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ATOMIC DECOMPOSITION OF HARDY–MORREY SPACES WITH VARIABLE EXPONENTS

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Abstract. The Hardy–Morrey spaces with variable exponents are introduced in terms of maximal functions. The atomic decomposition of Hardy–Morrey spaces with variable exponents is established. This decomposition extends and unifies several atomic decompositions of Hardy type spaces such as the Hardy–Morrey spaces and the Hardy spaces with variable exponents. Some applications of this atomic decomposition on singular integral are presented.

1. Introduction

The main result of this paper is the atomic decomposition of Hardy–Morrey spaces with variable exponents. The atomic decomposition is one of the remarkable features for Hardy type spaces. It had been generalized to some non-classical Hardy spaces such as the weighted Hardy spaces [6, 24, 56], the Hardy–Orlicz spaces [59] and the Hardy–Lorentz spaces [1].

Recently, the atomic decomposition has been further extended to several different Hardy type spaces. The atomic decomposition of Hardy–Morrey spaces is developed in [34]. We also have the atomic decomposition for weighted Hardy–Morrey spaces in [33]. Furthermore, the atomic decomposition for Hardy spaces with variable exponents is established in [46, 54].

The main result of this paper further generalizes the atomic decomposition to Hardy–Morrey spaces with variable exponents. It is an extension and unification of the atomic decompositions in [34, 33, 46]. Roughly speaking, we follow the approach for developing the atomic decomposition for classical Hardy spaces [55, Chapter III, Section 2]. We find that to establish the atomic decomposition in Hardy–Morrey spaces with variable exponents, we need the Fefferman–Stein vector-valued maximal inequalities in Morrey spaces with variable exponents.

The vector-valued maximal inequalities on L^p, $1 < p < \infty$, are well known [19]. There are several generalizations of these inequalities. For instance, the vector-valued maximal inequalities on classical Morrey spaces are proved in [51, Theorem 2.4] and [57]. The vector-valued maximal inequalities on rearrangement-invariant quasi-Banach function spaces and its associate Morrey type spaces are given in [27].

The vector-valued maximal inequalities on Morrey spaces with variable exponents are presented and proved in Section 3. The Hardy–Morrey spaces with variable exponents are defined via the maximal functions in Section 4. The introduction of the Hardy–Morrey spaces with variable exponents is inspired by the studies of the Hardy–Morrey spaces and the variable exponent analysis.

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The Hardy–Morrey spaces are introduced in [53] by using Littlewood–Paley characterization. The reader may also consult [27, 52, 63] for the studies of Triebel–Lizorkin–Morrey spaces which is a generalization of the Hardy–Morrey space. The reader is also referred to [40] for another approach based on Peetre's maximal functions.

In addition, the maximal function characterization of the Hardy–Morrey space is given in [34]. It also contains the atomic decomposition of Hardy–Morrey spaces. The Littlewood-Paley characterization of the weighted Hardy–Morrey spaces is presented and proved in [32]. The Hardy–Morrey spaces are members of the family of Hardy–Morrey spaces with variable exponents studied in this paper.

The introduction of Hardy–Morrey spaces with variable exponents is motivated a second inspiration, the Morrey spaces with variable exponents [46]. Recently, there are a number of researches on extending the classical results from the Lebesgue space to the Lebesgue spaces with variable exponent. One of the important results in the variable exponent analysis is the identification of the class of exponent functions for which the Hardy–Littlewood maximal function is bounded on the corresponding Lebesgue spaces with variable exponent [10, 11, 13, 15, 16, 36, 38, 39, 47, 48, 49].

The variable exponent analysis also covers the studies of Morrey spaces [2, 26, 30, 32, 35], Besov spaces and Triebel–Lizorkin spaces [3, 18, 22, 61, 62]. Moreover, as mentioned previously, the Hardy spaces with variable exponents are also introduced and studied in [46]. It contains the atomic decomposition and some of its important applications such as the boundedness of some singular integral operators.

In Section 5, the main result of this paper, the atomic decomposition of Hardy–Morrey spaces with variable exponents, Theorems 5.1, 5.2 and 5.3, is presented and proved. The proof of this decomposition follows the idea from the classical Hardy spaces [55, Chapter III, Section 2]. It is also inspired by the results given in [34, 46, 56]. For instance, we follow the ideas in [46] to obtain the boundedness of some singular integral operators on Hardy–Morrey spaces with variable exponents at the end of this section as an application of our main results. Furthermore, we extend an inequality in [56, Chapter VIII, Lemma 5] to the Morrey spaces with variable exponents which is an essential supporting result to establish our main theorems.

However, the techniques given in [34, 46, 56, 55] are insufficient to establish our main results, some new ideas are needed. We find that to obtain some crucial inequalities for the Morrey spaces with variable exponents, we have to identify the associate spaces of the Morrey spaces with variable exponents which poses a main obstacle for our studies.

We overcome these difficulties by considering a pre-dual of Morrey space, the block spaces with variable exponents, instead of the associate spaces of the Morrey spaces with variable exponents. For details, the reader is referred to Section 5. Additionally, some indices introduced in [30, Definition 2.6] for the study of vector-valued singular integral operators on Morrey type spaces are needed in our main theorems to give a precise characterization of the atomic decomposition.

The Hardy–Morrey spaces are members of the family of Hardy–Morrey spaces with variable exponents studied in this paper. Moreover, the main results, Theorems 5.1, 5.2 and 5.3, in this paper extend the atomic decomposition for Hardy–Morrey spaces in [34].

The Hardy–Morrey spaces with variable exponents include the Hardy spaces with variable exponents as special cases. In addition, our atomic decomposition also extends the atomic decomposition in [46, Theorems 4.5 and 4.6] in the sense that the exponent functions are not restricted to satisfy the local and global log-Hölder conditions [46, (2.5)-(2.6)]. Our results apply to some exponent functions for which the local and global log-Hölder conditions are not necessarily fulfilled [12, 38, 47].

This paper is organized as follows. Section 2 presents some definitions and notions from variable exponent analysis. This section also gives the definition of the Morrey spaces with variable exponents used in this paper. The vector-valued maximal inequalities on Morrey spaces with variable exponents are proved in Section 3. The Hardy–Morrey spaces with variable exponents are introduced in Section 4. The atomic decomposition of Hardy–Morrey spaces with variable exponents is established in Section 5.

2. Preliminaries

We begin with the definition of Lebesgue spaces with variable exponent and some of theirs properties. Let $p(\cdot)\colon \mathbf{R}^n \to (0,\infty]$ be a Lebesgue measurable function. Define

$$p_- = \operatorname{ess\,inf}\{p(x)\colon x\in\mathbf{R}^n\} \quad \text{and} \quad p_+ = \operatorname{ess\,sup}\{p(x)\colon x\in\mathbf{R}^n\}.$$

Definition 2.1. Let $p(\cdot)\colon \mathbf{R}^n \to (0,\infty]$ be a Lebesgue measurable function with $0<p_-\le p_+\le\infty$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbf{R}^n)$ consists of all Lebesgue measurable functions $f\colon \mathbf{R}^n\to\mathbf{C}$ satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} = \inf\{\lambda>0\colon \rho(|f(x)|/\lambda)\le 1\}<\infty$$

where

$$\rho(f) = \int_{\{x\in\mathbf{R}^n\colon p(x)\neq\infty\}} |f(x)|^{p(x)}\,dx + \|f\chi_{\{x\in\mathbf{R}^n\colon p(x)=\infty\}}\|_{L^\infty}.$$

We call $p(\cdot)$ the exponent function of $L^{p(\cdot)}(\mathbf{R}^n)$.

When $|\{x\in\mathbf{R}^n\colon p(x)=\infty\}|=0$, we find that

$$\|f\|_{L^{p(\cdot)/a}(\mathbf{R}^n)} = \||f|^{\frac{1}{a}}\|^a_{L^{p(\cdot)}(\mathbf{R}^n)}. \tag{2.1}$$

That is, $L^{p(\cdot)/a}(\mathbf{R}^n)$ is the a-th power of $L^{p(\cdot)}(\mathbf{R}^n)$ (the $\frac{1}{a}$-convexification of $L^{p(\cdot)}(\mathbf{R}^n)$), see [50, Section 2.2] or [41, Volume II, pp. 53–54]. Therefore, if $0<p_-\le p_+\le\infty$, then $L^{p(\cdot)}(\mathbf{R}^n)$ is the p_--convexification of $L^{p(\cdot)/p_-}(\mathbf{R}^n)$. Since $L^{p(\cdot)/p_-}(\mathbf{R}^n)$ is a Banach space [37, Theorem 2.5], $L^{p(\cdot)}(\mathbf{R}^n)$ is a quasi-Banach space [50, Proposition 2.22].

Let $B(z,r)=\{x\in\mathbf{R}^n\colon |x-z|<r\}$ denote the open ball with center $z\in\mathbf{R}^n$ and radius $r>0$. Let $\mathbf{B}=\{B(z,r)\colon z\in\mathbf{R}^n,\ r>0\}$. We now give the definition of the Morrey space with variable exponent used in this paper.

Definition 2.2. Let $p(\cdot)\colon \mathbf{R}^n\to(0,\infty)$ and $u(x,r)\colon \mathbf{R}^n\times(0,\infty)\to(0,\infty)$. The *Morrey space with variable exponent* $\mathcal{M}_{p(\cdot),u}$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathcal{M}_{p(\cdot),u}} = \sup_{z\in\mathbf{R}^n,R>0}\frac{1}{u(z,R)}\|\chi_{B(z,R)}f\|_{L^{p(\cdot)}(\mathbf{R}^n)}<\infty.$$

By using (2.1), we also have the corresponding result for Morrey space. That is, the a-th power of $\mathcal{M}_{p(\cdot),u}$ is $\mathcal{M}_{\frac{p(\cdot)}{a},u^a}$. More precisely, we have

$$\|f\|_{\mathcal{M}_{\frac{p(\cdot)}{a},u^a}} = \||f|^{\frac{1}{a}}\|^a_{\mathcal{M}_{p(\cdot),u}}. \tag{2.2}$$

Thus, $\mathcal{M}_{p(\cdot),u}$ is a quasi-Banach space.

For any $p\colon \mathbf{R}^n \to [1,\infty]$, the conjugate function p' is defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We have the following duality result for $L^{p(\cdot)}(\mathbf{R}^n)$.

Theorem 2.1. *If $1 \le p(\cdot) \le \infty$, then $L^{p(\cdot)}(\mathbf{R}^n)$ is a Banach function space and the associate space of $L^{p(\cdot)}(\mathbf{R}^n)$ is $L^{p'(\cdot)}(\mathbf{R}^n)$. Moreover, if $1 < p_- \le p_+ < \infty$, $L^{p(\cdot)}(\mathbf{R}^n)$ is reflexive.*

The above theorem is given in [12, Theorem 2.34, Corollary 2.81 and Section 2.10.3] and [17, Theorems 3.2.13 and 3.4.7].

With the preceding theorems, we have the subsequent property for $L^{p(\cdot)}(\mathbf{R}^n)$ when the exponent function $p(\cdot)$ satisfying $0 < p_- \le p_+ < \infty$. The reader is referred to [4, Chapter I, Section 1, Definition 3.1] for the definition of absolutely continuous norm and [27, Definition 2.4] for the definition of absolutely continuous quasi-norm.

Corollary 2.2. *Let $p(\cdot)\colon \mathbf{R}^n \to (0,\infty)$ be a Lebesgue measurable function with $0 < p_- \le p_+ < \infty$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbf{R}^n)$ has absolutely continuous quasi-norm.*

Proof. From the definition of absolutely continuous quasi-norm, the a-power of X possesses absolutely continuous quasi-norm provided that the quasi-Banach space X has. Therefore, our result follows from the fact that the $\frac{p_-}{2}$-power of $L^{p(\cdot)}(\mathbf{R}^n)$ is a reflexive Banach space, and, by Theorem 2.1, the $\frac{p_-}{2}$-power of $L^{p(\cdot)}(\mathbf{R}^n)$ possesses absolutely continuous norm. □

Let $\mathcal{P}$ denote the set of all Lebesgue measurable functions $p(\cdot)\colon \mathbf{R}^n \to (1,\infty)$ such that $1 < p_- \le p_+ < \infty$. We now define several classes of exponent functions. They are all related to the boundedness of the Hardy–Littlewood maximal operator.

Definition 2.3. Let $\overline{\mathcal{B}}$ consists of all Lebesgue measurable functions $p(\cdot)\colon \mathbf{R}^n \to [1,\infty]$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$. Write $p(\cdot) \in \overline{\mathcal{B}}'$ if $p'(\cdot) \in \overline{\mathcal{B}}$. Let $\mathcal{B}$ denote the set of all $p(\cdot)$ belonging to $\mathcal{P}$ such that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$.

An important subset of $\mathcal{B}$ is the class of globally log-Hölder continuous functions $p \in C^{\log}(\mathbf{R}^n)$ with $1 < p_- \le p_+ < \infty$, see [18, Definition 2.1]. Recall that $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ if it satisfies

$$|p(x) - p(y)| \le C\frac{1}{\log(1/|x-y|)}, \quad |x-y| \le \frac{1}{2} \tag{2.3}$$

$$|p(x) - p(y)| \le C\frac{1}{\log(e+|x|)}, \quad |y| \ge |x|. \tag{2.4}$$

According to [15, Theorem 8.1], we have the following characterization of $\mathcal{B}$.

Theorem 2.3. *Let $p(\cdot) \in \mathcal{P}$. Then the following conditions are equivalent:*

(1) $p(\cdot) \in \mathcal{B}$.
(2) $p'(\cdot) \in \mathcal{B}$.

(3) $p(\cdot)/q \in \mathcal{B}$ *for some* $1 < q < p_-$.
(4) $(p(\cdot)/q)' \in \mathcal{B}$ *for some* $1 < q < p_-$.

We recall the indices given in [30, Definition 2.6] for the study of vector-valued singular integral operators on Morrey type spaces.

Definition 2.4. For any $p(\cdot) \in \overline{\mathcal{B}}$, let $\kappa_{p(\cdot)}$ denote the supremum of those $q \geq 1$ such that $p(\cdot)/q \in \overline{\mathcal{B}}$. For any $p(\cdot) \in \overline{\mathcal{B}}'$, define $e_{p(\cdot)}$ to be the conjugate of $\kappa_{p'(\cdot)}$.

