z^n - w^2$, which is invariant under the transformations $w(z) \mapsto \eta w(\eta z)$, $\eta^{n+2} = 1$. There are exactly two solutions that are invariant under these transformations, namely those which either have a pole or else a zero at the origin. These solutions are generic, hence there are $n+2$ mutually distinct exceptional solutions. They are obtained from a single one, w_0, say, by rotating the plane:

$$w_\nu(z) = e^{\frac{2\nu\pi i}{n+2}} w_0\big(e^{\frac{2\nu\pi i}{n+2}} z\big);$$

w_ν has a single string of poles along every Stokes ray s_μ except those that bound the Stokes sector S_ν.

In the general case (R) the solutions w_ν need not be mutually distinct.

Example 4. The eigenvalue problem $f'' + (z^4 - \lambda)f = 0$, $f \in L^2(\mathbf{R})$, has infinitely many solutions (λ_k, f_k) $(0 < \lambda_k \to \infty)$, see Titchmarsh [13]. The eigenfunctions f_k have only finitely many non-real zeros. For every eigenpair $(\lambda, f) = (\lambda_k, f_k)$, $u(z) = f(e^{-i\pi/6}z)$ satisfies $u'' - (z^4 + e^{-i\pi/3}\lambda)u = 0$, and $w = u'/u$ solves

$$w' = z^4 + e^{-i\pi/3}\lambda - w^2.$$

Up to finitely many the poles of the exceptional solution $w = w_2 = w_5$ belong to the rays $\arg z = \frac{\pi}{6}$ and $\arg z = \frac{7}{6}\pi$, hence w has the symbol $[1, -1, -\mathbf{1}, -1, 1, \mathbf{1}]$.

Example 5. Eremenko and Gabrielov [2] considered the linear equation

$$y'' - (z^3 - az + \lambda)y = 0.$$

For certain real parameters a and λ it has solutions with infinitely many zeros, only finitely many of them are non-real or real and positive. Thus $w' = z^3 - az + \lambda - w^2$ has a solution w with symbol $[1, \mathbf{1}, 1, \mathbf{1}, 1]$, hence $w = w_1 = w_{-1}$, and mutually distinct solutions w_0, w_{-2}, and w_2 with symbols $[1, -1, -\mathbf{1}, -1, 1]$, $[-\mathbf{1}, -1, 1, -1, 1]$, and $[1, -1, 1, -1, -\mathbf{1}]$, respectively, each having three strings of poles.

5. Poles close to a single line

Several papers (Eremenko and Merenkov [3], Eremenko and Gabrielov [2], Gundersen [4, 5], Shin [11]) are devoted to the question whether or not the linear differential equation

(10) $\quad y'' - P(z)y = 0 \quad (P(z) = a_n z^n + \cdots$ a polynomial of degree n, $|a_n| = 1)$

has solutions with all but finitely many zeros on the real axis. From Theorem 5 we obtain (see also [3, 4]):

Theorem 7. *Suppose that equation* (10) *has a solution whose zeros are asymptotic to the real axis. Then the following is true:*
If n is even, then either

- *y has only finitely many zeros, or else*
- *$n \equiv 0 \mod 4$, $a_n = -1$, y has exactly one string of zeros asymptotic to the negative and positive real axis, and $y'/y \approx \mp i z^{n/2}$ holds on the upper and lower half-plane, respectively.*

If $n = 2m + 1$ is odd, then either

- *$a_n = 1$, y has exactly one string of poles asymptotic to the negative real axis with asymtotics $y'/y \approx (-1)^{m+1} z^{n/2}$ on $|\arg z| < \pi$, or else*
- *$a_n = -1$, y has exactly one string of poles asymptotic to the positive real axis with asymtotics $y'/y \approx (-1)^{m+1}(-z)^{n/2}$ on $|\arg(-z)| < \pi$.*

If P is real, then in each case all but finitely many zeros are real and y is a (multiple of a) *real entire function.*

6. The Schwarzian derivative

In [10] Nevanlinna considered the locally univalent meromorphic functions f of finite order. They are characterised by the fact that their Schwarzian derivative $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$ is a polynomial $2P$, say. Moreover, f is the quotient $y(z;0)/y(z;\infty)$ of two linearly independent solutions to the linear differential equation

$$y'' + P(z)y = 0,$$

which is equivalent to the Riccati equation $w' = -P(z) - w^2$ via $w = y'/y$. The generic solutions have counting function of poles and Nevanlinna characteristic $T(r, w) \sim Cr^\varrho$ with $\varrho = 1 + \frac{1}{2}\deg P$; $C > 0$ is some known constant. Every exceptional solution w_ν, however, has counting function and Nevanlinna characteristic $T(r, w_\nu) \sim C\,\frac{n+2-2d_\nu}{n+2}\,r^\varrho$, where d_ν is some positive integer such that $\sum_\nu d_\nu = n + 2$. Since the zeros of $f - a$ are the same as the zeros of $y(z;a) = y(z;0) - ay(z;\infty)$, hence coincide with the poles of $w(z;a) = y'(z;a)/y(z;a)$, it follows that f has Nevanlinna deficiencies $\delta(a_\nu) = \frac{2d_\nu}{n+2}$ $(w_\nu(z) = w(z;a_\nu))$ with $\sum_\nu \delta(a_\nu) = 2$.

7. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. From

$$w_h'(\mathfrak{z}) = h^{-n}a(h + h^{-n/2}\mathfrak{z}) + w_h(\mathfrak{z})^2$$

and $z^{-n}a(z) \to 1$ as $z \to \infty$ it follows that

$$|w_h'(\mathfrak{z})| \le 2 + |w_h(\mathfrak{z})|^2$$

holds on $|\mathfrak{z}| < R$, $|h| > \eta_R$. Thus the family $(w_h^\sharp)_{|h|\ge 1}$ of *spherical derivatives*

$$w_h^\sharp = \frac{|w_h'|}{1+|w_h|^2}$$

is bounded on $|\mathfrak{z}| < R$ by $M(R) = \sup\{w_h^\sharp(\mathfrak{z})\colon |\mathfrak{z}| < R,\ 1 < |h| < \eta_R\} + 2$, say. The limit function $\mathfrak{w} = \lim_{h_k\to\infty} w_{h_k} \equiv \infty$ does not occur since otherwise $u_{h_k} = 1/w_{h_k}$ would tend to zero, this contradicting $u_{h_k}' = 1 - h_k^{-n} a(h_k + h_k^{-n/2}\mathfrak{z}) u_{h_k}^2 \to 1$. Thus every limit function $\mathfrak{w}$ satisfies (5) outside the set $\mathfrak{P}$ of poles of $\mathfrak{w}$. □

Proof of Theorem 2. From Theorem 1 and Hurwitz' Theorem it follows that given $\epsilon > 0$ and $R > 0$ there exists some $r_0 > 0$, such that the disc

$$\triangle_R(p) = \{z\colon |z-p| < R|p|^{-n/2}\}$$

about any pole p with $|p| > r_0$ contains the poles $\tilde{p}_k$ with

$$|\tilde{p}_k - (p + k\pi i p^{-n/2})| < \epsilon |p|^{-n/2} \quad (-k_1(p) \le k \le k_2(p)),$$

and no others; the numbers k_1 and k_2 are bounded by a number only depending on R (for example, $k_1 = k_2 = 318$ if $R = 1000$ and r_0 is sufficiently large). Thus up to finitely many every pole is contained in a unique string of poles (p_k) satisfying (6). Then $z_k = p_k^\varrho$ ($\varrho = n/2 + 1$) satisfies

$$z_{k+1} = z_k + \omega\varrho + o(1)$$

with $\omega = \pm\pi i$ fixed, hence $z_k = \omega\varrho k + o(k)$, $p_k = (\omega\varrho k)^{1/\varrho}(1 + o(1))$, and

$$\frac{n+2}{2}\arg p_k = \arg\omega + o(1) = \pm\frac{\pi}{2} + o(1) \mod 2\pi,$$

that is, $\arg p_k = \theta_\nu + o(1) = \frac{2\nu\pi+1}{n+2} + o(1)$ holds for some ν. The counting function of σ equals $n(r,\sigma) = \frac{r^\varrho}{\pi\varrho} + o(r^\varrho)$, and from $n(r,w) = O(r^\varrho)$ it follows that there are only finitely many strings of poles. □

8. Proof of Theorem 3

Let w be any solution to (R) and $S\colon |\arg z - \phi_0| < \eta$ any sector that is 'pole-free' for w. From Theorem 1 then it follows that $w(z)z^{-n/2}$ tends to either $+1$ or else -1 as $z \to \infty$; the convergence to $+1$, say, is uniform on each closed sub-sector $S(\delta)\colon |\arg z - \phi_0| \le \eta - \delta$ (take any sequence $h_k \to \infty$ in $S(\delta)$ such that $\lim_{h_k\to\infty} |w(h_k)h_k^{-n/2} - 1| = \limsup_{z\to\infty} |w(z)z^{-n/2} - 1|$ on $S(\delta)$). If $n = 2m$ is even we set $v(z) = z^{-m}w(z)$ to obtain

