SPACE-FILLING VS. LUZIN'S CONDITION (N)

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Abstract. Let us assume that we are given two metric spaces (X, d_X) and (Y, d_Y) where the Hausdorff dimension s of X is strictly smaller than that of Y. Suppose that X is σ-finite with respect to $\mathcal{H}^s$. Then we show that for quite general metric spaces, if $f\colon X \to Y$ is a measurable surjection, there is a set $N \subset X$ with $\mathcal{H}^s(N) = 0$ and $\mathcal{H}^s(f(N)) > 0$. If f is continuous, then we investigate whether N can be chosen to be perfect. We also study more general situations where the measures on X and Y are not necessarily the same and not necessarily Hausdorff measures.

1. Introduction

Around 1877, Cantor realized that there exists a bijection between the unit interval $[0,1]$ and the unit square $[0,1]^2$. Motivated by this result, there has been an interest in understanding how mappings can increase the dimension. Netto proved that it is impossible to find a continuous bijection between the interval and the square. However, Peano's work showed that one can construct a continuous surjection from the interval onto the square. We refer the reader to the book of Sagan for a more detailed account of the history of space-filling curves, [Sag94]. Denoting[1] by $p\colon [0,1] \twoheadrightarrow [0,1]^2$ the space-filling curve from Peano's construction, we can define $f\colon [0,1]^2 \twoheadrightarrow [0,1]^2$ by $f(x,y) = p(x)$ and see that there are one-dimensional sets, the intervals $[0,1] \times \{x\}$, $x \in [0,1]$, for example, that are mapped onto two-dimensional sets. This violates *Luzin's condition (N)*, which requires that sets of measure zero be mapped to sets of measure zero. Luzin's condition (N) is important in applications such as elasticity, see e.g. [MS95]. It is also a requirement for various area- and change of variables-formulas to hold, see for example Proposition 1.1 in [Mal94].

We take the following question as starting point for our inquiry: "Given a continuous surjection $f\colon [0,1] \twoheadrightarrow [0,1]^2$ is there a set $N \subset [0,1]$ such that $\mathcal{H}^1(N) = 0$ and $\mathcal{H}^1(f(N)) > 0$?" The fact that the target is higher dimensional than the domain means philosophically that some subsets of the domain are "blown up". Luzin's condition (N) requires that *small sets stay small*. However, space-filling maps could *still* satisfy Luzin's condition (N) by mapping *only* sets of positive $\mathcal{H}^1$-measure onto two-dimensional sets.

In [WZ12, WZ09a, WZ09b], we (Wildrick and the author) have studied space-fillings and Luzin's condition (N) with respect to so called[2] *Sobolev–Lorentz spaces*. These are generalizations of Sobolev spaces, spaces of mappings that possess some

[1]We use the symbol $\twoheadrightarrow$ to indicate that the mapping under consideration is a surjection.

[2]These spaces have also been studied in [RM09] and [Rom08] in the metric setting and in [KKM99] in the Euclidean setting.

sort of derivatives that have a certain integrability. We further argued that a space-filling in a Sobolev–Lorentz space cannot satisfy Luzin's condition (N), the reason being that we can partition the space in a null set and countably many sets where the restrictions of the mapping are Lipschitz. As Lipschitz mappings do not increase the dimension, it is necessary that the null set is mapped to a set of higher dimension. The following criterion is basically Theorem 1 in [Tro00]:

Lemma 1.1. [Tro00, Theorem 1] *Let $f\colon X \to Y$ be a mapping between two metric spaces. Suppose that X is locally compact and separable and that μ is a Radon measure on X. Then the following conditions are equivalent:*

(a) *There exists a measurable function $w\colon X \to [0,\infty]$ that is finite almost everywhere and such that*

$$d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)(w(x_1) + w(x_2))$$

for all x_1 and x_2 in X.

(b) *There exists a monotone sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset X$ such that $f \upharpoonright K_k$ is Lipschitz and $\mu(X \setminus \bigcup_{k=1}^{\infty} K_k) = 0$.*

In this article, we further investigate the dichotomy between Luzin's condition (N) and space-fillings on a more abstract level than in our previous work. Especially, we look also at settings where (b) of Lemma 1.1 is not available.

More precisely, we will look at mappings of which we me merely know that they are continuous or sometimes even only measurable. To give a flavor of what kind of results we obtain, we state now one of the results of this article and a corollary. The terms will be explained in the course of the article.

Theorem 1.2. *Let $(X, \mathfrak{T})$ be a regular, second countable topological space. Let μ be a Borel measure on X that is $\mathcal{G}_\delta$-regular and such that X is σ-finite with respect to μ. We further require that X can be written as countable union of compact sets and a set of measure zero. Suppose that (Y, d) is a metric space with Hausdorff measure $\mathcal{H}^h$ that is of finite order, $h(0) = 0$, and assume that Y is not σ-finite with respect to $\mathcal{H}^h$. If $f\colon X \twoheadrightarrow Y$ is a measurable surjection, then there exists a set $N \subset X$ with $\mu(N) = 0$ and $\mathcal{H}^h(f(N)) > 0$.*

As corollary, we obtain the following:

Corollary 1.3. *Assume that $s > 0$ and (X, d_X) is separable and σ-finite with respect to $\mathcal{H}^s$ and can be written as countable union of compact sets and a set of measure zero. Suppose that the metric space (Y, d_Y) has dimension t. If $f\colon X \twoheadrightarrow Y$ is a measurable surjection, then there exists a set $N \subset X$ such that $\mathcal{H}^s(N) = 0$ and $\dim_H f(N) = t$.*

Based on [HT08] and [WZ12], we will give a construction scheme for space-fillings between rather general metric measure spaces that map a perfect set of measure zero to the whole target space, see Theorem 9.1. This leads to the question if we can always find a (perfect) set of measure zero that is mapped to the whole space. However, for example in the case $f\colon [0,1] \twoheadrightarrow [0,1]^2$, this fails if f is 1/2-Hölder, which may happen:

Theorem 1.4. [Buc96, Theorem 3] *There exist Peano curves $F\colon [0,1] \to [0,1]^2$ that are α-Hölder continuous for $\alpha = 1/2$, but no such curve is α-Hölder continuous for $\alpha > 1/2$.*

For related results, see also [Shc10], where Hölder continuous surjections between cubes are studied. Other counter examples can be constructed from the following result from [HS05]:

Theorem 1.5. [HS05, Thread Theorem] *For each $n \geq 2$ there exists a continuous, one-to-one mapping $\varphi \colon [0, 1) \to (0, 1)^n$ such that $\mathcal{L}^1(\varphi^{-1}(B)) = \mathcal{L}^n(B)$ for all Borel subsets B of $[0, 1]^n$.*

For each $n \geq 2$, we obtain a continuous bijection $f \colon [0, 1) \to f([0, 1)) \subset (0, 1)^n$ such that $\mathcal{L}^1(B) = \mathcal{L}^n(f(B))$ for every Borel set $B \subset [0, 1)$, and thus $\mathcal{L}^1(N) = 0$ if and only if $\mathcal{L}^n(f(N)) = 0$. Note that in the present case $\mathcal{L}^n(f([0, 1))) > 0$.

The following example was explained to the author by Wildrick, who learned it from Hencl.

Example 1.6. If $f \colon [0, 1] \twoheadrightarrow [0, 1]^2$ is a continuous surjection, then there exists a set $N \subset [0, 1]$ with $\mathcal{H}^1(N) = 0$ and $\mathcal{H}^1(f(N)) > 0$. Moreover, N can be chosen to be closed.

Proof. Suppose that the first statement in the example is not true. Hence, for every set $N \subset [0, 1]$ with $\mathcal{H}^1(N) = 0$, we obtain $\mathcal{H}^1(f(N)) = 0$. Equivalently put, if $\mathcal{H}^1(f(E)) > 0$, then $\mathcal{H}^1(E) > 0$ for all sets $E \subset [0, 1]$. To arrive at a contradiction, it suffices to find uncountably many pairwise disjoint closed sets E_α with $\mathcal{H}^1(f(E_\alpha)) > 0$. Our assumption tells us then that $\mathcal{H}^1(E_\alpha) > 0$, and hence $\mathcal{H}^1([0, 1]) = \infty$. We can clearly find uncountably many pairwise disjoint closed sets $F_\alpha \subset [0, 1]^2$ with $\mathcal{H}^1(F_\alpha) > 0$. The sets $E_\alpha := f^{-1}(F_\alpha)$ do the job. $\qquad\square$

We will construct in Theorem 9.1 continuous surjections. In these constructions, it is easily observed that N can be chosen to be a perfect set. The following result puts this observation into perspective. Note that it implies that each closed set in a T_1-space can be written as union of a perfect and of a countable set.

Theorem 1.7. [Kur72, Cantor–Bendixson, Theorem XIV.5.3] *Every T_1-space[3] with a countable base is the union of two disjoint sets, one dense in itself and closed (i.e. perfect) and the other countable.*

Structure of the paper. The paper can roughly be divided into three parts. We start in Section 2 with notation, the measure theoretic background, and some properties of metric spaces. The main content of Section 3 is to give an abstract version of Theorem 1.6. An important property that we need in this abstract result is the existence of a certain amount of pairwise disjoint sets. In Section 4, we give a first result guaranteeing the needed amount of pairwise disjoint subsets. The following section concludes the first part of the article by providing a first result about blowing up sets.

Unlike in the first part, where we focussed on conditions on Y to obtain sets that are blown up, in the second part, we impose more conditions on X. We do this by searching conditions such that X can be written as union of compact sets with finite measure and a set of measure zero, such that the restrictions of the space-filling under question to the compact sets are continuous. In Section 6, we turn our focus to such a partition of X. In Section 7, our thoughts center around Luzin's theorem to improve above partition such that the restriction of the mapping to the sets of

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[3]A topological space is called a T_1-*space* if each single element is closed.

positive measure are continuous. This second part is concluded with the proof of Theorem 1.2 in Section 8.

The following sections constitute the last part. They complement the findings in the first two parts. Having assumed the existence of space-fillings in the first two parts, we give in Section 9 a result showing that there are plenty of space-fillings. Section 10 complements the existence of blown up sets by considering space-fillings that do satisfy Luzin's condition (N). In some respect, these results show the sharpness of some of the assumptions in our main results. We conclude the article with Section 11, where we describe some applications of our results.

2. Notation, measure theoretic background, and some properties of metric spaces

In this section, we lay out some measure theoretic facts. As sources, we mainly use [Rog70] and [How95] as references for the Hausdorff measures and Fremlin's opus on measure theory [Fre].

We denote by *measure* what some other authors, Fremlin for example, call *outer measure*. If we cite a result from one of his volumes, then we replace a possible occurrence of a σ-algebra by the σ-algebra of the measurable sets.

