ENERGY ESTIMATES FOR VARIATIONAL MINIMIZERS OF A PARABOLIC DOUBLY NONLINEAR EQUATION ON METRIC MEASURE SPACES

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Abstract. In this paper a variational approach is taken to study a doubly nonlinear parabolic equation. We consider energy estimates for parabolic minimizers related to this equation. These energy estimates play a fundamental role in obtaining Harnack estimates. Our treatment is done in general metric measure spaces with a doubling measure and a Poincaré inequality.

1. Introduction

This paper is a continuation to a recent paper [4]. We study parabolic minimizers which are related to the doubly nonlinear parabolic equation

$$(1.1) \qquad \frac{\partial(|u|^{p-2}u)}{\partial t} - \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$

Our study is carried out in the context of a general complete metric measure space (X, d, μ) with a doubling measure μ. We shall also assume that X supports a weak Poincaré inequality [1].

A real-valued function u on $\Omega \times (0, T)$, where Ω is a non-empty open set in X, is a parabolic minimizer if it satisfies the inequality

$$p \int_{\mathrm{supp}(\phi)} |u|^{p-2} u \frac{\partial \phi}{\partial t} \, d\mu \, dt + \int_{\mathrm{supp}(\phi)} g_u^p \, d\mu \, dt \le \int_{\mathrm{supp}(\phi)} g_{u+\phi}^p \, d\mu \, dt,$$

for every compactly supported $\phi \in \mathrm{Lip}(\Omega \times (0, T))$. Here we write g_u for the minimal p-weak upper gradient of u [1]. In the Euclidean case, we can take g_u to be $|\nabla u|$ almost everywhere, and it can be shown that every weak solution to (1.1) is a parabolic minizer and, conversely, every parabolic minimizer is a weak solution of (1.1). Parabolic minimizers have been studied in [6] and [7]. The regularity theory for parabolic minizers on metric measure spaces has been considered in [3], [4], and [5].

In [4] it is shown, that if X is a complete metric space equipped with a doubling measure μ such that the space supports also a weak Poincaré inequality, then a positive parabolic minimizer which is both bounded and bounded away from zero satisfies certain Caccioppoli type energy estimates and, moreover, Harnack estimates

hold if X is assumed to be geodesic. In the present note, we remove these extra technical assumptions concerning the lower and upper bound of a positive minimizer. More precisely, we show that any positive parabolic minimizer u satisfies the same fundamental energy estimates which are used in the article [4] as a basis for proving the Harnack inequality. The approach is purely variational and, in particular, does not base on equation (1.1).

More importantly, our technique for removing the extra assumptions bases on the dampening of the growth rate of powers of a minimizer. The traditional truncation argument is problematic in the context of parabolic (quasi)minimizers. Indeed, it is not known whether parabolic super- or subminimizers are produced by truncating minimizers. The dampening technique presented here circumvents this, and therefore, seems to be interesting in its own right and new.

Lastly, we mention that it seems feasible that the technical assumption $u \geq \rho > 0$ in the estimates for weak super- and subsolutions to the doubly nonlinear equation in Kinnunen–Kuusi [2] can be removed by using the dampening technique presented in the present note.

2. Energy estimates

The notation and parabolic setting in this paper are as in [4]. We refrain from giving a detailed discussion on the metric measure space context (doubling condition, upper gradients, weak Poincaré inequalities etc.) and content ourselves with refering to [4] or for a detailed study to a recent monograph [1]. Our standing assumptions in this paper are the following. By the triplet (X, d, μ) we denote a complete metric space X, where d is the metric and μ a Borel measure on X. The measure μ is supposed to be doubling. We also assume that X supports a weak $(1, p)$-Poincaré inequality, where $1 < p < \infty$. For any open set Ω of X we write $\Omega_T = \Omega \times (0, T)$, for $T > 0$, and the product measure $d\mu\, dt$ is written as $d\nu$. A minimal p-weak upper gradient of a real-valued function u on X is written as g_u. The reader not familiar with the metric space set-up may consider g_u to be the modulus of the distributional gradient $|\nabla u|$ and the Newtonian space $N^{1,p}$ to be the classical Sobolev space $W^{1,p}$.

