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ISOMETRY ON LINEAR n-NORMED SPACES

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Abstract. This paper generalizes the Aleksandrov problem, the Mazur–Ulam theorem and Benz theorem on n-normed spaces. It proves that a one-distance preserving mapping is an n-isometry if and only if it has the zero-distance preserving property, and two kinds of n-isometries on n-normed spaces are equivalent.

1. Introduction

Let X and Y be metric spaces. A mapping $f\colon X \to Y$ is called an isometry if it satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, assume that f preserves the distance r, i.e., for all $x, y \in X$ with $d_X(x, y) = r$, it holds that $d_Y(f(x), f(y)) = r$. Then r is called a conservative (or preserved) distance for the mapping f.

Mazur and Ulam [13] proved a theorem which tells that *every isometry of a real normed space onto a real normed space is a linear mapping up to a translation.*

Aleksandrov [1] posed the following problem: *Examine whether the existence of a single conservative distance for some mapping f implies that f is an isometry.*

Benz [2] proved the following result that is related to Mazur–Ulam theorem. *Let X and Y be real linear normed spaces such that* $\dim X \geq 2$ *and Y is strictly convex. Suppose that $\rho > 0$ is a fixed real number and that $N > 1$ is a fixed integer. Finally, let $f\colon X \to Y$ be a mapping such that for all $x, y \in X$ $\|x-y\| = \rho \Rightarrow \|f(x)-f(y)\| \leq \rho$, and $\|x - y\| = N\rho \Rightarrow \|f(x) - f(y)\| \geq N\rho$. Then f is an affine isometry.*

Rassias and Šemrl et al. [16, 8, 9] proved a series of results on the Aleksandrov problem on normed spaces. Chu et al. and Park et al. [4, 5, 15] in linear n-normed spaces, defined the concept of a w-n-isometry and n-isometry that are suitable to represent the notion of a volume-preserving mapping, and generalized the Aleksandrov problem to n-normed spaces.

In this paper, we prove that all conditions given in [4, 5, 7, 10, 11, 12, 14, 15] are equivalent; i.e., if f has the w-n-DOPP, then the following properties are equivalent:

(1) f preserves w-n-0-distance (n-collinear);

(2) f is a w-n-Lipschitz;

(3) f preserves 2-collinearity;

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(4) f is affine;

(5) f is an n-isometry;

(6) f is n-Lipschitz;

(7) f preserves n-0-distance;

(8) f is a w-n-isometry.

In the end, we generalize Benz's theorem [3] to n-normed spaces.

2. Terminology

Definition 2.1. [5] Assume that X is a real linear space with $\dim X \geq n$ and $\|\cdot, \ldots, \cdot\| \colon X^n \to \mathbf{R}$ is a function which satisfies

(1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent,
(2) $\|x_1, \ldots, x_n\| = \|x_{j_1}, \ldots, x_{j_n}\|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$,
(3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$,
(4) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$

for any $\alpha \in \mathbf{R}$ and all $x_1, \ldots, x_n \in X$. Then the function $\|\cdot, \ldots, \cdot\|$ is called the n-norm on X and $(X, \|\cdot, \ldots, \cdot\|)$ is called a linear n-normed space.

Remark 2.2. [5] Let X be a real linear n-normed space. Then

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i + x_j, \ldots, x_j, \ldots, x_n\|$$

for $x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n \in X$.

Definition 2.3. [6] Let X be a real linear n-normed space. A sequence $\{x_k\}$ is said to converge to $x \in X$ (in the n-norm) if

$$\lim_{k \to \infty} \|x_k - x, y_2, \ldots, y_n\| = 0$$

for every $y_2, \ldots, y_n \in X$.

Some concepts on w-n-distance:

Definition 2.4. [15] Let X and Y be two real linear n-normed spaces. A mapping $f \colon X \to Y$ is said to be a w-n-isometry if

$$\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| = \|x_1 - x_0, \ldots, x_n - x_0\|$$

for all $x_0, x_1, \ldots, x_n \in X$.