Definition 2.1. (Measure, σ-finite, (Borel) measurable) Let X be a set.

(a) If $\mu\colon \mathcal{P}(X) \to [0,\infty]$ is such that
  - $\mu(\emptyset) = 0$,
  - $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ if $A \subset \bigcup_{i=1}^{\infty} A_i$,

  then we say that μ is a *measure*, and we call (X,μ) a *measure space*.

(b) If μ is a measure on X, a set M is said to be μ-*measurable*[4] if for all sets A, B with $A \subset M$ and $B \subset X \setminus M$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$. We say that μ is *Borel measurable* if each Borel set is measurable.

(c) We say that a set S in a measure space (X,μ) is σ-*finite* with respect to μ if S can be written as countable union of measurable subsets with finite measure.

(d) Let (X,μ) be a measure space and $(Y,\mathfrak{S})$ a topological space. We say that a mapping $f\colon X \to Y$ is *measurable* if for any open set $O \in \mathfrak{S}$, its preimage $f^{-1}(O)$ is measurable; if $f^{-1}(O)$ is a Borel set, we say that f is *Borel measurable*.

The following fact can be found for example in Section 112C in [Fre].

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[4]If the measure under consideration is clear, then we also speak simply of *measurable*. We will also use the equivalent formulation that M is measurable if and only if for every set F, we have $\mu(F) = \mu(F \cap M) + \mu(F \setminus M)$.

Lemma 2.2. *Let (X, μ) be a measure space. If $(M_n)_n$ is a non-decreasing sequence of measurable sets (that is, $M_n \subset M_{n+1}$, $n \in \mathbf{N}$), then*

$$\mu\left(\bigcup_n M_n\right) = \lim_{n \to \infty} \mu(M_n) = \sup_n \mu(M_n).$$

We take the following definition from Definition 9 in [Rog70] and 411B and 411D in [Fre]:

Definition 2.3. ($\mathcal{R}$-regular, inner regular, and outer regular) If $\mathcal{R}$ is a class of sets, a measure μ is said to be

(a) $\mathcal{R}$-*regular*, if for each E in X there is a set R in $\mathcal{R}$ with $E \subset R$ and $\mu(E) = \mu(R)$,

(b) *inner regular with respect to $\mathcal{R}$* if

$$\mu(M) = \sup\{\mu(R) \colon R \in \mathcal{R}, \ R \subset M, \text{ and } R \text{ measurable}\}$$

for every measurable set M,

(c) *outer regular with respect to $\mathcal{R}$* if

$$\mu(M) = \inf\{\mu(R) \colon R \in \mathcal{R}, \ R \supset M, \text{ and } R \text{ measurable}\}$$

for every measurable set M.

Definition 2.4. (Premeasure, finite order) A *premeasure* ξ on Y is a function mapping the subsets of Y to the non-negative reals satisfying

(a) $\xi(\emptyset) = 0$,

(b) if $U \subset V$, then $\xi(U) \leq \xi(V)$ for all $U, V \subset Y$.

We will say that the premeasure ξ is of *finite order* if and only if for some constant η, we have

(c) $\xi(\widehat{U}) \leq \eta \xi(U)$ for all $U \subset Y$,

(d) $\inf\{\xi(B(y, \delta)) \colon \delta > 0\} \leq \eta \xi(\{y\})$ for all $y \in Y$,

where

$$\widehat{U} = \bigcup\{E \subset Y \colon E \cap U \neq \emptyset \text{ and } \operatorname{diam} E \leq \operatorname{diam} U\}$$

and $B(y, \delta)$ denotes the open ball with center y and radius δ.

Definition 2.5. (δ-cover) We say that a sequence $(U_i)_i$ of subsets of Y is a δ-*cover* of a set E if and only if $E \subset \bigcup U_i$ and

$$\operatorname{diam} U_i \leq \delta, \quad i \in \mathbf{N}.$$

We use $\Omega_\delta(E)$ to denote the family of all such (countable) δ-covers of E.

Definition 2.6. (Hausdorff measure) The measures Λ_δ^ξ are defined for $\delta > 0$ by

$$\Lambda_\delta^\xi(E) = \inf\left\{\sum_{i=1}^\infty \xi(U_i) \colon (U_i)_i \in \Omega_\delta(E)\right\},$$

with the convention that $\inf \emptyset = \infty$. The *Hausdorff ξ-measure* Λ^ξ is then defined as $\Lambda^\xi(E) := \sup_{\delta > 0} \Lambda_\delta^\xi(E)$.

Definition 2.7. (Hausdorff function, finite order) A function h, defined for all non-negative real numbers, is a *Hausdorff function* if and only if the following conditions are satisfied:

(a) $h(t) > 0$ for all $t > 0$,

(b) $h(t) \geq h(s)$ for all $t \geq s$,

(c) h is continuous from the right for all $t \geq 0$.

For such a function and a positive constant Θ, we define a premeasure, ξ say, on X by

$$\begin{aligned} \xi(U) &= \min\{h(\operatorname{diam} U), h(\Theta)\}, \quad U \neq \emptyset, \\ \xi(\emptyset) &= 0. \end{aligned} \tag{1}$$

If the premeasure ξ is defined by some Hausdorff function h, we will use $\mathcal{H}^h$ for Λ^ξ. If

$$\limsup_{t \to 0+} \frac{h(3t)}{h(t)} < \infty,$$

we say that h is *of finite order*.

Remark 2.8.

- The constant Θ ensures that the premeasure assigns finite values to all sets. We can allow for Hausdorff functions that admit the value ∞ if it is possible to choose $0 < \Theta$ so small that $h(t) < \infty$ for $t \leq \Theta$.
- If h is of finite order, then the induced premeasure is of finite order for small enough Θ. When we speak of finite order, then we assume that Θ has been chosen to be so small that ξ is of finite order.
- Assume that h is of finite order, continuous from the right in 0, increasing, $0 < h(t) < \infty$ for $t > 0$, and $h(0) = 0$. If we set $H(0) = 0$ and for $t > 0$

$$H(t) := \frac{1}{t} \int_t^{2t} h(s)\, ds,$$

  then H is a continuous Hausdorff function of finite order comparable to h for small arguments, see for example Section 1 in [Edg07] for more information.

We take the definition of the Hausdorff–Besicovitch dimension from Section 4 in [How95].

Definition 2.9. (Hausdorff–Besicovitch dimension) We define the *Hausdorff–Besicovitch dimension*[5] of a metric space, (X, d), to be the supremum of all non-negative s for which $\Lambda^{h(s)} > 0$, where $h(s)$ is defined on all non-negative t by $h(s)(t) = t^s$. We denote the Hausdorff–Besicovitch dimension of X by $\dim_H(X)$.

We can define a partial strict ordering on the set of Hausdorff functions:

Definition 2.10. We say that

$$g \prec h$$

for two Hausdorff functions g and h if

$$\lim_{t \to 0+} \frac{h(t)}{g(t)} = 0.$$

The following lemma is stated on p. 79 in [Rog70] as corollary and enables us to talk about sort of *generalized dimension*.

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[5]It is also known as *Hausdorff dimension*. Sometimes we also simply omit both, Hausdorff and Besicovitch, and just talk about dimension.

Lemma 2.11. *Let f, g, and h be Hausdorff functions with $f \prec g \prec h$. If a set E in a metric space has σ-finite positive $\mathcal{H}^g$-measure, then E has zero $\mathcal{H}^h$-measure and non-σ-finite $\mathcal{H}^f$-measure.*

Theorem 27 on p. 50 in [Rog70] tells us about regularity properties of Hausdorff measures:

Theorem 2.12. *A Hausdorff measure $\mathcal{H}^h$ is a regular, $\mathcal{G}_\delta$-regular[6] metric[7] measure, all Borel sets are $\mathcal{H}^h$-measurable, and each $\mathcal{H}^h$-measurable set of finite $\mathcal{H}^h$-measure contains an $\mathcal{F}_\sigma$-set with the same measure.*

Let us turn our attention towards metric spaces. The aim is to introduce the needed definitions in order to state Howroyd's result on the existence of disjoint subsets of positive measure.

Definition 2.13. (Analytic, Souslin) A Hausdorff space is *analytic*, also called *Souslin*, if it is either empty or a continuous image of $\mathbf{N}_0^{\mathbf{N}_0}$.

Definition 2.14. (Souslin's operation) Let S be the set $\bigcup_k \mathbf{N}_0^k$. If $\mathcal{E}$ is a family of sets, we write $\mathbf{S}(\mathcal{E})$ for the family of sets expressible in the form[8]

$$\bigcup_{\phi \in \mathbf{N}_0^{\mathbf{N}_0}} \bigcap_{k \geq 1} E_{\phi \restriction k}$$

for some family $(E_\sigma)_{\sigma \in S}$ in $\mathcal{E}$. A family $(E_\sigma)_{\sigma \in S}$ is called a *Souslin scheme*; the operation

$$(E_\sigma)_{\sigma \in S} \mapsto \bigcup_{\phi \in \mathbf{N}_0^{\mathbf{N}_0}} \bigcap_{k \geq 1} E_{\phi \restriction k}$$

is *Souslin's operation*. Thus $\mathbf{S}(\mathcal{E})$ is the family of sets obtainable from sets in $\mathcal{E}$ by Souslin's operation. If $\mathcal{E} = \mathbf{S}(\mathcal{E})$, we say that $\mathcal{E}$ is *closed under Souslin's operation*.

Definition 2.15. (Polish space) A topological space X is *Polish* if it is separable and its topology can be defined from a metric under which X is complete.

Remark 2.16. By 423B in [Fre], Polish spaces are analytic.

Definition 2.17. (Souslin-F) Let X be a topological space. A subset of X is a *Souslin-F* set in X if it is obtainable from closed subsets of X by Souslin's operation; that is, it is the projection of a closed subset of $\mathbf{N}_0^{\mathbf{N}_0} \times X$. For a subset of $\mathbf{R}^n$, or, more generally, of any Polish space, it is common to say 'Souslin set' for 'Souslin-F set'.

The next theorem is a version of [Fre, Theorem 423E].

Theorem 2.18. *Let $(X, \mathfrak{T})$ be an analytic Hausdorff space. For a subset A of X, the following are equivalent:*

(i) *A is analytic,*

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[6]A set is termed $\mathcal{G}_\delta$ if it can be written as countable intersection of open sets. If it has a representation as countable union of closed sets, then it is called $\mathcal{F}_\sigma$-set.

[7]A metric measure μ is such that $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever A and B have a positive distance.

[8]As Fremlin in [Fre], we can regard a member of $\mathbf{N}_0$ as the set of its predecessors, so that $\mathbf{N}_0^k$ can be identified with the set of functions from k to $\mathbf{N}_0$, and if $\phi \in \mathbf{N}_0^{\mathbf{N}_0}$ and $k \in \mathbf{N}_0$, we can speak of the restriction $\phi \restriction k \in \mathbf{N}_0^k$.