We say that $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a *parabolic minimizer* related to the doubly nonlinear equation if the inequality

$$(2.1) \qquad p \int_{\mathrm{supp}(\phi)} |u|^{p-2} u \frac{\partial \phi}{\partial t}\, d\nu + \int_{\mathrm{supp}(\phi)} g_u^p\, d\nu \leq \int_{\mathrm{supp}(\phi)} g_{u+\phi}^p\, d\nu,$$

holds for all $\phi \in \mathrm{Lip}_0(\Omega_T) = \{f \in \mathrm{Lip}(\Omega_T) \colon \mathrm{supp}(f) \subset \Omega_T\}$. If (2.1) holds for all nonnegative $\phi \in \mathrm{Lip}_0(\Omega_T)$ a function $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a *parabolic supermin-imizer*; and a *parabolic subminimizer* if (2.1) holds for all nonpositive $\phi \in \mathrm{Lip}_0(\Omega_T)$.

Obtaining energy estimates for parabolic minimizers is based on testing u with a suitably chosen test function which depends on u itself, and then performing various basic techniques from analysis to obtain an estimate of the desired form. One of these techniques is partial integration with respect to time. In order to be able to do

this later on, we use a certain time regularization procedure to alter inequality (2.1) to a more convenient form for our purposes.

Let u be a nonnegative parabolic (sub- or super)minimizer, and let $\phi \in \mathrm{Lip}_0(\Omega_T)$ be a valid test function. For sufficiently small $\delta > 0$, we take a standard mollifier $\xi_\delta \in C_0^\infty(\mathbf{R})$ satisfying $\mathrm{supp}(\xi_\delta) \subset [-\delta, \delta]$, $\xi_\delta \geq 0$, and $\|\xi_\delta\|_{L^1(\mathbb{R})} = 1$, and denote $(\phi)_\delta(x,t) := \int_{-\delta}^{\delta} \phi(x, t-s)\xi_\delta(s)ds$ in Ω_T.

We test u with $\phi(x, t+s)$, where s is small enough. By (2.1), after conducting a change of variable in the first integral on the left hand side, then multiplying both sides of the obtained inequality with the standard mollifyer $\xi_\delta(s)$, and integrating the resulting expression with respect to s gives us, after using Fubini's theorem, the inequality

$$(2.2) \quad p \int_{\mathrm{supp}(\phi)} \left(u^{p-1}\right)_\delta \frac{\partial \phi}{\partial t} \, d\nu + \int_{\mathrm{supp}(\phi)} (g_u^p)_\delta \, d\nu \leq \int_{\mathrm{supp}(\phi)} \left(g_{u(x,t-s)+\phi(x,t)}^p\right)_\delta \, d\nu.$$

As will be seen below, this inequality is now in a more convenient form for establishing energy estimates.

Strictly speaking, in order to use Fubini's theorem above, one needs to know that the parabolic minimal p-weak upper gradient $g_u(x,t)$ of a parabolic minimizer u is measurable on Ω_T with respect to the product measure ν. In the metric measure space case it seems uncertain if one can prove this. In the Euclidean space product measurablility is known, and hence the difficulty is not present. Also, in the metric space setting one can bypass a use of mollification and Fubini's theorem by assuming more time regularity on a parabolic minimizer u. Since in this paper we do not make any additional assumptions on the time regularity of u and, on the other hand, the underlying space is a general metric measure space, the reader should keep in mind that in our treatment the difficulty related to the product measurability is in fact omitted.

We shall next conduct further general preparative work for establishing energy estimates, by studying how a cutoff in time affects the estimate in (2.2).

