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THE OBSTACLE AND DIRICHLET PROBLEMS ASSOCIATED WITH p-HARMONIC FUNCTIONS IN UNBOUNDED SETS IN $\mathbf{R}^n$ AND METRIC SPACES

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Abstract. The obstacle problem associated with p-harmonic functions is extended to unbounded open sets, whose complement has positive capacity, in the setting of a proper metric measure space supporting a (p,p)-Poincaré inequality, $1 < p < \infty$, and the existence of a unique solution is proved. Furthermore, if the measure is doubling, then it is shown that a continuous obstacle implies that the solution is continuous, and moreover p-harmonic in the set where it does not touch the obstacle. This includes, as a special case, the solution of the Dirichlet problem for p-harmonic functions with Sobolev type boundary data.

1. Introduction

The classical Dirichlet problem is the problem of finding a function that is harmonic (i.e., a solution of the Laplace equation) in a given domain $\Omega \subset \mathbf{R}^n$ and takes prescribed boundary values. According to Dirichlet's principle, this is equivalent to minimizing the Dirichlet energy integral,

$$\int_\Omega |\nabla u|^2 \, dx,$$

among all admissible functions.

A more general and nonlinear Dirichlet problem considers the p-Laplace equation,

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

where $1 < p < \infty$, which reduces to the usual Laplace equation when $p = 2$. This is the Euler–Lagrange equation for the p-energy integral,

$$\int_\Omega |\nabla u|^p \, dx.$$

A minimizer/weak solution is said to be *p-harmonic* if it is continuous.

The nonlinear potential theory of p-harmonic functions has been studied since the 1960s. Initially for $\mathbf{R}^n$, and later generalized to weighted $\mathbf{R}^n$, Riemannian manifolds, and other settings. The interested reader may consult the monograph Heinonen–Kilpeläinen–Martio [20] for a thorough treatment in weighted $\mathbf{R}^n$.

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It is not clear how to employ partial differential equations in a general metric measure space. However, by using the notion of minimal p-weak upper gradients as substitutes for the modulus of the usual gradients, the variational approach of minimizing the p-energy integral becomes available. This has led to the more recent development of nonlinear potential theory on complete metric spaces equipped with a doubling measure supporting a p-Poincaré inequality.

The purpose of this paper is to extend the so-called *obstacle problem* associated with p-harmonic functions to allow for *unbounded* open sets with Dirichlet boundary data. The obstacle problem has been studied for *bounded* sets in (weighted) $\mathbf{R}^n$ (see, e.g., Heinonen–Kilpeläinen–Martio [20] and the references therein), and later also for bounded sets in more general metric spaces (see, e.g., Björn–Björn [3], [4], [5], Björn–Björn–Mäkäläinen–Parviainen [6], Björn–Björn–Shanmugalingam [8], Kinnunen–Martio [25], Kinnunen–Shanmugalingam [26], and Shanmugalingam [31]). The double obstacle problem has also been studied (see, e.g., Farnana [13, 14, 15, 16] and Eleuteri–Farnana–Kansanen–Korte [12]).

The setting in which we will study the obstacle problem will be that of a proper metric measure space X supporting a (p,p)-Poincaré inequality where we let Ω be a nonempty (possibly unbounded) open subset of X such that the capacity of the complement is positive (which is needed for the boundary data to make sense). Furthermore, we let the obstacle ψ be an extended real-valued function and we assume the boundary data f to be a function in $D^p(\Omega)$ (see Section 2 for definitions). In this setting, we prove Theorem 3.4 which asserts that there exists a unique (up to sets of capacity zero) solution of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem whenever the set of admissible functions is nonempty. Thus, instead of merely studying the Dirichlet problem, we solve the associated obstacle problem of minimizing the p-energy integral among all functions that satisfy the given boundary condition and are greater than or equal to ψ in Ω (perhaps with the exception of a set of capacity zero). This problem reduces to the Dirichlet problem for p-harmonic functions with Sobolev type boundary data when $\psi \equiv -\infty$.

The existence part of the proof of Theorem 3.4 starts with a minimizing sequence of admissible functions $\{u_j\}_{j=1}^\infty$ that minimizes the p-energy integral. We let $w_j = u_j - f$ extended to zero outside Ω and exhaust X by an increasing sequence of balls. The sequence $\{w_j\}_{j=1}^\infty$ is shown to be bounded on every ball by using Maz′ya's inequality and the boundedness of the corresponding sequence of minimal p-weak upper gradients. Hence, by Mazur's lemma, we can find convex combinations of functions from $\{w_j\}_{j=1}^\infty$ that converge to an L^p-function in the smallest ball. Repeatedly, we can then find new convex combinations of the previous convex combinations that converge to L^p-functions on larger and larger balls. Using these L^p-functions, we construct a function u and show that the diagonal sequence of the sequences of convex combinations converges to $u - f$. By estimating integrals of minimal p-weak upper gradients from the diagonal sequence using Jensen's inequality and induction, we can use a consequence of Mazur's lemma to show that $u - f \in D_0^p(\Omega)$ and hence that u is an admissible function. In a similar way, using the obtained estimations, we conclude the existence part by showing that u is indeed a minimizer.

The paper is organized as follows. In the next section, we establish notation, review some basic definitions and results relating to Sobolev type spaces on metric spaces. In Section 3, we define the obstacle problem that allow for unbounded open sets and prove the main result of this paper (Theorem 3.4). We also obtain some further results. Finally, in Section 4, assuming X to be a complete p-Poincaré space with a doubling measure, we show the existence of a unique lsc-regularized solution, and, moreover, that if the obstacle is continuous, then the solution is continuous and furthermore p-harmonic in the set where it does not touch the obstacle. As a special case, this implies that there exists a unique solution of the Dirichlet problem for p-harmonic functions with boundary values in $D^p(\Omega)$ taken in Sobolev sense. To the best of the author's knowledge, these results are new also for $\mathbf{R}^n$.

2. Notation and preliminaries

We assume throughout the paper that $(X, \mathcal{M}, \mu, d)$ is a metric measure space (which we will refer to as X) equipped with a metric d and a measure μ such that

$$0 < \mu(B) < \infty$$

for all balls $B \subset X$ (we make the convention that balls are nonempty and open). The σ-algebra $\mathcal{M}$ on which μ is defined is the completion of the Borel σ-algebra.

We start with the assumption that $1 \leq p < \infty$. However, in the next section (and for the rest of the paper), we will assume that $1 < p < \infty$. The measure μ is said to be *doubling* if there exists a constant $C_\mu \geq 1$ such that

$$0 < \mu(2B) \leq C_\mu \mu(B) < \infty$$

for all balls $B \subset X$. We use the notation that if B is a ball with radius r, then the ball with radius λr that is concentric with B is denoted by λB.

Recall that the characteristic function χ_E of a set E is defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. The set E is compactly contained in A if $\overline{E}$ (the closure of E) is a compact subset of A. We denote this by $E \Subset A$. The extended real number system is denoted by $\overline{\mathbf{R}} := [-\infty, \infty]$. Recall also that $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, and hence that $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

By a *curve* in X, we mean a rectifiable nonconstant continuous mapping γ from a compact interval into X. Since our curves have finite length, they may be parametrized by arc length, and we will always assume that this has been done.

Unless otherwise stated, the letter C will be used to denote various positive constants whose exact values are unimportant and may vary with each usage.

We follow Heinonen–Koskela [21], [22] in introducing upper gradients (Heinonen and Koskela, however, called them very weak gradients).

Definition 2.1. A Borel function $g\colon X \to [0, \infty]$ is said to be an *upper gradient* of a function $f\colon X \to \overline{\mathbf{R}}$ whenever

$$|f(x) - f(y)| \leq \int_\gamma g\, ds \tag{2.1}$$

holds for all pairs of points $x, y \in X$ and every curve γ in X joining x and y. We make the convention that the left-hand side is infinite if both terms are.