(ii) *A is Souslin-F,*

(iii) *A can be obtained by Souslin's operation from the family of Borel subsets of X.*

Definition 2.19. (Finite structural dimension) We say that (Y, d) has *finite structural dimension* if and only if for all positive ξ, there exist $N \in \mathbf{N}$ such that every subset of Y of sufficiently small diameter δ can be covered by N sets of diameter not greater than $\xi\delta$.

Remark 2.20. Recall that a space is *doubling* if there is a number N such that each ball can be covered by N balls of half the radius of the original ball. Doubling spaces have finite structural dimension. On the other hand, if a space has finite structural dimension and is compact, then it is doubling.

Definition 2.21. (Ultrametric) The metric space (Y, d) is said to be *ultrametric* if and only if

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all x, y, and z in Y.

Howroyd lists different kind of spaces for which his result holds. We collect these spaces under a common condition:

Definition 2.22. Assume that a space $A = (A, d, \mathcal{H}^h)$ is given.

(H) We say that A has property (H) if and only if (A, d) is an analytic subspace of a complete, separable metric space (Y, d), and h is a continuous Hausdorff function with $h(0) = 0$ such that at least one of the following properties holds:
   (a) h is of finite order,
   (b) Y has finite structural dimension,
   (c) Y is ultrametric.

(H∞) We say that A has property (H∞) if and only if A satisfies property (H) and is not σ-finite with respect to $\mathcal{H}^h$.

Assume $X = (X, \mu)$ is a measure space and κ a cardinal.

(Aκ) We say that X satisfies property (Aκ) if and only if it can be written as union of κ-many measurable sets of finite measure and is further such that the union of κ-many sets of measure zero has measure zero as well.

Remark 2.23. (Continuum Hypothesis (CH)) Sometimes we will assume the Continuum Hypothesis (CH), i.e. that every infinite subset of $\mathbf{R}$ has either the same cardinality as $\mathbf{R}$ or as $\mathbf{Q}$.

3. An abstract result

It is surprising how few properties of $[0, 1]$, $[0, 1]^2$ and their standard metric and measure, we actually need for the argument in Example 1.6 to work. We start by reviewing the example in a very abstract fashion. More precisely, we look at $[0, 1]$ and $[0, 1]^2$ as topological spaces. We further replace the Hausdorff measures by more general measures, for example Borel measures. In Example 1.6, we have that the target Y is higher dimensional than the domain X. This is encoded by the fact that $[0, 1]$ has finite $\mathcal{H}^1$-measure and $[0, 1]^2$ is not σ-finite with respect to $\mathcal{H}^1$. In the new setting, we do not compare the "dimensions" of the domain and the target. However, we express the fact that the domain is "smaller" than the target by requiring that the

domain can be written as union of, say, κ-many measurable sets of finite measure, whereas Y cannot. This also permits to look at situations where the image is lower dimensional than the target. If we look for example at a continuous surjection from $[0,1]^3$ onto $[0,1]^2$, then there is a set of measure zero in $[0,1]^3$ that is mapped onto a set of dimension two.

Let us finally state a quite abstract result:

Lemma 3.1. *Assume that $\kappa < \lambda$ are two cardinals and if η is a cardinal with $\kappa \cdot \eta = \lambda$, then $\aleph_0 < \eta$. Suppose that (X, μ) is a measure space satisfying $(A\kappa)$. Assume $(Y, \nu, \mathfrak{S})$ is a measure space equipped with topology $\mathfrak{S}$ that contains λ-many pairwise disjoint Borel sets Y_j that have positive measure. Further, we stipulate the existence of a measurable surjection $f \colon X \twoheadrightarrow Y$. Then there is a set $N \subset X$ such that $\mu(N) = 0$ and $\nu(f(N)) > 0$. Moreover, N can be chosen to be the preimage of one of the sets Y_j.*

Proof. Without loss of generality, we may assume that $\mu(X) > 0$ for otherwise we can choose $N = f^{-1}(Y_1)$. We first write X as union

$$X = \bigcup_{i \in I} X_i,$$

where each X_i is measurable, has finite measure, and card $I \leq \kappa$. We let $\{Y_j\}_j$ be a collection of λ-many pairwise disjoint Borel sets Y_j with $\nu(Y_j) > 0$. We set $A_j := f^{-1}(Y_j)$ and note that the sets A_j are measurable. We further argue that the sets in $\{A_j\}_j$ are pairwise disjoint. If not, then there are distinct j and k and a point $x \in X$ contained in $A_j \cap A_k$. But then, as f is surjective, $f(x) \in f(A_j) \cap f(A_k) = Y_j \cap Y_k$—a contradiction.

Assume by contradiction that $\mu(N) = 0$ implies $\nu(f(N)) = 0$ for all sets $N \subset X$. Otherwise stated, if $\nu(f(E)) > 0$, then $\mu(E) > 0$ for each measurable set $E \subset X$. This establishes that each set A_j has positive measure.

For each set A_j, there is a set X_i such that $\mu(X_i \cap A_j) > 0$ for otherwise the measure of A_j would be zero. Having the different cardinalities in mind, we find a set X_n such that there are uncountably many A_j such that $\mu(X_n \cap A_j) > 0$. There exists $m \in \mathbf{N}$ and a countably infinite set L such that for the above chosen set X_n

$$\mu(A_l \cap X_n) > \frac{1}{m}$$

for $l \in L$. This implies that

$$\infty > \mu(X_n) \geq \sum_{l \in L} \mu(X_n \cap A_l) \geq \sum_{l \in L} \frac{1}{m} = \infty.$$

This contradiction gives the proof. $\qquad\square$

Remark 3.2. Ostaszewski shows in Theorem 2 of [Ost74] that Martin's axiom implies that the union of less than $2^{\aleph_0}$ sets of μ-measure zero is of μ-measure zero if μ is a measure of Hausdorff type. He attributes the proof for the case when μ is a Hausdorff measure $\mathcal{H}^h$ with Hausdorff function h satisfying $h(0) = 0$ to Martin and Solovay, [MS70]. Further studies of the connection between Martin's axiom and Hausdorff measures can be found in [Zin].

The conclusion of Lemma 3.1 can hold in cases where the assumptions are not satisfied:

Example 3.3. Let $Y = \{y_0\}$ be a set with one element. Define the Borel measure ν on Y by $\nu(Y) = \infty$. Then Y is not σ-finite with respect to ν, but does not contain uncountably many pairwise disjoint sets. If (X, μ) is a measure space that contains a point $\{x_0\}$ that has measure zero and $f \colon X \twoheadrightarrow Y$ is a surjection, then there exists a set $N \subset X$ with $\mu(N) = 0$ and $0 < \nu(f(N)) = \infty$. For example, we can take $N = \{x_0\}$.

However, we will see in Example 10.10 that we need some conditions in order that the conclusion of Lemma 3.1 holds.[9]

4. A first test for the existence of disjoint sets

In Lemma 3.1, we require the existence of a certain amount of pairwise disjoint Borel sets B_α in Y with $\nu(B_\alpha) > 0$. Later, we will assume that Y is a metric space and ν a Hausdorff measure; but before, we provide an abstract criterion for the existence of the desired sets. We will employ it later in concrete situations.

The idea is the following: we assume that $\nu(Y) = \infty$ and that Y cannot be written as, let us say, a countable union $\bigcup_n Y_n$ of Borel sets Y_n with $\nu(Y_n) < \infty$. We find a Borel set $B_1 \subset Y$ with $0 < \nu(B_1) < \infty$. In the next step, we look at $Y \setminus B_1$ and extract a Borel set $B_2 \subset Y \setminus B_1$ with $0 < \nu(B_2) < \infty$. Continuing this process, we end up with countably many pairwise disjoint Borel sets B_n with $0 < \nu(B_n)$. To prove the existence of a desired collection with uncountably many elements, we resort to Zorn's Lemma. We prove that there has to be a maximal collection of pairwise disjoint Borel sets, and that this collection has the desired cardinality.

Let us recall Zorn's Lemma, which is equivalent to the axiom of choice:

Lemma 4.1. (Zorn's Lemma) *Let $(P, \leq)$ be a nonempty partially ordered set, i.e. for all a, b, $c \in P$*

(a) $a \leq a$,
(b) $a \leq b$ and $b \leq a$ implies $a = b$,
(c) $a \leq b$ and $b \leq c$ implies $a \leq c$.

A chain Q in P is a subset of P such that for all a, $b \in Q$, $a \leq b$ or $b \leq a$. If each chain in P has an upper bound in P, then P has a maximal element.

In the following lemma, we can for example think of $\mathcal{A}$ as the analytic sets, I as $\mathbf{N}$ and $\mathcal{C}$ as the collection of the Borel or the closed sets.

Lemma 4.2. *Let (Y, ν) be a measure space and $\mathcal{C}$ be a collection of subsets of Y. Assume there exists a collection $\mathcal{A}$ of subsets of Y and a cardinal $\kappa \geq \aleph_0$ with the following properties*

(a) $Y \in \mathcal{A}$,
(b) *Y cannot be written as union of κ-many sets in $\mathcal{A}$ with finite measure,*
(c) *$\bigcup_{j \in J} C_j$ and $Y \setminus \bigcup_{j \in J} C_j$ lie in $\mathcal{A}$ provided $\operatorname{card}(J) \leq \kappa$ and $C_j \in \mathcal{C}$,*
(d) *If $\nu(A) = \infty$ for some $A \in \mathcal{A}$, then there exists a set $C \in \mathcal{C}$ with $C \subset A$ and $0 < \nu(C) < \infty$.*

Under these conditions, there exists a collection of pairwise disjoint sets $C_\gamma \in \mathcal{C}$ with $0 < \nu(C_\gamma)$ whose cardinality is strictly larger than κ.

---

[9]At least when we assume the Continuum Hypothesis.

Proof. We let

$$\mathcal{D} := \{\{D_\delta\}_{\delta \in J}\}$$

be the set of all possible collections $\{D_\delta\}_{\delta \in J}$ of pairwise disjoint sets in the collection $\mathcal{C}$ with $0 < \nu(D_\delta) < \infty$. We say that $\{D_\delta\}_{\delta \in J} \leq \{\widetilde{D}_{\tilde\delta}\}_{\tilde\delta \in \tilde{J}}$ if $\{D_\delta\}_{\delta \in J} \subset \{\widetilde{D}_{\tilde\delta}\}_{\tilde\delta \in \tilde{J}}$. Since $\subset$ is an ordering relation, $(\mathcal{D}, \leq)$ is a partially ordered set.

Let us verify the assumptions of Zorn's Lemma and thus the existence of a maximal collection in $\mathcal{D}$.