Definition 2.5. [15] Let X and Y be two real linear n-normed spaces. A mapping $f \colon X \to Y$ is said to have the w-n-distance one preserving property (w-n-DOPP) if $\|x_1 - x_0, \ldots, x_n - x_0\| = 1$ implies $\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| = 1$ for all $x_0, x_1, \ldots, x_n \in X$.

Definition 2.6. [15] Let X and Y be two real linear n-normed spaces. A mapping $f \colon X \to Y$ is said to be w-n-Lipschitz if

$$\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| \leq \|x_1 - x_0, \ldots, x_n - x_0\|$$

for all $x_0, x_1, \ldots, x_n \in X$.

Some concepts on n-distance:

Definition 2.7. [15] Let X and Y be two real linear n-normed spaces. A mapping $f\colon X \to Y$ is said to be an n-isometry if

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = \|x_1 - y_1, \dots, x_n - y_n\|$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 2.8. [15] Let X and Y be two real linear n-normed spaces. A mapping $f\colon X \to Y$ is said to have the n-distance one preserving property (n-DOPP) if $\|x_1 - y_1, \dots, x_n - y_n\| = 1$ implies $\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 1$ for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 2.9. [15] Let X and Y be two real linear n-normed spaces. A mapping $f\colon X \to Y$ is said to be n-Lipschitz if

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| \leq \|x_1 - y_1, \dots, x_n - y_n\|$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 2.10. Let X and Y be two real linear n-normed spaces. A mapping $f\colon X \to Y$ is said to preserve the 2-collinearity if for all $x, y, z \in X$, the existence of $t \in \mathbf{R}$ with $z - x = t(y - x)$ implies the existence of $s \in \mathbf{R}$ with $f(z) - f(x) = s(f(y) - f(x))$.

Definition 2.11. [5] Let X and Y be two real linear n-normed spaces. The points $x_0, x_1, \dots, x_n$ of X are called n-collinear if for every i, $\{x_j - x_i \colon 0 \leq j \neq i \leq n\}$ is linearly dependent.

Definition 2.12. [5] Let X and Y be two real linear n-normed spaces. A mapping $f\colon X \to Y$ is said to preserve the n-collinearity if n-collinearity of $f(x_0), f(x_1), \dots, f(x_n)$ follows from the n-collinearity of $x_0, x_1, \dots, x_n$.

Remark 2.13. Let X and Y be two real linear n-normed spaces. A mapping f preserves the n-collinearity means that f preserves w-0-distance ($\|x_1 - x_0, \dots, x_n - x_0\| = 0$ implies $\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0$).

3. Main results on two isometries

One of remarkable differences between normed spaces and n-normed spaces is that $\|x - y\| = 0$ implies $\|f(x) - f(y)\| = 0$ for any mapping f from normed space X to Y. However, it is not true for n-normed spaces.

Lemma 3.1. *Let X and Y be two real n-normed spaces. Suppose that f satisfies w-n-DOPP and $\|x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\| = 0$ implies $\|f(x_1) - f(x_0), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0$. Then f preserves 2-collinearity.*

Proof. We first show that f is injective. Let x_0 and x_1 be any distinct points in X. Since $\dim X \geq n$, there are $x_2, \dots, x_n \in X$ such that $x_1 - x_0, \dots, x_n - x_0$ are linearly independent. Thus, $\|x_1 - x_0, \dots, x_n - x_0\| \neq 0$.

Set $z_2 = x_0 + \frac{x_2 - x_0}{\|x_1 - x_0, \dots, x_n - x_0\|}$. Then we have

$$\|x_1 - x_0, z_2 - x_0, x_3 - x_0, \dots, x_n - x_0\| = 1.$$

Since f has the w-n-DOPP, we get

$$\|f(x_1) - f(x_0), f(z_2) - f(x_0), f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)\| = 1$$

and it follows that $f(x_0) \neq f(x_1)$. Hence, f is injective.

For $n=2$, f is obviously 2-collinear by the condition that $\|x_1-x_0, x_2-x_0\|=0$ implies $\|f(x_1)-f(x_0), f(x_2)-f(x_0)\|=0$.