Recall that a Borel function $g\colon X \to Y$ is a function such that the inverse image $g^{-1}(G) = \{x \in X : g(x) \in G\}$ is a Borel set for every open subset G of Y.

Observe that upper gradients are not unique (if we add a nonnegative Borel function to an upper gradient of f, then we obtain a new upper gradient of f) and that $g \equiv \infty$ is an upper gradient of all functions. Note also that if g and $\tilde{g}$ are upper gradients of u and $\tilde{u}$, respectively, then $g - \tilde{g}$ is not in general an upper gradient of $u - \tilde{u}$. However, upper gradients are subadditive, that is, if g and $\tilde{g}$ are upper gradients of u and $\tilde{u}$, respectively, and $a \in \mathbf{R}$, then $|a|g$ and $g + \tilde{g}$ are upper gradients of au and $u + \tilde{u}$, respectively.

A drawback of upper gradients is that they are not preserved by L^p-convergence. Fortunately, it is possible to overcome this problem by relaxing the condition, and we therefore follow Koskela–MacManus [27] in introducing p-weak upper gradients. To do this, we need the following definition.

Definition 2.2. Let Γ be a family of curves in X. The *p-modulus* of Γ is

$$\operatorname{Mod}_p(\Gamma) := \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that

$$\int_\gamma \rho \, ds \geq 1$$

for all curves $\gamma \in \Gamma$. Whenever a property holds for all curves except for a curve family of zero p-modulus, it is said to hold for *p-almost every* (*p-a.e.*) curve.

The p-modulus (as the module of order p of a system of measures) was defined and studied by Fuglede [17]. Heinonen–Koskela [22] defined the p-modulus of a curve family in a metric measure space and observed that the corresponding results by Fuglede carried over directly.

The p-modulus has the following properties (as observed in [22]): $\operatorname{Mod}_p(\varnothing) = 0$, $\operatorname{Mod}_p(\Gamma_1) \leq \operatorname{Mod}_p(\Gamma_2)$ whenever $\Gamma_1 \subset \Gamma_2$, and $\operatorname{Mod}_p\bigl(\bigcup_{j=1}^\infty \Gamma_j\bigr) \leq \sum_{j=1}^\infty \operatorname{Mod}_p(\Gamma_j)$. If Γ_0 and Γ are two curve families such that every curve $\gamma \in \Gamma$ has a subcurve $\gamma_0 \in \Gamma_0$, then $\operatorname{Mod}_p(\Gamma) \leq \operatorname{Mod}_p(\Gamma_0)$. For proofs of these properties and all other results in this section, we refer to Björn–Björn [4]. Some of the references that we mention below may not provide a proof in the generality considered here, but such proofs are given in [4].

Definition 2.3. A measurable function $g\colon X \to [0,\infty]$ is said to be a *p-weak upper gradient* of a function $f\colon X \to \overline{\mathbf{R}}$ if (2.1) holds for all pairs of points $x, y \in X$ and p-a.e. curve γ in X joining x and y.

Note that a p-weak upper gradient, as opposed to an upper gradient, is *not* required to be a Borel function. It is convenient to demand upper gradients to be Borel functions, since then the concept of upper gradients becomes independent of the measure, and all considered curve integrals will be defined. The situation is a bit different for p-weak upper gradients, as the curve integrals need only be defined for p-a.e. curve, and therefore, it is in fact enough to require that p-weak upper gradients are measurable functions. There is no disadvantage in assuming only measurability, since the concept of p-weak upper gradients would depend on the measure anyway (as the p-modulus depends on the measure). The advantage is that some results become more appealing (see, e.g., Björn–Björn [4]).

Since the p-modulus is subadditive, it follows that p-weak upper gradients share the subadditivity property with upper gradients.

Definition 2.4. The *Dirichlet space* on X, denoted by $D^p(X)$, is the space of all extended real-valued functions on X that are everywhere defined, measurable, and have upper gradients in $L^p(X)$.

If E is a measurable set, then we can consider E to be a metric space in its own right (with the restriction of d and μ to E). Thus, the Dirichlet space $D^p(E)$ is also given by Definition 2.4.

The local Dirichlet space is defined analogously to the local space $L^p_{\mathrm{loc}}(X)$, and hence we say that a function f on X belongs to $D^p_{\mathrm{loc}}(X)$ if for every $x \in X$ there is a ball B such that $x \in B$ and $f \in D^p(B)$.

Lemma 2.4 in Koskela–MacManus [27] asserts that if g is a p-weak upper gradient of a function f, then for all q such that $1 \le q \le p$, there is a decreasing sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $\|g_j - g\|_{L^q(X)} \to 0$ as $j \to \infty$. This implies that a measurable function belongs to $D^p(X)$ whenever it (merely) has a p-weak upper gradient in $L^p(X)$.

If $u \in D^p(X)$, then u has a *minimal p-weak upper gradient*, denoted by g_u, which is minimal in the sense that $g_u \le g$ a.e. on X for all p-weak upper gradients g of u. This was proved for $p > 1$ by Shanmugalingam [31] and for $p \ge 1$ by Hajłasz [18]. The minimal p-weak upper gradient g_u is a true substitute for $|\nabla u|$ in metric spaces.

One of the important properties of minimal p-weak gradients is that they are local in the sense that if two functions $u, v \in D^p(X)$ coincide on a set E, then $g_u = g_v$ a.e. on E. Moreover, if $U = \{x \in X\colon u(x) > v(x)\}$, then $g_u\chi_U + g_v\chi_{X\setminus U}$ is a minimal p-weak upper gradient of $\max\{u, v\}$, and $g_v\chi_U + g_u\chi_{X\setminus U}$ is a minimal p-weak upper gradient of $\min\{u, v\}$. These results are from Björn–Björn [2].

It is well-known that the restriction of a minimal p-weak upper gradient to an open subset remains minimal with respect to that subset. As a consequence, the results above about minimal p-weak upper gradients can be extended to functions in $D^p_{\mathrm{loc}}(X)$ having minimal p-weak upper gradients in $L^p_{\mathrm{loc}}(X)$.

With the help of upper gradients, it is possible to define a type of Sobolev space on the metric space X. This was done by Shanmugalingam [30]. We will, however, use a slightly different (semi)norm. The reason for this is that when we define the capacity in Definition 2.6, it will then be subadditive.

Definition 2.5. The *Newtonian space* on X is

$$N^{1,p}(X) := \{u \in D^p_{\mathrm{loc}}(X)\colon \|u\|_{N^{1,p}(X)} < \infty\},$$

where $\|\cdot\|_{N^{1,p}(X)}$ is the seminorm defined by

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p}.$$

We emphasize the fact that our Newtonian functions are defined everywhere, and not just up to equivalence classes of functions that agree almost everywhere. This is essential for the notion of upper gradients to make sense.

The associated normed space defined by $\widetilde{N}^{1,p}(X) = N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space (see Shanmugalingam [30]). Note that some authors denote the space of the everywhere defined functions by $\widetilde{N}^{1,p}(X)$, and then define the Newtonian space, which they denote by $N^{1,p}(X)$, to be the corresponding space of equivalence classes.

The local space $N^{1,p}_{\rm loc}(X)$ and the space $N^{1,p}(E)$ when E is a measurable set are defined analogously to the Dirichlet spaces.

Recall that a metric space is said to be *proper* if all bounded closed subsets are compact. In particular, this is true if it is complete and the measure is doubling. If X is proper and Ω is an open subset of X, then $f \in L^p_{\rm loc}(\Omega)$ if and only if $f \in L^p(\Omega')$ for all open $\Omega' \Subset \Omega$. This is the case also for $D^p_{\rm loc}$ and $N^{1,p}_{\rm loc}$.

