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QUASI-LIPSCHITZ EQUIVALENCE OF SUBSETS OF AHLFORS–DAVID REGULAR SETS

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Abstract. In the paper, it is proved that for any Ahlfors–David s-regular sets E and F in Euclidean spaces, there exist subsets $E' \subset E$ and $F' \subset F$ such that $\dim_H E' = \dim_H F' = s$ and E', F' are quasi-Lipschitz equivalent.

1. Introduction

For $E \subset \mathbf{R}^n$ and $F \subset \mathbf{R}^m$, a bijection $f\colon E \to F$ is said to be *bilipschitz* if there is a positive number L such that

$$L^{-1}|x-y| \le |f(x)-f(y)| \le L|x-y| \quad \text{for all } x, y \in E.$$

We say that sets E and F in Euclidean spaces are *bilipschitz equivalent* if there exists a bilipschitz bijection from E onto F and denote by $E \sim F$. We say that E can be *bilipschitz embedded* into F if there exists a subset F' of F such that $E \sim F'$ and denote by $E \hookrightarrow F$.

Definition 1. [8] A compact set F is said to be *Ahlfors–David s-regular* (s-regular for short), if there is a Borel measure ν supported on E and a constant C_F such that

$$C_F^{-1} r^s \le \nu(B(x,r)) \le C_F r^s \tag{1.1}$$

for all $x \in F$ and $0 < r \le |F|$, where $|F|$ is the diameter of F and $B(x,r)$ is the closed ball with center x and radius r.

Remark 1. Any s-regular set has Hausdorff dimension s.

Remark 2. Any $C^{1+\gamma}(\gamma > 0)$ self-conformal set F satisfying the open set condition is s-regular, where $s = \dim_H F$ and $\nu = \mathcal{H}^s|_F$. In particular, any self-similar set satisfying the open set condition is regular.

Suppose that A and B are regular with $\dim_H A < \dim_H B$. Mattila and Saaranen [9] proved that for any $\epsilon > 0$, there exists a regular subset A' of A with $|\dim_H A' - \dim_H A| < \epsilon$ such that $A' \hookrightarrow B$, where A' is bilipschitz equivalent to a generalized

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Cantor set, which is self-similar. They also obtained that if $\dim_H A < 1$, then $A \hookrightarrow B$. However, for $\dim_H A = 1$, Deng etc. [2] pointed out that if $A = [0,1]$, any subset $A' \subset [0,1]$ with positive Lebesgue measure can not be bilipschitz embedded into any self-similar set satisfying the strong separation condition (SSC).

The above works raise the following question: *For two regular subsets A and B of Euclidean spaces satisfying* $\dim_H A = \dim_H B$, *what kind of good subsets A' of A can be bilipschitz embedded into B?* Here we hope that the good subset A' is close to A, for example, $|\dim_H A' - \dim_H A|$ is small enough or $\mathcal{H}^s(A') > 0$ with $s = \dim_H A$.

Llorente and Mattila [7] assumed open set condition and then proved that for self-conformal sets E and F with the same dimension s, if there exist subsets $E' \subset E$ and $F' \subset F$ with $\mathcal{H}^s(E'), \mathcal{H}^s(F') > 0$ such that $E' \sim F'$, then $E \sim F$. For self-similar sets with the *same* dimension satisfying SSC, Deng etc. [2] obtained the similar result. However, Falconer and Marsh [3] pointed out that the self-similar sets (satisfying SSC) with the *same* dimension need *not* be bilipschitz equivalent. Then the results of [7, 2] imply that for two self-similar sets with $\dim_H E = \dim_H F = s$ but $E \not\sim F$, we can not find subsets $E'(\subset E)$, $F'(\subset F)$ with positive $\mathcal{H}^s$ measure such that $E' \sim F'$.

We will introduce a notion weaker than bilipschitz equivalence.

Definition 2. [18] The compact subsets E and F of Euclidean spaces are said to be *quasi-Lipschitz equivalent*, if there is a bijection $f\colon E \to F$ such that for all $x_1, x_2 \in E$,

$$\frac{\log|f(x_1) - f(x_2)|}{\log|x_1 - x_2|} \to 1 \quad \text{uniformly as } |x_1 - x_2| \to 0. \tag{1.2}$$

We say that E can be *quasi-Lipschitz embedded* into F if E is quasi-Lipschitz equivalent to a subset of F.

Remark 3. It is proved in [18] that two self-conformal sets E, F satisfying SSC are quasi-Lipschitz equivalent if and only if they have the same Hausdorff dimension. This result fails for bilipschitz equivalence, e.g. self-similar sets satisfying SSC as shown in [3, 19]. [5] and [12] discussed the quasi-Lipschitz equivalence of Moran sets and regular sets.