By (a), the set Y is in $\mathcal{A}$ and by (b) we obtain $\nu(Y) = \infty$. Assumption (d) gives a set $D \in \mathcal{C}$ with $0 < \nu(D) < \infty$ implying $\{D\} \in \mathcal{D} \neq \emptyset$.

Let $\{\{D_\delta^j\}_{\delta \in J_j}\}_{j \in J}$ be a chain in $\mathcal{D}$. We set

$$\mathcal{U} := \{D_\delta^j : \delta \in J_j, \, j \in J\}$$

and claim that $\mathcal{U}$ is an upper bound of the chain in $\mathcal{D}$. From the definition of $\mathcal{U}$, it follows at once that its elements are sets D_δ^j in $\mathcal{C}$ with $0 < \nu(D_\delta^j) < \infty$. Now let us show that the sets in $\mathcal{U}$ are pairwise disjoint. Let us select two distinct sets in $\mathcal{U}$. The chain property allows us to assume that both sets are in the same element of the chain forcing their disjointness. We conclude that $\mathcal{U}$ is indeed in $\mathcal{D}$, and it is easy to see that $\mathcal{U}$ is an upper bound for the chain.

Zorn's Lemma 4.1 guarantees the existence of a maximal collection $\{M_\gamma\}_{\gamma \in C}$ of pairwise disjoint sets in $\mathcal{C}$ with $0 < \nu(M_\gamma) < \infty$.

Assume by contradiction that above maximal collection has cardinality at most κ. We set

$$M := \bigcup_{\gamma \in C} M_\gamma.$$

By (c), M and $Y \setminus M$ lie in $\mathcal{A}$. According to (b), the measure of $Y \setminus M$ is infinite. Appealing to (d), there exists a set $\widetilde{M} \subset Y \setminus M$ in $\mathcal{C}$ with $0 < \nu(\widetilde{M}) < \infty$, and it is clearly disjoint to every element in $\{M_\gamma\}_{\gamma \in C}$. But then the collection $\{M_\gamma\}_{\gamma \in C} \cup \{\widetilde{M}\} \in \mathcal{D}$ contradicts the maximality of $\{M_\gamma\}_{\gamma \in C}$, and we are done. $\qquad\square$

Remark 4.3. For related theorems, see [Dav68] (or the comment before Corollary 5.3) and Theorem 2 in [Lar74].

5. Existence of disjoint sets in the metric setting and first results

We give now a version of Corollary 7 in [How95] that we will use to verify (d) in Lemma 4.2.

Theorem 5.1. (Howroyd) *Suppose $(A, d, \mathcal{H}^h)$ satisfies property (H). Then for all real l with[10] $l < \Lambda^\xi(A)$, there exists a (compact) subset K of A such that*

$$l < \mathcal{H}^h(K) < \infty.$$

Corollary 5.2. *Suppose $(Y, d, \mathcal{H}^h)$ satisfies $(H\infty)$. Then there are uncountably many pairwise disjoint compact subsets of Y with positive measure.*

Proof. In Lemma 4.2, we let $\mathcal{C}$ be the compact and $\mathcal{A}$ the analytic subsets of Y. Further, we let $\kappa = \aleph_0$. By Theorem 2.18, Borel sets are analytic. Requirement (d) in Lemma 4.2 follows from Theorem 5.1, and the other assertions in the lemma are easily verified. $\qquad\square$

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[10]See (1) for the relation between ξ and h.

If we assume that the space Y is compact, then we can be more precise about the number of pairwise disjoint sets in Lemma 4.2. The Theorem in [Dav68] roughly states that if (Y, d) is a compact space of non-σ-finite $\mathcal{H}^h$-measure, and every closed subset of Y has subsets of finite measure, then Y contains a system of $2^{\aleph_0}$ disjoint closed subsets each of non-σ-finite measure.

Corollary 5.3. *If $(Y, d, \mathcal{H}^h)$ is a compact space that satisfies (H∞), then there are $2^{\aleph_0}$ pairwise disjoint compact subsets each of non-σ-finite measure.*

We recall the definition of a perfect set.

Definition 5.4. (Perfect) A set is *perfect* if it is closed, and each open set that meets it at all, meets it in an infinite set.

Theorem 5.5. *Suppose $X = (X, \mu)$ is σ-finite and μ is Borel. Suppose further that $(Y, d, \mathcal{H}^h)$ satisfies (H∞). If $f: X \twoheadrightarrow Y$ is a measurable surjection, then there is a set $N \subset X$ such that $\mu(N) = 0$ and $\mathcal{H}^h(f(N)) > 0$. Moreover, if X is additionally equipped with a topology with a countable basis and such that points are closed, and we further stipulate that f is such that for every closed set $F \subset Y$, the preimage $f^{-1}(F)$ can be written as countable union of closed sets, then N can be chosen to be perfect.*

Proof. We want to apply Lemma 3.1. The assumptions on (X, μ) are satisfied with $\kappa = \aleph_0$. Using Corollary 5.2, we obtain the existence of some cardinal $\lambda > \kappa$ along with λ-many pairwise disjoint compact sets Y_j with positive measure. Applying Lemma 3.1 gives the existence of N, and it can be chosen to be the preimage of a closed set.

For the moreover part, note that N can be written as union of countably many closed sets F_n. If $\mathcal{H}^h(f(F_n))$ would be zero for all $n \in \mathbf{N}$, then $\mathcal{H}^h(f(N))$ as well. Hence one of the sets F_n is mapped to a set of positive measure. We extract a suitable perfect set by Cantor–Bendixson's Theorem 1.7. $\qquad\qquad\square$

If we know that the target is compact, we can relax the assumption that X is σ-finite a little bit due to Corollary 5.3. We skip the proof of the following result.

Theorem 5.6. *Let us assume that $\kappa < 2^{\aleph_0}$ is a cardinal. Suppose that (X, μ) satisfies (Aκ), and that $(Y, d, \mathcal{H}^h)$ satisfies (H∞) and is compact. If $f: X \to Y$ is a measurable surjection, then there is a set $N \subset X$ such that $\mu(N) = 0$ and $\mathcal{H}^h(f(N)) > 0$. Moreover, if $(X, \mu, \mathfrak{T})$ is a topological space where points are closed and with a countable basis, and f is such that for every closed set F, the preimage $f^{-1}(F)$ can be written as countable union of closed sets, then N can be chosen to be perfect.*

Question 5.7. Does Theorem 5.6 also hold if we drop the assumption that Y is compact?

We have encountered in Theorem 1.4 a (continuous) mapping $f: [0, 1] \twoheadrightarrow [0, 1]^2$ that maps every set N of $\mathcal{H}^1$-measure zero to a set of $\mathcal{H}^2$-measure zero. In contrast to Theorem 5.5, there is no (perfect) set P of $\mathcal{H}^1$-measure zero with $\mathcal{H}^2(f(P)) > 0$. However, we may try to replace $\mathcal{H}^2(f(P)) > 0$ by the slightly weaker condition $\dim_H f(P) = 2$.

Actually, the next result will be an important ingredient in the proof; however it is stated in greater generality than necessary for the existence of above set. We skip the proof.

Lemma 5.8. *Let $t > 0$ and define*

$$h(x) = \begin{cases} 0, & x = 0, \\ x^t \log(\frac{1}{x^{t/2}}), & 0 < x \le e^{-2/t}, \\ e^{-2}, & x > e^{-2/t}. \end{cases}$$

Then h is a continuous Hausdorff function of finite order and $x^{t'} \prec h(x) \prec x^t$ for all $0 < t' < t$.

Corollary 5.9. (Corollary of Theorem 5.5) *Suppose (X, d) is σ-finite with respect to $\mathcal{H}^s$. Assume (Y, d) is t-dimensional and an analytic subspace of a complete, separable space. If $f \colon X \twoheadrightarrow Y$ is a measurable surjection, then there exists a set $N \subset X$ such that $\mathcal{H}^s(N) = 0$ and $\dim_H f(N) = t$. If additionally t and $\mathcal{H}^t(Y)$ are positive and f is such that for each closed set $F \subset Y$, its preimage can be written as countable union of closed sets, then N can be chosen to be perfect.*

Proof. If $t = 0$, then $N = \emptyset$ does the job. Otherwise, using Theorem 5.5, we choose for each n with $t - 1/n > 0$ a set N_n such that $\mathcal{H}^s(N_n) = 0$ and $\mathcal{H}^{t-1/n}(f(N))$ is positive. The union of the sets N_n is as required. For the second statement, we may by Theorem 5.5 assume that Y is σ-finite with respect to $\mathcal{H}^t$. We choose h as in Lemma 5.8. By Lemma 2.11, $(Y, d, \mathcal{H}^h)$ satisfies (H∞). Theorem 5.5 provides us with a set N (perfect under the additional assumption on f) with $\mathcal{H}^s(N) = 0$ and $\mathcal{H}^h(f(N)) > 0$, thus with $\dim_H f(N) = t$ by Lemma 5.8 and Lemma 2.11. $\quad\square$

We look now at targets that are not separable. These spaces are so large that it is no problem to find uncountably many of the desired disjoint sets.

For the following theorem, see for example Proposition 1.14 in [Lév79]:

Theorem 5.10. (Tarski) *The axiom of choice is equivalent to the following statement: for every infinite set A, there is a bijection between A and $A \times A$.*

Lemma 5.11. *Every uncountable set A can be written as union of $\operatorname{card}(A)$-many pairwise disjoint uncountable sets.*

Proof. Assume that A is uncountable. By Tarski's Theorem 5.10, we find a bijection $f \colon A \times A \twoheadrightarrow A$. We define

$$\mathcal{A} := \{A_\alpha := f(A \times \{\alpha\}), \ \alpha \in A\}.$$

We claim that $\mathcal{A}$ is a partition as described in the statement of the lemma. Given a set A_α, its cardinality is the same as the one of $A \times \{\alpha\}$ and thus of A. Hence A_α is uncountable. Given distinct α and β, then $A \times \{\alpha\}$ and $A \times \{\beta\}$ are disjoint, and since f is injective, their images under f are disjoint as well. Finally, we see that the cardinality of $\mathcal{A}$ is the same as the one of A. $\quad\square$

Lemma 5.12. *Assume that κ is a cardinal with $\aleph_0 \le \kappa$. Suppose that (Y, d) is a metric space that does not contain any dense set of cardinality less or equal than κ. Then there exists a collection of pairwise disjoint closed sets that have infinite measure for every Hausdorff measure $\mathcal{H}^h$, and the cardinality λ of the collection is strictly larger than κ.*

Proof. First, we want to construct a set whose cardinality is larger than κ, and whose elements are separated so that the measure of the set is infinite. We do this with the help of Zorn's Lemma 4.1.