We replace in (2.2) the test function $\phi \in \mathrm{Lip}_0(\Omega_T)$ with the function

$$\phi(x,t)(\eta_{h,\sigma}(t) - \eta_{h,\tau}(t)) \in \mathrm{Lip}_0(\Omega_T),$$

where $0 < \sigma < \tau < T$ and the function $\eta_{h,\tau}$ is the piecewise linear cutoff function

$$\eta_{h,\tau}(t) = \begin{cases} 0 & \text{if } 0 < t \leq \tau, \\ (t-\tau)/h & \text{if } \tau < t < \tau + h, \\ 1 & \text{if } \tau + h \leq t \leq T. \end{cases}$$

After letting $h \to 0+$, we obtain by Lebesgue's differentiation theorem and by integrating by parts

$$(2.3) \quad -p \int_\sigma^\tau \int_\Omega \frac{\partial \left(u^{p-1}\right)_\delta}{\partial t} \phi \, d\mu \, dt + \int_\sigma^\tau \int_\Omega (g_u^p)_\delta \, d\mu \, dt \leq \int_\sigma^\tau \int_\Omega \left(g_{u(x,t-s)+\phi(x,t)}^p\right)_\delta \, d\mu \, dt,$$

for every valid test function $\phi \in \mathrm{Lip}_0(\Omega_T)$ and almost every $0 < \sigma < \tau < T$.

We are now ready to prove the energy estimates for parabolic super- and subminimizers. It is not obvious to see how the interplay between the time regularization and the dampening function $H_\lambda(u)$ defined below should be organized for parabolic minimizers related to the doubly nonlinear equation. That is why instead of presenting only a formal proof, we also take into account the time regularization.

We begin by proving the estimate for positive superminimizers; cp. Lemma 3.1 in [4].

Lemma 2.1. *Let u be a positive parabolic superminimizer and $0 < \varepsilon \neq p - 1$. Then*

$$
(2.4) \quad
\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^{p-1-\varepsilon}\varphi^p \, d\mu + \int_{\operatorname{supp}(\varphi)} g_u^p u^{-1-\varepsilon}\varphi^p \, d\nu \\
&\leq \left(1 + \frac{p}{\varepsilon}\right)\left(C_1 \int_{\operatorname{supp}(\varphi)} u^{p-1-\varepsilon} g_\varphi^p \, d\nu + C_2 \int_{\operatorname{supp}(\varphi)} u^{p-1-\varepsilon}|(\varphi^p)_t| \, d\nu\right)
\end{aligned}
$$

for every $\varphi \in \operatorname{Lip}_0(\Omega_T)$, $0 \leq \varphi \leq 1$, where

$$
C_1 = \left(\frac{p}{\varepsilon}\right)^{p-1} \quad \text{and} \quad C_2 = \frac{p-1}{|p-1-\varepsilon|}.
$$

Proof. For $\lambda > 0$ we define the function $H_\lambda(s)$ for nonnegative s by setting

$$
H_\lambda(s) = \begin{cases} \lambda^{-\varepsilon} + \varepsilon\lambda^{-\varepsilon-1}(\lambda - s) & \text{if } 0 \leq s \leq \lambda, \\ s^{-\varepsilon} & \text{if } s > \lambda. \end{cases}
$$

Then H_λ is continuously differentiable, and

$$
H_\lambda'(s) = \begin{cases} -\varepsilon\lambda^{-\varepsilon-1} & \text{if } 0 \leq s \leq \lambda, \\ -\varepsilon s^{-\varepsilon-1} & \text{if } s > \lambda. \end{cases}
$$

We note that $|H'(s)| \leq \varepsilon\lambda^{-\varepsilon-1}$. We set $\alpha = \frac{1}{\varepsilon}\lambda^{\varepsilon+1}$, and so $\alpha|H'(s)| \leq 1$ for all $s \geq 0$. Moreover, in order to be handle the time regularization below, we define $\widetilde{H_\lambda}(s) = H_\lambda(s^{1/(p-1)})$. We also define the function $h_\lambda(s)$ for $s \geq 0$ by setting

$$
h_\lambda(s) = \begin{cases} \int_0^s \widetilde{H_\lambda}(r) \, dr & \text{if } \varepsilon < p - 1, \\ \int_s^\infty \widetilde{H_\lambda}(r) \, dr & \text{if } \varepsilon > p - 1, \end{cases}
$$

and let $\iota = \operatorname{sgn}(p - 1 - \varepsilon)$.