The index $e_{p(\cdot)}$ is used to study the Morrey spaces with variable exponents, see Definitions 3.1 and 2.2. The index $\kappa_{p(\cdot)}$ is a crucial ingredient to formulate the atomic decomposition in Theorem 5.3.

When $p(\cdot) \in \mathcal{B}$, we have $p(\cdot), p'(\cdot) \in \overline{\mathcal{B}} \cap \overline{\mathcal{B}}'$. According to Theorem 2.3, $\kappa_{p(\cdot)}$, $\kappa_{p'(\cdot)}$, $e_{p(\cdot)}$ and $e_{p'(\cdot)}$ are well defined and

$$1 < \kappa_{p(\cdot)} \leq p_-. \tag{2.5}$$

Moreover, $p_+ \leq e_{p(\cdot)}$. In particular, if $p(\cdot) \in C^{\log}(\mathbf{R}^n)$, then $p_+ = e_{p(\cdot)}$ and $p_- = \kappa_{p(\cdot)}$.

We present a remarkable feature satisfied by the Lebesgue spaces with variable exponents. It is a special case of a result of Banach function spaces [29, Proposition 2.2]. It also plays a crucial role on the establishment of the Fefferman–Stein vector-valued inequalities on the Morrey spaces with variable exponents in the next section.

Proposition 2.4. *Let* $p(\cdot) \in \overline{\mathcal{B}} \cup \overline{\mathcal{B}}'$. *We have a constant* $C > 0$ *so that for any* $B \in \mathbf{B}$,

$$|B| \leq \|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \leq C|B|. \tag{2.6}$$

The proof of Proposition 2.4 is given in [31, Lemma 3.2].

With the above proposition, we have the following estimate for the $L^{p(\cdot)}(\mathbf{R}^n)$ norm of the characteristic function of $B \in \mathbf{B}$. Let $p\colon \mathbf{R}^n \to (0, \infty]$ be a Lebesgue measurable function with $p_- > 0$. For any $B \in \mathbf{B}$, define $\overline{p}_B$ by

$$\frac{1}{\overline{p}_B} = \frac{1}{|B|} \int_B \frac{1}{p(x)}\, dx.$$

Proposition 2.5. *Let* $p(\cdot) \in \mathcal{B}$ *and* $1 < p_- \leq p_+ < \infty$. *There exist* $C_1, C_2 > 0$ *so that for any* $B \in \mathbf{B}$,

$$C_1 |B|^{\frac{1}{\overline{p}_B}} \leq \|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)} \leq C_2 |B|^{\frac{1}{\overline{p}_B}}. \tag{2.7}$$

For the proofs of the above propositions, the reader is referred to [32, Propositions 1.5 and 1.6].

Proposition 2.6. *Let* $p(\cdot)\colon \mathbf{R}^n \to [1, \infty]$ *be a Lebesgue measurable function.*

(1) *If* $p(\cdot) \in \bar{\mathcal{B}}$ *and* $1 < \kappa_{p(\cdot)}$, *then for any* $1 < q < \kappa_{p(\cdot)}$, *there exists constant* $C_1 > 0$ *such that for any* $x_0 \in \mathbf{R}^n$ *and* $r > 0$, *we have*

$$\frac{\|\chi_{B(x_0, 2^j r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x_0, r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \leq C_1 2^{\frac{jn}{q}}, \quad \forall j \in \mathbf{N}. \tag{2.8}$$

(2) *If $p(\cdot) \in \bar{\mathcal{B}}'$ and $1 < \kappa_{p'(\cdot)}$, then for any $1 < s < \kappa_{p'(\cdot)}$, there exists constant $C_2 > 0$ such that for any $x_0 \in \mathbf{R}^n$ and $r > 0$, we have*

$$C_2 2^{jn(1-\frac{1}{s})} \leq \frac{\|\chi_{B(x_0,2^j r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}, \quad \forall j \in \mathbf{N}. \tag{2.9}$$

Proof. For any $B = B(x_0, r) \in \mathbf{B}$ and $j \in \mathbf{N}$, we have a constant $C > 0$ such that

$$C 2^{-jn} \leq \mathrm{M}(\chi_B)(x)$$

when $x \in B(x_0, 2^j r)$, $j \in \mathbf{N}$. Thus, for any $1 < q < \kappa_{p(\cdot)}$, there exists a $q < \tilde{q}$ so that $p(\cdot)/\tilde{q} \in \overline{\mathcal{B}}$. Subsequently,

$$2^{-jn}\|\chi_{B(x_0,2^j r)}\|_{L^{p(\cdot)/\tilde{q}}} \leq C\|\mathrm{M}(\chi_B)\|_{L^{p(\cdot)/\tilde{q}}} \leq C\|\chi_B\|_{L^{p(\cdot)/\tilde{q}}}.$$

Since, for any $B \in \mathbf{B}$ and $q > 0$, $\|\chi_B\|_{L^{p(\cdot)/q}} = \|\chi_B\|^q_{L^{p(\cdot)}}$, we obtain (2.8).

Similarly, as $1 < \kappa_{p'(\cdot)}$, for any $1 < s < \kappa_{p'(\cdot)}$, there exists a $1 < s < \tilde{s}$ so that $p'(\cdot)/\tilde{s} \in \overline{\mathcal{B}}$. Thus, for any $1 < s < \kappa_{p'(\cdot)}$, we also have

$$\frac{\|\chi_{B(x_0,2^j r)}\|^s_{L^{p'(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x_0,r)}\|^s_{L^{p'(\cdot)}(\mathbf{R}^n)}} = \frac{\|\chi_{B(x_0,2^j r)}\|_{L^{p'(\cdot)/s}(\mathbf{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p'(\cdot)/s}(\mathbf{R}^n)}} \leq C_1 2^{jn}, \quad \forall j \in \mathbf{N}.$$

Therefore, Proposition 2.4 yields (2.9). □

The above result can be considered as a generalization of the notion of Boyd's indices to variable Lebesgue spaces. The Boyd indices gives a control on the operator norm of the dilation operator $D_s(f)(x) = f(sx)$ on rearrangement-invariant Banach function spaces (see [4, Chapter 3, Section 5]). Even though the Boyd indices is not necessarily well defined on variable Lebesgue spaces, the above proposition provides a pivotal estimate to obtain our main result in the following section.

For a generalization of Boyd's indices to Banach function space, the reader is referred to [29].

3. Vector-valued maximal inequalities

We are now ready to state and prove the Fefferman–Stein vector-valued maximal inequalities on Morrey spaces with variable exponents. Even though Theorem 3.1 is proved in order to establish the atomic decomposition of Hardy–Morrey spaces with variable exponents, it has its own independent interest. It includes the corresponding inequalities on Morrey spaces given in [57]. It also covers the vector-valued maximal inequalities on variable Lebesgue spaces shown in [9].

Theorem 3.1. *Let $p(\cdot) \in \mathcal{B}$, $1 < q < \infty$ and $u\colon \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If there exists a constant $C > 0$ such that for any $x \in \mathbf{R}^n$ and $r > 0$, u fulfills*

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} u(x, 2^{j+1}r) < C u(x,r), \tag{3.1}$$

then there exists $C > 0$ such that for any $f = \{f_i\}_{i\in\mathbf{Z}}$, $f_i \in L_{\mathrm{loc}}(\mathbf{R}^n)$, $i \in \mathbf{Z}$,

$$\|\|\mathbf{M}(f)\|_{l^q}\|_{\mathcal{M}_{p(\cdot),u}} \leq C\|\|f\|_{l^q}\|_{\mathcal{M}_{p(\cdot),u}} \tag{3.2}$$

where $\mathbf{M}(f) = \{\mathrm{M}(f_i)\}_{i\in\mathbf{Z}}$.

Proof. Let $f=\{f_i\}_{i\in\mathbf{Z}}\subset L_{\rm loc}(\mathbf{R}^n)$. For any $z\in\mathbf{R}^n$ and $r>0$, write $f_i(x)=f_i^0(x)+\sum_{j=1}^\infty f_i^j(x)$, where $f_i^0=\chi_{B(z,2r)}f_i$ and $f_i^j=\chi_{B(z,2^{j+1}r)\backslash B(z,2^jr)}f_i$, $j\in\mathbf{N}\backslash\{0\}$. Applying the vector-valued maximal inequalities in [9, Corollary 2.1] to $f^0=\{f_i^0\}_{i\in\mathbf{Z}}$, we find that $\|\|\mathbf{M}(f^0)\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\le C\|\|f^0\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}$. Indeed,

$$\begin{aligned}\frac{1}{u(z,r)}\|\chi_{B(z,r)}\|\mathbf{M}(f^0)\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}&\le C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)}\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le C\sup_{\substack{y\in\mathbf{R}^n\\ r>0}}\frac{1}{u(y,r)}\|\chi_{B(y,r)}\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\end{aligned}$$

because inequality (3.1) yields $u(z,2r)<Cu(z,r)$ for some constant $C>0$ independent of $z\in\mathbf{R}^n$ and $r>0$.

Furthermore, there is a constant $C>0$ such that, for any $j\ge1$ and $i\in\mathbf{Z}$,

$$\chi_{B(z,r)}(x)(\mathrm{M}f_i^j)(x)\le C2^{-jn}r^{-n}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}|f_i(y)|\,dy.$$

Since l^q is a Banach lattice, we find that

$$\chi_{B(z,r)}(x)\|\{(\mathrm{M}f_i^j)(x)\}_{i\in\mathbf{Z}}\|_{l^q}\le C2^{-jn}r^{-n}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbf{Z}}\|_{l^q}\,dy. \tag{3.3}$$

Using the generalized Hölder inequality given in [37, Theorem 2.1], we obtain

$$\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbf{Z}}\|_{l^q}\,dy\le C\big\|\chi_{B(z,2^{j+1}r)}\|\{f_i\}_{i\in\mathbf{Z}}\|_{l^q}\big\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}$$

for some $C>0$.

Applying the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbf{R}^n)}$ on both sides of (3.3), we have

$$\begin{aligned}&\|\chi_{B(z,r)}\|\{(\mathrm{M}f_i^j)\}_{i\in\mathbf{Z}}\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le C2^{-jn}r^{-n}\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\big\|\chi_{B(z,2^{j+1}r)}\|\{f_i\}_{i\in\mathbf{Z}}\|_{l^q}\big\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

Proposition 2.4 guarantees that

$$\begin{aligned}&\|\chi_{B(z,r)}\|\{(\mathrm{M}f_i^j)\}_{i\in\mathbf{Z}}\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le C\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\big\|\chi_{B(z,2^{j+1}r)}\|\{f_i\}_{i\in\mathbf{Z}}\|_{l^q}\big\|_{L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\chi_{B(z,r)}\|\mathbf{M}f^j\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le C\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\frac{u(z,2^{j+1}r)}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le C\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}u(z,2^{j+1}r)\sup_{\substack{y\in\mathbf{R}^n\\ R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

Hence, using inequality (3.1), we obtain

$$\begin{aligned}\frac{1}{u(z,r)}\|\chi_{B(z,r)}\|\mathbf{M}f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)} &\leq \frac{1}{u(z,r)}\sum_{j=0}^{\infty}\|\chi_{B(z,r)}\|\mathbf{M}f^j\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\ &\leq C\sup_{\substack{y\in\mathbf{R}^n\\ R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbf{R}^n)},\end{aligned}$$

where the constant $C>0$ is independent of r and z. Taking supreme over $z\in\mathbf{R}^n$ and $r>0$ gives (3.2). □

Theorem 3.1 also extends the boundedness results of maximal operator from Lebesgue spaces with variable exponents [9, 10, 13, 47] to Morrey spaces with variable exponent on $\mathbf{R}^n$. For the corresponding result of Morrey spaces with variable exponents on bounded domains, the reader is referred to [2, 35]. Notice that the Morrey spaces with variable exponents defined in [2] are different from [35, Definition 1.2] and Definition 2.2. The Morrey spaces with variable exponents studied in [2] are defined via the modular form

$$\rho(f)=\int_{\{x\in\mathbf{R}^n:p(x)\neq\infty\}}|f(x)|^{p(x)}\,dx+\|f\chi_{\{x\in\mathbf{R}^n:p(x)=\infty\}}\|_{L^\infty}$$

while our Morrey spaces with variable exponents and the one introduced in [35, Definition 1.2] are defined by the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbf{R}^n)}$.

The above theorem also provides the boundedness result for the Hardy–Littlewood operator on the scalar version which gives an generalization on the result for the boundedness of the maximal operator on Morrey spaces with variable exponents [26, Theorem 5.8].

Theorem 3.2. *Let $p(\cdot)\in\mathcal{B}$. If u satisfy (3.1), then*

$$\|\mathrm{M}(f)\|_{\mathcal{M}_{p(\cdot),u}}\leq C\|f\|_{\mathcal{M}_{p(\cdot),u}}. \tag{3.4}$$

If $p(\cdot)\in\bar{\mathcal{B}}'$, $\kappa_{p'(\cdot)}>1$ and there exists a $0\leq\rho<1$ such that for any $x\in\mathbf{R}^n$ and $r>0$,

$$\frac{u(x,2^{j+1}r)}{u(x,r)}\leq C\left(\frac{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^{\rho},\quad \forall j\in\mathbf{N}$$

for some $C>0$, then Proposition 2.6 assures that inequality (3.1) holds. Some further discusses on condition (3.1) are provided in [32]. Moreover, condition (3.1) is also used in [33] for the study of weighted Hardy–Morrey spaces. With the above motivation, we define the weight function for Morrey spaces with variable exponents in the following.

Definition 3.1. Let $0<q\leq\infty$. A Lebesgue measurable function $u(x,r)\colon\mathbf{R}^n\times(0,\infty)\to(0,\infty)$ is said to be a *Morrey weight function* if there exist a $0\leq\lambda<\frac{1}{q}$ and constants $C_1,C_2>0$ so that for any $x\in\mathbf{R}^n$, $u(x,r)>C_1$, $r\geq 1$,

$$\frac{u(x,2r)}{u(x,r)}\leq 2^{n\lambda},\quad r>0, \tag{3.5}$$

$$C_2^{-1}\leq\frac{u(x,t)}{u(x,r)}\leq C_2,\quad 0<r\leq t\leq 2r. \tag{3.6}$$

We denote the class of Morrey weight functions by $\mathcal{W}_q$.