Various definitions of capacities for sets can be found in the literature (see, e.g., Kinnunen–Martio [24] and Shanmugalingam [30]). We will use the following definition.

Definition 2.6. The (*Sobolev*) *capacity* of a set $E \subset X$ is

$$C_p(E) := \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

Whenever a property holds for all points except for points in a set of capacity zero, it is said to hold *quasieverywhere* (*q.e.*). Note that we follow the custom of refraining from making the dependence on p explicit here.

Trivially, we have $C_p(\varnothing) = 0$, and $C_p(E_1) \leq C_p(E_2)$ whenever $E_1 \subset E_2$. Furthermore, the proof in Kinnunen–Martio [24] for capacities for Hajłasz–Sobolev spaces on metric spaces can easily be modified to show that C_p is countably subadditive, that is, $C_p\bigl(\bigcup_{j=1}^\infty E_j\bigr) \leq \sum_{j=1}^\infty C_p(E_j)$. Note that C_p is finer than μ in the sense that the capacity of a set may be positive even when the measure of the same set equals zero.

Shanmugalingam [30] showed that if two Newtonian functions are equal almost everywhere, then they are in fact equal quasieverywhere. This result extends to functions in $D^p_{\rm loc}(X)$.

For $E \subset X$, we let Γ_E denote the family of all curves in X that intersect E. Lemma 3.6 in Shanmugalingam [30] asserts that $\operatorname{Mod}_p(\Gamma_E) = 0$ whenever $C_p(E) = 0$. This implies that two functions have the same set of p-weak upper gradients whenever they are equal quasieverywhere.

To be able to compare boundary values of Dirichlet functions (and of Newtonian functions), we introduce the following spaces.

Definition 2.7. The *Dirichlet space with zero boundary values in* $A \setminus E$, for subsets E and A of X, where A is measurable, is

$$D^p_0(E; A) := \{f|_{E\cap A} \colon f \in D^p(A) \text{ and } f = 0 \text{ in } A \setminus E\}.$$

The *Newtonian space with zero boundary values in* $A \setminus E$, denoted by $N^{1,p}_0(E; A)$, is defined analogously. We let $D^p_0(E)$ and $N^{1,p}_0(E)$ denote $D^p_0(E; X)$ and $N^{1,p}_0(E; X)$, respectively.

The assumption "$f = 0$ in $A \setminus E$" can in fact be replaced by "$f = 0$ q.e. in $A \setminus E$" without changing the obtained spaces.

It is easy to verify that the function spaces that we have introduced are vector spaces and lattices. This means that if both u and v belong to one of these spaces and $a, b \in \mathbf{R}$, then $au + bv$, $\max\{u, v\}$, and $\min\{u, v\}$ belong to the same space, and furthermore, as a direct consequence, the same is true for u_+, u_-, and $|u|$.

The following lemma is useful for asserting that certain functions belong to a Dirichlet space with zero boundary values.

Lemma 2.8. *Let $E \subset X$ be measurable and let $u \in D^p(E)$. If there exist two functions $u_1, u_2 \in D^p_0(E)$ such that $u_1 \le u \le u_2$ q.e. in E, then $u \in D^p_0(E)$.*

This was proved for Newtonian functions in open sets in Björn–Björn [3], and with trivial modifications, it provides a proof for the version of the lemma that we need in this paper. For the reader's convenience, we give the proof here.

Proof. Let $v_1, v_2 \in D^p(X)$ be such that $v_1|_E = u_1$, $v_2|_E = u_2$, and $v_1 = v_2 = 0$ outside E. Moreover, let $g_1, g_2 \in L^p(X)$, be upper gradients of v_1 and v_2, respectively. Let $g \in L^p(E)$ be an upper gradient of u and define

$$v = \begin{cases} u & \text{in } E, \\ 0 & \text{in } X \setminus E, \end{cases} \qquad \text{and} \qquad \tilde{g} = \begin{cases} g_1 + g_2 + g & \text{in } E, \\ g_1 + g_2 & \text{in } X \setminus E. \end{cases}$$

To complete the proof, it suffices to show that $\tilde{g}$, which belongs to $L^p(X)$, is a p-weak upper gradient of v.

Let $E' \subset E$ with $C_p(E') = 0$ be such that $u_1 \le u \le u_2$ in $E \setminus E'$. Let γ be an arbitrary curve in $X \setminus E'$ with endpoints x and y. Then $\operatorname{Mod}_p(\Gamma_{E'}) = 0$, so the following argument asserts that $\tilde{g}$ is a p-weak upper gradient of v.

If $\gamma \subset E \setminus E'$, then

$$|v(x) - v(y)| = |u(x) - u(y)| \le \int_\gamma g\, ds \le \int_\gamma \tilde{g}\, ds.$$

On the other hand, if $x, y \in X \setminus E$, then

$$|v(x) - v(y)| = 0 \le \int_\gamma \tilde{g}\, ds.$$

Hence, by splitting γ into two parts, and possibly reversing the direction, we may assume that $x \in E \setminus E'$ and $y \in X \setminus E$. It follows that

$$\begin{aligned} |v(x) - v(y)| &= |u(x)| \le |v_1(x)| + |v_2(x)| = |v_1(x) - v_1(y)| + |v_2(x) - v_2(y)| \\ &\le \int_\gamma g_1\, ds + \int_\gamma g_2\, ds \le \int_\gamma \tilde{g}\, ds. \end{aligned}$$ □

Proposition 2.9. *If $\Omega \subset X$ is open, then $D^p_0(\Omega) = D^p_0(\Omega; \overline{\Omega})$.*

The proof is very similar to the proof of Lemma 2.8 (see, e.g., Proposition 2.39 in Björn–Björn [4] for a corresponding proof for Newtonian functions).

The following two results from Björn–Björn–Parviainen [7] (Lemma 3.2 and Corollary 3.3), following from Mazur's lemma (see, e.g., Theorem 3.12 in Rudin [29]), will play a major role in the existence proof for the obstacle problem.

Lemma 2.10. *Assume that $1 < p < \infty$. If g_j is a p-weak upper gradient of u_j, $j = 1, 2, \dots$, and both $\{u_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are bounded in $L^p(X)$, then there exist functions $u, g \in L^p(X)$, convex combinations $v_j = \sum_{i=j}^{N_j} a_{j,i} u_i$ with p-weak upper gradients $\tilde{g}_j = \sum_{i=j}^{N_j} a_{j,i} g_i$, $j = 1, 2, \dots$, and a subsequence $\{u_{j_k}\}_{k=1}^\infty$, such that*

(a) *both $u_{j_k} \to u$ and $g_{j_k} \to g$ weakly in $L^p(X)$ as $k \to \infty$;*
(b) *both $v_j \to u$ and $\tilde{g}_j \to g$ in $L^p(X)$ as $j \to \infty$;*
(c) *$v_j \to u$ q.e. as $j \to \infty$;*
(d) *g is a p-weak upper gradient of u.*

Recall that $a_1v_1 + \dots + a_nv_n$ is said to be a convex combination of $v_1, \dots, v_n$ whenever $a_k \geq 0$ for all $k = 1, \dots, n$ and $a_1 + \dots + a_n = 1$.

Corollary 2.11. *Assume that $1 < p < \infty$. If $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(X)$ and $u_j \to u$ q.e. on X as $j \to \infty$, then $u \in N^{1,p}(X)$ and*

$$\int_X g_u^p \, d\mu \leq \liminf_{j\to\infty} \int_X g_{u_j}^p \, d\mu.$$

In general, the upper gradients of a function give no control over the function. This is obviously so when there are no curves. Requiring a Poincaré inequality to hold is one possibility of gaining such a control by making sure that there are enough curves connecting any two points.