$$z^{-m}v' + mz^{-m-1}v = a(z)z^{-2m} - v^2. \tag{11}$$

If, however, $n = 2m+1$ is odd we set $v(z) = z^{-n}w(z^2)$ to obtain

$$z^{-n-1}v' + nz^{-n-2}v = 2a(z^2)z^{-2n} - 2v^2. \tag{12}$$

From (11) resp. (12) and the fact that $v(z) \to \pm 1$ on some sector S we have to conclude $v \sim \pm 1 + \sum_{k=1}^\infty c_k z^{-k}$ on S. For definiteness we will consider equation (11) with $v(z) \to 1$ on S. If we assume that

$$v(z) = 1 + \sum_{k=1}^n c_k z^{-k} + o(|z|^{-n}) = \psi_n(z) + o(|z|^{-n})$$

has already been proved (this is true for $n=0$) we obtain from

$$v'(z)=\psi_n'(z)+o(|z|^{-n-1})$$

and (11)

$$a(z)z^{-2m}-v^2=z^{-m}\psi_n'(z)+mz^{-m-1}\psi_n(z)+o(|z|^{-n-m-1}).$$

The algebraic equation

$$a(z)z^{-2m}-y^2=z^{-m}\psi_n'(z)+mz^{-m-1}\psi_n(z)$$

has a unique solution $y=1+\sum_{k=1}^\infty c_k'z^{-k}$ about $z=\infty$, and from $v+y=2+o(1)$ and $(v-y)(v+y)=v^2-y^2=o(|z|^{-n-m-1})$ it follows that

$$v=y+o(|z|^{-n-m-1})=1+\sum_{k=1}^{n+1}c_k'z^{-k}+o(|z|^{-n-1})=\psi_{n+1}(z)+o(|z|^{-n-1}).$$

It is obvious that $c_k=c_k'$ holds for $0\le k\le n$, and this proves the existence part. The proof is the same in all other cases.

To prove the uniqueness part of Theorem 3 we assume that w_1 and w_2 have the same asymptotic expansion on the sector S. Then $u=w_1-w_2$ solves

$$u'=-(w_1(z)+w_2(z))u=-2\varepsilon z^{n/2}(1+O(|z|^{-\frac{1}{2}}))u,$$

hence $u=C\exp(-\frac{2\varepsilon}{\varrho}z^\varrho+O(|z|^{\varrho-\frac{1}{2}}))$ holds. Our hypothesis $\varepsilon\mathrm{Re}\,z^\varrho<0$ and $u\to0$ on $S'\subset S$ then gives $u=C=0$, and this proves Theorem 3 completely. □

9. Proof of Theorem 5

Since all but finitely many poles of w are simple with residue 1, the Residue Theorem gives

$$n(r,w)=\frac{1}{2\pi i}\int_{\Gamma_r}w(z)\,dz+O(1), \tag{13}$$

where the simple closed curve Γ_r is obtained from the circle $C_r\colon|z|=r$ by replacing the intersection of C_r with any disc $\triangle_\epsilon(p)=\{z\colon|z-p|<\epsilon|p|^{-n/2}\}$ ($\epsilon>0$ sufficiently small, p any pole of w) by an appropriate sub-arc of $\partial\triangle_\epsilon(p)$. From $w=O(|z|^{n/2})=O(|z|^{\varrho-1})$ on Γ_r (this following from the normality of the family $w_h(\mathfrak{z})=h^{-n/2}w(h+h^{-n/2}\mathfrak{z})$) and the fact that $\Gamma_r\cap\{z\colon|\arg z-\theta_\nu|<\delta\}$ has length at most $2\pi\delta r$ as $\delta\to0$, it follows that the contribution of the Stokes sector S_ν to the counting function of poles equals

$$(-1)^\nu\varepsilon_\nu\frac{r^\varrho}{\pi\varrho}+o(r^\varrho)\quad(\varrho=n/2+1).$$

In particular, w has $\sum_\nu(-1)^\nu\varepsilon_\nu$ strings of poles. Integrating w along the line segment σ from $r_0e^{i(\theta_\nu-\delta)}$ ($\delta>0$ small, $r_0>0$ large) to $re^{i(\theta_\nu-\delta)}$ gives

$$\frac{1}{2\pi i}\int_\sigma w(z)\,dz=\frac{\varepsilon_\nu}{2\pi i\varrho}r^\varrho e^{i\varrho(\theta_\nu-\delta)}+o(r^\varrho)=(-1)^\nu\frac{\varepsilon_\nu}{2\pi\varrho}e^{-i\varrho\delta}r^\varrho+o(r^\varrho).$$