For any $B = B(x,r) \in \mathbf{B}$, we write $u(B) = u(x,r)$.

Lemma 3.3. *Let $p \in \bar{\mathcal{B}}'$ with $e_{p(\cdot)} > 1$. If $u \in \mathcal{W}_{e_{p(\cdot)}}$, then u fulfills (3.1).*

Proof. By the definition of $e_{p(\cdot)}$, for the given $\lambda < \frac{1}{e_{p(\cdot)}}$, we have $\lambda < 1 - \frac{1}{k_{p'(\cdot)}}$. Therefore, there exists a $1 < s < \kappa_{p'(\cdot)}$ such that $\lambda < 1 - \frac{1}{s}$. According to (2.9) and (3.5), we obtain

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \frac{u(x,2^{j+1}r)}{u(x,r)} \le C \sum_{j=0}^{\infty} 2^{nj(\lambda-(1-\frac{1}{s}))} < C$$

for some $C > 0$ independent of x and r. Thus, u fulfills (3.1). □

For instance, the function $u(x,r) = r^{n\lambda(x)}$ where $0 \le \lambda(x) \le \lambda_+ < 1/e_{p(\cdot)}$ fulfills $u \in \mathcal{W}_{e(\cdot)}$ (see [2, Section 3]). We give an extension of the notion of exponent function $p(\cdot)$ when the range of $p(\cdot)$ is a subset of $(0,\infty)$.

Definition 3.2. Let $\tilde{\mathcal{B}}$ be the set of all Lebesgue measurable functions $p\colon \mathbf{R}^n \to (0,\infty)$ satisfying $0 < p_- \le p_+ < \infty$ and $p(\cdot)/a \in \mathcal{B}$ for some $a > 0$. For any $p(\cdot) \in \tilde{\mathcal{B}}$, define

$$m_{p(\cdot)} = \sup\{a > 0\colon p(\cdot)/a \in \mathcal{B}\}, \tag{3.7}$$

$$h_{p(\cdot)} = \sup\{a e_{p(\cdot)/a}\colon a > 0 \text{ and } p(\cdot)/a \in \mathcal{B}\}. \tag{3.8}$$

As $e_{p(\cdot)/a} \ge 1$, we have $m_{p(\cdot)} \le h_{p(\cdot)}$. When $p(\cdot) \in \mathcal{B}$, Theorem 2.3 guarantees that $m_{p(\cdot)} = \kappa_{p(\cdot)} > 1$. The use of the indices $m_{p(\cdot)}$ and $h_{p(\cdot)}$ is revealed by the following properties. For any $0 < q < \infty$, we have

$$h_{\frac{p(\cdot)}{q}} = \frac{1}{q} h_{p(\cdot)} \tag{3.9}$$

and

$$u \in \mathcal{W}_{h_{p(\cdot)}} \iff u^q \in \mathcal{W}_{h_{p(\cdot)}/q}. \tag{3.10}$$

When $p(\cdot) \in \mathcal{B}$, we have

$$e_{p(\cdot)} \le h_{p(\cdot)} \quad \text{and} \quad \mathcal{W}_{h_{p(\cdot)}} \subseteq \mathcal{W}_{e_{p(\cdot)}}. \tag{3.11}$$

Notice that (3.9) is an essential feature possessed by $h_{p(\cdot)}$ that does not share with $e_{p(\cdot)}$. The index $h_{p(\cdot)}$ can be considered as the homogenization of the index $e_{p(\cdot)}$. Especially, (3.9) is also a crucial property in connection with the a-power of $\mathcal{M}_{p(\cdot),u}$, see (2.2). The properties (3.9), (3.10) and (3.11) are applied frequently on the proofs of our main results, Theorems 4.1, 5.1 and 5.2. The following proposition reveals the reason why we use a stronger condition (3.5) instead of (3.1).

Proposition 3.4. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u(x,r) \in \mathcal{W}_{h_{p(\cdot)}}$. Then $\chi_E \in \mathcal{M}_{p(\cdot),u}$ for any bounded Lebesgue measurable set E. Moreover, for any $f \in \mathcal{M}_{p(\cdot),u}$, f is finite almost everywhere.*

Proof. Let E be a bounded Lebesgue measurable set with $E \subseteq B(0,R)$, $R > 0$. For any $x_0 \in \mathbf{R}^n$ and $k \in \mathbf{Z}$, write $D_k = B(x_0, 2^{-k})$. When $k \in \mathbf{N}$, Proposition 2.6, Definitions 3.1 and 3.2 offer a $s > 1$ and $a > 0$ satisfying $s < \kappa_{(p(\cdot)/a)'}$ and $\lambda < \frac{1}{a}(1-\frac{1}{s})$

so that

$$\frac{\|\chi_{D_k\cap E}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{u(x_0,2^{-k})} \le \frac{\|\chi_{D_k}\|^{\frac{1}{a}}_{L^{p(\cdot)/a}(\mathbf{R}^n)}}{u(x_0,2^{-k})} \le C\frac{\|\chi_{D_0}\|^{\frac{1}{a}}_{L^{p(\cdot)/a}(\mathbf{R}^n)}}{u(x_0,1)} 2^{-kn(\frac{1}{a}(1-\frac{1}{s})-\lambda)}$$
$$\le C\|\chi_{B(0,R+2)}\|^{\frac{1}{a}}_{L^{p(\cdot)/a}(\mathbf{R}^n)}$$

for some $C>0$ independent of k and x_0. If $k<0$, then

$$\frac{\|\chi_{D_k\cap E}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{u(x_0,2^{-k})} \le \frac{\|\chi_{E}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{u(x_0,2^{-k})} \le C$$

for some $C>0$ independent of k. As u satisfies (3.6), the above inequalities assure that $\chi_E\in\mathcal{M}_{p(\cdot),u}$.

Let $f\in\mathcal{M}_{p(\cdot),u}$. Assume that $F=\{x\in\mathbf{R}^n\colon |f|=\infty\}$ satisfies $|F|>0$. We find that there exists a $R>0$ such that $G=F\cap B(0,R)$ satisfies $|G|>0$. Hence, G is a bounded Lebesgue measurable set.

For any $n\in\mathbf{N}$, $n\chi_G\le|f|$. Since $\mathcal{M}_{p(\cdot),u}$ is a lattice, we have $n\|\chi_G\|_{\mathcal{M}_{p(\cdot),u}}\le\|f\|_{\mathcal{M}_{p(\cdot),u}}$ which contradicts to the assumption that $f\in\mathcal{M}_{p(\cdot),u}$. Thus, $|F|=0$ and f is finite almost everywhere. □

We have the subsequent result for the boundedness of Hardy–Littlewood maximal operator on the a-th power of $\mathcal{M}_{p(\cdot),u}$.

Proposition 3.5. *Let $p(\cdot)\in\tilde{\mathcal{B}}$ and $u(x,r)\in\mathcal{W}_{h_{p(\cdot)}}$. If $0<b<m_{p(\cdot)}$, then* M *is bounded on $\mathcal{M}_{\frac{p(\cdot)}{b},u^b}$. In particular, $p(\cdot)/b\in\mathcal{B}$.*

Proof. By the definition of $m_{p(\cdot)}$, there exists a $b<\alpha<m_{p(\cdot)}$ such that $p(\cdot)/\alpha\in\mathcal{B}$. Let $\theta=\alpha/b$. For any $f\in\mathcal{M}_{\frac{p(\cdot)}{b},u^b}$, Jensen's inequality reveals that

$$(\mathrm{M}(f))^\theta\le\mathrm{M}(|f|^\theta). \tag{3.12}$$

In view of (2.2), (3.10), (3.11) and Lemma 3.3, Theorem 3.2 guarantees that

$$\|\mathrm{M}(f)\|_{\mathcal{M}_{\frac{p(\cdot)}{b},u^b}} = \|(\mathrm{M}(f))^\theta\|^{\frac{1}{\theta}}_{\mathcal{M}_{\frac{p(\cdot)}{\alpha},u^\alpha}} \le \|\mathrm{M}(|f|^\theta)\|^{\frac{1}{\theta}}_{\mathcal{M}_{\frac{p(\cdot)}{\alpha},u^\alpha}}$$
$$\le C\||f|^\theta\|^{\frac{1}{\theta}}_{\mathcal{M}_{\frac{p(\cdot)}{\alpha},u^\alpha}} = C\|f\|_{\mathcal{M}_{\frac{p(\cdot)}{b},u^b}}$$

for some $C>0$. The belonging $p(\cdot)/b\in\mathcal{B}$ follows by taking $u\equiv1$ in the above inequalities. □

We now offer an application of the Fefferman–Stein vector-valued maximal inequalities to study the variable Triebel–Lizorkin–Morrey spaces and the associate sequence spaces via the Littlewood–Paley function. Let $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$ denote the class of Schwartz functions and tempered distributions, respectively. For any $d\in\mathbf{N}$, let $\mathcal{P}_d$ denote the class of polynomials in $\mathbf{R}^n$ of degree less than or equal to d. Let $\mathcal{P}_\infty=\bigcup_{d\in\mathbf{N}}\mathcal{P}_d$.

Definition 3.3. Let $-\infty<\alpha<\infty$, $0<q<\infty$, $p(\cdot)\in\tilde{\mathcal{B}}$ and $u\in\mathcal{W}_{h_{p(\cdot)}}$. The variable Triebel–Lizorkin–Morrey spaces $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ consists of those $f\in\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}_\infty$ such that

$$\|f\|_{\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}(\varphi)} = \left\|\left(\sum_{\nu\in\mathbf{Z}}2^{\nu\alpha q}|\varphi_\nu*f|^q\right)^{1/q}\right\|_{\mathcal{M}_{p(\cdot),u}}<\infty,$$

where $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\nu \in \mathbf{Z}$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ satisfies

$$\text{supp}\, \hat{\varphi} \subseteq \{x \in \mathbf{R}^n : 1/2 \le |x| \le 2\} \quad \text{and} \quad |\hat{\varphi}(\xi)| \ge C, \quad 3/5 \le |x| \le 5/3 \tag{3.13}$$

for some $C > 0$.

Obviously, $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ is a quasi-Banach space. If $p(x) \equiv p$ is a constant function and $p > 1$, then $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ reduces to the homogeneous version of the Triebel–Lizorkin–Morrey spaces considered in [52, 57, 60]. When $u \equiv 1$, $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ becomes the variable Triebel–Lizorkin spaces studied in [18, 61, 62]. On one hand, the variable Triebel–Lizorkin–Morrey spaces is a generalization of the Triebel–Lizorkin–Morrey spaces introduced in [52, 57, 60]. On the other hand, it gives an extension of the variable Triebel–Lizorkin spaces considered in [18, 61, 62] to the Morrey spaces setting.

Another family of variable Triebel–Lizorkin–Morrey spaces is introduced in [30]. They are complementary of each others. In Definition 3.3, $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ is defined for any $-\infty < \alpha < \infty$ while in [30], it is only defined for $0 < \alpha < \infty$. On the other hand, the variable Triebel–Lizorkin–Morrey spaces studied in [30] can be defined when α and q are functions of $x \in \mathbf{R}^n$.

For any $j \in \mathbf{Z}$ and $k = (k_1, k_2, \dots, k_n) \in \mathbf{Z}^n$, $Q_{j,k} = \{(x_1, x_2 \dots, x_n) \in \mathbf{R}^n : k_i \le 2^j x_i \le k_i + 1, i = 1, 2, \dots, n\}$. Let $|Q|$ and $l(Q)$ be the Lebesgue measure of Q and the side length of Q, respectively. We denote the set of dyadic cubes $\{Q_{j,k} : j \in \mathbf{Z}, k \in \mathbf{Z}^n\}$ by $\mathcal{Q}_d$. By following the idea given in [20], we introduce the sequence space associated with $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$.

Definition 3.4. Let $-\infty < \alpha < \infty$, $0 < q < \infty$, $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. The sequence space $\dot{e}^{\alpha,q}_{p(\cdot),u}$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}_d}$ such that

$$\|s\|_{\dot{e}^{\alpha,q}_{p(\cdot),u}} = \left\| \left(\sum_{Q \in \mathcal{Q}_d} (2^{\nu\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{\mathcal{M}_{p(\cdot),u}} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$.

Identity (2.1) and Theorem 3.1 assure that for any $p \in \tilde{\mathcal{B}}$ and $0 < q < \infty$, there exists an $a > 0$ so that

$$\| \|\{\mathrm{M}(f_i)\}_{i \in \mathbf{Z}}\|_{l^{q/a}} \|_{L^{p(\cdot)/a}(\mathbf{R}^n)} \le C \| \|\{f_i\}_{i \in \mathbf{Z}}\|_{l^{q/a}} \|_{L^{p(\cdot)/a}(\mathbf{R}^n)}$$

for some $C > 0$. Using the terminology given in [27], the pair $(l^q, L^{p(\cdot)}(\mathbf{R}^n))$ is a-admissible. Moreover, the variable Triebel–Lizorkin spaces are member of Littlewood–Paley spaces [27, Definition 2.1].

We follow the general approach provided in [27] and obtain the following results. For simplicity, we refer the reader to [20, 21] for the definitions of the ϕ-ψ transform and smooth N-atom, $N \in \mathbf{N}$.

Theorem 3.6. *The ϕ-transform S_ϕ is a bounded linear operator from $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ to $\dot{e}^{\alpha,q}_{p(\cdot),u}$. The ψ-transform T_ψ is a bounded linear operator from $\dot{e}^{\alpha,q}_{p(\cdot),u}$ to $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$. Moreover, for any φ_1 and φ_2 satisfying the conditions given in Definition 3.3, the quasi-norms $\|\cdot\|_{\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}(\varphi_1)}$ and $\|\cdot\|_{\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}(\varphi_2)}$ are mutually equivalent.*

The above theorem guarantees that the function space $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ is well defined. Furthermore, the definition of $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ is independent of the function φ used in Definition

3.3. The smooth atomic decomposition for $\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$ is obtained as a special case of the smooth atomic decomposition of Littlewood–Paley spaces [28, Theorem 2.1].