Fix a natural number $n \in \mathbf{N}$. We let

$$\mathbf{M}_n := \{M_\alpha\}_\alpha$$

be the collection of all sets M_α such that if x and y are distinct points in M_α, then $d(x,y) \geq 1/n$. We define an order $\leq$ on $\mathbf{M}_n$ simply by set inclusion: $C \leq D$ if and only if $C \subset D$. It is clear that Y is not empty, and we can take $y_0 \in Y$. Then the set $\{y_0\}$ is in $\mathbf{M}_n$ showing that $\mathbf{M}_n$ is not empty. Assume that $\{A_\alpha\}_{\alpha \in J}$ is a chain in $\mathbf{M}_n$. We claim that

$$\mathcal{U} := \bigcup_{\alpha \in J} A_\alpha$$

is an upper bound for the chain in $\mathbf{M}_n$. Given two distinct points x and y in $\mathcal{U}$, they belong a priori to two different elements A_α and A_β. However, by the chain property, one set is contained in the other, and hence we can assume that x and y belong to the same set, and hence $d(x,y) \geq 1/n$. That $\mathcal{U}$ is larger than any element in the chain is clear. Applying Zorn's Lemma 4.1, we conclude the existence of a maximal set $\mathcal{M}_n$, whose elements are all at least $1/n$ apart. We set

$$\mathcal{M} := \bigcup_n \mathcal{M}_n.$$

We assume by contradiction that $\mathcal{M}$ and hence every $\mathcal{M}_n$ has cardinality at most κ. The contradiction will follow as soon as we have shown that $\mathcal{M}$ is dense in Y. Let $\varepsilon > 0$ and $y \in Y$. We choose a natural number $n \in \mathbf{N}$ such that $1/n < \varepsilon$. If y would be such that $d(x,y) \geq 1/n$ for all $x \in \mathcal{M}_n$, then we could add y to $\mathcal{M}_n$ contradicting the maximality of $\mathcal{M}_n$. Hence, there exists $x \in \mathcal{M}_n$ with $d(x,y) < 1/n < \varepsilon$. Since y and $\varepsilon > 0$ were arbitrary, it follows that $\mathcal{M}$ is a dense subset of Y with cardinality bounded from above by κ leading to a contradiction. Hence the cardinality of $\mathcal{M}$ is strictly larger than κ. We can conclude that this is true as well for one of the sets $\mathcal{M}_n$. In Lemma 5.11, we have verified that we can write $\mathcal{M}_n$ as union of card$(\mathcal{M}_n)$-many pairwise disjoint uncountable sets $(N_\alpha)_\alpha$. Since any N_α contains uncountably many points, which are separated, it does not have a countable cover with sets of diameter smaller than $1/n$, and hence any Hausdorff measure of N_α is infinite. Furthermore, since N_α consists only of separated points, it is closed. $\square$

Theorem 5.13. *Suppose that $\kappa \geq \aleph_0$ is a cardinal. We assume that $X = (X, \mu)$ is a measure space satisfying $(A\kappa)$. We stipulate that $(Y, d, \mathcal{H}^h)$ is a metric measure space, where $\mathcal{H}^h$ is a Hausdorff measure. Suppose that any dense set in Y has cardinality strictly larger than κ. If $f \colon X \twoheadrightarrow Y$ is a measurable surjection, then there is a set $N \subset X$ such that $\mu(N) = 0$ and $\mathcal{H}^h(f(N)) > 0$. Moreover, if X is additionally equipped with a topology with a countable basis and such that points are closed, and f is further such that for every closed set F, the preimage $f^{-1}(F)$ can be written as countable union of closed sets, then N can be chosen to be perfect.*

Proof. The proof is essentially as the one of Theorem 5.5. In Lemma 3.1, λ is as indicated in Lemma 5.12, and again the sets Y_j are closed. The moreover-part is proven exactly as in the proof of Theorem 5.5. $\square$

6. Some topological considerations

Until now, we required Y to be an analytic subset of a complete, separable metric space. We want to get rid of this restriction. However, the price we pay is the introduction of some additional regularity conditions on X. They enable us to find a countable partition of X by *compact* sets K_n (and a set of measure zero) such that the restrictions of f to the sets K_n are continuous. Therefore, the images $Y_n = f(K_n)$ are compact as well.

Our goal in this section is to show that a σ-finite, σ-compact space can be written as countable union of compact sets with finite measure and a set of measure zero. To achieve this, we put some restrictions on the measure and the topology.

The main point in the following proposition is the fact that we can choose the compact sets to have *finite* measure. We will later on study the existence of the desired $\mathcal{F}_\sigma$-sets.

Proposition 6.1. *Let $(X, \mathfrak{T})$ be a topological Hausdorff space and μ a Borel measure on X that is σ-finite and has the property that each measurable set of finite measure contains an $\mathcal{F}_\sigma$-set with the same measure. Suppose that X can be written as countable union of compact sets and a set of measure zero. Then we can write X as countable union of compact sets with* finite *measure and a set of measure zero.*

Proof. The nature of the statement allows us to assume that X is σ-finite and σ-compact. By definition, we can decompose $X = \bigcup_n X_n$, where all X_n are measurable and have finite measure. By assumption, we can write $X_n = \widehat{Z}_n \cup N_n$, where $\widehat{Z}_n$ is an $\mathcal{F}_\sigma$-set and N_n has measure zero. Hence

$$X = \bigcup_n Z_n \cup N,$$

where N has measure zero and the sets Z_n are $\mathcal{F}_\sigma$-sets with finite measure. Hence, each Z_n has the form

$$Z_n = \bigcup_l C_l^n,$$

where the sets C_l^n are closed.

On the other hand, we can write

$$X = \bigcup_m K_m,$$

where the sets K_m are compact. Now,

$$Z_n = Z_n \cap \left(\bigcup_m K_m \right) = \bigcup_m (Z_n \cap K_m),$$

$$Z_n \cap K_m = \left(\bigcup_l C_l^n \right) \cap K_m = \bigcup_l (C_l^n \cap K_m).$$

In topological Hausdorff spaces, compact sets are closed. As closed subsets of compact sets, the sets $C_l^n \cap K_m$ are compact. It follows that the sets $Z_n \cap K_m$ and hence Z_n are σ-compact. $\square$

Question 6.2. Can we weaken the condition in Proposition 6.1 that each measurable set of finite measure contains an $\mathcal{F}_\sigma$-set with the same measure and still obtain the same conclusion?

We further want to study the condition concerning the $\mathcal{F}_\sigma$-sets in the last proposition. The proof of the following result is as the one of Theorem 22 on p. 35 in [Rog70]:

Theorem 6.3. *Let μ be a Borel measure that is $\mathcal{G}_\delta$-regular on a topological space $(X, \mathfrak{T})$ such that every open set is a $\mathcal{F}_\sigma$-set. If $E \subset X$ is measurable with $\mu(E) < \infty$, then there exists an $\mathcal{F}_\sigma$-set $H \subset E$ with $\mu(H) = \mu(E)$.*

We now give conditions on a topological space implying that every open set is an $\mathcal{F}_\sigma$-set.

Definition 6.4. (Regular) A topological space $(X, \mathfrak{T})$ is called *regular* if points are closed and if for any $x \in X$ and closed set $C \subset X$ that does not contain x, there exist disjoint open subsets U and V of X such that $x \in U$ and $C \subset V$.

Remark 6.5. Note that a regular space is Hausdorff.

Definition 6.6. (Second countable) A topological space is *second countable* if it has a countable basis.

Lemma 6.7. *Let $(X, \mathfrak{T})$ be a regular, second countable topological space. Then every open set is an $\mathcal{F}_\sigma$-set.*

Proof. Let O be an open set and choose $x \in O$. We note that $C := X \setminus O$ is closed. Since X is regular, there exist disjoint open sets U and V with $x \in U$ and $C \subset V$. Now $D := X \setminus V \subset X \setminus C$ is closed, and since x is not a point in V, it lies in D. If $y \in U$, then $y \notin V$ and hence $y \in D$. Thus

$$x \in U \subset D \subset X \setminus C = X \setminus (X \setminus O) = O.$$

Consequently, we can choose for every $x \in O$ an open set U_x with $x \in U_x \subset \overline{U_x} \subset O$. Let

$$\mathcal{O} := \{O_i\}$$

be a countable basis for the topology. Hence, for every $x \in O$ and U_x, we find an open set O_i with $x \in O_i \subset \overline{O_i} \subset \overline{U_x} \subset O$. We collect all occurring indices in the set I. Then

$$O = \bigcup_{i \in I} O_i \subset \bigcup_{i \in I} \overline{O_i} \subset O,$$

and the claim follows. $\qquad\qquad\square$

7. From measurable to continuous

In Howroyd's Theorem 5.1, we have the requirement that Y is an analytic subset of a complete, separable metric space. We would like to get rid of some assumptions on Y and add in turn more requirements on X. In this section, we show that Y inherits the desired topological properties from X. For example if X is compact and separable, and $f : X \twoheadrightarrow Y$ is continuous, then $Y = f(X)$ is compact (thus complete), and separable as well. Thus, we can use the machinery that we developed before.

The following is from Definition 411M in [Fre]; however, we have allowed ourselves to change the transcription of Luzin:

Definition 7.1. (Almost continuous, Luzin measurable) Let $(X, \mu, \mathfrak{T})$ be a measure space with topology $\mathfrak{T}$ and $(Y, \mathfrak{S})$ another topological space. We say that a

mapping $f\colon X \to Y$ is *almost continuous* or *Luzin measurable* if μ is inner regular with respect to the family of subsets A of X such that $f\restriction_A$ is continuous.

The main point in is the verification that our setting permits the application of the following version of Theorem 451S in [Fre], referenced there to an article by Fremlin and one by Koumoullis and Prikry:

Theorem 7.2. (Luzin's theorem) *Let (X, μ) be a measure space with topology $\mathfrak{T}$ such that*

(a) *whenever a set is measurable with infinite measure, there exists a measurable subset with positive and finite measure (μ is semi-finite),*

(b) *every point of X has a neighborhood of finite measure (μ is locally finite),*

(c) *if E is not measurable, then there exists a measurable set F of finite measure such that the intersection $E \cap F$ is not measurable (if additionally μ is semi-finite, then this is called locally determined),*

(d) *the topology is Hausdorff,*

(e) *μ is inner regular with respect to the compact sets.*

Assume Y is a metrizable space. Then a function $f\colon X \to Y$ is measurable if and only if it is almost continuous.

With the help of Luzin's theorem, we obtain a nice partition of our space:

Proposition 7.3. *Let $(X, \mathfrak{T})$ be a compact space where the topology $\mathfrak{T}$ is Hausdorff. Assume that μ is a Borel measure on X that is inner regular with respect to the compact sets and $\mu(X) < \infty$. Suppose $(Y, \mathfrak{S})$ is metrizable. If $f\colon X \to Y$ is measurable, then we can write*

$$X = \bigcup_n K_n \cup N,$$

where K_n is compact, $\mu(N) = 0$, and $f\restriction_{K_n}$ is continuous.