Let φ be a function in $\operatorname{Lip}_0(\Omega_T)$ such that $0 \leq \varphi \leq 1$. We choose the nonnegative function $\phi = \alpha\widetilde{H_\lambda}((u^{p-1})_\delta)\varphi^p$ in (2.3), and note that defined this way ϕ is a valid test function. The first term on the left hand side of (2.3) becomes, after integrating by parts

$$
\begin{aligned}
-p \int_{\tau_1}^{\tau_2} \int_\Omega \frac{\partial (u^{p-1})_\delta}{\partial t} \phi \, d\mu \, dt &= -\iota\alpha p \int_{\tau_1}^{\tau_2} \int_\Omega \frac{\partial h_\lambda((u^{p-1})_\delta)}{\partial t}\varphi^p \, d\mu \, dt \\
&= -\iota\alpha p \left[\int_{\Omega\times\{t\}} h_\lambda((u^{p-1})_\delta)\varphi^p \, d\mu\right]_{t=\tau_1}^{\tau_2} + \iota\alpha p \int_{\tau_1}^{\tau_2}\int_\Omega h_\lambda((u^{p-1})_\delta)(\varphi^p)_t \, d\mu \, dt.
\end{aligned}
$$

Letting then $\delta \to 0$, it can shown by Fubini's theorem and the properties of time mollifications that

$$
\lim_{\delta\to 0}\left|\iota\alpha p \int_{\tau_1}^{\tau_2}\int_\Omega h_\lambda((u^{p-1})_\delta)(\varphi^p)_t \, d\mu \, dt\right| \leq \alpha p \int_{\tau_1}^{\tau_2}\int_\Omega |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\mu \, dt,
$$

$$
\lim_{\delta\to 0}\left(-\iota\alpha p\left[\int_{\Omega\times\{t\}} h_\lambda((u^{p-1})_\delta)\varphi^p \, d\mu\right]_{t=\tau_1}^{\tau_2}\right) = -\iota\alpha p\left[\int_{\Omega\times\{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu\right]_{t=\tau_1}^{\tau_2}.
$$

For the upper gradient terms in (2.3) we have

$$\lim_{\delta \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega (g_u^p)_\delta \, d\mu \, dt = \int_{\tau_1}^{\tau_2} \int_\Omega g_u^p \, d\mu \, dt,$$

and

(2.5)
$$\limsup_{\delta \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \left(g_{u(\cdot, \cdot -s) + \alpha \widetilde{H}_\lambda((u^{p-1})_\delta)\varphi^p}^p \right)_\delta \, d\mu \, dt$$
$$\leq \int_{\tau_1}^{\tau_2} \int_\Omega g_{u + \alpha \widetilde{H}_\lambda(u^{p-1})\varphi^p}^p \, d\mu \, dt = \int_{\tau_1}^{\tau_2} \int_\Omega g_{u + \alpha H_\lambda(u)\varphi^p}^p \, d\mu \, dt.$$

To verify that the inequality in (2.5) holds, we refer to [5, Section 6]. In order not to stray away from our main focus, we omit the proof. Combining the above results into (2.3), we obtain the inequality

(2.6)
$$- \iota \alpha p \left[\int_{\Omega \times \{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_\Omega g_u^p \, d\mu \, dt$$
$$\leq \int_{\tau_1}^{\tau_2} \int_\Omega g_{u + \alpha H_\lambda(u)\varphi^p}^p \, d\mu \, dt + \alpha p \int_{\tau_1}^{\tau_2} \int_\Omega |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\mu \, dt.$$

for almost every $0 < \tau_1 < \tau_2 < T$. Issues related to time regularity have now been dealt with, and we can continue proving the energy estimate. We begin by observing that