Definition 2.12. Let $q \geq 1$. We say that X supports a (q,p)-*Poincaré inequality* (or that X is a (q,p)-Poincaré space) if there exist constants $C_{\mathrm{PI}} > 0$ and $\lambda \geq 1$ (dilation constant) such that

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leq C_{\mathrm{PI}} \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu\right)^{1/p} \tag{2.2}$$

for all balls $B \subset X$, all integrable functions u on X, and all upper gradients g of u.

In (2.2), we have used the convenient notation,

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

We write p-*Poincaré inequality* instead of $(1,p)$-Poincaré inequality for short, and if X supports a p-Poincaré inequality, we say that X is a p-Poincaré space. By using Hölder's inequality, one can show that if X supports a (q,p)-Poincaré inequality, then X supports a $(\tilde{q},\tilde{p})$-Poincaré inequality for all $\tilde{q} \leq q$ and $\tilde{p} \geq p$. From the next section on, we will assume that X supports a (p,p)-Poincaré inequality. Then we have the following useful assertion which implies that a function can be controlled by its minimal p-weak upper gradient. This was proved for Euclidean spaces by Maz′ya (see, e.g., [28]), and later J. Björn [9] observed that the proof goes through also for metric spaces. The following version is from Björn–Björn [4] (Theorem 5.53).

Theorem 2.13. (Maz′ya's inequality) *If X supports a (p,p)-Poincaré inequality, then there exists a constant $C_{\mathrm{MI}} > 0$ such that*

$$\int_{2B} |u|^p \, d\mu \leq \frac{C_{\mathrm{MI}}(\operatorname{diam}(B)^p + 1)\mu(2B)}{C_p(B \cap S)} \int_{2\lambda B} g_u^p \, d\mu,$$

whenever $B \subset X$ is a ball, $u \in N^{1,p}_{\mathrm{loc}}(X)$, and $S = \{x \in X : u(x) = 0\}$.

The following result from Björn–Björn [4] (Proposition 4.14) is also a useful consequence of the (p,p)-Poincaré inequality.

Proposition 2.14. *If X supports a (p,p)-Poincaré inequality and $\Omega \subset X$ is open, then $D^p_{\mathrm{loc}}(\Omega) = N^{1,p}_{\mathrm{loc}}(\Omega)$.*

3. The obstacle problem

In this section, we assume that $1 < p < \infty$, that X is proper and supports a (p,p)-Poincaré inequality with dilation constant λ, and that $\Omega \subset X$ is nonempty, open, and such that $C_p(X \setminus \Omega) > 0$.

Kinnunen–Martio [25] defined an obstacle problem for Newtonian functions in open sets in a complete p-Poincaré space with a doubling measure. They proved that there exists a unique solution whenever the set is bounded and such that the complement has positive measure and the set of feasible solutions is nonempty (Theorem 3.2 in [25]). Shanmugalingam [30] had earlier solved the Dirichlet problem (i.e., the obstacle problem with obstacle $\psi \equiv -\infty$).

Roughly, Kinnunen and Martio defined their obstacle problem as follows.

Definition 3.1. Let V be a nonempty *bounded* open subset of X with $C_p(X \setminus V) > 0$. For $\psi\colon V \to \overline{\mathbf{R}}$ and $f \in N^{1,p}(V)$, define

$$\mathcal{K}^{\mathrm{B}}_{\psi,f}(V) = \{v \in N^{1,p}(V)\colon v - f \in N^{1,p}_0(V) \text{ and } v \geq \psi \text{ q.e. in } V\}.$$

A function u is a *solution of the* $\mathcal{K}^{\mathrm{B}}_{\psi,f}(V)$*-obstacle problem* if $u \in \mathcal{K}^{\mathrm{B}}_{\psi,f}(V)$ and

$$\int_V g_u^p \, d\mu \leq \int_V g_v^p \, d\mu$$

for all $v \in \mathcal{K}^{\mathrm{B}}_{\psi,f}(V)$.

They required that $\mu(X \setminus V) > 0$ and merely that $v \geq \psi$ a.e. instead of q.e., which does not matter if the obstacle ψ is in $D^p_{\mathrm{loc}}(V)$, since then $v \geq \psi$ a.e. implies that $v \geq \psi$ q.e. This follows from Corollary 3.3 in Shanmugalingam [30]; see also Corollary 1.60 in Björn–Björn [4]. However, the distinction may be important. For example, if K is a compact subset of V such that $C_p(K) > \mu(K) = 0$, then the solution of the $\mathcal{K}^{\mathrm{B}}_{\chi_K,0}(V)$-obstacle problem takes the value 1 on K, whereas the solution of the corresponding obstacle problem defined by Kinnunen–Martio [25] is the trivial solution (because their candidate solutions do not "see" this obstacle). Moreover, it is possible to have no solution of the $\mathcal{K}^{\mathrm{B}}_{\psi,f}(V)$-obstacle problem when there is a solution of the corresponding obstacle problem defined in [25] (see, e.g., the discussion following Definition 3.1 in Farnana [13]).

See also Farnana [13, 14, 15, 16] for the double obstacle problem, and Björn–Björn [5] for the obstacle problem on nonopen sets.

Now we define the obstacle problem without the boundedness requirement.

Definition 3.2. Let $V \subset X$ be nonempty, open, and such that $C_p(X \setminus V) > 0$. (Note that V is allowed to be unbounded.) For $\psi\colon V \to \overline{\mathbf{R}}$ and $f \in D^p(V)$, define

$$\mathcal{K}_{\psi,f}(V) = \{v \in D^p(V)\colon v - f \in D^p_0(V) \text{ and } v \geq \psi \text{ q.e. in } V\}.$$

We say that a function u is a *solution of the* $\mathcal{K}_{\psi,f}(V)$*-obstacle problem* (*with obstacle* ψ *and boundary values* f) if $u \in \mathcal{K}_{\psi,f}(V)$ and

$$\int_V g_u^p \, d\mu \leq \int_V g_v^p \, d\mu$$

for all $v \in \mathcal{K}_{\psi,f}(V)$. We let $\mathcal{K}_{\psi,f}(\Omega)$ be denoted by $\mathcal{K}_{\psi,f}$ for short when $V = \Omega$.

Observe that we only define the obstacle problem for V with $C_p(X \setminus V) > 0$. Otherwise the condition $u - f \in D^p_0(V)$ becomes empty, since then $D^p_0(V) = D^p(V)$.

Note also that we solve the obstacle problem for boundary data $f \in D^p(V)$. Because such a function is not defined on ∂V, we do not really have boundary *values*, and hence the definition should be understood in a weak Sobolev sense.

Remark 3.3. If $f \in N^{1,p}(V)$ and V is bounded, then Proposition 2.7 in Björn–Björn [5] asserts that $D_0^p(V) = N_0^{1,p}(V)$, and hence we have $\mathcal{K}_{\psi,f}(V) = \mathcal{K}^{\mathrm{B}}_{\psi,f}(V)$. Thus, Definition 3.2 is a generalization of Definition 3.1 to Dirichlet functions and to unbounded sets.

The main result in this paper shows that the $\mathcal{K}_{\psi,f}$-obstacle problem has a unique solution under the natural condition of $\mathcal{K}_{\psi,f}$ being nonempty.

Theorem 3.4. *Let $\psi\colon \Omega \to \overline{\mathbf{R}}$ and let $f \in D^p(\Omega)$. Then there exists a unique* (*up to sets of capacity zero*) *solution of the $\mathcal{K}_{\psi,f}$-obstacle problem whenever $\mathcal{K}_{\psi,f} \neq \varnothing$.*

The standing assumption that X is proper is needed only in the end of the existence part of the proof.