Thus, if γ_r^ν denotes the simple closed curve which consists of the line segment σ, the part of Γ_r from $re^{i(\theta_\nu-\delta)}$ to $re^{i(\theta_\nu+\delta)}$, the line segment from $re^{i(\theta_\nu+\delta)}$ to $r_0e^{i(\theta_\nu+\delta)}$, and the circular arc on $|z|=r_0$ from $r_0e^{i(\theta_\nu+\delta)}$ to $r_0e^{i(\theta_\nu-\delta)}$ we obtain

$$\frac{1}{2\pi i}\int_{\gamma_r^\nu}w(z)\,dz=(-1)^\nu\frac{r^\varrho}{2\pi\varrho}[\varepsilon_\nu-\varepsilon_{\nu+1}]+O(\delta r^\varrho)+o(r^\varrho)$$

$(r \to \infty,\ \delta \to 0)$. Now the integral on the left hand side equals the number of poles inside γ_r^ν, while $(-1)^\nu \frac{1}{2}[\varepsilon_\nu - \varepsilon_{\nu+1}]$ coincides with the number of strings of poles along the Stokes ray s_ν: $\arg z = \theta_\nu$. From this the assertions (a), (b), and (c) in Theorem 5 immediately follow. □

10. Proof of Theorem 6

It is easily seen that equation (11) resp. (12), written as

$$z^{-q} v' = f(z, v) \quad (q = m \text{ resp. } q = n+1) \tag{14}$$

has a formal solution $\varepsilon_\nu + \sum_{\nu=1}^\infty c_\nu z^{-\nu}$ with $\varepsilon_\nu = -(-1)^\nu$. Since $\lim_{z\to\infty} f_v(z, \varepsilon_\nu) = -2\varepsilon_\nu \neq 0$, Theorem 12.1 in Wasow's monograph [14] applies to the corresponding equation for $v - \varepsilon_\nu$. Hence to every sector $|\arg z - \theta_0| < \frac{\pi}{2q+2}$ there exists a solution to equation (14) with asymptotic expansion $v \sim \varepsilon_\nu + \sum_{\nu=1}^\infty c_\nu z^{-\nu}$. In particular, for every ν we obtain a (unique) solution $w = w_\nu$ to (R) with the desired asymptotic expansion (9) on the Stokes sector S_ν. □

11. Proof of Theorem 7

If $y(z) = P_1(z)e^{P_2(z)}$ has only finitely many zeros, then $n = 2\deg P_2 - 2$ is even, and not much more can be said (of course, P can be computed explicitly from P_1 and P_2). From now on we assume that y has infinitely many zeros. The change of variables $w(z) = \eta y'(\eta z)/y(\eta z)$ with $\eta^{n+2} a_n = 1$ transforms equation (10) into equation (R) with $a(z) = \eta^2 P(\eta z) = z^n + \cdots$, hence the question whether or not there are solutions y to (10) having infinitely many zeros, 'most' of them close to the real axis is transformed into the question for solutions w to (R) having just one string of poles asymptotic to some Stokes ray s_ν: $\arg z = \theta_\nu = \frac{(2\nu+1)\pi}{n+2}$ if n is odd, and asymptotic to the Stokes rays s_ν and $s_{\nu+m}$ if $n = 2m$ is even, respectively. This yields $\bar\eta = \pm e^{i\theta_\nu}$ up to an arbitrary root of unity of order $n+2$, and we are free to choose $\eta = e^{-i\frac{\pi}{n+2}}$ and $\nu = 0$ if n is even, and $\eta = \pm 1$ and $\nu = m+1$ if $n = 2m+1$ is odd. In the first case we obtain $a_n = -1$, and from Theorem 5 it follows that $\epsilon_0 - \epsilon_1 = 2$ and $(-1)^{m+1}(\epsilon_{m+1} - \epsilon_{m+2}) = 2$, hence $\epsilon_0 = 1$ and $\epsilon_1 = -1$, this implying $\epsilon_2 = \cdots = \epsilon_{m+1} = \epsilon_1 = -1$, $\epsilon_{m+2} = \cdots = \epsilon_{2m+1} = \epsilon_0 = 1$, $m = 2k$ and $n = 4k$. This proves the first part of Theorem 6.

In the second case we have $a_n = +1$ and $a_n = -1$ with zeros asymptotic to the negative and positive real axis, respectively, and asymptotic expansions $y'/y \approx (-1)^{m+1} z^{n/2}$ on $|\arg z| < \pi$ resp. $y'/y \approx (-1)^{m+1}(-z)^{n/2}$ on $|\arg(-z)| < \pi$ (note that $z^{n/2}$ means $(\sqrt{z})^n$).

Now y is uniquely determined up to a constant factor. Thus if P is a real polynomial, then the zeros of $y^*(z) = \overline{y(\bar z)}$ are also asymptotic to the real axis, hence y and y^* are linearly dependent, and y is a multiple of a real function with all but finitely many zeros real. □

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