(2.7)
$$g_{u + \alpha H_\lambda(u)\varphi^p} \leq (1 + \alpha H_\lambda'(u)\varphi^p)g_u + \alpha p \varphi^{p-1} H_\lambda(u) g_\varphi.$$

That (2.7) holds is not evident, but since $|\alpha H_\lambda'(u)| < 1$, the inequality can be shown to be true using the pathwise properties of upper gradients, completely analogously to what is done in the proof of Lemma 3.1 in [4]. The inequality in (2.7) can be rewritten as

$$g_{u + \alpha H_\lambda(u)\varphi^p} \leq (1 - \alpha|H_\lambda'(u)|\varphi^p)g_u + \alpha|H_\lambda'(u)|\varphi^p \frac{p H_\lambda(u)}{\varphi|H_\lambda'(u)|}g_\varphi,$$

and since $0 \leq \alpha|H_\lambda'(u)|\varphi(x,t)^p \leq 1$, we get by the convexity of $t \mapsto t^p$:

$$g_{u + H_\lambda(u)\varphi^p}^p \leq (1 - \alpha|H_\lambda'(u)|\varphi^p)g_u^p + \alpha p^p \frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}}g_\varphi^p.$$

Plugging this into (2.6) gives us

(2.8)
$$- \iota \alpha p \left[\int_{\Omega \times \{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} + \alpha \int_{\tau_1}^{\tau_2} \int_\Omega |H_\lambda'(u)|\varphi^p g_u^p \, d\mu \, dt$$
$$\leq \alpha p^p \int_{\tau_1}^{\tau_2} \int_\Omega \frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}}g_\varphi^p \, d\mu \, dt + \alpha p \int_{\tau_1}^{\tau_2} \int_\Omega |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\mu \, dt.$$

The inequality in (2.8) is independent of the scaling factor α so we can divide it out. Since the constants on the right hand side of the estimate (2.8) do not depend on the choice of τ, the inequality (2.8) implies

(2.9)
$$\operatorname*{ess\,sup}_{0 < t < T} p \int_\Omega h_\lambda(u(x,t)^{p-1})\varphi(x,t)^p \, d\mu$$
$$\leq p^p \int_{\Omega_T} \frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}}g_\varphi^p \, d\nu + p \int_{\Omega_T} |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\nu.$$

On the other hand, the inequality (2.8) gives us also

$$(2.10) \quad \int_{\Omega_T} |H_\lambda'(u)| \varphi^p g_u^p \, d\nu \le p^p \int_{\Omega_T} \frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}} g_\varphi^p \, d\nu + p \int_{\Omega_T} |h_\lambda(u^{p-1})| |(\varphi^p)_t| \, d\nu.$$

Inequalities (2.9) and (2.10) are valid for all $0 < \varepsilon \ne p-1$. Letting λ decrease to 0, we see that $H_\lambda(u)$, $|H_\lambda'(u)|$ and $h_\lambda(u)$ converge pointwise in a monotonously increasing way. In particular, we check by using a change of variable and then the theorem of dominated convergence that in the case $\varepsilon < p-1$

$$\lim_{\lambda \to 0} h_\lambda(u^{p-1}) = \lim_{\lambda \to 0} \int_0^{u^{p-1}} H_\lambda(r^{\frac{1}{p-1}}) \, dr = \lim_{\lambda \to 0} (p-1) \int_0^u s^{p-2} H_\lambda(s) \, ds$$

$$= (p-1) \int_0^u s^{p-2-\varepsilon} \, ds = \frac{p-1}{p-1-\varepsilon} u^{p-1-\varepsilon},$$

and by a similar reasoning in the case $\varepsilon > p-1$

$$\lim_{\lambda \to 0} h_\lambda(u^{p-1}) = -\frac{p-1}{p-1-\varepsilon} u^{p-1-\varepsilon}.$$