Proof. Note that by Theorem 211L in [Fre], (X, μ) is locally determined. The setting is such that we can apply Theorem 7.2 to obtain a sequence $(A_m)_m$ of measurable sets in X such that the restriction $f\restriction_{A_m}$ of f to A_m is continuous and

$$\mu(X) < \mu(A_m) + \frac{1}{2m}.$$

Using the inner regularity of μ with respect to compact sets, we obtain compact sets K_m contained in A_m such that

$$\mu(X) < \mu(A_m) + \frac{1}{2m} < \mu(K_m) + \frac{1}{m}.$$

We set $N := X \setminus \bigcup_m K_m$ and deduce that

$$\mu(N) = \mu\left(X \setminus \bigcup_m K_m \right) = \mu\left(\bigcap_m (X \setminus K_m) \right) \leq \mu(X \setminus K_{m_0}) \leq \mu(X) - \mu(K_{m_0}) < \frac{1}{m_0}$$

for every $m_0 \in \mathbf{N}$. The claim follows. $\qquad\square$

8. Proofs of Theorem 1.2 and of Corollary 1.3

Proof of Theorem 1.2. First, we want to write X as countable union of compact sets with finite measure and a set of measure zero. According to Proposition 6.1,

it suffices to verify that each measurable set of finite measure contains an $\mathcal{F}_\sigma$-set of the same measure (regular spaces are Hausdorff). Theorem 6.3 reduces the this task to the verification that open sets are F_σ-sets. Under our assumptions, this follows from Lemma 6.7. Summarizing, we may now apply Proposition 6.1 (we noted in Remark 6.5 that X is Hausdorff; the same is true for the sets K_n in the following decomposition), i.e. we may write

$$X = \bigcup_n K_n \cup N,$$

where the sets K_n are compact with finite measure, and N has measure zero. We want to replace each compact set K_n by a union of compact sets and a set of measure zero such that f restricted to each of these new compact sets is continuous. In view of Proposition 7.3, let us verify the inner regularity with respect to compact sets of the restriction of μ to K_n denoted by $\mu\!\restriction_{K_n}$. If $E \subset K_n$ is measurable, then we have argued before that it contains an $\mathcal{F}_\sigma$-set $F \subset E$ with the same measure. Hence F can be written as countable union of closed, and since X is compact, of compact sets. Let us denote these compact sets by L_m and note that, by replacing L_m by the union $\bigcup_{r=1}^m L_r$ if necessary, the inclusion $L_m \subset L_{m+1}$ holds. By Lemma 2.2, we have

$$\mu(E) = \mu(F) = \sup_m \mu(L_m),$$

verifying the inner regularity of $\mu\!\restriction_{K_n}$ with respect to compact sets. By Proposition 7.3, we can now write

$$X = \bigcup_n M_n \cup M_0$$

where $\mu(M_0) = 0$, the sets M_n are compact with $\mu(M_n) < \infty$, and $f\!\restriction_{M_n}$ is continuous. If $\mathcal{H}^h(f(M_0)) > 0$, then we are done. Otherwise, by the fact that $\mathcal{H}^h$ is G_δ-regular as noted in Theorem 2.12, there is a Borel set $B \supset f(M_0)$ with $\mathcal{H}^h(B) = 0$. Note that

$$Y = \bigcup_n f(M_n) \cup B$$

is, as the sets $f(M_n)$ are compact, a countable union of Borel sets. If each set $f(M_n)$ would have a representation as countable union of measurable sets with finite measure, then this would also be true for Y. By our assumption on Y, this is not possible. Hence, there is a set M_n such that $f\!\restriction_{M_n} : M_n \twoheadrightarrow f(M_n)$ is such that we can apply Theorem 5.5. The statement follows. $\qquad\square$

Question 8.1. Can the set N in Theorem 1.2 be chosen to be perfect? The problem lies in the case where $\mathcal{H}^h(f(M_n)) > 0$ implies that $n = 0$.

Proof of Corollary 1.3. The proof is basically as the one of Corollary 5.9. We just apply Theorem 1.2 instead of Theorem 5.5. $\qquad\square$

9. Space-fillings

We have been talking about surjections from one space onto another one. Here, we give a result concerning their existence. As blueprints for their constructions, the proofs of Theorem 1.3 in [HT08] and of Theorem 5.1 in [WZ12] were used. See also Section 2 in [AS12].

Theorem 9.1. *Let (X, d) be a locally compact metric space. Suppose h is a Hausdorff function with $\lim_{t \to 0} h(t) = 0$. Let Y be any non-empty length-compact[11] metric space. Suppose that $P \subset X$ is a non-empty perfect set. Then there exists a compact, perfect set $P' \subset P$ and a continuous surjection $f \colon X \twoheadrightarrow Y$ such that $\mathcal{H}^h(P') = 0$ and $f(P') = Y$.*

Proof. We suppose that Y is equipped with the path metric. Without loss of generality, we can assume that $\operatorname{diam} Y \leq 1$.

As building blocks of the continuous surjection, we will use mappings from annuli[12] to paths. Thus, the ingredients for the construction of the continuous surjection are as follows: a system of annuli/balls, a system of paths, and bump functions. As the space X is locally compact, it suffices to construct the continuous surjection in a compact ball such that the surjection is constant in a neighborhood of the boundary. Consequently, in what follows, we assume that X is a compact ball and that P has positive distance to the boundary. We will also tacitly assume that every chosen ball in X is contained in this compact ball.[13] We start by looking at the bump functions. Given $x_0 \in X$ and $0 < \delta < \varepsilon$, we can construct continuous functions $\eta \colon X \to [0, 1]$ satisfying

(i) $\operatorname{supp} \eta$ is a compact subset of $B(x_0, \varepsilon)$,
(ii) $\eta(x) = 1$ for all $x \in B(x_0, \delta)$,
(iii) η is Lipschitz.

Let us turn our focus to the system of paths in Y. Since Y is length-compact, we may find for each non-negative integer n a finite set $Y_n = \{y_i^n\}_{i=1}^{k_n}$ with the property that each $y \in Y$ can be connected to a point in Y_n by a path of length no greater than 2^{-n}. Then $\bigcup_n Y_n$ is dense in Y.

For each positive integer $n \geq 1$, we may partition Y_n into sets $\mathcal{C}(y_i^{n-1})$ so that if $y_j^n \in \mathcal{C}(y_i^{n-1})$, then there is a 1-Lipschitz path $\gamma_j^n \colon [0, 2^{-(n-1)}] \to Y$ satisfying $\gamma_j^n(0) = y_i^{n-1}$ and $\gamma_j^n(2^{-(n-1)}) = y_j^n$.

To each path, we need to assign a corresponding annulus. Let us mix the construction of the annuli and the mapping f. Actually, we will work with globally defined mappings and modify them in balls.

Let $f_0 \colon X \to Y$ be the constant mapping $f_0(x) = y_1^0$ for all $x \in X$. Fix some point $x_0 \in P$. As P is perfect and non-empty, it is infinite, and so we may find a collection $\mathcal{C}(x_1^0)$ of k_1 distinct points $\{x_i^1\}_{i=1}^{k_1} \subset P$. Choose $\varepsilon_1 > 0$ so small and balls $\{B(x_i^1, \varepsilon_1)\}_{i=1}^{k_1}$ such that their centers have distance at least $3\varepsilon_1$ from each other (hence the balls are pairwise disjoint), and $k_1 h(2\varepsilon_1) < 1$. We can fix a number $0 < \delta_1 < \varepsilon_1$ and for each of the points x_i^1 a corresponding bump function $\eta_i^1 \colon X \to [0, 1]$ as[14] above.

As the collection $\{B(x_i^1, \varepsilon_1)\}_{i=1}^{k_1}$ consists of pairwise disjoint balls, we may define the mapping $f_1 \colon X \to Y$ by

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[11]This means that Y is compact with respect to the path metric.

[12]Maybe, it would be more accurate to speak about balls instead of annuli. In each step, we modify the mapping in balls. However, if we look what additional part stays fixed when we go from one step to the next, then we obtain a system of annuli.

[13]Fixing a point $x \in P$ and a closed ball $\overline{B}(x, r)$ that is compact, we note that $B(x, r/2) \cap P$ is uncountable. By the Cantor–Bendixson Theorem 1.7, we can replace the original perfect set by one in $\overline{B}(x, 3r/4) \cap P$.

[14]In the construction of the bump function, we let $\delta = \delta_1$, $\varepsilon = \varepsilon_1$, and $x_0 = x_i^1$.

$$f_1(x) = \begin{cases} \gamma_i^1 \circ \eta_i^1(x), & x \in B(x_i^1, \varepsilon_1), \\ f_0(x), & x \notin \cup_{i=1}^{k_1} B(x_i^1, \varepsilon_1). \end{cases}$$

Note that $f_1(x_i^1) = \gamma_i^1(1) = y_i^1$. It is easily checked that f_1 is Lipschitz continuous.

Now, we continue inductively. For each $n \in \mathbf{N}$, we may find a set of points $\{x_i^{n+1}\}_{i=1}^{k_{n+1}} \subset P$, pairwise disjoint balls $B(x_i^{n+1}, \varepsilon_{n+1})$, and a continuous mapping[15] $f_{n+1} \colon X \to Y$ defined by

$$f_{n+1}(x) = \begin{cases} \gamma_i^{n+1}(\frac{1}{2^n}\eta_i^{n+1}(x)), & x \in B(x_i^{n+1}, \varepsilon_{n+1}), \\ f_n(x), & x \notin \cup_i^{k_{n+1}} B(x_i^{n+1}, \varepsilon_{n+1}), \end{cases}$$

such that

(a) $k_{n+1}h(2\varepsilon_{n+1}) < \frac{1}{n+1}$,

(b) $x_j^n \notin B(x_i^{n+1}, \varepsilon_{n+1}) \subset B(x_j^n, \delta_n)$ if $x_i^{n+1} \in \mathcal{C}(x_j^n)$. Further, each $B(x_j^n, \delta_n)$ contains at least two balls[16] $B(x_i^{n+1}, \varepsilon_{n+1})$, the centers of the balls have distance bounded from below by $3\varepsilon_{n+1}$, and each ball contains infinitely many points of P,

(c) for each $y_{n+1} \in \mathcal{C}(y_n^i)$, there is a $x_{n+1}^i \in \mathcal{C}(x_n^i)$,

(d) $f_{n+1}(x) = f_n(x)$ for all $x \notin \bigcup_{i=1}^{k_{n+1}} B(x_i^{n+1}, \varepsilon_{n+1})$,

(e) $f_{n+1}(x) = y_1^0$ if $x \notin \bigcup_{i=1}^{k_1} B(x_i^1, \varepsilon_1)$,

(f) $f_m(x_i^{n+1}) = y_i^{n+1}$ for all integers $m \geq n+1$ and $i = 1, \ldots, k_n$,

(g) $d_Y(f_{n+1}(x), f_n(x)) \leq 2^{-n}$ for all $x \in X$,

(h) f_{n+1} is continuous.