In the uniqueness part of the proof, we use the fact that $L^p(\Omega)$ is strictly convex. Clarkson [11] introduced the notions of strict convexity and uniform convexity (the latter being a stronger condition), and proved that all L^p-spaces, $1 < p < \infty$, are uniformly convex. A Banach space Y (with norm $\|\cdot\|$) is *strictly convex* if $x = cy$ for some constant $c > 0$ whenever x and y are nonzero and $\|x + y\| = \|x\| + \|y\|$. In particular, $x = y$ whenever $\|x\| = \|y\| = \bigl\|\frac{1}{2}(x + y)\bigr\| = 1$.

The idea used in the uniqueness part of the proof comes from Cheeger [10].

Proof. (*Existence*) We start by choosing a ball $B \subset X$ such that $C_p(B \setminus \Omega) > 0$ and $B \cap \Omega \neq \varnothing$. Clearly, we have $B \subset 2B \subset 3B \subset \cdots \subset X = \bigcup_{t=1}^\infty tB$. Let

$$I = \inf_v \int_\Omega g_v^p \, d\mu,$$

where the infimum is taken over all functions in $\mathcal{K}_{\psi,f}$. Clearly, we have $0 \le I < \infty$. Let $\{u_j\}_{j=1}^\infty \subset \mathcal{K}_{\psi,f}$ be a minimizing sequence such that

$$I_j := \int_\Omega g_{u_j}^p \, d\mu \searrow I \quad \text{as } j \to \infty.$$

Let $w_j \in D^p(X)$ be such that $w_j = u_j - f$ in Ω and $w_j = 0$ outside Ω, $j = 1, 2, \ldots$. We claim that both $\{w_j\}_{j=1}^\infty$ and $\{g_{w_j}\}_{j=1}^\infty$ are bounded in $L^p(tB)$ for each $t \ge 1$. To show that, we first observe that $g_{w_j} \le (g_{u_j} + g_f)\chi_\Omega$ a.e., and hence

$$\|g_{w_j}\|_{L^p(X)} \le \|g_{u_j}\|_{L^p(\Omega)} + \|g_f\|_{L^p(\Omega)} \le \|g_{u_1}\|_{L^p(\Omega)} + \|g_f\|_{L^p(\Omega)} =: C' < \infty.$$

Let $t \ge 1$ be arbitrary and let $S = \bigcap_{j=1}^\infty \{x \in X : w_j(x) = 0\}$. Then

$$C_p(tB \cap S) \ge C_p(tB \setminus \Omega) \ge C_p(B \setminus \Omega) > 0.$$

Maz′ya's inequality (Theorem 2.13) asserts that there exists a constant $C_{tB} > 0$ such that

$$\int_{2tB} |w_j|^p \, d\mu \le C_{tB}^p \int_{2\lambda tB} g_{w_j}^p \, d\mu.$$

This implies that we also have

$$\|w_j\|_{L^p(tB)} \le C_{tB} \|g_{w_j}\|_{L^p(X)} \le C_{tB} C' =: C'_{tB} < \infty, \tag{3.1}$$

and the claim follows.

Consider the ball B. Lemma 2.10 asserts that we can find a function $\varphi_1 \in L^p(B)$ and convex combinations

$$\varphi_{1,j} = \sum_{k=j}^{N_{1,j}} a_{1,j,k} w_k \in D^p(X), \quad j = 1, 2, \dots, \tag{3.2}$$

such that $\varphi_{1,j} \to \varphi_1$ q.e. in B as $j \to \infty$. Because $\varphi_{1,j} = 0$ outside Ω, we must have $\varphi_1 = 0$ q.e. in $B \setminus \Omega$, and hence we may choose φ_1 such that $\varphi_1 = 0$ in $B \setminus \Omega$. Let $v_{1,j} = f + \varphi_{1,j}|_\Omega$. Then

$$v_{1,j} = f + \sum_{k=j}^{N_{1,j}} a_{1,j,k} w_k|_\Omega = \sum_{k=j}^{N_{1,j}} a_{1,j,k}(f + w_k|_\Omega) = \sum_{k=j}^{N_{1,j}} a_{1,j,k} u_k \geq \psi \quad \text{q.e. in } \Omega.$$

We also have

$$g_{v_{1,j}} \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{u_k} \quad \text{a.e. in } \Omega \quad \text{and} \quad g_{\varphi_{1,j}} \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{w_k} \quad \text{a.e. on } X.$$

A sequence of convex combinations of functions taken from a bounded sequence must also be bounded, and therefore we can apply Lemma 2.10 repeatedly here. Hence, for every $n = 2, 3, 4, \dots$, we can find a function $\varphi_n \in L^p(nB)$ such that $\varphi_n = 0$ in $nB \setminus \Omega$ and convex combinations

$$\varphi_{n,j} = \sum_{k=j}^{N_{n,j}} a_{n,j,k} \varphi_{n-1,k} \in D^p(X), \quad j = 1, 2, \dots, \tag{3.3}$$

such that $\varphi_{n,j} \to \varphi_n$ q.e. in nB as $j \to \infty$. Let $v_{n,j} = f + \varphi_{n,j}|_\Omega$. Then

$$v_{n,j} = \sum_{k=j}^{N_{n,j}} a_{n,j,k}(f + \varphi_{n-1,k}|_\Omega) = \sum_{k=j}^{N_{n,j}} a_{n,j,k} v_{n-1,k} \geq \psi \quad \text{q.e. in } \Omega,$$

and also

$$g_{v_{n,j}} \leq \sum_{k=j}^{N_{n,j}} a_{n,j,k} g_{v_{n-1,k}} \quad \text{a.e. in } \Omega \quad \text{and} \quad g_{\varphi_{n,j}} \leq \sum_{k=j}^{N_{n,j}} a_{n,j,k} g_{\varphi_{n-1,k}} \quad \text{a.e. on } X.$$

Let $u = f + \varphi|_\Omega$, where φ is the function on X defined by

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x) \chi_{nB \setminus (n-1)B}(x), \quad x \in X.$$

We shall now show that u truly is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. To do that, we first establish that $u \in \mathcal{K}_{\psi,f}$, and then show that u is indeed a minimizer. Because $\varphi = u - f$ in Ω and $\varphi = 0$ outside Ω, it suffices to show that $\varphi \in D^p(X)$ in order to establish that $u - f \in D^p_0(\Omega)$ and $u \in D^p(\Omega)$.

Consider the diagonal sequences $\{v_{n,n}\}_{n=1}^\infty$ and $\{\varphi_{n,n}\}_{n=1}^\infty$. Observe that the latter is bounded in $L^p(tB)$ for each $t \geq 1$ since $\|\varphi_{n,j}\|_{L^p(tB)} \leq C'_{tB}$ for all n and j, by (3.1), (3.2), and (3.3).

We claim that $\varphi_{n,n} \to \varphi$ q.e. on X as $n \to \infty$. To prove that, we start by fixing an integer $n \geq 1$ and consider nB. Then

$$\begin{aligned}|\varphi_{n+1} - \varphi_n| &\leq |\varphi_{n+1} - \varphi_{n+1,j}| + |\varphi_{n+1,j} - \varphi_n| \\ &\leq |\varphi_{n+1} - \varphi_{n+1,j}| + \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} |\varphi_{n,k} - \varphi_n| \to 0\end{aligned}$$

q.e. in nB as $j \to \infty$. Thus, $\varphi_{n+1} = \varphi_n$ q.e. in nB for $n = 1, 2, \ldots$.