This paper focuses an alternative question: *For regular sets A and B in Euclidean spaces with* $\dim_H A = \dim_H B$, *what kinds of good subsets of A can be quasi-Lipschitz embedded into B?*

Now we give our main theorem.

Theorem 1. *Suppose that $s > 0$. For s-regular sets E and F in Euclidean spaces, there exist subsets $E' \subset E$ and $F' \subset F$ with* $\dim_H E' = \dim_H F' = s$ *such that E' and F' are quasi-Lipschitz equivalent.*

Frostman's lemma shows that if $E \subset \mathbf{R}^d$ is compact and $\mathcal{H}^t(E) > 0$, then there is a Borel measure μ supported on E such that

$$\mu(B(x,r)) \le r^t \tag{1.3}$$

for all $x \in \mathbf{R}^d$, $r > 0$. Let E' be the support of the above measure μ. Can we obtain a constant $c > 0$ such that

$$cr^t \le \mu(B(x,r)) \le r^t \tag{1.4}$$

for all $x \in E'$ and $r \leq |E'|$? If inequality (1.4) holds, then E contains an Ahlfors–David t-regular subset E'.

Then a natural question is *whether E with $\dim_H E = s$ always contains a t-regular subset with $t \in (0, s]$*. The following proposition offers a negative answer.

Proposition 1. *For any given $s \in (0,1)$, there exists an s-Hausdorff dimensional Moran set $F \subset \mathbf{R}^1$ such that F does not contain any regular subset.*

For $s = 1$, Example 5.3 in [9] gave a set with positive $\mathcal{L}^1$ measure which contains no regular subset. In fact, the key point is that the set in [9] does not contain any uniformly perfect subset. Inspired by this, for any given $s \in (0,1)$, we will obtain a *Moran set* [13, 14] with the structure $(I, \{n_k\}, \{c_k\})$, where I is the closed interval $[0,1]$, $n_k \to \infty$ and $c_k = n_k^{-1/s}$. Then this Moran set, with s-Hausdorff dimension, contains no regular subsets. In fact, it is the key that none of its subsets can be uniformly perfect.

When is a Moran set Ahlfors–David regular? We note that the above Moran set

$$c_* = \inf_k c_k = 0,$$

where $c_{k,1} = \cdots = c_{k,n_k} = c_k$. For the Moran set with structure $(J, \{n_k\}_{k\geq 1}, \{c_{k,j}\}_{k\geq 1, j\leq n_k})$ [13, 14], under the condition

$$c_* = \inf_{k,j} c_{k,j} > 0,$$

the following Proposition 2 gives a necessary and sufficient condition for a Moran set on $\mathbf{R}^1$ to be regular.

Proposition 2. *Suppose a Moran set F is defined as in (2.4) on $\mathbf{R}^1$ satisfying that $c_* = \inf_{k,j} c_{k,j} > 0$. Then F is s-regular if and only if there are constants $0 < \alpha, \beta < \infty$ such that*

$$\alpha \leq \prod_{k=1}^{N} \sum_{j=1}^{n_k} c_{k,j}^s \leq \beta \quad \text{for all } N > 0. \tag{1.5}$$

Remark 4. For the Moran set in the proof of Proposition 1, let $c_{i,1} = \cdots = c_{i,n_i} = c_i$ for all i, we have $\prod_{k=1}^{N} \sum_{j=1}^{n_k} c_{k,j}^s = 1$ for all k. Then the condition $c_* > 0$ is necessary.

The paper is organized as follows. In Section 2, we prove Theorem 1 by constructing a special homogeneous Moran subset, which is quasi Ahlfors–David s-regular and quasi uniformly disconnected. The proof is based on Lemma 1 from [11]. In Section 3, we prove Proposition 1 using uniform perfectness [10] and Proposition 2 using the measure in [1, 6].

2. Moran subsets with full dimension

2.1. Moran sets. Suppose that $J \subset \mathbf{R}^d$ is a compact set with nonempty interior. Let $\{n_k\}_{k\geq 1}$ be a given positive integer sequence satisfying $n_k \geq 2$ for all k. Let $\psi = \psi_k$ be a finite positive real vector sequence, where

$$\psi_k = (c_{k,1}, \cdots, c_{k,n_k}), \quad 0 < c_{k,j} < 1, \ k \in \mathbf{N}, \ 1 \leq j \leq n_k.$$

The set of finite words is denoted by $\mathcal{D}^\infty = \bigcup_{k=0}^\infty \mathcal{D}^k$, where

$$\mathcal{D}^k = \{i_1 \cdots i_k \colon i_j \in \mathbf{N} \cap [1, n_j] \text{ for all } j\}$$

and $D^0 = \{\emptyset\}$ and $\emptyset$ is the empty word. Given $\sigma = i_1 \cdots i_k \in \mathcal{D}^k$, $\tau = j_1 \cdots j_l \in \mathcal{D}^l$, denote the word $\sigma * \tau = i_1 \cdots i_k j_1 \cdots j_l$. The length of the word $\sigma \in \mathcal{D}^k$ is denoted by $|\sigma| (= k)$.