Theorem 3.7. *Let $N \in \mathbf{N}$. If $f \in \dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}$, then there exists a sequence $\{s_Q\}_{Q\in\mathcal{Q}_d} \in \dot{e}^{\alpha,q}_{p(\cdot),u}$ and a family of smooth N-atoms $\{a_Q\}_{Q\in\mathcal{Q}_d}$ such that $f = \sum_{Q\in\mathcal{Q}_d} s_Q a_Q$ and $\|\{s_Q\}_{Q\in\mathcal{Q}_d}\|_{\dot{e}^{\alpha,q}_{p(\cdot),u}} \leq C\|f\|_{\dot{\mathcal{E}}^{\alpha,q}_{p(\cdot),u}}$, where C is a positive constant independent of f.*

The preceding theorem extends the smooth atomic decompositions of variable Triebel–Lizorkin spaces obtained in [18, Theorem 3.11] and [62, Theorem 1] to variable Triebel–Lizorkin–Morrey spaces.

4. Hardy–Morrey spaces with variable exponents

The Hardy–Morrey spaces with variable exponents are defined in this section by using the maximal function characterizations. One of the remarkable and fundamental properties of Hardy type spaces are the equivalence of definitions of Hardy type spaces by different maximal functions such as the nontangential maximal function characterizations and the grand maximal function characterizations. The equivalence of these maximal function characterizations are established in this section.

We begin with some well-known notions and notations for studying Hardy type spaces. Recall that a $f \in \mathcal{S}'(\mathbf{R}^n)$ is said to be a bounded tempered distribution if $\varphi * f \in L^\infty(\mathbf{R}^n)$ for any $\varphi \in \mathcal{S}(\mathbf{R}^n)$. For any $N \in \mathbf{N}$, define

$$\mathfrak{N}_N(\phi) = \sup_{x\in\mathbf{R}^n} (1+|x|)^N \sum_{|\gamma|\leq N+1} |\partial^\gamma \phi(x)|, \quad \forall \phi \in \mathcal{S}(\mathbf{R}^n).$$

Write

$$\mathcal{F}_N = \{\phi \in \mathcal{S}(\mathbf{R}^n) \colon \mathfrak{N}_N(\phi) \leq 1\}.$$

For any $t > 0$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$, write $\Phi_t(x) = t^{-n}\Phi(x/t)$.

Let $a, b > 0$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$. For any $f \in \mathcal{S}'(\mathbf{R}^n)$, define

$$M(f,\Phi)(x) = \sup_{t>0} |(\Phi_t * f)(x)|, \tag{4.1}$$

$$M_a^*(f,\Phi)(x) = \sup_{t>0} \sup_{y\in\mathbf{R}^n, |y-x|\leq at} |(\Phi_t * f)(y)|, \tag{4.2}$$

$$M_b^{**}(f,\Phi)(x) = \sup_{t>0} \sup_{y\in\mathbf{R}^n} \frac{|(\Phi_t * f)(x-y)|}{(1+t^{-1}|y|)^b} \tag{4.3}$$

and

$$(\mathcal{M}_N f)(x) = \sup_{\phi\in\mathcal{F}_N} M_1^*(f,\Phi)(x).$$

Let

$$P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$$

be the Poisson kernel.

Definition 4.1. Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. The Hardy–Morrey space with variable exponent $\mathcal{H}_{p(\cdot),u}$ consists of all bounded $f \in \mathcal{S}'(\mathbf{R}^n)$ satisfying

$$\|f\|_{\mathcal{H}_{p(\cdot),u}} = \|M(f,P)\|_{\mathcal{M}_{p(\cdot),u}} < \infty.$$

We prove one of the fundamental theorem for Hardy–Morrey spaces with variable exponents. It is an extension for the corresponding theorem for Hardy spaces [25, Theorem 6.4.4] [55, Chpater III, Section 1], Hardy–Morrey spaces [34, Section 2] and Hardy spaces with variable exponents [46, Section 3].

Theorem 4.1. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$.*

(1) *There exists a $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Phi(x)\,dx \neq 0$ and a constant $C > 0$ such that*

$$\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} \leq C\|f\|_{\mathcal{H}_{p(\cdot),u}} \tag{4.4}$$

for all bounded tempered distribution $f \in \mathcal{S}'(\mathbf{R}^n)$.

(2) *For every $a > 0$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$, there exists a constant $C > 0$ such that*

$$\|M_a^*(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} \leq C\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}}, \quad \forall f \in \mathcal{S}'(\mathbf{R}^n). \tag{4.5}$$

(3) *For every $a > 0$ and $b > n/m_{p(\cdot)}$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$, there exists a constant $C > 0$ such that*

$$\|M_b^{**}(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} \leq C\|M_a^*(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}}, \quad \forall f \in \mathcal{S}'(\mathbf{R}^n). \tag{4.6}$$

(4) *For every $b > 0$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Phi(x)\,dx \neq 0$, there exists a constant $C > 0$ such that if $N = [b] + 1$, then*

$$\|(\mathcal{M}_N f)\|_{\mathcal{M}_{p(\cdot),u}} \leq C\|M_b^{**}(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}}, \quad \forall f \in \mathcal{S}'(\mathbf{R}^n). \tag{4.7}$$

(5) *For every $N \in \mathbf{N}$, there exists a constant $C > 0$ such that if $f \in \mathcal{S}'(\mathbf{R}^n)$ satisfies $\|(\mathcal{M}_N f)\|_{\mathcal{M}_{p(\cdot),u}} < \infty$, then f is a bounded tempered distribution, $f \in \mathcal{H}_{p(\cdot),u}$ and*

$$\|f\|_{\mathcal{H}_{p(\cdot),u}} \leq C\|\mathcal{M}_N(f)\|_{\mathcal{M}_{p(\cdot),u}}. \tag{4.8}$$

Proof. The proof of the above theorem is similar to [25, Theorem 6.4.4] and [55, Chapter III, Theorem 1]. For simplicity, we just outline the inequalities used there and applying them to our result. For a complete and detail account on the proof, the reader is referred to [25, 55].

Part (1). By the proof of [25, Theorem 6.4.4(a)], we have the pointwise inequalities

$$M(f,\Phi)(x) \leq CM(f,P)(x), \quad \forall x \in \mathbf{R}^n. \tag{4.9}$$

Thus, (4.4) follows from (4.9).

Part (2). Similar to the proof [25, Theorem 6.4.4(b)], we present the proof for $M_1^*(f,\Phi)$ only as the general case follows similarly. For any $\epsilon > 0$ and any sufficiently large $N \in \mathbf{N}$, write

$$M_1^*(f,\Phi)^{\epsilon,N}(x) = \sup_{0<t<\frac{1}{\epsilon}} \sup_{|y-x|\leq t} |(\Phi_t * f)(y)| \left(\frac{t}{t+\epsilon}\right)^N \frac{1}{(1+\epsilon|y|)^N}.$$

We show that if $M(f,\Phi)\in\mathcal{M}_{p(\cdot),u}$, then $M_1^*(f,\Phi)^{\epsilon,N}\in\mathcal{M}_{p(\cdot),u}\cap L^\infty(\mathbf{R}^n)$. From [25, p. 45], we have

$$
\begin{aligned}
M_1^*(f,\Phi)^{\epsilon,N}(x) &\le \frac{C}{(1+\epsilon|x|)^{N-m}}\\
&\le C(\chi_{B(0,1)}(x)+\sum_{k=1}^{\infty}2^{-k(N-m)}\chi_{B(0,2^k)\setminus B(0,2^{k-1})}(x))
\end{aligned} \tag{4.10}
$$

where $C>0$ is independent of $x\in\mathbf{R}^n$. From the proof of Proposition 3.4, we find that

$$
\|\chi_{B(0,2^k)\setminus B(0,2^{k-1})}\|_{\mathcal{M}_{p(\cdot),u}}\le C\max(\|\chi_{B(0,2^{k+1})}\|^{\frac{1}{a}}_{L^{p(\cdot)/a}(\mathbf{R}^n)},\|\chi_{B(0,2^k)}\|_{L^{p(\cdot)}(\mathbf{R}^n)})
$$

for some $a>0$ independent of k. Thus, Proposition 2.5 yields

$$
\|\chi_{B(0,2^k)\setminus B(0,2^{k-1})}\|_{\mathcal{M}_{p(\cdot),u}}\le C2^{\frac{n(k+1)}{p_-}} \tag{4.11}
$$

for some $C>0$ independent of k. Therefore, when N is large enough, (4.10) and (4.11) conclude that $M_1^*(f,\Phi)^{\epsilon,N}\in\mathcal{M}_{p(\cdot),u}\cap L^\infty(\mathbf{R}^n)$.

Next, we recall two auxiliary functions used in [25, pp. 45–47]. For any $L>0$, write

$$
U(f,\Phi)^{\epsilon,N}(x)=\sup_{0<t<\frac{1}{\epsilon}}\sup_{|y-x|\le t}t|\nabla(\Phi_t*f)(y)|\left(\frac{t}{t+\epsilon}\right)^N\frac{1}{(1+\epsilon|y|)^N}
$$

and

$$
V(f,\Phi)^{\epsilon,N,L}(x)=\sup_{0<t<\frac{1}{\epsilon}}\sup_{y\in\mathbf{R}^n}|(\Phi_t*f)(y)|\left(\frac{t}{t+\epsilon}\right)^N\frac{1}{(1+\epsilon|y|)^N}\left(\frac{t}{t+|x-y|}\right)^L.
$$

From [25, p. 46], for any $0<q<\infty$, we obtain

$$
|(\Phi_t*f)(y)|\left(\frac{t}{t+\epsilon}\right)^N\frac{1}{(1+\epsilon|y|)^N}\left(\frac{t}{t+|x-y|}\right)^L\le\left(\mathrm{M}[M_1^*(f,\Phi)^{\epsilon,N}]^q\right)^{\frac{1}{q}}(x).
$$

According to Theorem 2.3 and the definition of $\tilde{\mathcal{B}}$, we have a $0<q<\infty$ such that $p(\cdot)/q\in\mathcal{B}$. Additionally, (3.9)–(3.11) assert that

$$
u^q\in\mathcal{W}_{h_{p(\cdot)/q}}=\mathcal{W}_{h_{p(\cdot)/q}}\subseteq\mathcal{W}_{e_{p(\cdot)/q}}. \tag{4.12}
$$

In view of Lemma 3.3, we are allowed to apply Theorem 3.2 to $\mathcal{M}_{\frac{p(\cdot)}{q},u^q}$. Hence, by (2.2), we obtain

$$
\begin{aligned}
\|V(f,\Phi)^{\epsilon,N,L}\|_{\mathcal{M}_{p(\cdot),u}}&\le C\|\mathrm{M}[M_1^*(f,\Phi)^{\epsilon,N}]^q\|^{\frac{1}{q}}_{\mathcal{M}_{\frac{p(\cdot)}{q},u^q}}\\
&\le C\|[M_1^*(f,\Phi)^{\epsilon,N}]^q\|^{\frac{1}{q}}_{\mathcal{M}_{\frac{p(\cdot)}{q},u^q}}\le C\|M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}}
\end{aligned} \tag{4.13}
$$

for some $C>0$ independent of f. Moreover, according to [25, (6.4.23)], we also have the pointwise inequality

$$
U(f,\Phi)^{\epsilon,N}(x)\le CV(f,\Phi)^{\epsilon,N,L}(x),\quad\forall x\in\mathbf{R}^n \tag{4.14}
$$

for some $C>0$ independent of f. Hence, (4.13) and (4.14) yield

$$
\|U(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}}\le C_0\|M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \tag{4.15}
$$

for some $C_0 > 0$ independent of f.

Define
$$E_\epsilon = \{x \in \mathbf{R}^n : U(f,\Phi)^{\epsilon,N}(x) \le 2C_0 M_1^*(f,\Phi)^{\epsilon,N}(x)\},$$
where C_0 is given by (4.15). We obtain
$$\|\chi_{(E_\epsilon)^c} M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \le \frac{1}{2C_0}\|\chi_{(E_\epsilon)^c} U(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \le \frac{1}{2C_0}\|U(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}}.$$
By using (4.15), we find that
$$\|\chi_{(E_\epsilon)^c} M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \le \frac{1}{2}\|M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}}. \tag{4.16}$$
Furthermore, by [25, (6.4.27)], for any $0 < q < \infty$ and $x \in E_\epsilon$, we have
$$M_1^*(f,\Phi)^{\epsilon,N}(x) \le C\big(\mathrm{M}[M(f,\Phi)]^q\big)^{\frac{1}{q}}(x)$$
where $C > 0$ is independent of f and ϵ.

Similar to the proof of (4.13), by applying Theorem 3.2 with sufficiently small $q > 0$ to $\mathcal{M}_{\frac{p(\cdot)}{q},u^q}$, we get
$$\|\chi_{E_\epsilon} M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \le C\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} \tag{4.17}$$
for some $C > 0$ independent of $\epsilon > 0$ and f. Inequalities (4.16) and (4.17) assure that
$$\|M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}} \le C\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} + \frac{1}{2}\|M_1^*(f,\Phi)^{\epsilon,N}\|_{\mathcal{M}_{p(\cdot),u}}.$$
Since $M_1^*(f,\Phi)^{\epsilon,N} \in \mathcal{M}_{p(\cdot),u}$, by applying Lebesgue monotone theorem, we obtain
$$\|M_a^*(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} \le C\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}}$$
for some $C > 0$ depending on N. This guarantees that
$$\|M(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} < \infty \implies \|M_a^*(f,\Phi)\|_{\mathcal{M}_{p(\cdot),u}} < \infty.$$
With this assertion, (4.5) can be established by repeating the above arguments with $U(f,\Phi)^{\epsilon,N}$ and $V(f,\Phi)^{\epsilon,N,L}$ replaced by two new auxiliary functions
$$U(f,\Phi)(x) = \sup_{0<t<\infty} \sup_{|y-x|\le t} t|\nabla(\Phi_t * f)(y)|$$
and
$$V(f,\Phi)^L(x) = \sup_{0<t<\infty} \sup_{y\in\mathbf{R}^n} |(\Phi_t * f)(y)| \left(\frac{t}{t+|x-y|}\right)^L.$$
For brevity, we skip the details. For a detail account of the above procedures, the reader is referred to [25, pp. 44–50] or [55, pp. 95–98].