Let $n>2$. Assume that x_0, x_1, x_2 are distinct points of X which are 2-collinear. Then x_1-x_0, x_2-x_0 are linearly dependent and $f(x_0), f(x_1), f(x_2)$ are also distinct by the injectivity of f.

Since $\dim X \geq n$, there exist $y_1, y_2, \dots, y_n \in X$ such that $y_1-x_0, y_2-x_0, \dots, y_n-x_0$ are linearly independent. Hence, it holds that

$$\|y_1-x_0, y_2-x_0, \dots, y_n-x_0\| \neq 0.$$

Let $z_1 = x_0 + \frac{y_1-x_0}{\|y_1-x_0, y_2-x_0, \dots, y_n-x_0\|}$. Then we have

$$\|z_1-x_0, y_2-x_0, \dots, y_n-x_0\| = 1.$$

Since f has the w-n-DOPP,

$$\|f(z_1)-f(x_0), f(y_2)-f(x_0), \dots, f(y_n)-f(x_0)\| = 1.$$

Hence, the set $A=\{f(x)-f(x_0)\colon x\in X\}$ contains n linearly independent vectors.

Since for any $x_3, \dots, x_n \in X$

$$\|x_1-x_0, x_2-x_0, x_3-x_0, \dots, x_n-x_0\| = 0$$

and f preserves the n-collinearity, we have

$$\|f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_n)-f(x_0)\| = 0, \tag{3.1}$$

i.e., $f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_n)-f(x_0)$ are linearly dependent. If there exist $x_3, \dots, x_{n-1}$ such that $f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_{n-1})-f(x_0)$ are linearly independent, then

$$\begin{aligned} A &= \{f(x_n)-f(x_0)\colon x_n \in X\} \\ &\subset \operatorname{span}\{f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_{n-1})-f(x_0)\}, \end{aligned}$$

which contradicts the fact that A contains n linearly independent vectors.

Then, for any $x_3, \dots, x_{n-1}$, $f(x_3)-f(x_0), \dots, f(x_{n-1})-f(x_0)\}$ are linearly dependent. If there exist $x_3, \dots, x_{n-2}$ such that $f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_{n-2})-f(x_0)$ are linearly independent, then

$$\begin{aligned} A &= \{f(x_{n-1})-f(x_0)\colon x_{n-1} \in X\} \\ &\subset \operatorname{span}\{f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \dots, f(x_{n-2})-f(x_0)\}, \end{aligned}$$

which contradicts the fact that A contains n linearly independent vectors.

And so on, $f(x_1)-f(x_0)$ and $f(x_2)-f(x_0)$ are linearly dependent, i.e., $f(x_0)$, $f(x_1)$, $f(x_2)$ are 2-collinear. Therefore, f preserves the 2-collinearity. □

Corollary 3.2. *Let X and Y be two real linear n-normed spaces. If f is w-n-Lipschitz and satisfies w-n-DOPP, then f preserves 2-collinearity.*

Proof. Because f is w-n-Lipschitz, then $\|x_1-x_0, x_2-x_0, \dots, x_n-x_0\| = 0$ implies $\|f(x_1)-f(x_0), f(x_2)-f(x_0), \dots, f(x_n)-f(x_0)\| = 0$. Hence f preserves 2-collinearity by Lemma 3.1. □

Lemma 3.3. [11] *Let X and Y be two real n-normed spaces. Suppose that $f\colon X \to Y$ satisfies n-DOPP and preserves 2-collinearity, then f preserves w-n-distance $\frac{1}{k}$ for each $k \in \mathbf{N}$.*

Lemma 3.4. *Let X and Y be two real n-normed spaces. If $f\colon X\to Y$ satisfies w-n-DOPP and preserves 2-collinearity, then f is affine.*