We say that the family $\mathcal{F} = \{J_\sigma : \sigma \in \mathcal{D}^\infty\}$ of subsets of $\mathbf{R}^d$ has Moran structure, if the following three conditions hold:

(i) for any $\sigma \in \mathcal{D}^\infty$, J_σ is geometrically similar to J, where we denote by $J_\emptyset = J$;

(ii) for any $k \geq 0$ and $\sigma \in \mathcal{D}^{k-1}$,

$$J_{\sigma*1}, \cdots, J_{\sigma*n_k} \subset J_\sigma \tag{2.1}$$

satisfying

$$\mathrm{int}(J_{\sigma*i}) \cap \mathrm{int}(J_{\sigma*j}) = \varnothing \quad \text{whenever } i \neq j, \tag{2.2}$$

where int denotes the interior of the set;

(iii) for any $k \geq 1$, $\sigma \in \mathcal{D}^{k-1}$ and $1 \leq j \leq n_k$, it holds that

$$\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_{k,j}. \tag{2.3}$$

Then we call the following compact set

$$F = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \mathcal{D}^k} J_\sigma \tag{2.4}$$

a *Moran set* in $\mathbf{R}^d$ with the structure $(J, \{n_k\}, \{\psi_k\}) = (J, \{n_k\}, \{c_{k,j}\})$. The members of the family $\{J_\sigma : \sigma \in \mathcal{D}^k\}$ are called *basic elements* of rank k.

A Moran set F defined in (2.4) is said to be *homogeneous* with the structure $(J, \{n_k\}, \{c_k\})$, if $c_{k,1} = \cdots = c_{k,n_k} = c_k$ for any $k \geq 1$.

When we talk about a Moran set on $\mathbf{R}^1$, for convenience as in [13, 14], we always assume that the initial set J is a closed interval. The members of the family $\{J_\sigma : \sigma \in \mathcal{D}^k\}$ are called *basic intervals* of rank k.

2.2. Result on quasi-Lipschitz equivalence. Recall the notions of quasi uniform disconnectedness and quasi Ahlfors–David regularity in [11].

Definition 3. We say that a subset F of metric space X is *quasi uniformly disconnected* if there is a function $\rho : (0, \infty) \to (0, \infty)$ with $\lim_{t \to 0} \frac{\log \rho(t)}{\log t} = 1$ such that for any $x \in F$, $r > 0$, there is a subset $B \subset F$ such that

$$F \cap B(x, \rho(r)) \subset B \subset B(x, r) \quad \text{and} \quad \mathrm{dist}(B, F \backslash B) > \rho(r), \tag{2.5}$$

where $\mathrm{dist}(A_1, A_2)$ denotes the least distance between A_1 and A_2.

Definition 4. A compact set F is said to be *quasi Ahlfors–David s-regular*, if there exists a Borel measure ν supported on F and a non-decreasing function $h : (0, |F|) \to (0, +\infty)$ with $\lim_{t \to 0} h(t) = 0$, such that for all $x \in F$ and $0 < r \leq |F|$,

$$s(1 - h(r)) \leq \frac{\log \nu(B(x, r))}{\log r} \leq s(1 + h(r)). \tag{2.6}$$

In fact, any quasi s-regular set has Hausdorff dimension s. Inequality (2.6) means that as $r \to 0$,

$$\frac{\log \nu(B(x, r))}{\log r} \to s \quad \text{uniformly for all } x \in F.$$

The reference [11] points out the following result on quasi-Lipschitz equivalence.

Lemma 1. *Suppose A and B are compact and quasi uniformly disconnected in metric spaces. If A and B are quasi s-regular and quasi t-regular respectively, then they are quasi-Lipschitz equivalent if and only if $s=t$.*

2.3. Construction of Moran subsets. We will construct subsets of full dimension and obtain their quasi-Lipschitz equivalence by using Lemma 1.