Point (g) above shows that $(f_n)_n$ is a Cauchy sequence of mappings in the supremum norm. The sequence $(f_n)_n$ converges uniformly to a continuous function $f \colon X \to Y$.

We consider

$$P' := \bigcap_{n \in \mathbf{N}} \bigcup_{i=1}^{k_n} \overline{B}(x_i^n, \varepsilon_n)$$

and argue in the following that it is a as required in the theorem. Let us show the compactness of P'. As intersection of closed sets it is closed as well and as closed subset of a compact set, it is itself compact.

To conclude that P' is not empty, we consider a sequence $(x_n)_n$ of centers of balls such that x_{n+1} is in $B(x_n, \varepsilon_n)$. It follows that $x_{n+m} \in \overline{B}(x_n, \varepsilon_n)$ for all $m \in \mathbf{N}$. By the compactness of $\overline{B}(x_n, \varepsilon_n)$, a subsequence of $(x_m)_m$ converges in $\overline{B}(x_n, \varepsilon_n)$. However, since the original sequence is a Cauchy sequence, the original sequence converges. Since n was arbitrary, the existence of the limit in the intersection follows.

We continue by verifying that P' is perfect. We have already verified that it is closed. For showing the last required property in the definition of perfect, we assume by contradiction that there is an open set O that hits P' but only in finitely many points. Let $x \in O \cap P'$. Then there is some $r > 0$ such that $B(x, r) \subset O$ and $B(x, r)$ hits P' only in finitely many, say N, points. There is a ball $\overline{B}(x_n, \varepsilon_n)$ that contains

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[15]For η_i^{n+1}, we choose $\delta = \delta_{n+1}$ and $\varepsilon = \varepsilon_{n+1}$ for the radii, and we further set $x_0 = x_i^{n+1}$ in the construction of the bump function detailed before.

[16]The requirement that we need at least two balls is a small nuisance. We will need it in order to find a perfect set P' that is blown up. However, it is no problem as we can always add additional paths.

x and lies in $B(x, r)$. We find a generation $n + m$ that has more than $N + 1$ disjoint balls in $B(x_n, \varepsilon_n)$. Continuing similarly as when we showed that P' is not empty, we obtain the existence of $N + 1$ distinct points in $P' \cap B(x, r)$—a contradiction.

Now, we verify that $\mathcal{H}^h(P') = 0$. Let $\delta > 0$. We can choose n so large that $2\varepsilon_n < \delta$. Thus

$$\mathcal{H}^h_\delta(P') \leq k_n h(2\varepsilon_n) < \frac{1}{n}$$

for all large n. We see that $\mathcal{H}^h_\delta(P') = 0$ and since $\delta > 0$ was arbitrary, the claim follows.

Let us conclude the proof by showing that $f(P') = Y$. Let $y \in Y$. We find a sequence $(y_n)_n \subset Y$ converging to y such that each y_n is some $y^n_{j(n)}$. Thus, for each y_n, there is $x_n := x^n_{j(n)}$ with $f(x_n) = y_n$. We can find a subsequence of $(x_n)_n$ that converges to some $x \in P'$ with $f(x) = y$. $\square$

Remark 9.2. In the above theorem, we cannot require $\mathcal{H}^h(P') = 0$ simultaneously for all Hausdorff functions with $h(0) = 0$, since in this case P' is countable, see Corollary 3 on p. 67 in [Rog70].

Remark 9.3. (Assumes (CH)) According to Theorem 7.5 in [Kee86], assuming the Continuum Hypothesis and that n is a natural number, there exists a separable metric space X_n such that $0 < \mathcal{H}^n(X_n) < \infty$ but with the property that there is no continuous map $f \colon X_n \to [0, 1]$ that is onto.

10. Some space-fillings satisfying Luzin's condition (N)

Previously, we have assumed that X can be written as union of a certain amount of sets of finite measure, whereas Y cannot. Actually, we have also excluded the case where Y has σ-finite measure. We have done so with good reason:

Example 10.1. Let us consider the differentiable bijection $\tan \colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \twoheadrightarrow \mathbf{R}$. Let us assume that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbf{R}$ are equipped with the Euclidean distance and Lebesgue measure. Hence $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has finite measure and $\mathbf{R}$ is σ-finite. Note that as locally Lipschitz continuous function, the tangent $\tan$ satisfies Luzin's condition (N).

The following example enlightens that different dimensions do not always force sets to been blown up. Note that there are compact 0-dimensional spaces that are not countable.

Example 10.2. Assume that $(X, d_X, \mathcal{H}^0)$ is 0-dimensional, and (Y, d_Y, ν) is a metric measure space. If $f \colon X \twoheadrightarrow Y$ is a surjection, then it satisfies Luzin's condition (N).

Proof. As $\mathcal{H}^0$ is the counting measure, the only subset of X with measure zero is the empty set, which is mapped onto the empty set as well. $\square$

Remark 10.3. Why does above example not violate our results? Let $\nu = \mathcal{H}^h$ as specified in our results. As f is a surjection, we have $\operatorname{card} Y \leq \operatorname{card} X$ violating the conditions of Theorem 5.5 and Theorem 5.13, and if X is countable also of Theorem 5.6. In the remaining case, Y is assumed to be compact and thus separable. By Section 2 in [Fre73], $\operatorname{card}(Y) \in \{\aleph_0, 2^{\aleph_0}\}$—none of the alternatives is in the scope of Theorem 5.6.

The two preceding Examples 10.1 and 10.2 are quite special. In the first example, the domain and the target have the same dimension, while in in the second one, the domain is zero-dimensional and has therefore no non-trivial sets of measure zero. Our goal is to construct space-fillings whose domains have dimension larger than zero, and whose dimensions of the targets do not necessarily agree with the ones of the domains. In most constructions, we assume the Continuum Hypothesis for the existence of certain sets in the domain.

Before we can start with constructing such sets, we need to deal with cardinalities of families of certain subsets in metric spaces.

Lemma 10.4. *Let (X, d) be a separable metric space with at least[17] $2^{\aleph_0}$ elements. Assume that h is a Hausdorff function with $h(0) = 0$. Let*

$$\mathcal{O} := \{O \subset X \colon O \text{ open}\},$$
$$\mathcal{C} := \{C \subset X \colon C \text{ closed}\},$$
$$\mathcal{E} := \{E \subset X \colon E \text{ is a } G_\delta\text{-set that is } \sigma\text{-finite with respect to } \mathcal{H}^h\}.$$

Then $\operatorname{card} \mathcal{O} = \operatorname{card} \mathcal{C} = \operatorname{card} \mathcal{E} = 2^{\aleph_0}$.

Proof. By Theorem XIV.3.1 in [Kur72], the cardinality of the family of all open sets is bounded from above by $2^{\aleph_0}$. The same upper bound applies to the closed sets. Since points are closed, both families $\mathcal{O}$ and $\mathcal{C}$ have cardinality $2^{\aleph_0}$. As points are G_δ-sets with zero measure, it suffices to prove that $\operatorname{card} \mathcal{E} \leq 2^{\aleph_0}$ to obtain that $\operatorname{card} \mathcal{E} = 2^{\aleph_0}$. For each set $E \in \mathcal{E}$ we find a sequence O_i of open sets whose intersection is E. Thus the cardinality of $\mathcal{E}$ is bounded from above by the cardinality of all sequences of real numbers. Cardinal arithmetics, see for example Sections 3 and 4 in [Jec03], gives the wished upper bound. $\qquad\square$

Lemma 10.5. *Assume $(X, d, \mathcal{H}^h)$ is compact and satisfies ($H\infty$). Then X contains exactly $2^{\aleph_0}$ perfect sets of non-σ-finite measure. Moreover, we can find $2^{\aleph_0}$ perfect sets of non-σ-finite measure that are pairwise disjoint.*

Proof. Let us first determine the cardinality of the family of the closed sets that do not have σ-finite $\mathcal{H}^h$-measure. Note that X is separable, thus the desired upper bound for the cardinality of the collection of the closed sets of Lemma 10.4 applies. From Corollary 5.3, we know of the existence of the required amount of pairwise disjoint compact sets with the right size.

Let us now tackle the cardinality of the perfect sets. Since perfect sets are closed, the same upper bound applies. Separable metric spaces have a countable basis, and thus Cantor–Bendixson's Theorem 1.7 gives the desired amount of perfect sets. $\quad\square$

The following construction is inspired by an example by Besicovitch, see Chapter II in [Bes33]; parts of Besicovitch's example are also used in Example 10.8:

Proposition 10.6. *(Assumes (CH)) Assume that $(X, d, \mathcal{H}^h)$ is compact, satisfies ($H\infty$), and that (CH) holds. Then there exists a set $G \subset X$ such that*

(a) *the cardinality of G is $2^{\aleph_0}$,*
(b) *if $E \subset G$ is σ-finite with respect to $\mathcal{H}^h$, then $G \cap E$ is countable,*
(c) *G is non-σ-finite with respect to $\mathcal{H}^h$,*
(d) *each subset $A \subset G$ is $\mathcal{H}^h$-measurable.*

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[17]The conclusion shows that in this case, we actually have exactly $2^{\aleph_0}$ elements.

Proof. We set

$\mathcal{P} := \{P \subset X \colon P \text{ is perfect but not } \sigma\text{-finite with respect to } \mathcal{H}^h\}$,

$\mathcal{E} := \{E \subset X \colon E = \bigcap_{n \in \mathbf{N}} O_n, O_n \text{ open, and } E \text{ is } \sigma\text{-finite with respect to } \mathcal{H}^h\}$.

Using Lemmas 10.4 and 10.5, we obtain that

$$\operatorname{card} \mathcal{P} = \operatorname{card} \mathcal{E} = 2^{\aleph_0}.$$

Let us denote the ordinal corresponding to the real numbers by $\mathbf{c}$. There exist bijections between $\mathbf{c}$ and $\mathcal{P}$, and between $\mathbf{c}$ and $\mathcal{E}$, respectively. Given $i \in \mathbf{c}$, we denote by P_i and E_i the corresponding element in $\mathcal{P}$ and $\mathcal{E}$, respectively. Let us choose an arbitrary point x_1 out of the set $P_1 \setminus E_1$. Since P_1 is not σ-finite, but E_1 is, such a point certainly exists.