Part (3). From the proof of [25, Theorem 6.4.4(c)], we have the pointwise inequality
$$M_b^{**}(f,\Phi)(x) \le \max(1, a^{-n})\left(\mathrm{M}(M_a^*(f,\Phi)^{\frac{n}{b}})(x)\right)^{\frac{b}{n}}, \quad \forall x \in \mathbf{R}^n.$$
Therefore, whenever $b > \frac{n}{m_{p(\cdot)}}$, (2.2), (3.7) and Proposition 3.5 give (4.6).

Part (4). The proof of [25, Theorem 6.4.4(d)] asserts that
$$(\mathcal{M}_N f)(x) \le C M_b^{**}(f,\Phi)(x), \quad \forall x \in \mathbf{R}^n$$

for some $C > 0$ independent of f. Obviously, (4.7) follows.

Part (5). Let $f \in \mathcal{S}'(\mathbf{R}^n)$ satisfy $\|\mathcal{M}_N f\|_{\mathcal{M}_{p(\cdot),u}} < \infty$ for some $N \in \mathbf{N}$. For any fixed $\varphi \in \mathcal{S}(\mathbf{R}^n)$, we have a constant $c > 0$ such that $c\varphi \in \mathcal{F}_N$. Therefore, $M_1^*(f, c\varphi) \le \mathcal{M}_N(f)$. Moreover, Proposition 3.4 yields for any $x \in \mathbf{R}^n$

$$c|(\varphi * f)(x)| \le \inf_{|y-x|\le 1} M_1^*(f, c\varphi)(y) \le \frac{\|\chi_{B(x,1)} M_1^*(f, c\varphi)\|_{\mathcal{M}_{p(\cdot),u}}}{\|\chi_{B(x,1)}\|_{\mathcal{M}_{p(\cdot),u}}}$$

$$\le C\|\mathcal{M}_N f\|_{\mathcal{M}_{p(\cdot),u}} < \infty, \tag{4.18}$$

which guarantee that f is a bounded tempered distribution. Finally, by the proof of [25, Theorem 6.4.4(e)], we have

$$\sup_{t>0} |(P_t * f)(x)| \le C(\mathcal{M}_N f)(x), \quad \forall x \in \mathbf{R}^n$$

for some $C > 0$ independent of f and, hence, (4.8) follows. □

The preceding theorem also gives several maximal functions characterizations of Hardy–Morrey spaces with variable exponents such as the Poisson characterizations, the nontangential maximal function characterizations and the grand maximal function characterizations. As shown in Part (5) of Theorem 4.1, for any $\varphi \in \mathcal{S}(\mathbf{R}^n)$, we have

$$|(\varphi * f)(x)| \le C\|\mathcal{M}_N f\|_{\mathcal{M}_{p(\cdot),u}} \le C\|f\|_{\mathcal{H}_{p(\cdot),u}}, \quad \forall x \in \mathbf{R}^n.$$

These inequalities guarantee that whenever f_j converges to f in $\mathcal{H}_{p(\cdot),u}$, f_j also converges to f in $\mathcal{S}'(\mathbf{R}^n)$.

For simplicity, for the rest of the paper, we denote the grand maximal function by $\mathcal{M}$.

5. Atomic decomposition

The main results of this paper are presented in this section. The atomic decomposition for the Hardy–Morrey spaces with variable exponents is established. We start with the definition of cubes and atoms. For any $z = (z_1, \cdots, z_n) \in \mathbf{R}^n$ and $r > 0$, let $Q(z, r) = \{(y_1, \cdots, y_n) \in \mathbf{R}^n \colon \max_{1\le i\le n} |y_i - z_i| \le r/2\}$ denote the cube with center z and side length $l(Q(z, r)) = r$. Write $\mathcal{Q} = \{Q(z, r) \colon z \in \mathbf{R}^n \text{ and } r > 0\}$. For any $Q = Q(z, r) \in \mathcal{Q}$ and $k > 0$, we define $kQ = Q(z, kr)$.

Definition 5.1. Let $p(\cdot) \in \tilde{\mathcal{B}}$, $p_+ < q \le \infty$ and $1 \le q \le \infty$. Let $d \in \mathbf{N}$ satisfy

$$d \ge d_{p(\cdot)} = \min\{k \in \mathbf{N} \colon m_{p(\cdot)}(n + k + 1) > n\}.$$

A function a is a $(p(\cdot), q, d)$ atom if there exists $Q \in \mathcal{Q}$ such that

$$\operatorname{supp} a \subset 3Q, \tag{5.1}$$

$$\|a\|_{L^q} \le \frac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}}, \tag{5.2}$$

$$\int_{\mathbf{R}^n} x^\gamma a(x)dx = 0, \quad \forall \gamma \in \mathbf{N}^n \text{ satisfying } |\gamma| \le d. \tag{5.3}$$

The above definition for $(p(\cdot), q, d)$ atom follows from the definition of atoms for the classical Hardy spaces [55]. The major modification is on the size condition. Precisely, the modification is on the denominator of the right hand side of (5.2), see [46, Definition 1.4]. We have a further modification on the index $d_{p(\cdot)}$. We replace the index p_- from [46, Definition 1.4] by $m_{p(\cdot)}$. From the proof of the subsequent atom decomposition, we see that it is necessary to made such modification.

We present the main result of this paper in the following. We spilt it into two theorems, the first one is the decomposition theorem and the second one is the reconstruction theorem.

Theorem 5.1. *Let $p(\cdot) \in \tilde{\mathcal{B}}$, $p_+ < q \le \infty$, $1 \le q \le \infty$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. For any $f \in \mathcal{H}_{p(\cdot),u}$, there exist a family of $(p(\cdot), q, d)$ atoms $\{a_j\}_{j\in\mathbf{N}}$ with $\operatorname{supp} a_j \subset 3Q_j$, $Q_j \in \mathcal{Q}$ and a sequence of scalars $\{\lambda_j\}_{j\in\mathbf{N}}$ such that*

$$f = \sum_{j\in\mathbf{N}} \lambda_j a_j \quad \text{in} \quad \mathcal{S}'(\mathbf{R}^n) \tag{5.4}$$

$$\left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}} \le C \|f\|_{\mathcal{H}_{p(\cdot),u}}, \quad \forall 0 < s < \infty \tag{5.5}$$

for some $C > 0$ independent of f.

Theorem 5.2. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. There exists a $q_0 > 1$ such that for any family of $(p(\cdot), q, d)$ atoms $\{a_j\}_{j\in\mathbf{N}}$ with $q > q_0$, $\operatorname{supp} a_j \subset Q_j$ and sequence of scalars $\{\lambda_j\}_{j\in\mathbf{N}}$ satisfying*

$$\left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}} < \infty \tag{5.6}$$

for some $0 < s < \min(1, m_{p(\cdot)})$, the series

$$f = \sum_{j\in\mathbf{N}} \lambda_j a_j$$

converges in $\mathcal{S}'(\mathbf{R}^n)$ and $f \in \mathcal{H}_{p(\cdot),u}$ with

$$\|f\|_{\mathcal{H}_{p(\cdot),u}} \le C \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}} \tag{5.7}$$

for some $C > 0$ independent of f.

Notice that Theorem 5.2 does not cover the important boundary case when $s = m_{p(\cdot)}$. To establish the reconstruction theorem for this case, we need to impose a stronger condition on the exponent function.

Theorem 5.3. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$ satisfy $m_{p(\cdot)} \le 1$ and the conditions in Theorem 5.2 with $s = m_{p(\cdot)}$. If $p(\cdot)/m_{p(\cdot)} \in \bar{\mathcal{B}}'$ and $\kappa_{(p(\cdot)/m_{p(\cdot)})'} > 1$, then the conclusions of Theorem 5.2 are valid with $s = m_{p(\cdot)}$.*

Theorems (5.1), (5.2) and (5.3) generalize and unify the atomic decompositions for Hardy spaces, Hardy–Morrey spaces and Hardy space with variable exponents in [34, 46, 55]. The reader may have a wrong impression that it is unnecessary to extend the range of s to $0 < s \le m_{p(\cdot)}$. In fact, the atomic decompositions of the classical Hardy spaces, the Hardy–Morrey spaces and the Hardy spaces with variable exponents are special cases of Theorem 5.3, not Theorem 5.2.

Although Theorem 5.3 induces the atomic decompositions in [34, 46, 55], it only applies to a restrictive class of exponent functions. Theorem 5.2 extends the atomic decomposition of $\mathcal{H}_{p(\cdot),u}$ to a larger class of exponent functions.

When $p(\cdot) = p$, $0 < p \le 1$, is a constant function and $u \equiv 1$, the Hardy–Morrey space with variable exponent reduces to the classical Hardy space. Moreover, $L^{p(\cdot)}(\mathbf{R}^n) = \mathcal{M}_{p(\cdot),u} = L^p$ and $m_{p(\cdot)} = p$. Thus, $p(\cdot)/m_{p(\cdot)} \in \bar{\mathcal{B}}'$ and $\kappa_{(p(\cdot)/m_{p(\cdot)})'} = \infty$. When $s = p$, (5.5) and (5.6) (with respect to Theorem 5.3) offer

$$\Big(\sum_{j\in\mathbf{N}} |\lambda_j|^p\Big)^{\frac{1}{p}} = \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^p}}\right)^p \chi_{Q_j} \right\|_{L^1}^{\frac{1}{p}} = \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^p \chi_{Q_j} \right\|_{\mathcal{M}_{1,u^p}}^{\frac{1}{p}}$$
$$= \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}} < \infty$$

which gives the condition imposed on the sequences for the atomic decomposition of classical Hardy spaces [55, Chapter III, Section 2.2, Theorem 2].

In case $p(\cdot) = q$, $0 < q \le 1$, is a constant function and $u(x,r) = |B(x,r)|^{\frac{1}{p}-\frac{1}{q}}$, $q \le p < \infty$, $\mathcal{H}_{p(\cdot),u}$ becomes the Hardy–Morrey spaces studied in [34]. Additionally, (5.5) gives

$$\sup_{Q\in\mathcal{Q}} \left(|Q|^{1-\frac{q}{p}} \sum_{j\in\mathbf{N}} \frac{|\lambda_j|^q |Q_j \cap Q|}{|Q_j|}\right)^{\frac{1}{q}} \le \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^q \chi_{Q_j} \right\|_{\mathcal{M}_{1,u^q}}^{\frac{1}{q}} \tag{5.8}$$
$$\le \left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}} < C\|f\|_{\mathcal{H}_{p(\cdot),u}}$$

when $0 < s \le q = m_{p(\cdot)}$. Notice that whenever the sequence $\{\lambda_j\}_{j\in\mathbf{N}}$ satisfies (5.8), it also fulfills [34, (1.2)].

The normalization on atoms given in [34, Definition 1.4] is different from us. Our normalization of atoms follows from the classical approach given in [55, Chapter III, Section 2.2]. When $p(\cdot)$ satisfies $0 < p_- \le p_+ < \infty$ and the globally log-Hölder continuous conditions (2.3)-(2.4) and $u \equiv 1$, $\mathcal{H}_{p(\cdot),u}$ is the Hardy space with variable exponents studied in [46]. In addition, $m_{p(\cdot)} = p_-$ and $\kappa_{(p(\cdot)/p_-)'} = (p_+/p_-)' > 1$. Thus, Theorems 5.1 and 5.3 apply.

In fact, even for the special case $u \equiv 1$, Theorems 5.1 and 5.3 extend the atomic decomposition of Hardy space with variable exponents [46, Theorems 4.5 and 4.6] to the case when $p(\cdot) \in \tilde{\mathcal{B}}$ does not satisfy (2.3) and (2.4). For examples of function $p(\cdot) \in \mathcal{B}$ that do not satisfy (2.3) and (2.4), the reader is referred to [38, 47].

To prove Theorem 5.1, we state a well known preliminary supporting result for the atomic decomposition [55, Chapter III, Section 2.1] and [56, Chapter VIII, Lemma 3]. We use the presentation given in [46, Lemma 4.7].

Proposition 5.4. *Let $d \in \mathbf{N}$ and $\sigma > 0$. For any $f \in \mathcal{S}'(\mathbf{R}^n)$, there exist $g \in \mathcal{S}'(\mathbf{R}^n)$, $\{b_k\}_{k\in\mathbf{N}} \subset \mathcal{S}'(\mathbf{R}^n)$, a collection of cubes $\{Q_k\}_{k\in\mathbf{N}} \subseteq \mathcal{Q}$ and a family of smooth functions with compact supports $\{\eta_k\}$ such that*

(1) $f = g + b$ *where* $b = \sum_{k\in\mathbf{N}} b_k$,

(2) *the family* $\{Q_k\}_{k\in\mathbf{N}}$ *has the bounded intersection property and*

$$\bigcup_{k\in\mathbf{N}} Q_k = \{x \in \mathbf{R}^n \colon (\mathcal{M}f)(x) > \sigma\},$$

(3) $\operatorname{supp}\eta_k \subset Q_k$, $0 \le \eta_k \le 1$ *and*

$$\sum_{k\in\mathbf{N}} \eta_k = \chi_{\{x\in\mathbf{R}^n:(\mathcal{M}f)(x)>\sigma\}},$$

(4) *the tempered distribution* g *satisfies*

$$\begin{aligned}(\mathcal{M}g)(x) \le&(\mathcal{M}f)(x)\chi_{\{x\in\mathbf{R}^n:(\mathcal{M}f)(x)\le\sigma\}}(x)\\ &+\sigma\sum_{k\in\mathbf{N}}\frac{l(Q_k)^{n+d+1}}{(l(Q_k)+|x-x_k|)^{n+d+1}},\end{aligned}$$

where x_k *denotes the center of the cube* Q_k,

(5) *the tempered distribution* b_k *is given by* $b_k = (f - c_k)\eta_k$ *where* $c_k \in \mathcal{P}_d$ *satisfying*

$$\int_{\mathbf{R}^n} b_k(x)q(x)\,dx = 0, \quad \forall q \in \mathcal{P}_d,$$

and

$$(\mathcal{M}b_k)(x) \le C(\mathcal{M}f)(x)\chi_{Q_k}(x) + \sigma\frac{l(Q_k)^{n+d+1}}{|x-x_k|^{n+d+1}}\chi_{\mathbf{R}^n\setminus Q_k}(x)$$

for some $C > 0$.