Thus, by the monotone convergence theorem, we have

$$\lim_{\lambda \to 0} \operatorname*{ess\,sup}_{0 < t < T} p \int_\Omega h_\lambda(u^{p-1}) \varphi^p \, d\mu = \operatorname*{ess\,sup}_{0 < t < T} \frac{p(p-1)}{|p-1-\varepsilon|} \int_\Omega u^{p-\varepsilon-1} \varphi^p \, d\mu,$$

$$(2.11) \quad \lim_{\lambda \to 0} \int_{\Omega_T} |H_\lambda'(u)| \varphi^p g_u^p \, d\nu = \varepsilon \int_{\Omega_T} g_u^p u^{-1-\varepsilon} \varphi^p \, d\nu,$$

$$\lim_{\lambda \to 0} p \int_{\Omega_T} |h_\lambda(u^{p-1})| |(\varphi^p)_t| \, d\nu = \frac{p(p-1)}{|p-1-\varepsilon|} \int_{\Omega_T} u^{p-1-\varepsilon} |(\varphi^p)_t| \, d\nu.$$

It remains to verify the convergence of the last integral in (2.9) and (2.10) as $\lambda \to 0$. If the integral

$$(2.12) \quad \int_{\operatorname{supp}(\phi)} u^{p-1-\varepsilon} g_\phi^p \, d\nu$$

was infinite, then the estimate in (2.4) is trivially true and the proof is complete. Assume thus that the integral in (2.12) is finite. At the points $(x,t) \in \Omega_T$ where $0 < u(x,t) < \lambda$, we have

$$\frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}} = \begin{cases} \left(\varepsilon^{\frac{1}{p}-1} \lambda^{1-\frac{\varepsilon+1}{p}} + \varepsilon^{\frac{1}{p}} (\lambda - u) \lambda^{-\frac{\varepsilon+1}{p}} \right)^p, & 0 < u(x,t) < \lambda, \\ \varepsilon^{1-p} u^{p-1-\varepsilon}, & u(x,t) \ge \lambda. \end{cases}$$

From this expression we see by the theorem of dominated convergence that

$$(2.13) \quad \lim_{\lambda \to 0} p^p \int_{\Omega_T} \frac{H_\lambda(u)^p}{|H_\lambda'(u)|^{p-1}} g_\varphi^p \, d\nu = p^p \varepsilon^{1-p} \int_{\Omega_T} u^{p-1-\varepsilon} g_\varphi^p \, d\nu.$$

Indeed, if $\varepsilon < p-1$ then we can use $H_1(u)^p / |H_1'(u)|^{p-1} g_\varphi^p$ to dominate, and in the case in which $\varepsilon > p-1$ the term $\varepsilon^{1-p} u^{p-1-\varepsilon} g_\varphi^p$ dominates the integrand. Combining (2.11) with (2.13) through (2.9) and (2.10), and summing up the resulting inequalities completes the proof. $\qquad\square$

We then prove the energy estimate for positive subminimizers; cp. Lemma 4.1 in [4].

Lemma 2.2. *Let* u *be a nonnegative parabolic subminimizer and let* $\varepsilon \geq 1$. *Then*

$$\operatorname*{ess\,sup}_{0<t<T} \int_{\Omega} u^{p-1+\varepsilon} \varphi^p \, d\mu + \int_{\operatorname{supp}(\varphi)} g_u^p u^{-1+\varepsilon} \varphi^p \, d\nu$$

$$\leq \left(1 + \frac{p}{\varepsilon}\right) \left(C_1 \int_{\operatorname{supp}(\varphi)} u^{p-1+\varepsilon} g_\varphi^p \, d\nu + C_2 \int_{\operatorname{supp}(\varphi)} u^{p-1+\varepsilon} |(\varphi^p)_t| \, d\nu\right)$$

for every $\varphi \in \operatorname{Lip}_0(\Omega_T)$, $0 \leq \varphi \leq 1$, *where*

$$C_1 = \left(\frac{p}{\varepsilon}\right)^{p-1} \quad \text{and} \quad C_2 = \frac{p-1}{|p-1+\varepsilon|}.$$