Suppose that $E \subset \mathbf{R}^d$ is an s-regular set with the measure ν supported on E such that

$$C_E^{-1} r^s \le \nu(B(x,r)) \le C_E r^s$$

for all $x \in E$ and $0 < r \le |E|$, where $C_E > 0$ is a constant. Now, we will construct recursively a full dimensional homogeneous Moran subset E' of E such that E' is quasi s-regular and quasi uniformly disconnected.

Given $\varepsilon > 0$ small enough, let $R_0 = 1$ and

$$R_k = \varepsilon^{1+2+\cdots+k} \quad \text{for all } k \ge 1. \tag{2.7}$$

Then

$$\frac{R_k}{R_{k-1}} = \varepsilon^k \to 0 \quad \text{and} \quad \frac{\log R_k}{\log R_{k-1}} \to 1 \quad \text{as } k \to \infty. \tag{2.8}$$

For any compact subset A of $\mathbf{R}^d$, let $M_\varepsilon(A)$ and $N_\varepsilon(A)$ be the maximum number of disjoint ε-balls with centers in A and the minimum number of ε-balls needed to cover A respectively. By [14], we have

$$C_d N_\varepsilon(A) \le N_{2\varepsilon}(A) \le M_\varepsilon(A) \le N_\varepsilon(A), \tag{2.9}$$

where $C_d > 0$ is a constant depending on the space $\mathbf{R}^d$.

Fix $x_\emptyset \in E$ for empty word $\emptyset$. Since ε is small enough, we can take $n_1 = 2$ and $x_1, x_2 \in B(x_\emptyset, 1/2) \cap E$ such that $B(x_1,\varepsilon) \cap B(x_2,\varepsilon) = \varnothing$.

By induction, assume we obtain points $\{x_{i_1\cdots i_{k-1}}\}_{i_1\cdots i_{k-1}} \subset E$ satisfying

(1) $x_{i_1\cdots i_{k-2}i_{k-1}} \in B(x_{i_1\cdots i_{k-2}}, R_{k-2}/2) \cap E$ for all $i_1\cdots i_{k-2}i_{k-1}$;
(2) $B(x_{i_1\cdots i_{k-2}i_{k-1}}, R_{k-1}) \cap B(x_{i_1\cdots i_{k-2}j_{k-1}}, R_{k-1}) = \varnothing$ if $i_{k-1} \ne j_{k-1}$.

Given point $x := x_{i_1\cdots i_{k-1}} \in E$, suppose that $\{B(y_i, R_k/2)\}_{i=1}^{N_k(x)}$ is a covering of $B(x, R_{k-1}/2) \cap E$, where $y_i \in B(x, R_{k-1}/2)$ and $N_k(x) := N_{R_k/2}(B(x,R_{k-1}/2) \cap E)$. Using the definition of $N_\varepsilon(\cdot)$, we can take $z_i \in B(x, R_{k-1}/2) \cap E$ such that

$$B(x, R_{k-1}/2) \cap E \subset \bigcup_{i=1}^{N_k(x)} B(z_i, R_k).$$

Therefore, we have

$$\begin{aligned} 2^{-s} \cdot C_E^{-1} \cdot R_{k-1}^s &\le \nu(B(x,R_{k-1}/2) \cap E) \le \nu\Big(\bigcup_{i=1}^{N_k(x)} B(z_i,R_k)\Big) \\ &\le \sum_{i=1}^{N_k(x)} \nu(B(z_i,R_k)) = N_k(x) \cdot C_E \cdot R_k^s. \end{aligned}$$

Let $M_k(x) := M_{R_k}(B(x,R_{k-1}/2) \cap E)$. It follows from (2.9) that

$$M_k(x) \ge C_d^2 N_k(x),$$

which implies

$$M_k(x) \ge D \cdot (R_{k-1}/R_k)^s = D \cdot \varepsilon^{-ks}, \tag{2.10}$$

where $D = C_d^2 \cdot 2^{-s} \cdot C_E^{-2}$.

Therefore, by (2.10) we can take $n_k = [D\varepsilon^{-ks}]$ points

$$\{x_{i_1\cdots i_{k-1}i_k}\}_{i_k=1}^{n_k} \subset B(x_{i_1\cdots i_{k-1}}, R_{k-1}/2) \cap E \tag{2.11}$$

satisfying

$$B(x_{i_1\cdots i_{k-1}i_k}, R_k) \cap B(x_{i_1\cdots i_{k-1}j_k}, R_k) = \varnothing \quad \text{for any } i_k \neq j_k, \tag{2.12}$$

where $[a]$ is the integral part of a. Then

$$E' = \bigcap_{k\ge 1} \bigcup_{i_1\cdots i_k} B(x_{i_1\cdots i_k}, R_k). \tag{2.13}$$

is a homogeneous Moran subset of $E(\subset \mathbf{R}^d)$ with structure $(B(x_\emptyset, 1), \{n_k\}, \{c_k\})$ where

$$n_1 = 2,\ n_k = [D\varepsilon^{-ks}] \ \text{ for } k \ge 2 \ \text{ and } \ c_k = R_k/R_{k-1} = \varepsilon^k.$$