Roughly speaking, the proof of Theorem 5.1 follows the idea of atomic decomposition for the classical Hardy spaces in [55, Chapter III, Section 2].

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. It suffices to establish the atomic decomposition for $(p(\cdot),\infty,d)$ atoms since $(p(\cdot),\infty,d)$ atoms are $(p(\cdot),q,d)$ atoms for any $1 \le q < \infty$. According to Proposition 5.4, for any $\sigma = 2^j$, $j \in \mathbf{Z}$, we have distributions g^j, b^j satisfying (1)–(5) in Proposition 5.4 and $f = g^j + b^j$. Write

$$O^j = \{x \in \mathbf{R}^n \colon (\mathcal{M}f)(x) > 2^j\} = \bigcup_{k\in\mathbf{N}} Q_k^j \tag{5.9}$$

where $\bigcup_{k\in\mathbf{N}} Q_k^j$ is the decomposition of O^j given by Item (2) of Proposition 5.4.

Let $\{\eta_k^j\}$ be the family of smooth functions with respect to the decomposition $O^j = \bigcup_{k\in\mathbf{N}} Q_k^j$ given by Proposition 5.4, Item (3). In addition, as $\mathcal{M}f \in \mathcal{M}_{p(\cdot),u}$, according to Proposition 3.4, we have $O^{j+1} \subset O^j$ and $\bigcap_{j\in\mathbf{N}} O^j = \emptyset$. For any $\varphi \in \mathcal{S}(\mathbf{R}^n)$, we have a constant $c > 0$ such that $c\varphi \in \mathcal{F}_N$. Proposition 5.4 yields

$$\begin{aligned}&c|\varphi * g^j(x)| \le (\mathcal{M}g^j)(x)\\ &\le (\mathcal{M}f)(x)\chi_{\{x\in\mathbf{R}^n:(\mathcal{M}f)(x)\le 2^j\}}(x) + 2^j\sum_{k\in\mathbf{N}}\frac{l(Q_k^j)^{n+d+1}}{(l(Q_k^j)+|x-x_k^j|)^{n+d+1}} \le C2^j\end{aligned}$$

for some $C > 0$ where x_k^j is the center of Q_k^j. That is, $g^j \to 0$ in $\mathcal{S}'(\mathbf{R}^n)$ as $j \to -\infty$.

Next, we show that $b^j \to 0$ in $\mathcal{S}'(\mathbf{R}^n)$ as $j \to \infty$. By Propositions 3.5 and Item (5) of Proposition 5.4, for any $n/(n + d_{p(\cdot)} + 1) < r < \min(1, m_{p(\cdot)})$ and $Q \in \mathcal{Q}$, we have $p(\cdot)/r \in \mathcal{B}$ and

$$\begin{aligned}
&\int_Q |(\mathcal{M}b^j)(x)|^r\,dx \\
&\le C\int_Q \sum_{k\in\mathbf{N}} |(\mathcal{M}f)(x)|^r \chi_{Q_k^j}(x)\,dx + C2^{jr}\int_Q \sum_{k\in\mathbf{N}} \left(\frac{l(Q_k^j)^{n+d_{p(\cdot)}+1}\chi_{\mathbf{R}^n\setminus Q_k^j}(x)}{(l(Q_k^j)+|x-x_k^j|)^{n+d_{p(\cdot)}+1}}\right)^r dx \\
&\le C\int_{Q\cap O^j} |(\mathcal{M}f)(x)|^r\,dx + C2^{jr}\sum_{k\in\mathbf{N}} \int_{\mathbf{R}^n} \chi_Q(x)((\mathrm{M}\,\chi_{Q_k^j})(x))^{r(n+d_{p(\cdot)}+1)/n}\,dx.
\end{aligned}$$

By using [23, Chapter II, Theorem 2.12], we obtain

$$\begin{aligned}
\int_{\mathbf{R}^n} ((\mathrm{M}\,\chi_{Q_k^j})(x))^{r(n+d_{p(\cdot)}+1)/n}\chi_Q(x)\,dx &\le \int_{\mathbf{R}^n} (\chi_{Q_k^j}(x))^{r(n+d_{p(\cdot)}+1)/n}(\mathrm{M}\,\chi_Q)(x)\,dx \\
&= \int_{\mathbf{R}^n} \chi_{Q_k^j}(x)(\mathrm{M}\,\chi_Q)(x)\,dx \\
&= \int_{Q_k^j} (\mathrm{M}\,\chi_Q)(x)\,dx,
\end{aligned}$$

because $r(n + d_{p(\cdot)} + 1)/n > 1$. Consequently, the above inequalities, (5.9) and the bounded intersection property satisfied by $\{Q_k^j\}_{k\in\mathbf{N}}$ yield

$$\int_Q |(\mathcal{M}b^j)(x)|^r\,dx \le C\int_{O^j} |(\mathcal{M}f)(x)|^r(\mathrm{M}\,\chi_Q)(x)\,dx$$

for some $C > 0$. For any $\varphi \in \mathcal{S}(\mathbf{R}^n)$, by the above inequalities and (4.18), we find that

$$\begin{aligned}
|b^j * \varphi(x)|^r &\le C\frac{1}{|Q(x,1)|}\int_{Q(x,1)} |M_1^*(b^j,\varphi)(y)|^r\,dy \le C\int_{Q(x,1)} |(\mathcal{M}b^j)(y)|^r\,dy \\
&\le C\int_{O^j} |(\mathcal{M}f)(y)|^r(\mathrm{M}\,\chi_{Q(x,1)})(y)\,dy \\
&\le C\int_{O^j} |(\mathcal{M}f)(y)|^r(1+|x-y|)^{-n}\,dy
\end{aligned}$$

for some $C > 0$.

As $p(\cdot)/r \in \mathcal{B}$, by using the Hölder inequality for the pair $L^{p(\cdot)/r}(\mathbf{R}^n)$ and $L^{(p(\cdot)/r)'}(\mathbf{R}^n)$ and Proposition 2.4, we find that

$$\begin{aligned}
&\int_{\mathbf{R}^n} |(\mathcal{M}f)(y)|^r(1+|x-y|)^{-n}\,dy \le C\sum_{k=0}^{\infty} 2^{-kn}\int_{\mathbf{R}^n} |(\mathcal{M}f)(y)|^r\chi_{B^k}(y)\,dy \\
&\le C\sum_{k=0}^{\infty} \frac{1}{|B(x,2^k)|}\|(\mathcal{M}f)(y)^r\|_{L^{p(\cdot)/r}(\mathbf{R}^n)}\|\chi_{B(x,2^k)}\|_{L^{(p(\cdot)/r)'}(\mathbf{R}^n)} \\
&\le C\sum_{k=0}^{\infty} \frac{u(x,2^k)^r}{\|\chi_{B(x,2^k)}\|_{L^{p(\cdot)/r}(\mathbf{R}^n)}}\|\mathcal{M}f\|^r_{\mathcal{M}_{p(\cdot),u}}
\end{aligned}$$

where $B^k = B(x, 2^k)\backslash B(x, 2^{k-1})$ when $k \ge 1$ and $B^0 = B(x, 1)$.

Since $u^r \in \mathcal{W}_{h_{p(\cdot)/r}}$, Lemma 3.3 gives

$$\int_{\mathbf{R}^n} |(\mathcal{M}f)(y)|^r (1+|x-y|)^{-n}\, dy \le C\|\mathcal{M}f\|^r_{\mathcal{M}_{p(\cdot),u}}.$$

In view of the fact that $O^j \downarrow \emptyset$, the dominated convergence theorem yields that

$$\lim_{j\to\infty} |b^j * \varphi(x)|^r \le C \lim_{j\to\infty} \int_{O^j} |(\mathcal{M}f)(y)|^r (1+|x-y|)^{-n}\, dy = 0.$$

Thus, $b^j * \varphi \to 0$ pointwisely. That is, $b^j \to 0$ in $\mathcal{S}'(\mathbf{R}^n)$ when $j \to \infty$.

The convergence of g^j and b^j assert that the telescoping sum

$$f = \sum_{j\in\mathbf{Z}} (g^{j+1} - g^j)$$

converges in $\mathcal{S}'(\mathbf{R}^n)$. By using Item (5) of Proposition 5.4, we also have

$$g^{j+1} - g^j = b^{j+1} - b^j = \sum_{k\in\mathbf{N}} ((f - c_k^{j+1})\eta_k^{j+1} - (f - c_k^j)\eta_k^j)$$

where $c_k^j \in \mathcal{P}_d$ satisfies

$$\int_{\mathbf{R}^n} (f(x) - c_k^j(x)) q(x) \eta_k^j(x)\, dx = 0, \quad \forall q \in \mathcal{P}_d.$$

Moreover, we have $f = \sum_{j,k} A_k^j$, where

$$A_k^j = (f - c_k^j)\eta_k^j - \sum_{l\in\mathbf{N}} (f - c_l^{j+1})\eta_l^{j+1}\eta_k^j + \sum_{l\in\mathbf{N}} c_{k,l}\eta_l^{j+1}$$

and $c_{k,l} \in \mathcal{P}_d$ satisfies

$$\int_{\mathbf{R}^n} ((f(x) - c_l^{j+1}(x))\eta_k^j(x) - c_{k,l}(x)) q(x) \eta_l^{j+1}(x)\, dx = 0, \quad \forall q \in \mathcal{P}_d.$$

Write

$$a_k^j = \lambda_{j,k}^{-1} A_k^j \quad \text{and} \quad \lambda_{j,k} = c2^j \|\chi_{Q_k^j}\|_{L^{p(\cdot)}(\mathbf{R}^n)},$$

where c is a constant determined by the family $\{A_k^j\}_{j,k}$ and most importantly, it is independent of j and k, see [55, pp. 108–109]. Therefore, similar to the proof for the classical Hardy space [55, Chapter III, Section 2], a_k^j is a $(p(\cdot), \infty, d)$ atom.

According to the definition of Q_k^j and following from the fact that the family $\{Q_k^j\}_{k\in\mathbf{N}}$ has the finite intersection property, we find that for any $0 < s < \infty$

$$\sum_{k\in\mathbf{N}} \left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_k^j}(x) \le C2^{sj}\chi_{O^j}(x)$$

for some $C > 0$. Consequently,

$$\sum_{j,k} \left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_k^j}(x) \le C \sum_{j\in\mathbf{Z}} 2^{sj}\chi_{O^j}(x) \le C(\mathcal{M}f)(x)^s.$$

By applying the quasi-norm $\|\cdot\|^{1/s}_{\mathcal{M}_{p(\cdot)/s,u^s}}$ on both sides of the above inequality, (2.2) yields

$$\Bigg\| \sum_{j,k} \left(\frac{|\lambda_{j,k}|}{\|\chi_{Q_k^j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_k^j} \Bigg\|^{\frac{1}{s}}_{\mathcal{M}_{p(\cdot)/s,u^s}} \le C\|f\|_{\mathcal{H}_{p(\cdot),u}}, \quad 0 < s < \infty$$

for some $C > 0$ independent of f. □

To prove Theorem 5.2, we need several supporting results. We use a duality result for $\mathcal{M}_{p(\cdot),u}$ to provide some crucial estimates. Therefore, we introduce the block space for variable exponents in the following.

We now present the definition of block spaces with variable exponents given in [7]. It is inspired by the classical block spaces defined in [5].

Definition 5.2. Let $p(\cdot)\colon \mathbf{R}^n \to [1,\infty]$ and $u(x,r)\colon \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. A Lebesgue measurable function b is a $(u,p(\cdot))$-block if $\operatorname{supp} b \subseteq B(x_0,r)$, $x_0 \in \mathbf{R}^n$, $r > 0$, and

$$\|b\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le \frac{1}{u(x_0,r)}. \tag{5.10}$$

We write $b \in \mathfrak{b}_{p(\cdot),u}$ if b is a $(u,p(\cdot))$-block. Define $\mathfrak{B}_{p(\cdot),u}$ by

$$\mathfrak{B}_{p(\cdot),u} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k \colon \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is an } (u,p(\cdot))\text{-block} \right\}. \tag{5.11}$$

The space $\mathfrak{B}_{p(\cdot),u}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}_{p(\cdot),u}} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \right\}. \tag{5.12}$$

We call $\mathfrak{B}_{p(\cdot),u}$ the block space with variable exponent.

A simple consequence of the above definition is that for any $b \in \mathfrak{b}_{p(\cdot),u}$, we have

$$\|b\|_{\mathfrak{B}_{p(\cdot),u}} \le 1. \tag{5.13}$$

We first present the Hölder inequality for $\mathcal{M}_{p(\cdot),u}$ and $\mathfrak{B}_{p'(\cdot),u}$.

Lemma 5.5. *Let $p(\cdot)\colon \mathbf{R}^n \to [1,\infty]$ and $u(x,r)\colon \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. We have*

$$\int |f(x)g(x)|\,dx \le C\|f\|_{\mathcal{M}_{p(\cdot),u}} \|g\|_{\mathfrak{B}_{p'(\cdot),u}}$$

for some $C > 0$ independent of $f \in \mathcal{M}_{p(\cdot),u}$ and $g \in \mathfrak{B}_{p'(\cdot),u}$.

Proof. For any $b \in \mathfrak{b}_{p'(\cdot),u}$ with $\operatorname{supp} b \subseteq B$, by using the Hölder inequality for $L^{p(\cdot)}(\mathbf{R}^n)$ [37, Theorem 2.1] and (5.10), we have

$$\begin{aligned}\int_{\mathbf{R}^n} |f(x)b(x)|\,dx &\le C\|\chi_B f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|b\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \\ &\le C\frac{1}{u(B)}\|\chi_B f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C_1\|f\|_{\mathcal{M}_{p(\cdot),u}}.\end{aligned} \tag{5.14}$$

For any $g \in \mathfrak{B}_{p'(\cdot),u}$, we have a family of $(u, p'(\cdot))$-blocks $\{b_k\}_{k\in\mathbf{N}}$ and sequence $\{\lambda_k\}_{k\in\mathbf{N}}$ such that $f = \sum_{k\in\mathbf{N}} \lambda_k b_k$ and

$$\sum_{k=1}^{\infty} |\lambda_k| < 2\|g\|_{\mathfrak{B}_{p'(\cdot),u}}. \tag{5.15}$$

Therefore, (5.14) and (5.15) give

$$\int |f(x)g(x)|\,dx \le \sum_{k\in\mathbf{N}} |\lambda_k| \int_{\mathbf{R}^n} |f(x)b_k(x)|\,dx \le C\|f\|_{\mathcal{M}_{p(\cdot),u}} \|g\|_{\mathfrak{B}_{p'(\cdot),u}}. \qquad \square$$

The following is the norm conjugate formula for $\mathcal{M}_{p(\cdot),u}$ and $\mathfrak{B}_{p(\cdot),u}$.