Proof. For $\lambda > 0$ we define the function $H_\lambda(s)$ by setting

$$H_\lambda(s) = \begin{cases} s^\varepsilon & \text{if } 0 \leq s \leq \lambda, \\ \lambda^\varepsilon + \varepsilon\lambda^{\varepsilon-1}(s-\lambda) & \text{if } s > \lambda. \end{cases}$$

Then $H_\lambda(s)$ is continuously differentiable, and we note that $H_\lambda'(s) \leq \varepsilon\lambda^{\varepsilon-1}$, and we define the scaling factor $\alpha = \frac{1}{\varepsilon}\lambda^{-\varepsilon+1}$; whence $\alpha H'(u) \leq 1$ for all $u \geq 0$. In order to handle the time regularization, we define $\widetilde{H_\lambda}(s) = H_\lambda(s^{1/(p-1)})$. We also define the function h_λ by setting

$$h_\lambda(s) = \int_0^s \widetilde{H_\lambda}(r) \, dr.$$

Let φ be a function in $\operatorname{Lip}_0(\Omega_T)$ such that $0 \leq \varphi \leq 1$. As a nonpositive test function in (2.3) we choose $\phi = -\alpha\widetilde{H_\lambda}((u^{p-1})_\delta)\varphi^p$. After handling the time regularization as in the proof of Lemma 2.1, we arrive to

$$(2.14) \quad \begin{aligned} &-\iota\alpha p \int_{\Omega\times\{\tau\}} h_\lambda(u^{p-1})\varphi^p \, d\mu + \int_0^\tau \int_\Omega g_u^p \, d\mu \, dt \\ &\leq \int_0^\tau \int_\Omega g_{u+\alpha H_\lambda(u)\varphi^p}^p \, d\mu \, dt + \alpha p \int_\tau^T \int_\Omega |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\mu \, dt. \end{aligned}$$

Since $0 \leq \alpha|H_\lambda'(u)|\varphi^p \leq 1$, as in the proof of Lemma 3.1 in [4], we can show by using the pathwise properties of upper gradients and convexity that

$$g_{u-H_\lambda(u)\varphi^p}^p \leq (1 - \alpha H_\lambda'(u)\varphi^p)g_u^p + \alpha p^p \frac{H_\lambda(u)^p}{H_\lambda'(u)^{p-1}} g_\varphi^p.$$

From (2.14) we get the result in the same way as in Lemma 2.1, since it is easily verified that $H_\lambda(u)^p H_\lambda'(u)^{1-p} \leq \varepsilon^{1-p}u^{p-1+\varepsilon}$. $\qquad\square$

We prove the following logarithmic energy estimate for positive superminimizers which covers the case $\varepsilon = p - 1$ in Lemma 2.1; cp. Lemma 5.1 in [4].

Lemma 2.3. *Let* u *be a positive parabolic superminimizer. Then the inequality*

$$-p \left[\int_{\Omega\times\{t\}} (\log u)\varphi^p \, d\mu\right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_\Omega g_{\log u}^p \varphi^p \, d\mu \, dt$$

$$\leq \left(\frac{p}{p-1}\right)^p \int_{\tau_1}^{\tau_2} \int_\Omega g_\phi^p \, d\mu \, dt + p \int_{\tau_1}^{\tau_2} \int_\Omega |\log u||(\varphi^p)_t| \, d\mu \, dt$$

holds for every $\varphi \in \operatorname{Lip}_0(\Omega_T)$, $0 \leq \varphi \leq 1$, *and almost every* $0 < \tau_1 < \tau_2 < T$.