By definition, we have $\varphi = \varphi_1$ in B. Now assume that $\varphi = \varphi_n$ q.e. in nB for some positive integer n. By definition also, we have $\varphi = \varphi_{n+1}$ in $(n+1)B \setminus nB$, and because $\varphi_{n+1} = \varphi_n$ q.e. in nB, it follows that $\varphi = \varphi_{n+1}$ q.e. in $(n+1)B$. Hence, by induction, we have $\varphi = \varphi_n$ q.e. in nB for $n = 1, 2, \ldots$.

For $n = 1, 2, \ldots$, let E_n be the subset of nB where $\varphi_{n,j} \to \varphi_n = \varphi$ as $j \to \infty$ and let $E = \bigcup_{n=1}^{\infty} (nB \setminus E_n)$. By subadditivity, we have $C_p(E) \leq \sum_{n=1}^{\infty} C_p(nB \setminus E_n) = 0$. Let $x \in X \setminus E$ be given. Clearly, $x \in mB$ and $\varphi(x) = \varphi_m(x)$ for some positive integer m. Given $\varepsilon > 0$, choose J such that $j \geq J$ implies that

$$|\varphi_{m,j}(x) - \varphi_m(x)| < \varepsilon.$$

Assume that for some $n \geq m$, we have $|\varphi_{n,j}(x) - \varphi_m(x)| < \varepsilon$ for $j \geq J$. Then

$$|\varphi_{n+1,j}(x) - \varphi_m(x)| \leq \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} |\varphi_{n,k}(x) - \varphi_m(x)| < \varepsilon$$

for $j \geq J$. By induction, it follows that $|\varphi_{n,j}(x) - \varphi_m(x)| < \varepsilon$ for $n \geq m$ and $j \geq J$, and hence, for $n \geq \max\{m, J\}$, we have

$$|\varphi_{n,n}(x) - \varphi(x)| = |\varphi_{n,n}(x) - \varphi_m(x)| < \varepsilon.$$

We conclude that $\varphi_{n,n} \to \varphi$ q.e. on X, and also that $v_{n,n} \to u$ q.e. in Ω, as $n \to \infty$.

By using Jensen's inequality, we can see that

$$\int_\Omega g_{v_{1,j}}^p \, d\mu \leq \int_\Omega \biggl(\sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{u_k} \biggr)^p d\mu \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} \int_\Omega g_{u_k}^p \, d\mu \leq \int_\Omega g_{u_j}^p \, d\mu$$

and

$$\begin{aligned}\int_X g_{\varphi_{1,j}}^p \, d\mu &\leq \int_X \biggl(\sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{w_k} \biggr)^p d\mu \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} \int_\Omega (g_{u_k} + g_f)^p \, d\mu \\ &\leq 2^p \sum_{k=j}^{N_{1,j}} a_{1,j,k} \int_\Omega (g_{u_k}^p + g_f^p) \, d\mu \leq 2^p \int_\Omega (g_{u_j}^p + g_f^p) \, d\mu.\end{aligned}$$

Assume that for some positive integer n, it is true that

$$\int_\Omega g_{v_{n,j}}^p \, d\mu \leq \int_\Omega g_{u_j}^p \, d\mu \quad \text{and} \quad \int_X g_{\varphi_{n,j}}^p \, d\mu \leq 2^p \int_\Omega (g_f^p + g_{u_j}^p) \, d\mu.$$

Then

$$\int_\Omega g_{v_{n+1,j}}^p \, d\mu \le \int_\Omega \biggl(\sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} g_{v_{n,k}} \biggr)^p d\mu \le \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_\Omega g_{v_{n,k}}^p \, d\mu$$
$$\le \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_\Omega g_{u_k}^p \, d\mu \le \int_\Omega g_{u_j}^p \, d\mu$$

and

$$\int_X g_{\varphi_{n+1,j}}^p \, d\mu \le \int_X \biggl(\sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} g_{\varphi_{n,k}} \biggr)^p d\mu \le \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_X g_{\varphi_{n,k}}^p d\mu$$
$$\le 2^p \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_\Omega (g_f^p + g_{u_k}^p) \, d\mu \le 2^p \int_\Omega (g_f^p + g_{u_j}^p) \, d\mu.$$

By induction, and letting $j = n$, it follows that

$$\int_\Omega g_{v_{n,n}}^p \, d\mu \le \int_\Omega g_{u_n}^p \, d\mu \quad \text{and} \quad \int_X g_{\varphi_{n,n}}^p \, d\mu \le 2^p \int_\Omega (g_f^p + g_{u_n}^p) \, d\mu, \quad n = 1, 2, \dots .$$

Fix an integer $m \ge 1$. Since $\{\varphi_{n,n}\}_{n=1}^\infty$ and $\{g_{\varphi_{n,n}}\}_{n=1}^\infty$ are bounded in $L^p(mB)$ and $\varphi_{n,n} \to \varphi$ q.e. in mB as $n \to \infty$, Corollary 2.11 asserts that $\varphi \in N^{1,p}(mB)$. This implies that $\varphi \in D^p_{\mathrm{loc}}(X)$. Note that g_φ and $g_{\varphi_{n,n}}$ are minimal p-weak upper gradients of φ and $\varphi_{n,n}$, respectively, with respect to mB. Hence, by Corollary 2.11 again, it follows that

$$\int_{mB} g_\varphi^p \, d\mu \le \liminf_{n\to\infty} \int_{mB} g_{\varphi_{n,n}}^p \, d\mu \le \liminf_{n\to\infty} \int_X g_{\varphi_{n,n}}^p \, d\mu$$
$$\le 2^p \liminf_{n\to\infty} \int_\Omega (g_f^p + g_{u_n}^p) \, d\mu = 2^p \int_\Omega g_f^p \, d\mu + 2^p I.$$

Letting $m \to \infty$ yields

$$\int_X g_\varphi^p \, d\mu = \lim_{m\to\infty} \int_{mB} g_\varphi^p \, d\mu \le 2^p \int_\Omega g_f^p \, d\mu + 2^p I < \infty,$$

and hence $\varphi \in D^p(X)$. We conclude that $u - f \in D^p_0(\Omega)$ and $u \in D^p(\Omega)$.

Let $A_n = \{x \in \Omega \colon v_{n,n}(x) < \psi(x)\}$ for $n = 1, 2, ...$, and let $A = \bigcup_{n=1}^\infty A_n$. Then, since $v_{n,n} \to u$ q.e. in Ω as $n \to \infty$, it follows that $u \ge \psi$ q.e. in $\Omega \setminus A$. Because $v_{n,n} \ge \psi$ q.e. in Ω, we have $C_p(A_n) = 0$, and hence $C_p(A) = 0$ by the subadditivity of the capacity. Thus, $u \ge \psi$ q.e. in Ω, and we conclude that $u \in \mathcal{K}_{\psi,f}$.