2.4. The proof of Theorem 1. In fact, for any $x \in B(x_{i_1\cdots i_k i_{k+1}}, R_{k+1})$, we have

$$|x - x_{i_1\cdots i_k}| \le R_k/2 + R_{k+1}. \tag{2.14}$$

Given $i_1 \cdots i_k \neq j_1 \cdots j_k$, applying (2.14) we have

$$\begin{aligned} B(x_{i_1\cdots i_k i_{k+1}}, R_{k+1}) &\subset B(x_{i_1\cdots i_k}, R_k/2 + R_{k+1}) \ \text{ for all } i_{k+1}, \\ B(x_{j_1\cdots j_k j_{k+1}}, R_{k+1}) &\subset B(x_{j_1\cdots j_k}, R_k/2 + R_{k+1}) \ \text{ for all } j_{k+1}. \end{aligned} \tag{2.15}$$

We can take small ε in (2.7) such that $\frac{1}{2}\varepsilon^k + \varepsilon^{k+(k+1)} < 1$ and $\varepsilon^{k+1} < \frac{1}{6}$ for all $k \ge 1$, which implies

$$R_{k-1} > R_k/2 + R_{k+1} \ \text{ and } \ R_k > 6R_{k+1}.$$

Let $B = B(x_{i_1\cdots i_k}, R_k/2 + R_{k+1}))$. Since $B(x_{i_1\cdots i_k}, R_k) \cap B(x_{j_1\cdots j_k}, R_k) = \varnothing$, using (2.15) we have

$$\operatorname{dist}(B, E'\backslash B) \ge 2(R_k - (R_k/2 + R_{k+1})) > 2R_{k+1}. \tag{2.16}$$

Now, according to Lemma 1, we will check the properties of E'.

Lemma 2. *E' is quasi uniformly disconnected.*

Proof. Suppose that $2R_{k-1} < r \le 2R_{k-2}$ and $x \in B(x_{i_1\cdots i_k i_{k+1}}, R_{k+1}) \cap E'$. Let $\rho(r) = 2R_{k+1}$.

We take $B = E' \cap B(x_{i_1\cdots i_k}, R_k/2 + R_{k+1})$ as above. Using (2.16), we have

$$E' \cap B(x, 2R_{k+1}) \subset B. \tag{2.17}$$

Since $|x - x_{i_1\cdots i_k}| \le R_k/2 + R_{k+1}$ and $R_{k-1} > R_k/2 + R_{k+1}$, we have

$$B \subset B(x, 2R_{k-1}) \subset B(x, r). \tag{2.18}$$

By (2.8), we note that

$$1 \leftarrow \frac{\log(2R_{k+1})}{\log(2R_{k-1})} \le \frac{\log \rho(r)}{\log r} \le \frac{\log(2R_{k+1})}{\log(2R_{k-2})} \to 1. \tag{2.19}$$

Then quasi uniform disconnectedness follows from (2.16)–(2.19). □

Lemma 3. *E' is quasi Ahlfors–David s-regular.*

Proof. It is easy to check that

$$\lim_{k\to\infty} -\frac{\log n_1\cdots n_k}{\log c_1\cdots c_k} = \lim_{k\to\infty} \frac{\log n_1\cdots n_k}{\log(1/R_k)} = s.$$

Equipping the ball $B(x_{i_1\cdots i_k}, R_k)$ with mass $\frac{1}{n_1\cdots n_k}$, we obtain a mass distribution μ on F. In order to illustrate that (2.6) holds for F and μ, we only need to prove that

$$\frac{\log\mu(B(x,r))}{\log r} \to s \ \text{ uniformly.} \tag{2.20}$$

For this, we assume that $R_k/3 < r \le R_{k-1}/3$ and $x\in F$.