Proof. (1) Let $x=\frac{y+z}{2}$ for distinct $x,y,z\in X$. Then $y-x=-(z-x)$. Since f is injective and preserves 2-collinearity, there exists an $s\neq 0$ such that

$$f(y)-f(x)=s(f(z)-f(x)). \tag{3.2}$$

Since $\dim X\geq n$, there exist $x_1,x_2,\ldots,x_{n-1}\in X$ with $\|y-x,x_1-x,x_2-x,\ldots,x_{n-1}-x\|\neq 0$. Set $w=x+\frac{x_1-x}{\|y-x,x_1-x,x_2-x,\ldots,x_{n-1}-x\|}$. Then

$$\|y-x,w-x,x_2-x,\ldots,x_{n-1}-x\|=1 \tag{3.3}$$

and

$$\|f(y)-f(x),f(w)-f(x),f(x_2)-f(x),\ldots,f(x_{n-1})-f(x)\|=1.$$

Clearly, it follows from (3.2) that

$$\|f(z)-f(x),f(w)-f(x),f(x_2)-f(x),\ldots,f(x_{n-1})-f(x)\|=\frac{1}{|s|}. \tag{3.4}$$

Since $y-x=x-z$, (3.3) yields

$$\|z-x,w-x,x_2-x,\ldots,x_{n-1}-x\|=1,$$

and hence we have

$$\|f(z)-f(x),f(w)-f(x),f(x_2)-f(x),\ldots,f(x_{n-1})-f(x)\|=1. \tag{3.5}$$

Because f is injective, and comparing (3.4) with (3.5) we conclude that $s=-1$. Thus, $f(y)-f(x)=f(x)-f(z)$ and

$$f(\frac{y+z}{2})=\frac{f(y)+f(z)}{2}.$$

(2) Let $g(x)=f(x)-f(0)$. It is obvious that for any $x\in X$ and all rational numbers r,p, we have

$$g(rx)=rg(x),\quad g(rx+py)=rg(x)+pg(y). \tag{3.6}$$

(3) Next we show that g preserves any rational number n-distance. Suppose that

$$\|x_1-y_1,x_2-y_2,\ldots,x_n-y_n\|=\frac{t}{m}$$

for integers t,m. Then

$$\left\|\frac{1}{t}x_1-y_1,x_2-y_2,\ldots,x_n-y_n\right\|=\frac{1}{m},$$

according to Lemma 3.3 and (3.6), we have

$$\left\|\frac{1}{t}(g(x_1)-g(y_1)),g(x_2)-g(y_2),\ldots,g(x_n)-g(y_n)\right\|=\frac{1}{m}.$$

Thus

$$\|g(x_1)-g(y_1),g(x_2)-g(y_2),\ldots,g(x_n)-g(y_n)\|=\frac{t}{m}.$$

(4) For any $r\in\mathbf{R}$, since $g(0)$, $g(x)$, $g(rx)$ are also 2-collinear from $f(0)$, $f(x)$, $f(rx)$ are 2-collinear and $g(0)=0$. There exists a real number s such that

$$g(rx)=sg(x).$$

Let $\{r_k\}$ be a sequence of rational numbers with $\lim_{i\to\infty} r_k = s$. Then for any $y_2, \ldots, y_n \in Y$,

$$\lim_{k\to\infty} \|r_k g(x) - sg(x), y_2, \ldots, y_n\| = \lim_{k\to\infty} |r_k - s| \|g(x), y_2, \ldots, y_n\| = 0.$$

So $g(rx) = \lim_{k\to\infty} r_k g(x)$. This yields

$$\lim_{k\to\infty} \|g(r_k x) - g(rx), y_2, \ldots, y_n\| = 0.$$

Then for $x \neq 0$ and any k, we can find $x_2^k, \ldots, x_n^k$ which satisfy $\|x, x_2^k, \ldots, x_n^k\| > 1$ and $|r - r_k|\|x, x_2^k, \ldots, x_n^k\|$ is a rational number. This implies that

$$\begin{aligned} |r - r_k|\|x, x_2^k, \ldots, x_n^k\| &= \|(r - r_k)x, x_2^k, \ldots, x_n^k\| \\ &= ||rx - r_k x, x_2^k, \ldots, x_n^k\| \\ &= \|g(rx) - g(r_k x), g(x_2^k), \ldots, g(x_n^k)\|. \end{aligned}$$

Moreover, $\lim_{k\to\infty} \|g(r_k x) - g(rx), g(x_2^k), \ldots, g(x_n^k)\| = 0$ and $\|x, x_2^k, \ldots, x_n^k\| > 1$ imply that $\lim_{i\to\infty} r_k = r$. Thus $r = s$, and hence g is linear and f is affine. □

Lemma 3.5. *Let X and Y be two real n-normed spaces. Suppose that $f\colon X \to Y$ satisfies w-n-DOPP and f is affine. Then*

(1) *f preserves n-0-distance;*

(2) *f preserves n-1-distance (n-DOPP);*

(3) *f is an n-isometry.*

Proof. Set $g(x) = f(x) - f(0)$. Then $g(x)$ is linear.