Proposition 2.14 asserts that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$, and hence $f \in L^p(\Omega')$ for all open $\Omega' \Subset \Omega$. Let

$$\Omega_t = \Bigl\{ x \in tB \cap \Omega \colon \inf_y d(x,y) > \delta/t \Bigr\}, \quad 1 \le t < \infty,$$

where the infimum is taken over all $y \in \partial\Omega$ and $\delta > 0$ is chosen small enough so that $\Omega_1 \ne \varnothing$. Then we have $\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega = \bigcup_{t=1}^\infty \Omega_t$. Moreover, $\{v_{n,n}\}_{n=1}^\infty$ is bounded in $L^p(\Omega_t)$, since

$$\|v_{n,n}\|_{L^p(\Omega_t)} \le \|\varphi_{n,n}\|_{L^p(\Omega_t)} + \|f\|_{L^p(\Omega_t)} \le C'_{tB} + \|f\|_{L^p(\Omega_t)} < \infty.$$

Fix an integer $m \geq 1$. Since $\{v_{n,n}\}_{n=1}^{\infty}$ and $\{g_{v_{n,n}}\}_{n=1}^{\infty}$ are bounded in $L^p(\Omega_m)$, $v_{n,n} \to u$ q.e. in Ω_m as $n \to \infty$, and g_u and $g_{v_{n,n}}$ are minimal p-weak upper gradients of u and $v_{n,n}$, respectively, with respect to Ω_m, by Corollary 2.11, it follows that

$$\int_{\Omega_m} g_u^p \, d\mu \leq \liminf_{n\to\infty} \int_{\Omega_m} g_{v_{n,n}}^p \, d\mu \leq \liminf_{n\to\infty} \int_{\Omega} g_{v_{n,n}}^p \, d\mu \leq \liminf_{n\to\infty} \int_{\Omega} g_{u_n}^p \, d\mu = I.$$

Letting $m \to \infty$ completes the existence part of the proof by showing that

$$I \leq \int_{\Omega} g_u^p \, d\mu = \lim_{m\to\infty} \int_{\Omega_m} g_u^p \, d\mu \leq I.$$

(*Uniqueness*) Suppose that u' and u'' are solutions to the $\mathcal{K}_{\psi,f}$-obstacle problem. We begin this part by showing that $g_{u'} = g_{u''}$ a.e. in Ω. Clearly, $\frac{1}{2}(u' + u'') \in \mathcal{K}_{\psi,f}$, and hence

$$\begin{aligned} \|g_{u'}\|_{L^p(\Omega)} &\leq \|g_{\frac{1}{2}(u'+u'')}\|_{L^p(\Omega)} \leq \left\|\tfrac{1}{2}(g_{u'} + g_{u''})\right\|_{L^p(\Omega)} \\ &\leq \tfrac{1}{2}\|g_{u'}\|_{L^p(\Omega)} + \tfrac{1}{2}\|g_{u''}\|_{L^p(\Omega)} = \|g_{u''}\|_{L^p(\Omega)} = \|g_{u'}\|_{L^p(\Omega)}. \end{aligned}$$

Thus, $g_{u'} = g_{u''}$ a.e. in Ω by the strict convexity of $L^p(\Omega)$.

Now we show that $g_{u'-u''} = 0$ a.e. in Ω. Fix a real number c and let

$$u = \max\{u', \min\{u'', c\}\}.$$

The following shows that $u \in \mathcal{K}_{\psi,f}$. Clearly, $u \in D^p(\Omega)$. Furthermore, we have $u \geq u' \geq \psi$ q.e. in Ω, and $u - f \in D_0^p(\Omega)$ by Lemma 2.8, since

$$u - f \leq \max\{u', u''\} - f = \max\{u' - f, u'' - f\} \in D_0^p(\Omega)$$

and $u - f \geq u' - f \in D_0^p(\Omega)$.

Let $U_c = \{x \in \Omega\colon u'(x) < c < u''(x)\}$. Then we have $g_u = 0$ a.e. in U_c because $U_c \subset \{x \in \Omega\colon u(x) = c\}$. The minimizing property of $g_{u'}$ then implies that

$$\int_{\Omega} g_{u'}^p \, d\mu \leq \int_{\Omega} g_u^p \, d\mu = \int_{\Omega\setminus U_c} g_u^p \, d\mu = \int_{\Omega\setminus U_c} g_{u'}^p \, d\mu,$$

as $g_u = g_{u'} = g_{u''}$ a.e. in $\Omega \setminus U_c$. Hence, $g_{u'} = g_{u''} = 0$ a.e. in U_c for all $c \in \mathbf{R}$, and since

$$\{x \in \Omega\colon u'(x) < u''(x)\} \subset \bigcup_{c\in\mathbf{Q}} U_c,$$

we have $g_{u'} = g_{u''} = 0$ a.e. in $\{x \in \Omega\colon u'(x) < u''(x)\}$. Analogously, the same is true for $\{x \in \Omega\colon u'(x) > u''(x)\}$, and hence

$$g_{u'-u''} \leq (g_{u'} + g_{u''})\chi_{\{x\in\Omega\colon u'(x)\neq u''(x)\}} = 0 \quad \text{a.e. in } \Omega.$$

Because $u' - u'' = u' - f - (u'' - f) \in D_0^p(\Omega)$, there exists $w \in D^p(X)$ such that $w = u' - u''$ in Ω and $w = 0$ outside Ω. We have $g_w = g_{u'-u''}\chi_\Omega = 0$ a.e.

Let $\widetilde{S} = \{x \in X\colon w(x) = 0\}$ and let $t \geq 1$ be arbitrary. Then

$$C_p(tB \cap \widetilde{S}) \geq C_p(tB \setminus \Omega) \geq C_p(B \setminus \Omega) > 0.$$

Maz′ya's inequality (Theorem 2.13) applies to w, and hence there exists a constant $\widetilde{C}_{tB} > 0$ such that

$$\int_{tB\cap\Omega} |u' - u''|^p \, d\mu \leq \int_{2tB} |w|^p \, d\mu \leq \widetilde{C}_{tB} \int_{2\lambda tB} g_w^p \, d\mu = 0.$$

This implies that $u' = u''$ q.e. in $tB \cap \Omega$.

Let $V_m = \{x \in mB \cap \Omega\colon u'(x) \neq u''(x)\}$, $m = 1, 2, ...$, and let $V = \bigcup_{m=1}^{\infty} V_m$. Then $u' = u''$ in $\Omega \setminus V$. Since $C_p(V_m) = 0$ for all m, the subadditivity of the capacity implies that $C_p(V) = 0$, hence $u' = u''$ q.e. in Ω. We conclude that the solution of the $\mathcal{K}_{\psi,f}$-obstacle problem is unique (up to sets of capacity zero). $\square$

If $v = u$ q.e. in Ω and u is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem, then so is v. Indeed, $v = u$ q.e. implies that g_u is a p-weak upper gradient of v. Thus, $v \in D^p(\Omega)$ and $\int_\Omega g_v^p \, d\mu \leq \int_\Omega g_u^p \, d\mu$. Clearly, we have $v \geq \psi$ q.e., and since Lemma 2.8 asserts that $v - f \in D_0^p(\Omega)$, it follows that $v \in \mathcal{K}_{\psi,f}$.

The following criterion for the existence of a unique solution is easy to prove.

Proposition 3.5. *If $\psi, f \in D^p(\Omega)$, then $\mathcal{K}_{\psi,f} \neq \varnothing$ if and only if $(\psi - f)_+ \in D_0^p(\Omega)$.*

Proof. Suppose that $\mathcal{K}_{\psi,f} \neq \varnothing$ and let $v \in \mathcal{K}_{\psi,f}$. Since $(v - f)_+ \in D_0^p(\Omega)$ and

$$0 \leq (\psi - f)_+ \leq (v - f)_+ \quad \text{q.e. in } \Omega,$$

Lemma 2.8 asserts that $(\psi - f)_+ \in D_0^p(\Omega)$.

Conversely, suppose that $(\psi - f)_+ \in D_0^p(\Omega)$. Let $v = \max\{\psi, f\}$. Then we have $v \in D^p(\Omega)$, $v - f = (\psi - f)_+$, and $v \geq \psi$ in Ω. Thus, $v \in \mathcal{K}_{\psi,f}$. $\square$

The following comparison principle (for the version of the obstacle problem defined in Kinnunen–Martio [25]) was obtained in Björn–Björn [3]. Their proof (with trivial modifications) is valid also for our obstacle problem.