We suppose that $x \in B(x_{i_1\cdots i_{k-1}i_k i_{k+1}}, R_{k+1})$, then $x\in B(x_{i_1\cdots i_{k-1}}, R_{k-1})$. By (2.14) and $R_k < R_{k-1}/6$ for small ε, we have $B(x,r)\subset B(x_{i_1\cdots i_{k-1}}, R_{k-1}/2+R_k+r)\subset B(x_{i_1\cdots i_{k-1}}, R_{k-1})$. Thus

$$\mu(B(x,r)) \le \mu(B(x_{i_1\cdots i_{k-1}}, R_{k-1})) = \frac{1}{n_1\cdots n_{k-1}}.$$

On the other hand, since $2R_{k+1} < R_k/3(< r)$ when ε is small, we have $B(x_{i_1\cdots i_{k-1}i_k i_{k+1}}, R_{k+1})\subset B(x,r)$, which implies

$$\mu(B(x,r)) \ge \mu(B(x_{i_1\cdots i_{k-1}i_k i_{k+1}}, R_{k+1})) = \frac{1}{n_1\cdots n_{k+1}}.$$

Therefore, we have

$$\frac{\log n_1\cdots n_{k-1}}{\log(3/R_k)} \le \frac{\log\mu(B(x,r))}{\log r} \le \frac{\log n_1\cdots n_{k+1}}{\log(3/R_{k-1})},$$

where $\frac{\log n_1\cdots n_{k-1}}{\log(3/R_k)}, \frac{\log n_1\cdots n_{k+1}}{\log(3/R_{k-1})} \to s$ as $k\to\infty$. Then (2.20) follows. □

Since E' is quasi Ahlfors–David s-regular,

$$\dim_H E' = \dim_H E = s.$$

Using Lemmas 1–3, we obtain Theorem 1.

3. Regularity of Moran sets

In this section, we consider Moran subsets of $\mathbf{R}^1$ generated by the initial closed interval I. Without loss of generality, we always assume the diameter $|I| = |I_\emptyset| = 1$. If $\sigma = i_1\cdots i_k\in\mathcal{D}^k$, then each I_σ is similar to $I_\emptyset$ with ratio $c_{1,i_1}\cdots c_{k,i_k}$ and then

$$|I_\sigma| = c_{1,i_1}\cdots c_{k,i_k}. \tag{3.1}$$

Definition 5. A Moran set F defined as in (2.4) is called a *homogeneous uniform Cantor set* with the structure $(I,\{n_k\},\{c_k\})$, if where I is a closed interval and $\{c_k\}_{k\ge1}$ is a ratios sequence such that F satisfies, for all $\sigma\in\mathcal{D}^{k-1}$,

(1) $I_{\sigma*1}, I_{\sigma*2},\cdots, I_{\sigma*n_k}$ are subintervals of $I_{\sigma*n_k}$, arranged from left to right;
(2) I_σ and $I_{\sigma*1}$ share left end-points, and I_σ and $I_{\sigma*n_k}$ share right end-points;
(3) $\delta_{\sigma*1} = \cdots = \delta_{\sigma*(n_k-1)}$, where $\delta_{\sigma*j}$ is the length of gap between $I_{\sigma*j}$ and $I_{\sigma*(j+1)}$.

Recall that any I_σ with $\sigma\in\mathcal{D}^k$ is called a *basic intervals* of rank k.

Definition 6. A compact subset E of $\mathbf{R}^n$ is called *uniformly perfect* if there is a constant $0 < c < 1$ such that

$$E \cap \{y \colon cr \le |y - x| \le r\} \neq \varnothing \tag{3.2}$$

for all $0 < r < |E|$ and $x \in E$.

The uniform perfectness is an interesting invariant under bilipschitz mappings [10, 20, 17]. Using the definition of regularity, we obtain the following result directly.

Lemma 4. *Any Ahlfors–David regular set is uniformly perfect.*

3.1. A Moran set without regular subset. We will construct a Moran set such that none of its subsets can be uniformly perfect. Then Proposition 1 follows from Lemma 4.

For any $s \in (0, 1)$, let F be a homogeneous uniform Cantor set with the structure $(I, \{n_k\}, \{c_k\})$, where $I_\emptyset = [0, 1]$, $n_k \to \infty$, $n_{k+1}/n_k \to \infty$ and

$$c_k \equiv n_k^{-\frac{1}{s}} \quad \text{for all } k > 0. \tag{3.3}$$

Then $\dim_H(F) = \lim\limits_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k} = s$ (see [13]).

We note that the length of each gap of rank k

$$\delta_k = \frac{1 - n_k c_k}{n_k - 1} c_1 c_2 \cdots c_{k-1} = \frac{1 - n_k^{1-\frac{1}{s}}}{n_k - 1} c_1 c_2 \cdots c_{k-1}.$$

Since $n_{k+1}/n_k \to \infty$, we have $\delta_{k+1} < \delta_k$ for all k. Any basic interval of rank k has length $\lambda_k = c_1 c_2 \cdots c_k$. Therefore,

$$\lim_{k\to\infty} \frac{\lambda_k}{\delta_k} = \lim_{k\to\infty} c_k n_k = \lim_{k\to\infty} (n_k)^{1-\frac{1}{s}} = 0. \tag{3.4}$$

Suppose on the contrary that $E(\subset F)$ is uniformly perfect with constant c as in (3.2).