Lemma 5.6. *Let $p(\cdot)\colon \mathbf{R}^n \to [1,\infty]$ and $u(x,r)\colon \mathbf{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. We have constants $C_0, C_1 > 0$ such that*

$$C_0\|f\|_{\mathcal{M}_{p(\cdot),u}} \le \sup_{b\in\mathfrak{b}_{p'(\cdot),u}} \int_{\mathbf{R}^n} |f(x)b(x)|\,dx \le C_1\|f\|_{\mathcal{M}_{p(\cdot),u}}. \tag{5.16}$$

Proof. The inequality on the right hand side of (5.16) follows from (5.14). Next, we show the inequality on the left hand side of (5.16). According to the definition of $\mathcal{M}_{p(\cdot),u}$, there exists a $B \in \mathbf{B}$ such that

$$\frac{1}{2}\|f\|_{\mathcal{M}_{p(\cdot),u}} < \frac{1}{u(B)}\|\chi_B f\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

For this given $B \in \mathbf{B}$, the norm conjugate formula for $L^{p(\cdot)}(\mathbf{R}^n)$ (see [12, Propositions 2.34 and 2.37] and [17, Corollary 3.2.14]) yields a $g \in L^{p'(\cdot)}(\mathbf{R}^n)$ with $\|g\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \le 1$ such that

$$\frac{1}{2}\|f\|_{\mathcal{M}_{p(\cdot),u}} < \frac{1}{u(B)}\int_B |f(x)g(x)|\,dx = \int_{\mathbf{R}^n} |f(x)G(x)|\,dx,$$

where

$$G(x) = \frac{1}{u(B)}\chi_B(x)g(x).$$

Obviously, G is a $(u, p'(\cdot))$-block. Therefore, the inequality on the left hand side of (5.16) follows. $\square$

The subsequent lemma gives an estimate of the action of the Hardy–Littlewood operator on blocks. In addition, the q_0 appeared in Theorem 5.2 is determined by the following lemma and proposition.

Lemma 5.7. *Let $p(\cdot) \in \overline{\mathcal{B}}'$ and $u \in \mathcal{W}_{e_{p(\cdot)}}$. If $\kappa_{p'(\cdot)} > 1$, then there exists a $q_0 > 1$ such that for any $q > q_0$ and $b \in \mathfrak{b}_{p'(\cdot),u}$, we have*

$$\|(\mathrm{M}(|b|^{q'}))^{\frac{1}{q'}}\|_{\mathfrak{B}_{p'(\cdot),u}} \le C \tag{5.17}$$

for some $C > 0$ independent of b.

Proof. Let $b \in \mathfrak{b}_{p'(\cdot),u}$ with support $B(x_0, r)$, $x_0 \in \mathbf{R}^n$, $r > 0$. For any $k \in \mathbf{N}$, let $B_k = B(x_0, 2^k r)$. Write $m_k = \chi_{B_{k+1}\setminus B_k}(\mathrm{M}(|b|^{q'}))^{\frac{1}{q'}}$, $k \in \mathbf{N}\setminus\{0\}$ and $m_0 = \chi_{B_0}(\mathrm{M}(|b|^{q'}))^{\frac{1}{q'}}$. We have $\operatorname{supp} m_k \subseteq B_{k+1}\setminus B_k$ and

$$(\mathrm{M}(|b|^{q'}))^{\frac{1}{q'}} = \sum_{k=0}^{\infty} m_k.$$

Since $\kappa_{p'(\cdot)} > 1$, by using Jensen's inequality and (3.12), we have a $q_1 > 1$ such that when $q > q_1$, $p'(\cdot)/q'$ belongs to $\overline{\mathcal{B}}$. Therefore, by the boundedness of the Hardy–Littlewood maximal operator on $L^{p'(\cdot)/q'}(\mathbf{R}^n)$, we find that

$$\|m_0\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \leq C\| \mathrm{M}(|b|^{q'})\|^{\frac{1}{q'}}_{L^{p'(\cdot)/q'}(\mathbf{R}^n)} \leq C\|b\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \leq \frac{C}{u(x_0,r)}$$

for some $C > 0$ independent of x_0 and r. That is, m_0 is a constant-multiple of an $(u, p'(\cdot))$-block. The definition of Hardy–Littlewood maximal operator and the Hölder inequality for $L^{p(\cdot)}(\mathbf{R}^n)$ (see [37, Theroem 2.1]) assert that

$$\begin{aligned} |m_k|^{q'} &= \chi_{B_{k+1}\backslash B_k} |\,\mathrm{M}(|b|^{q'})| \leq \frac{\chi_{B_{k+1}\backslash B_k}}{2^{kn}r^n} \int_{B(x_0,r)} |b(x)|^{q'}\,dx \\ &\leq C\chi_{B_{k+1}\backslash B_k}\frac{1}{2^{kn}r^n}\||b|^{q'}\|_{L^{p'(\cdot)/q'}(\mathbf{R}^n)}\|\chi_{B(x_0,r)}\|_{L^{(p'(\cdot)/q')'}(\mathbf{R}^n)} \end{aligned} \tag{5.18}$$

for some $C > 0$ independent of k. Proposition 2.4 and (5.18) ensure that

$$\begin{aligned} \|m_k\|_{L^{p'(\cdot)}(\mathbf{R}^n)} &= \||m_k|^{q'}\|^{\frac{1}{q'}}_{L^{p'(\cdot)/q'}(\mathbf{R}^n)} \\ &\leq \left(\frac{\|\chi_{B_{k+1}\backslash B_k}\|_{L^{p'(\cdot)/q'}(\mathbf{R}^n)}}{2^{kn}r^n}\frac{r^n}{\|\chi_{B(x_0,r)}\|_{L^{p'(\cdot)/q'}(\mathbf{R}^n)}}\right)^{\frac{1}{q'}}\|b\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \\ &\leq C\frac{\|\chi_{B_{k+1}}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}}{2^{\frac{kn}{q'}}\|\chi_{B(x_0,r)}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}}\frac{u(x_0,2^{k+1}r)}{u(x_0,r)}\frac{1}{u(x_0,2^{k+1}r)}. \end{aligned}$$

Define $m_k = \sigma_k b_k$, where

$$\sigma_k = \frac{\|\chi_{B_{k+1}}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}}{2^{\frac{kn}{q'}}\|\chi_{B(x_0,r)}\|_{L^{p'(\cdot)}(\mathbf{R}^n)}}\frac{u(x_0,2^{k+1}r)}{u(x_0,r)}.$$

Consequently, b_k is a constant-multiple of an $(u, p'(\cdot))$-block and this constant does not depend on k. Hence, (5.13) yields

$$\|b_k\|_{\mathfrak{B}_{p'(\cdot),u}} \leq C \tag{5.19}$$

where C is independent of k.

Since $\kappa_{p'(\cdot)} > 1$ and $u \in \mathcal{W}_{e_{p(\cdot)}}$, Proposition 2.6 and Definition (3.1) yield, for any $1 < \beta < \kappa_{p'(\cdot)}$

$$\sum_{k=0}^{\infty}\sigma_k \leq C\sum_{k=0}^{\infty} 2^{\frac{kn}{\beta}+kn\lambda-\frac{kn}{q'}}$$

for some $C > 0$. As $\frac{1}{\kappa_{p'(\cdot)}} + \frac{1}{e_{p(\cdot)}} = 1$, for any fixed $\lambda < \frac{1}{e_{p(\cdot)}}$, we can choose a $\beta < \kappa_{p'(\cdot)}$ and $q_2 > 1$ such that for any $q > q_2$, we have $\frac{1}{\beta} + \lambda < \frac{1}{q'}$. Hence,

$$\sum_{k=0}^{\infty}\sigma_k \leq C\sum_{k=0}^{\infty} 2^{\frac{kn}{\beta}+kn\lambda-\frac{kn}{q'}} < C. \tag{5.20}$$

Since $\mathfrak{B}_{p'(\cdot),u}$ is a Banach lattice [7, Proposition 2.2], we find that

$$\|(\mathrm{M}(|b|^{q'}))^{\frac{1}{q'}}\|_{\mathfrak{B}_{p'(\cdot),u}} \leq \sum_{k=0}^{\infty}|\sigma_k|\|b_k\|_{\mathfrak{B}_{p'(\cdot),u}}.$$

Thus, (5.19) and (5.20) yield (5.17) with $q_0 = \max(q_1, q_1)$. □

In fact, the Hardy–Littlewood maximal operator is also bounded on $\mathfrak{B}_{p(\cdot),u}$. The reader is referred to [7, Theorem 3.2] for the details.

We apply the preceding results to generalize the inequalities shown in [56, Chapter VIII, Lemma 5] and [46, Lemma 4.11] to Morrey spaces with variable exponents in the following proposition.

Proposition 5.8. *Let $p(\cdot) \in \overline{\mathcal{B}}'$ and $u \in \mathcal{W}_{e_{p(\cdot)}}$. Let $\{\lambda_k\}_{k\in\mathbf{N}}$ be a sequence of scalars. If $\kappa_{p'(\cdot)} > 1$, then there exists a $q_0 > 1$ such that for any $q > q_0$ and $\{b_k\}_{k\in\mathbf{N}} \subset L^q$ with $\operatorname{supp} b_k \subseteq Q_k \in \mathcal{Q}$ and*

$$\|b_k\|_{L^q} \le \frac{|Q_k|^{\frac{1}{q}}}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}, \tag{5.21}$$

we have

$$\Big\|\sum_{k\in\mathbf{N}} \lambda_k b_k\Big\|_{\mathcal{M}_{p(\cdot),u}} \le C\Big\|\sum_{k\in\mathbf{N}} \frac{|\lambda_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{Q_k}\Big\|_{\mathcal{M}_{p(\cdot),u}} \tag{5.22}$$

for some $C > 0$ independent of $\{b_k\}_{k\in\mathbf{N}}$ and $\{\lambda_k\}_{k\in\mathbf{N}}$.

Proof. For any $g \in \mathfrak{b}_{p'(\cdot),u}$, we have

$$\left|\int_{\mathbf{R}^n} b_k(x)g(x)\,dx\right| \le \|b_k\|_{L^q}\|\chi_{Q_k}g\|_{L^{q'}} \le \frac{|Q_k|^{\frac{1}{q}}}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\left(\int_{Q_k}|g(x)|^{q'}\,dx\right)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Moreover,

$$\begin{aligned}\left|\int_{\mathbf{R}^n} b_k(x)g(x)\,dx\right| &\le \frac{|Q_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\left(\frac{1}{|Q_k|}\int_{Q_k}|g(x)|^{q'}\,dx\right)^{\frac{1}{q'}}\\ &\le C\frac{|Q_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\inf_{x\in Q_k}(\mathrm{M}(|g|^{q'})(x))^{\frac{1}{q'}}\\ &\le C\frac{1}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\int_{Q_k}(\mathrm{M}(|g|^{q'})(x))^{\frac{1}{q'}}\,dx\end{aligned}$$

for some $C > 0$.

The above inequalities yield

$$\begin{aligned}\left|\int_{\mathbf{R}^n}\Big(\sum_{k\in\mathbf{N}}\lambda_k b_k(x)\Big)g(x)\,dx\right| &\le C\sum_{k\in\mathbf{N}}\frac{|\lambda_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\int_{Q_k}(\mathrm{M}(|g|^{q'})(x))^{\frac{1}{q'}}\,dx\\ &\le C\int_{\mathbf{R}^n}\Big(\sum_{k\in\mathbf{N}}\frac{|\lambda_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{Q_k}(x)\Big)(\mathrm{M}(|g|^{q'})(x))^{\frac{1}{q'}}\,dx\\ &\le \Big\|\sum_{k\in\mathbf{N}}\frac{|\lambda_k|}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{Q_k}\Big\|_{\mathcal{M}_{p(\cdot),u}}\|(\mathrm{M}(|g|^{q'}))^{\frac{1}{q'}}\|_{\mathfrak{B}_{p'(\cdot),u}}\end{aligned}$$

where we use Lemma 5.5 for the last inequality. Therefore, Lemmas 5.6 and 5.7 yield (5.22). □

The above proposition plays a decisive role for estimating the sequence used to establish the atomic decomposition of $\mathcal{H}_{p(\cdot),u}$. With the above preparations, we now offer the proof for Theorem 5.2 in the following.

Proof of Theorem 5.2. Let $\{a_j\}_{j\in\mathbf{N}}$ be a family of $(p(\cdot),q,d)$ atoms with $\operatorname{supp} a_j \subseteq 3Q_j$ and $\{\lambda_j\}_{j\in\mathbf{N}}$ be a sequence of scalars satisfying (5.6). For any $\varphi\in\mathcal{S}(\mathbf{R}^n)$, by using [46, (4.21) and (4.22)], we have

$$M(a_j,\varphi)(x)\le C\left(\chi_{\tilde{Q}_j}(x)(\mathrm{M}\,a_j)(x)+\frac{(\mathrm{M}\,\chi_{Q_j})(x)^r}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right),$$

where $\tilde{Q}=2\sqrt{n}Q$ and $r=(n+d+1)/n$. Write $f=\sum_{j\in\mathbf{N}}\lambda_j a_j$, we find that

$$\begin{aligned}&\|M(f,\varphi)\|_{\mathcal{M}_{p(\cdot),u}}\\ &\le C\left(\Big\|\sum_{j\in\mathbf{N}}|\lambda_j|\chi_{\tilde{Q}_j}(\mathrm{M}\,a_j)\Big\|_{\mathcal{M}_{p(\cdot),u}}+\Big\|\sum_{j\in\mathbf{N}}|\lambda_j|(\mathrm{M}\,\chi_{Q_j})^r\|\chi_{Q_j}\|^{-1}_{L^{p(\cdot)}(\mathbf{R}^n)}\Big\|_{\mathcal{M}_{p(\cdot),u}}\right)=I+II.\end{aligned}$$

Since $0<s<1$, the s-inequality guarantees that

$$I\le\left\|\Big(\sum_{j\in\mathbf{N}}(|\lambda_j|\chi_{\tilde{Q}_j}(\mathrm{M}\,a_j))^s\Big)^{\frac{1}{s}}\right\|_{\mathcal{M}_{p(\cdot),u}}=\left\|\sum_{j\in\mathbf{N}}(|\lambda_j|\chi_{\tilde{Q}_j}(\mathrm{M}\,a_j))^s\right\|^{\frac{1}{s}}_{\mathcal{M}_{p(\cdot)/s,u^s}}.$$

Next, we apply Proposition 5.8 on $\mathcal{M}_{p(\cdot)/s,u^s}$ with $b_j=(\chi_{\tilde{Q}_j}(\mathrm{M}\,a_j))^s$ to estimate I. Therefore, we first verify the conditions given in Proposition 5.8.