(a) Suppose that

$$\|y_1 - x_1, \ldots, y_n - x_n\| = 0.$$

Then $\{y_1 - x_1, \ldots, y_n - x_n\}$ are linearly dependent. There are $a_1, a_2, \ldots, a_n$ which are not all zero such that

$$a_1(y_1 - x_1) + a_2(y_2 - x_2 \ldots, +a_n(y_n - x_n) = 0,$$

and

$$a_1(g(y_1) - g(x_1)) + a_2(g(y_2) - g(x_2)) \ldots, +a_n g((y_n) - g(x_n)) = 0.$$

Clearly, we have

$$\|g(y_1) - g(x_1), \ldots, g(y_n) - g(x_n)\| = 0,$$

which deduces

$$\|f(y_1) - f(x_1), \ldots, f(y_n) - f(x_n)\| = 0.$$

(b) Suppose that for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$,

$$\|y_1 - x_1, \ldots, y_n - x_n\| = 1.$$

For any $x_0 \in X$, set $z_i = x_0 + y_i - x_i$. Then

$$\|z_1 - x_0, \ldots, z_n - x_0\| = 1.$$

Since f satisfies w-n-DOPP, we have

$$\|f(z_1) - f(x_0), \ldots, f(z_n) - f(x_0)\| = 1.$$

Clearly,

$$\|g(z_1) - g(x_0), \ldots, g(z_n) - g(x_0)\| = 1,$$

and g is linear, which means

$$\|g(y_1) - g(x_1), \ldots, g(y_n) - g(x_n)\| = 1.$$

This implies

$$\|f(y_1) - f(x_1), \ldots, f(y_n) - f(x_n)\| = 1.$$

(c) Suppose that for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$,

$$\|y_1 - x_1, \ldots, y_n - x_n\| \neq 0,$$

and set

$$y = x_1 + \frac{y_1 - x_1}{\|y_1 - x_1, \ldots, y_n - x_n\|}. \tag{3.7}$$

This implies that $\|y - x_1, \ldots, y_n - x_n\| = 1$ and $\|f(y) - f(x_1), \ldots, f(y_n) - f(x_n)\| = 1$. Hence, it holds that

$$\|g(y) - g(x_1), g(y_2) - g(x_2), \ldots, g(y_n) - g(x_n)\| = 1. \tag{3.8}$$

Since g is linear, it follows from (3.7) and (3.8) that

$$\left\| \frac{g(y_1) - g(x_1)}{\|y_1 - x_1, \ldots, y_n - x_n\|}, g(y_2) - g(x_2), \ldots, g(y_n) - g(x_n) \right\| = 1.$$

This implies that

$$\left\| \frac{f(y_1) - f(x_1)}{\|y_1 - x_1, \ldots, y_n - x_n\|}, f(y_2) - f(x_2), \ldots, f(y_n) - f(x_n) \right\| = 1.$$

Hence, $\|f(y_1) - f(x_1), \ldots, f(y_n) - f(x_n)\| = \|y_1 - x_1, \ldots, y_n - x_n\|$, which shows that f is an n-isometry. □

Theorem 3.6. *Let X and Y be two real n-normed spaces. Suppose that f satisfies w-n-DOPP. Then the following properties are equivalent for f: w-n-Lipschitz, n-collinear (w-n-0-distance), 2-collinear, affine, n-isometry, n-Lipschitz, n-0-distance, w-n-isometry.*