Proof. As in the preceding proofs, for $\lambda > 0$ we define the function $H_\lambda(s)$ as

$$H_\lambda(s) = \begin{cases} \lambda^{1-p} + (p-1)\lambda^{-p}(\lambda - s) & \text{if } 0 \le s \le \lambda, \\ s^{1-p} & \text{if } s > \lambda, \end{cases}$$

and the scaling factor $\alpha = \frac{1}{p-1}\lambda^p$. Hence $\alpha|H'(s)| \le 1$ for all $s \ge 0$. Lastly, we define $\widetilde{H_\lambda}(s) = H_\lambda(s^{1/(p-1)})$ and the function h_λ by setting

$$h_\lambda(s) = \int_1^s \widetilde{H_\lambda}(r) \, dr.$$

The function $h_\lambda(s)$ is negative for $0 \le s < 1$. The positive and the negative part are written as $h_\lambda(s)_+$ and $h_\lambda(s)_-$. Let φ be a function in $\mathrm{Lip}_0(\Omega_T)$ such that $0 \le \varphi \le 1$. As a nonnegative test function in (2.3) we choose $\phi = \alpha \widetilde{H_\lambda}((u^{p-1})_\delta)\varphi^p$, and proceed as in the proof of Lemma 2.1. We obtain

(2.15)
$$-p \left[\int_{\Omega \times \{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu \right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_\Omega |H'_\lambda(u)|\varphi^p g_u^p \, d\mu \, dt$$
$$\le p^p \int_{\tau_1}^{\tau_2} \int_\Omega \frac{H_\lambda(u)^p}{|H'_\lambda(u)|^{p-1}} g_\varphi^p \, d\mu \, dt + p \int_{\tau_1}^{\tau_2} \int_\Omega |h_\lambda(u^{p-1})||(\varphi^p)_t| \, d\mu \, dt.$$

Consider the integrals

$$\int_{\Omega \times \{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu = \int_{\Omega \times \{t\}} h_\lambda(u^{p-1})_+ \varphi^p \, d\mu - \int_{\Omega \times \{t\}} h_\lambda(u^{p-1})_- \varphi^p \, d\mu$$
$$= (p-1) \int_{\Omega \times \{t\}} (\log u)_+ \varphi^p \, d\mu - \int_{\Omega \times \{t\}} h_\lambda(u^{p-1})_- \varphi^p \, d\mu.$$

The first term on the right-hand side is finite for almost all $t \in (\tau_1, \tau_2)$. As for the second term, we have the monotone convergence

$$0 \le h_\lambda(u^{p-1})_- \nearrow (p-1)(\log u)_-$$

as $\lambda \searrow 0$. Hence we may conclude that

$$\lim_{\lambda \to 0} \int_{\Omega \times \{t\}} h_\lambda(u^{p-1})\varphi^p \, d\mu = (p-1) \int_{\Omega \times \{t\}} (\log u)\varphi^p \, d\mu.$$

From this it is straightforward to verify that the desired estimate holds. $\qquad \square$

With these revised energy estimates at our disposal, we are able to carry out an iteration procedure similar to the one presented in [4]. We obtain, in particular, the weak Harnack inequality for positive parabolic superminimizers without the technical assumption, cp. [4, Lemma 6.1]. Finally, the following Harnack inequality for positive parabolic minimizers is obtained, cp. [4, Theorem 6.6].

Theorem 2.4. *Suppose that $1 < p < \infty$ and the measure μ in a geodesic metric space X is doubling with doubling constant C_μ, and the space supports a weak $(1,p)$-Poincaré inequality with constants C_p and Λ. Then a parabolic Harnack inequality is valid as follows: Let u be a positive parabolic minimizer in $Q_r \subset \Omega_T$. Let $0 < \delta < 1$. Then*

$$\operatorname*{ess\,sup}_{\delta Q^-} u \le C \operatorname*{ess\,inf}_{\delta Q^+} u,$$

where $0 < C(C_\mu, C_p, \Lambda, p, \delta, T) < \infty$.

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