Since $s<m_{p(\cdot)}$, Proposition 3.5 guarantees that $p(\cdot)/s\in\mathcal{B}$. Theorem 2.3 yields $(p(\cdot)/s)'\in\mathcal{B}$. Hence, (2.5) asserts that $\kappa_{p(\cdot)/s},\kappa_{(p(\cdot)/s)'}>1$.

As $s<1<q$, the boundedness of the Hardy–Littlewood maximal operator on $L^{q/s}$ asserts that

$$\|(\mathrm{M}\,a_j)^s\|_{L^{q/s}}\le C\|a_j\|^s_{L^q}\le C\frac{|Q_j|^{\frac{1}{q/s}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)/s}(\mathbf{R}^n)}}\le C\frac{|\tilde{Q}_j|^{\frac{1}{q/s}}}{\|\chi_{\tilde{Q}_j}\|_{L^{p(\cdot)/s}(\mathbf{R}^n)}}\tag{5.23}$$

for some $C>0$ independent of a_j where we use (2.8) for the last inequality. Moreover, (3.9)–(3.11) guarantee that

$$u^s\in\mathcal{W}_{h_{p(\cdot)}/s}=\mathcal{W}_{h_{p(\cdot)/s}}\subseteq\mathcal{W}_{e_{p(\cdot)/s}}.\tag{5.24}$$

Therefore, in view of (5.23) and (5.24), we are allowed to apply Proposition 5.8 on $\mathcal{M}_{p(\cdot)/s,u^s}$ with $b_j=(\chi_{\tilde{Q}_j}(\mathrm{M}\,a_j))^s$ and it offers a q_0 such that whenever $q>sq_0$, we have

$$I\le C\left\|\sum_{j\in\mathbf{N}}\frac{|\lambda_j|^s}{\|\chi_{\tilde{Q}_j}\|_{L^{p(\cdot)/s}(\mathbf{R}^n)}}\chi_{\tilde{Q}_j}\right\|^{\frac{1}{s}}_{\mathcal{M}_{p(\cdot)/s,u^s}}.$$

Let $\beta=s/2$. Since $\chi_{Q_j}\le\chi_{\tilde{Q}_j}\le C(\mathrm{M}\,\chi_{Q_j})^2$ for some $C>0$ independent of j, we infer that

$$I\le C\left\|\left(\sum_{j\in\mathbf{N}}\left(\frac{|\lambda_j|^\beta}{\|\chi_{Q_j}\|_{L^{p(\cdot)/\beta}(\mathbf{R}^n)}}(\mathrm{M}\,\chi_{Q_j})\right)^2\right)^{\frac{1}{2}}\right\|^{\frac{2}{s}}_{\mathcal{M}_{p(\cdot)/\beta,u^\beta}}.$$

As Proposition 3.5 asserts that $p(\cdot)/\beta\in\mathcal{B}$, by using (3.10) and (3.11), we have

$$u^\beta\in\mathcal{W}_{h_{p(\cdot)}/\beta}=\mathcal{W}_{h_{p(\cdot)/\beta}}\subseteq\mathcal{W}_{e_{p(\cdot)/\beta}}.$$

Theorem 3.1 and Lemma 3.3 ensure that

$$I \le C\left\| \left(\sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|^\beta}{\|\chi_{Q_j}\|_{L^{p(\cdot)/\beta}(\mathbf{R}^n)}} \chi_{Q_j} \right)^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}_{p(\cdot)/\beta, u^\beta}}^{\frac{2}{s}}$$

$$= C\left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s, u^s}}^{\frac{1}{s}}. \tag{5.25}$$

We next consider II. Since $m_{p(\cdot)}r > 1$, by the definition of $m_{p(\cdot)}$ and Proposition 3.5, we find that $p(\cdot)r \in \mathcal{B}$. Furthermore, (3.9)–(3.11) yield

$$u^{\frac{1}{r}} \in \mathcal{W}_{rh_{p(\cdot)}} = \mathcal{W}_{h_{p(\cdot)r}} \subseteq \mathcal{W}_{e_{p(\cdot)r}}.$$

In view of (2.2), Theorem 3.1 and Lemma 3.3 assure that

$$II \le C\left\| \left(\sum_{j\in\mathbf{N}} |\lambda_j| \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1} (\mathrm{M}\,\chi_{Q_j})^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_{p(\cdot)r, u^{1/r}}}^r$$

$$= C\left\| \left(\sum_{j\in\mathbf{N}} |\lambda_j| \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1} \chi_{Q_j}^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_{p(\cdot)r, u^{1/r}}}^r$$

$$= C\left\| \sum_{j\in\mathbf{N}} |\lambda_j| \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1} \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot), u}}$$

for some $C > 0$. As $0 < s < 1$, the s-inequality and (2.2) yield

$$II \le C\left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s, u^s}}^{\frac{1}{s}} \tag{5.26}$$

for some $C > 0$. Thus, (5.25) and (5.26) give

$$\|M(f,\varphi)\|_{\mathcal{M}_{p(\cdot),u}} \le C\left\| \sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^s \chi_{Q_j} \right\|_{\mathcal{M}_{p(\cdot)/s, u^s}}^{\frac{1}{s}}.$$

Hence, $f \in \mathcal{H}_{p(\cdot),u}$ and (5.7) is valid. Finally, the above estimates already show that for any fixed $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $M(f,\varphi) \in \mathcal{M}_{p(\cdot),u}$. In view of Proposition 3.4, for any $g \in \mathcal{M}_{p(\cdot),u}$, g is finite almost everywhere. Furthermore, we have $|f * \varphi| \le M(f,\varphi)$. Thus, $f * \varphi$ is also finite almost everywhere. This reveals that $f = \sum_{j\in\mathbf{N}} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbf{R}^n)$. □

Proof of Theorem 5.3. The proof of Theorem 5.3 follows from the proof of Theorem 5.2. The only modification is on checking the conditions for applying Proposition 5.8. In our case, since $\kappa_{(p(\cdot)/m_{p(\cdot)})'} > 1$, we can apply Proposition 5.8 for any $p(\cdot)/s$ with $0 < s \le m_{p(\cdot)}$. Hence, the range for s in which condition (5.6) applies can also be extended to $0 < s \le m_{p(\cdot)}$. The rest of the proof follows from the proof of Theorem 5.2. □

At the end of this paper, we present a major application of the atomic decomposition for $\mathcal{H}_{p(\cdot),u}$, we show the boundedness results of some singular integral operators on $\mathcal{H}_{p(\cdot),u}$. We find that the ideas given in [46, Section 5] for the studies of the boundedness of singular integral operator for Hardy spaces with variable exponent can be easily transferred to the corresponding studies for $\mathcal{H}_{p(\cdot),u}$. Therefore, we apply them to the Hardy–Morrey spaces with variable exponents in the following.

Theorem 5.9. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. Let $T\colon L^2 \to L^2$ be a bounded operator with its Schwartz kernel $K(x,y) = k(x-y)$ satisfying*

$$\sup_{x\in\mathbf{R}^n\setminus\{0\}} |x|^{n+m}|\nabla^m k(x)| < \infty, \quad \forall m \in \mathbf{N}\cup\{0\} \tag{5.27}$$

and

$$Tf(x) = \int_{\mathbf{R}^n} k(x-y)f(y)\,dy, \quad x \notin supp f$$

for any compact supported $f \in L^2$. Then, T can be extended to be a bounded operator from $\mathcal{H}_{p(\cdot),u}$ to $\mathcal{M}_{p(\cdot),u}$.

Proof. By Theorem 5.1, for any sufficient large q and $f \in \mathcal{H}_{p(\cdot),u}$, we have

$$f = \sum_{j\in\mathbf{N}} \lambda_j a_j$$

where $\{a_j\}$ is a family of $(p(\cdot), q, d)$ atoms and $\{\lambda_j\}_{j\in\mathbf{N}}$ satisfies (5.5). According to [46, Proposition 5.3], we have

$$|Tf(x)| \le C\sum_{j\in\mathbf{N}} |\lambda_j| \left(\chi_{\tilde{Q}_j}(x)|(Ta_j)(x)| + \frac{(\mathrm{M}\,\chi_{Q_j})(x)^r}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)$$

for some $C > 0$ where $r = (n+d+1)/n$. We can obtain our desired result by applying the ideas from the proof of Theorem 5.2 to the above inequality. For simplicity, we omit the detail and leave it to the readers. □

Next, we introduce the notion of molecule associated with $\mathcal{H}_{p(\cdot),u}$. The following definition for molecule is modified from the corresponding definition of molecule from [46].

Definition 5.3. Let $p(x) \in \tilde{\mathcal{B}}$, $p_+ < q \le \infty$ and $1 \le q \le \infty$. Let $d \in \mathbf{N}$ satisfy $d_{p(\cdot)} \le d$. A Lebesgue measurable function m is said to be a $(p(\cdot), q, d)$-molecule centered at a cube $Q \in \mathcal{Q}$ if it satisfies

$$\|\chi_{\tilde{Q}} m\|_{L^q} \le \frac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}},$$

$$|m(x)| \le \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\left(1 + \frac{|x-x_Q|}{l(Q)}\right)^{-2n-2q-3}, \quad x \in \mathbf{R}^n\setminus\tilde{Q},$$

$$\int_{\mathbf{R}^n} x^\gamma m(x)\,dx = 0, \quad \forall\gamma \in \mathbf{N}^n \text{ satisfying } |\gamma| \le d.$$

Similar to the atomic decomposition of $\mathcal{H}_{p(\cdot),u}$, we have the molecular characterization for $\mathcal{H}_{p(\cdot),u}$.

Theorem 5.10. *Let $p(\cdot) \in \tilde{\mathcal{B}}$ and $u \in \mathcal{W}_{h_{p(\cdot)}}$. There exists a $q_0 > 1$ such that for any family of $(p(\cdot), q, d)$ molecules centered at Q_j, $\{m_j\}_{j\in\mathbf{N}}$, with $q > q_0$ and* *sequence of scalars* $\{\lambda_j\}_{j\in\mathbf{N}}$ *satisfying*

$$\left\|\sum_{j\in\mathbf{N}}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^s\chi_{Q_j}\right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}}<\infty$$

for some $0<s<\min(1,m_{p(\cdot)})$, *the series*

$$f=\sum_{j\in\mathbf{N}}\lambda_j m_j$$

converges in $\mathcal{S}'(\mathbf{R}^n)$ *and* $f\in\mathcal{H}_{p(\cdot),u}$ *with*

$$\|f\|_{\mathcal{H}_{p(\cdot),u}}\le C\left\|\sum_{j\in\mathbf{N}}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^s\chi_{Q_j}\right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}}$$

for some $C>0$ *independent of* f.

Proof. According to [46, (5.2)], for any $\varphi\in\mathcal{S}(\mathbf{R}^n)$, we have

$$M(\sum_{j\in\mathbf{N}}\lambda_j m_j,\varphi)(x)\le C\sum_{j\in\mathbf{N}}|\lambda_j|\left(\chi_{\tilde{Q}_j}(x)(\mathrm{M}\,m_j)(x)+\frac{(\mathrm{M}\,\chi_{Q_j})(x)^r}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)$$

where $r=(n+d+1)/n$. Therefore, the rest of the proof is similar to the proof of Theorem 5.2. For brevity, we leave the detail to the reader. □

With the above molecular characterization of $\mathcal{H}_{p(\cdot),u}$, we obtain the boundedness of some Calderón–Zygmund operators for $\mathcal{H}_{p(\cdot),u}$.

Theorem 5.11. *Let* $p(\cdot)\in\tilde{\mathcal{B}}$ *and* $u\in\mathcal{W}_{h_{p(\cdot)}}$. *Let* $k\in\mathcal{S}(\mathbf{R}^n)$ *satisfy (5.27). Then,*

$$Tf(x)=(k*f)(x)$$

can be extended to be a bounded operator on $\mathcal{H}_{p(\cdot),u}$.

Proof. Theorem 5.1 guarantees that for any sufficient large q and $f\in\mathcal{H}_{p(\cdot),u}$, we have

$$f=\sum_{j\in\mathbf{N}}\lambda_j a_j$$

where $\{a_j\}$ is a family of $(p(\cdot),q,d)$ atoms and $\{\lambda_j\}_{j\in\mathbf{N}}$ satisfies (5.5).

Therefore, T can be extended as

$$Tf=\sum_{j\in\mathbf{N}}\lambda_j Ta_j.$$

In view of [46, Proposition 5.4], T maps a $(p(\cdot),q,d)$ atom associated with Q to a constant multiple of a $(p(\cdot),q,d)$ molecule centered at Q with the multiple constant independent of the atoms. Thus, $\{Ta_j\}_{j\in\mathbf{N}}$ is a family of $(p(\cdot),q,d)$ molecules.

Theorem 5.10 guarantees that

$$\|Tf\|_{\mathcal{H}_{p(\cdot),u}}\le C\left\|\sum_{j\in\mathbf{N}}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right)^s\chi_{Q_j}\right\|_{\mathcal{M}_{p(\cdot)/s,u^s}}^{\frac{1}{s}}\le C\|f\|_{\mathcal{H}_{p(\cdot),u}}$$

for some $C>0$ independent of f. □

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