Proof. w-n-DOPP and w-n-Lipschitz $\Rightarrow$ w-n-DOPP and n-collinear $\Rightarrow$ w-n-DOPP and 2-collinear $\Rightarrow$ w-n-DOPP and affine $\Rightarrow$ n-DOPP and n-0-distance $\Rightarrow$ n-isometry $\Rightarrow$ n-DOPP and n-Lipschitz or w-n-isometry $\Rightarrow$ w-n-DOPP and w-n-Lipschitz. □

Corollary 3.7. *Let X and Y be two real n-normed spaces. A mapping $f \colon X \to Y$ is a w-n-isometry if and only if f is an n-isometry.*

Proof. Obviously, if f is a w-n-isometry, then f preserves w-n-DOPP. □

Corollary 3.8. *Let X and Y be two real n-normed spaces. Suppose that f preserves w-ρ-distance for some fixed $\rho > 0$. Then the following properties are equivalent for f: w-n-Lipschitz, n-collinear (w-n-0-distance), 2-collinear, affine, n-isometry, n-Lipschitz, n-0-distance, w-n-isometry.*

Remark 3.9. Let X and Y be two real n-normed spaces. Suppose that f satisfies n-DOPP. Then the following properties are equivalent for f: w-n-Lipschitz, n-collinear (w-n-0-distance), 2-collinear, affine, n-isometry, n-Lipschitz, n-0-distance, w-n-isometry.

4. Main result on Benz Theorem

Theorem 4.1. *Let X and Y be two real linear n-normed spaces, $x_0, x_1, \dots, x_n \in X$, $\rho > 0$, $N = 1, 2, \dots$, and $f\colon X \to Y$ be a function satisfying the conditions*

(1) $\|x_1 - x_0, \dots, x_n - x_0\| = \rho$ *implies* $\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| \le \rho$,

(2) $\|x_1 - x_0, \dots, x_n - x_0\| = N\rho$ *implies* $\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| \ge N\rho$,

(3) f *is 2-collinear (or one of the equivalent conditions of Corollary 3.8 holds).*

Then f is an n-isometry.

Proof. We only need to show that f preserves w-ρ-distance. Let

$$\|x_1 - x_0, \dots, x_n - x_0\| = \rho.$$

Set $p_i = x_0 + i(x_1 - x_0)$, $i = 0, 1, \dots, N$. Clearly, we have $p_1 = x_1, p_0 = x_0, p_i - x_0 = i(x_1 - x_0), p_i - p_{i-1} = x_1 - x_0 = p_i - x_0$, and

$$\|p_i - p_{i-1}, x_2 - x_0, \dots, x_n - x_0\| = \|x_1 - x_0, \dots, x_n - x_0\| = \rho.$$

It follows from Remark 2.2 that $\|p_i - p_{i-1}, x_2 - p_i, \dots, x_n - p_i\| = \|p_i - p_{i-1}, x_2 - x_0, \dots, x_n - x_0\| = \rho$, and

$$\|p_N - x_0, \dots, x_n - x_0\| = N\rho. \tag{4.1}$$

By condition (1)

$$\|f(p_{i-1}) - f(p_i), f(x_2) - f(p_i), \dots, f(x_n) - f(p_i)\| \le \rho.$$

As p_i, p_{i-1}, x_0 are 2-collinear, $f(p_i), f(p_{i-1}), f(x_0)$ are 2-collinear, it is necessary from Remark 2.2 that

$$\begin{aligned}
&\|f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| \\
&= \|f(p_i) - f(p_{i-1}), f(x_2) - f(p_i) + f(p_i) - f(x_0), \dots, f(x_n) - f(p_i) + f(p_i) - f(x_0)\| \\
&= \|f(p_i) - f(p_{i-1}), f(x_2) - f(p_i), \dots, f(x_n) - f(p_i)\| \le \rho.
\end{aligned}$$

By (4.1) and condition (2),

$$\begin{aligned}
N\rho &\le \|f(p_N) - f(x_0), \dots, f(x_n) - f(x_0)\| \\
&\le \sum_1^N \|f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = N\rho.
\end{aligned}$$

Thus

$$\|f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = \rho.$$

This implies

$$\|f(x_1) - f(x_0), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = \rho.$$

It proves that f is an affine n-isometry by the Corollary 3.8. □

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