Fix a point $x \in E$. For any k, assume that x belongs to I_σ which is a basic interval of rank k. Note that $I_\sigma \subset \{y \colon \lambda_k \le |x - y|\}$. Then the construction of F implies that

$$F \cap \{y \colon 2\lambda_k \le |x - y| \le \delta_k/2\} = \varnothing,$$

which implies for all k,

$$0 < c \le \frac{2\lambda_k}{\delta_k/2}.$$

Letting $k \to \infty$, we obtain that $c = 0$. This a contradiction. Then Proposition 1 is proved.

3.2. Regular Moran set on $\mathbf{R}^1$. We begin the proof of Proposition 2.

" $\Longleftarrow$ " Suppose (1.5) holds, we will verify the regularity. In order to prove Proposition 2, we introduce the natural measure μ supported on Moran set F (see to [1]). Fix $s > 0$. Let

$$\mu(I_\emptyset) = 1, \tag{3.5}$$

where $\emptyset$ is the empty word. By induction, for $\sigma = i_1 \cdots i_k \in \mathcal{D}^k$, we write $\sigma^- = i_1 \cdots i_{k-1} \in \mathcal{D}^{k-1}$ and define

$$\mu(I_\sigma) = \frac{c_{k,i_k}^s}{\sum_{j=1}^{n_k} c_{k,j}^s} \mu(I_{\sigma^-}). \tag{3.6}$$

Using (3.6) again and again, we obtain that $\mu(I_\sigma) = \frac{(c_{k,i_k}c_{k-1,i_{k-1}}\cdots c_{1,i_1})^s}{\prod_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}^s}\mu(I_\emptyset)$. By (3.1) we have

$$\mu(I_\sigma) = \frac{|I_\sigma|^s}{\prod_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}^s} \tag{3.7}$$

More and more, we get a probability measure μ supported on F. Hence, by (1.5), it holds that

$$\beta^{-1}|I_\sigma|^s \le \mu(I_\sigma) \le \alpha^{-1}|I_\sigma|^s. \tag{3.8}$$

For any given point $x \in F$, fix $0 < r \le |F|$. The collection $\mathcal{W}_r$ of words is defined by

$$\mathcal{W}_r = \bigcup_{k=1}^{\infty} \left\{\sigma \in \mathcal{D}^k \colon I_\sigma \cap B(x,r) \neq \varnothing \text{ and } |I_\sigma| \le r < |I_{\sigma^-}|\right\}. \tag{3.9}$$

Let

$$\mathcal{A}_r = \{I_\sigma \mid \sigma \in \mathcal{W}_r\}. \tag{3.10}$$

For members of $\mathcal{A}_r$, since their interiors are pairwise disjoint and

$$|I_\sigma| \ge c_* r \quad \text{for any } I_\sigma \in \mathcal{A}_r, \tag{3.11}$$

we have $\#\mathcal{A}_r \le (2/c_* + 2)$. Notice that

$$B(x,r) \cap F \subset \bigcup_{I_\sigma \in \mathcal{A}_r} I_\sigma. \tag{3.12}$$

According to (3.8), we have

$$\begin{aligned}\mu(B(x,r)) = \mu(B(x,r) \cap F) &\le \sum_{I_\sigma \in \mathcal{A}_r} \mu(I_\sigma) \le \#\mathcal{A}_r \cdot \max_{I_\sigma \in \mathcal{A}} \mu(I_\sigma) \\ &\le (2/c_* + 2)\alpha^{-1} \cdot \max_{I_\sigma \in \mathcal{A}} |I_\sigma|^s \le (2/c_* + 2)\alpha^{-1} \cdot r^s\end{aligned}$$

On the other hand, since x is the center of $B(x,r)$, it is easy to find that there is always a word $\tau \in \mathcal{W}_r$ satisfying that $x \in I_\tau$ and $I_\tau \subset B(x,r)$ due to $|I_\tau| \le r$. Then it holds that, by (3.8),

$$\mu(B(x,r)) \ge \mu(I_\tau) \ge \beta^{-1}|I_\tau|^s \ge \beta^{-1}c_*^s r^s. \tag{3.13}$$

Therefore, we can get (1.1) for the measure μ and the constant $C_F = \max\{(2/c_* + 2)\alpha^{-1}, \beta c_*^{-s}\}$.