Take $i_0 \in \mathbf{c}$ and assume that for each $i < i_0$, we have already chosen a point $x_i \in P_i \setminus \bigcup_{k=1}^{i} E_k$, different from all the previously chosen points x_k for $k < i$. We choose now a point $x_{i_0} \in P_{i_0} \setminus \bigcup_{k=1}^{i_0} E_k$ different from all the already chosen points x_k. Arguing similarly as in the choice of x_1, such a point x_{i_0} certainly exists. We collect the points in the following set

$$G := \{x \in X \colon \text{there exists } i \in \mathbf{c} \text{ such that } x = x_i\}.$$

Before proving that G has the required properties, let us note that if $P \in \mathcal{P}$, then there is an index $i \in \mathbf{c}$ with $P = P_i$. Thus there exists a point $x_i \in G \cap P$ implying that $G \cap P$ is not empty.

(a) By construction.

(b) Assume that E is σ-finite with respect to $\mathcal{H}^h$, and we suppose first that it is a countable intersection of open sets. In this case, there is an index $i_0 \in \mathbf{c}$ such that $E = E_{i_0}$. For all $i > i_0$, we have

$$x_i \in P_i \setminus \bigcup_{k=1}^{i} E_k \subset P_i \setminus E_{i_0}.$$

Hence at most the points $x_1, x_2, \ldots, x_{i_0}$ lie in E_{i_0} and G. It follows that $G \cap E = G \cap E_{i_0}$ is countable.

If the σ-finite set E is not a countable intersection of open sets, then we write $E = \bigcup_{n \in \mathbf{N}} E_n$, where the sets E_n are measurable sets with finite measure. By Theorem 2.12, the Hausdorff measure is $\mathcal{G}_\delta$-regular. Thus the sets E_n are contained in sets F_n that can be written as countable intersection of open sets. Hence, by our considerations above, each $F_n \cap G$ is countable, and consequently this is true for each $E_n \cap G$ and finally for $E \cap G$.

(c) If G would be σ-finite, then by (b), it would be countable. This would contradict (a).

(d) Suppose $A \subset G$. Let $F \subset X$ be an arbitrary subset. We have to show that

$$\mathcal{H}^h(F) \geq \mathcal{H}^h(F \cap A) + \mathcal{H}^h(F \setminus A)$$

holds. If $\mathcal{H}^h(F) = \infty$ or $\mathcal{H}^h(F) = 0$, the inequality follows. Thus, we may assume that $0 < \mathcal{H}^h(F) < \infty$. It follows that $\mathcal{H}^h(F \cap A) < \infty$. But this implies by (b) that $F \cap A \subset G$ is countable, and our assumptions on h imply that $\mathcal{H}^h(F \cap A) = 0$. But then the inequality holds as well. □

Theorem 10.7. (Assumes (CH)) *Suppose* $(X, d, \mathcal{H}^h)$ *is compact and satisfies* *(H∞). We further stipulate that (CH) holds. Assume that* (Y, ν) *is a measure space such that* $\operatorname{card} Y = 2^{\aleph_0}$*, and* ν *is such that points have measure zero. Then there exists a measurable surjection* $f \colon X \twoheadrightarrow Y$ *such that whenever* $N \subset X$ *is* σ*-finite with respect to* $\mathcal{H}^h$*, then* $\nu(f(N)) = 0$*.*

Proof. Choose $G \subset X$ as in Proposition 10.6. According to Lemma 5.11, we can write G as union of $2^{\aleph_0}$ many, uncountable sets A_λ, $\lambda \in \mathbf{R}$, that are pairwise disjoint. We denote by $F \colon \mathbf{R} \twoheadrightarrow Y$ a bijection and fix a point $y_0 \in Y$. We define $f \colon X \to Y$ by $f(x) = F(\lambda)$ if $x \in A_\lambda$, and we set[18] $f(x) = y_0$ if $x \in X \setminus G$. By construction, f is a surjection.

Remember that each subset of G is measurable. The preimage of any subset of Y is either a subset of G or the union of $X \setminus G$ and a subset of G. In both cases, the preimage is measurable. This shows that f is measurable.

A set N of measure zero splits in a part lying in G and one lying in the complement of G. The part in the complement is mapped to the point y_0; its measure is zero. The part inside G has countably many points and is thus mapped to a set with the same cardinality upper bound. This gives the claim. $\square$

We may ask if the dichotomy between Luzin's condition (N) and space-fillings is still valid if we do not require the mapping to be measurable. To answer this question in Example 10.9, we first introduce a peculiar set constructed by Besicovitch, which has been the main source of inspiration for the construction of the set in Proposition 10.6. Here again, we assume that the Continuum Hypothesis (CH) holds.

Example 10.8. (Assumes (CH)) Besicovitch constructs in [Bes33, Chapter II] under the continuum hypothesis a set $H \subset [0,1]^2$, which has, amongst others, the following two properties

(a) the exterior plane measure of H is 1,
(b) any subset of H of plane measure zero is a countable set.

For example from Theorem 264I in [Fre], we know that the Hausdorff measure $\mathcal{H}^2$ and the Lebesgue measure $\mathcal{L}^2$ have the same measurable sets, the same null sets, and agree up to a factor on the σ-algebra of measurable sets. We claim that $\mathcal{H}^2(H) > 0$. Since $\mathcal{H}^2$ is a $\mathcal{G}_\delta$-regular Borel measure by Theorem 2.12, there exists a positive constant C and a Borel set B with $H \subset B$ and

$$\mathcal{H}^2(H) = \mathcal{H}^2(B) = C\mathcal{L}^2(B) \geq C\mathcal{L}^2(H) = C > 0.$$

In the following example, we construct (under the Continuum Hypothesis) a non-measurable space-filling that satisfies Luzin's condition (N). The main difference to the mapping constructed in Theorem 10.7 is that here, the domain has finite measure.

Example 10.9. (Assumes (CH)) If we have a surjection $f \colon H \twoheadrightarrow [0,1]^3$ (the existence follows since by the continuum hypothesis, H and $[0,1]^3$ have the same cardinality), then it satisfies Luzin's condition (N) with respect to $\mathcal{H}^2$ on domain and target, since the only sets of measure zero contained in H are countable and are thus mapped on countable sets in $[0,1]^3$ as well.

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[18]By Howroyd's Theorem 5.1, we see that $G \neq X$.

We can extend the mapping f to a surjection $F\colon [0,1]^2 \twoheadrightarrow [0,1]^3$ by letting $F(x) = 0$ whenever $x \notin H$ and $F(x) = f(x)$ otherwise. Theorem 1.2 tells us that F cannot be measurable.

Measurability of a mapping depends to a certain degree on the topology of the target. The following example contains a version of Example 10.9 where the topology of the target is trivial. It also shows that in Lemma 3.1, we cannot get rid of the assumption of the existence of the desired disjoint Borel sets.

Example 10.10. (Assumes (CH)) Let $(Y, \mathfrak{T})$ be a topological space, where the topology $\mathfrak{T}$ is given by $\mathfrak{T} = \{Y, \emptyset\}$. Suppose ν is a non-σ-finite Borel measure without atoms on Y and $\operatorname{card}(Y) = \operatorname{card}(\mathbf{R})$.

Let H be the set defined in Example 10.8. It is uncountable and by the Continuum Hypothesis, its cardinality is the same as that of $[0,1]^2$. We find a bijection $\tilde{f}\colon H \twoheadrightarrow Y$. Fix a point $y_0 \in Y$ and let $f\colon [0,1]^2 \twoheadrightarrow Y$ be defined by $f(x) = \tilde{f}(x)$ if $x \in H$ and $f(x) = y_0$ otherwise.

We assume that $[0,1]^2$ is equipped with the standard metric and measure. Let $N \subset [0,1]^2$ be a set of measure zero. Then

$$N_H := N \cap H$$

is of measure zero as well, and hence it is countable. Thus $f(N_H)$ is countable. The set $N_{[0,1]^2 \setminus H} := N \cap ([0,1]^2 \setminus H)$ is mapped to y_0. So $f(N_{[0,1]^2 \setminus H})$ and finally $f(N)$ are countable. Since ν has no atoms, $f(N)$ has measure zero. Thus f is a measurable space-filling satisfying Luzin's condition (N).

11. Applications

For example by Proposition 423B in [Fre], Polish spaces are analytic. Choosing for f the identity in Theorem 5.5 and by way of contradiction, we obtain the following result:

Corollary 11.1. (Corollary of Theorem 5.5) *Let (X, d, μ) be a complete, separable metric measure space, where μ is Borel. Suppose X is σ-finite with respect to μ. We assume further that h is a continuous Hausdorff function of finite order with $h(0) = 0$, and that $\mathcal{H}^h$ is absolutely continuous with respect to μ. Then X is σ-finite with respect to $\mathcal{H}^h$.*

From our results, we can obtain rigidity results:

Theorem 11.2. *Let (X, μ) be an analytic Hausdorff space such that μ is Borel with $\mu(X) < \infty$. Suppose Y is a complete and separable metric space and h a continuous Hausdorff function with $h(0) = 0$ and one of the following is satisfied:*

(a) *h is of finite order,*
(b) *Y has finite structural dimension,*
(c) *Y is ultrametric.*

Suppose that $f\colon X \to Y$ is Borel measurable and satisfies Luzin's condition (N) in the sense that $\mu(N) = 0$ implies that $\mathcal{H}^h(f(N)) = 0$. Then $f(X)$ is σ-finite with respect to h.

Proof. Under the given conditions, Y is analytic, see for example Proposition 423B in [Fre] and by Lemma 423G in the same source, we know that $f(X)$ is analytic. Now, f is surjective onto its image. If it would not be σ-finite with

respect to $\mathcal{H}^h$, then Theorem 5.5 would imply that f violates Luzin's condition (N). Thus the claim follows. □

We cite Theorem 1.3 in [CHM10]:

Theorem 11.3. *Let* $f \in W^{1,n-1}_{\mathrm{loc}}((-1,1)^n, \mathbf{R}^n)$ *be a homeomorphism*[19]. *Then for almost every* $y \in (-1,1)$ *the mapping* $f \upharpoonright_{(-1,1)^{n-1} \times \{y\}}$ *satisfies the* $((n-1)$-*dimensional) Luzin condition (N), i.e., for every* $A \subset (-1,1)^{n-1} \times \{y\}$, $\mathcal{H}^{n-1}(A) = 0$ *implies* $\mathcal{H}^{n-1}(f(A)) = 0$.

Combining above theorem with Theorem 11.2 and further exhausting the sets $(-1,1)^{n-1} \times \{y\}$ with compact sets, we obtain the following result:

Corollary 11.4. *Let* $f \in W^{1,n-1}_{\mathrm{loc}}((-1,1)^n, \mathbf{R}^n)$ *be a homeomorphism*[20]. *Then for almost every* $y \in (-1,1)$, *we obtain that* $f((-1,1)^{n-1} \times \{y\})$ *has dimension bounded from above by* $n-1$.

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[19]onto $f((-1,1)^n)$
[20]onto its image

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