" $\Longrightarrow$ " Suppose the Moran set is regular, we shall verify (1.5). We need the following lemma.

Lemma 5. *If F is s-regular, then there is a constant C such that*

$$C^{-1}|I_\sigma|^s \le \nu(I_\sigma) \le C|I_\sigma|^s, \quad \forall \sigma \in \mathcal{D}^\infty. \tag{3.14}$$

Proof. Suppose that there is a Borel probability measure ν supported on F and a constant C_F such that

$$C_F^{-1} r^s \le \nu(B(x,r)) \le C_F r^s. \tag{3.15}$$

For any given $\sigma \in \mathcal{D}^k$, let $\mathcal{P}$ be the set of all basic intervals of rank $(k+2)$ in I_σ, i.e.,

$$\mathcal{P} = \{I_{\sigma * j * h} \colon 1 \le j \le n_{k+1}, 1 \le h \le n_{k+2}\}.$$

Since $n_k \geq 2$ for all $k > 0$, it holds that $\#\mathcal{P} \geq 4$. Then we have

$$\mathcal{Q} = \mathcal{P} \backslash (J_- \cup J_+) \neq \varnothing, \tag{3.16}$$

where J_- is the most left member in $\mathcal{P}$ and J_+ is the most right one. Moreover, it is natural that

$$|J_-| \geq c_*^2 |I_\sigma| \quad \text{and} \quad |J_+| \geq c_*^2 |I_\sigma|. \tag{3.17}$$

Therefore, for any one point $x \in \mathcal{Q} \cap F$, we have $B(x, c_*^2|I_\sigma|) \subset I_\sigma$. Then it holds that, by (3.15),

$$\nu(I_\sigma) \geq \nu(B(x, c_*^2|I_\sigma|)) \geq C_F^{-1} \cdot c_*^{2s} |I_\sigma|^s \tag{3.18}$$

On the other hand, it is obvious that $I_\sigma \subset B(x, |I_\sigma|)$ for any $x \in F \cap I_\sigma$. Then

$$\nu(I_\sigma) \leq \nu(B(x, |I_\sigma|)) \leq C_F |I_\sigma|^s. \tag{3.19}$$

Therefore, let $C = \max\{C_F, c_*^{-2s} C_F\}$, we have (3.14). ☐

By (3.14), we have, $\forall k > 0$,

$$1 = \nu(I_\emptyset) = \sum_{\sigma \in D^k} \nu(I_\sigma) \geq C^{-1} \sum_{\sigma \in D^k} |I_\sigma|^s = C^{-1} \prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^s. \tag{3.20}$$

On the other hand, it is clear that

$$1 = \nu(I_\emptyset) = \sum_{\sigma \in D^k} \nu(I_\sigma) \leq C \sum_{\sigma \in D^k} |I_\sigma|^s = C \prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^s. \tag{3.21}$$

Let $\alpha = C^{-1}$ and $\beta = C$, (1.5) holds. Then Proposition 2 follows.

3.3. An example. For s-regular set E, by Theorem 5.7 of [8], we have

$$\overline{\dim}_B E = \underline{\dim}_B E = \dim_H E = s. \tag{3.22}$$

For Moran set with structure $(J, \{n_k\}, \{c_{k,j}\})$, the positive sequence $\{s_k\}_{k>0}$ is called the pre-dimension sequence of F, where s_k satisfies

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^{s_k} = 1.$$

Let $s_* = \underline{\lim}_{k\to\infty} s_k$ and $s^* = \overline{\lim}_{k\to\infty} s_k$. It was shown in [13, 14] that, if $c_* > 0$ for Moran set F as above, then

$$\dim_H F = s_* \quad \text{and} \quad \overline{\dim}_B F = s^*.$$

Therefore, if $s_* < s^*$, then F can not be regular.

Example 1. Let $n_k \equiv 2$ and $c_k \in \{1/3, 1/5\}$. Then $c_* > 0$. Take a sequence $\{c_k\}_k$ such that $a = \underline{\lim}_{k\to\infty} q_k < \overline{\lim}_{k\to\infty} q_k = b$, where

$$q_k = \frac{\#\{i \leq k \colon c_k = 1/3\}}{k}.$$

Then

$$\underline{\lim}_{k\to\infty} s_k = \frac{\log 2}{a \log 3 + (1-a) \log 5} \quad \text{and} \quad \overline{\lim}_{k\to\infty} s_k = \frac{\log 2}{b \log 3 + (1-b) \log 5},$$

which means $\dim_H F < \overline{\dim}_B F$ if Moran set F has the structure $\{[0,1], \{n_k\}, \{c_k\}\}$. Hence F can not be regular.

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