Lemma 3.6. *Let $\psi_j\colon \Omega \to \overline{\mathbf{R}}$ and $f_j \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi_j,f_j} \neq \varnothing$, and let u_j be a solution of the $\mathcal{K}_{\psi_j,f_j}$-obstacle problem for $j = 1, 2$. If $\psi_1 \leq \psi_2$ q.e. in Ω and $(f_1 - f_2)_+ \in D_0^p(\Omega)$, then $u_1 \leq u_2$ q.e. in Ω.*

Proof. Let $h = u_1 - f_1 - u_2 + f_2$. Then $h \in D_0^p(\Omega)$ and

$$\begin{aligned} -(f_1 - f_2)_+ - h_- &= -\max\{-(f_2 - f_1), 0\} - \max\{-h, 0\} \\ &= \min\{f_2 - f_1, 0\} + \min\{h, 0\} \leq \min\{f_2 - f_1, h\} \leq h. \end{aligned}$$

Since $-(f_1 - f_2)_+ - h_- \in D_0^p(\Omega)$, Lemma 2.8 asserts that $\min\{f_2 - f_1, h\} \in D_0^p(\Omega)$.

Let $u = \min\{u_1, u_2\}$. Then $u \in D^p(\Omega)$, and because $u_2 \geq \psi_2 \geq \psi_1$ q.e. in Ω, we clearly have $u \geq \psi_1$ q.e. in Ω. Moreover, as $u_1 - f_1 = u_2 - f_2 + h$, we have

$$\begin{aligned} u - f_1 &= \min\{u_1, u_2\} - f_1 = \min\{u_1 - f_1, u_2 - f_1\} \\ &= \min\{u_2 - f_2 + h, u_2 - f_1\} = u_2 - f_2 + \min\{h, f_2 - f_1\}. \end{aligned}$$

Hence, $u - f_1 \in D_0^p(\Omega)$, and we conclude that $u \in \mathcal{K}_{\psi_1,f_1}$.

Let $v = \max\{u_1, u_2\}$. Then $v \in D^p(\Omega)$ and $v \geq \psi_2$ q.e. in Ω. Because

$$\begin{aligned} v - f_2 &= \max\{u_1 - f_2, u_2 - f_2\} = \max\{u_1 - f_2, u_1 - f_1 - h\} \\ &= u_1 - f_1 + \max\{f_1 - f_2, -h\} = u_1 - f_1 - \min\{f_2 - f_1, h\}, \end{aligned}$$

we see that $v - f_2 \in D_0^p(\Omega)$, and hence $v \in \mathcal{K}_{\psi_2,f_2}$.

Let $E = \{x \in \Omega\colon u_2(x) \leq u_1(x)\}$. Since u_2 is a solution of the $\mathcal{K}_{\psi_2,f_2}$-obstacle problem, we have

$$\int_\Omega g_{u_2}^p \, d\mu \leq \int_\Omega g_v^p \, d\mu = \int_E g_{u_1}^p \, d\mu + \int_{\Omega \setminus E} g_{u_2}^p \, d\mu,$$

which implies that

$$\int_E g_{u_2}^p \, d\mu \le \int_E g_{u_1}^p \, d\mu.$$

By using the last inequality, we see that

$$\int_\Omega g_u^p \, d\mu = \int_E g_{u_2}^p \, d\mu + \int_{\Omega \setminus E} g_{u_1}^p \, d\mu \le \int_E g_{u_1}^p \, d\mu + \int_{\Omega \setminus E} g_{u_1}^p \, d\mu = \int_\Omega g_{u_1}^p \, d\mu.$$

Since $u \in \mathcal{K}_{\psi_1, f_1}$ and u_1 is a solution of the $\mathcal{K}_{\psi_1, f_1}$-obstacle problem, this inequality implies that also u is a solution. Thus, $u_1 = u$ q.e. in Ω, and we conclude that $u_1 \le u_2$ q.e. in Ω. □

The following local property of solutions of the obstacle problem can be useful. In some situations it may let us use results from the theory for bounded sets. This is the case when proving Theorems 4.4 and 4.5.

Proposition 3.7. *Let $\psi\colon \Omega \to \overline{\mathbf{R}}$ and $f \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi,f} \ne \varnothing$, and let u be a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. If $\Omega' \subset \Omega$ is open, then u is a solution of the $\mathcal{K}_{\psi,u}(\Omega')$-obstacle problem. Moreover, if $\Omega' \Subset \Omega$, then u is a solution also of the $\mathcal{K}^{\mathrm{B}}_{\psi,u}(\Omega')$-obstacle problem.*

Proof. Let $\Omega' \subset \Omega$ be open. Clearly, $u \in \mathcal{K}_{\psi,u}(\Omega')$. Let $v \in \mathcal{K}_{\psi,u}(\Omega')$ be arbitrary. To complete the first part of the proof, it is sufficient to show that

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu. \tag{3.4}$$

Let $E = \Omega \setminus \Omega'$ and extend v to Ω by letting $v = u$ in E. Then $v \ge \psi$ q.e. in Ω' and $v = u \ge \psi$ q.e. in E. Furthermore, because $v - u \in D^p_0(\Omega')$, we have $v = (v-u) + u \in D^p(\Omega)$ and $v - f = (v-u) + (u-f) \in D^p_0(\Omega)$, and hence we conclude that $v \in \mathcal{K}_{\psi,f}$.

Since u is a solution to the $\mathcal{K}_{\psi,f}$-obstacle problem, we have

$$\int_{\Omega'} g_u^p \, d\mu + \int_E g_u^p \, d\mu = \int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu = \int_{\Omega'} g_v^p \, d\mu + \int_E g_v^p \, d\mu. \tag{3.5}$$

As $u = v$ in E implies that $g_u = g_v$ a.e. in E, we have

$$\int_E g_v^p \, d\mu = \int_E g_u^p \, d\mu \le \int_\Omega g_u^p \, d\mu < \infty.$$

Subtracting the integrals over E in (3.5) yields (3.4).

For the second part, assume that $\Omega' \Subset \Omega$ and let $v \in \mathcal{K}^{\mathrm{B}}_{\psi,u}(\Omega')$ be arbitrary. Clearly, $v \in \mathcal{K}_{\psi,u}(\Omega')$. The first part of the proof asserts that u is a solution of the $\mathcal{K}_{\psi,u}(\Omega')$-obstacle problem and therefore (3.4) holds. By Proposition 2.14, we have $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, and hence $u \in N^{1,p}(\Omega')$. Thus, $u \in \mathcal{K}^{\mathrm{B}}_{\psi,u}(\Omega')$ and the proof is complete. □

There are many equivalent definitions of (super)minimizers in the literature (see Proposition 3.2 in A. Björn [1]). The first definition for metric spaces was given by Kinnunen–Martio [25]. We will follow Björn–Björn–Mäkäläinen–Parviainen [6], and we also follow the custom of not making the dependence on p explicit in the notation.

Definition 3.8. Let $V \subset X$ be nonempty and open. We say that $u \in N^{1,p}_{\mathrm{loc}}(V)$ is a *superminimizer* in V if

$$\int_{\varphi \neq 0} g_u^p \, d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \tag{3.6}$$

holds for all nonnegative $\varphi \in N_0^{1,p}(V)$. Furthermore, u is said to be a *minimizer* in V if (3.6) holds for all $\varphi \in N_0^{1,p}(V)$.

According to Proposition 3.2 in A. Björn [1], it is in fact only necessary to test (3.6) with (all nonnegative and all, respectively) $\varphi \in \mathrm{Lip}_c(V)$.

As a direct consequence of Proposition 3.7 together with Proposition 9.25 in Björn–Björn [4], we have the following result.

Proposition 3.9. *If u is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem, then u is a superminimizer in Ω.*

4. Lsc-regularized solutions and *p*-harmonic solutions

In this section, we make the rather standard assumptions that $1 < p < \infty$, that X is a complete p-Poincaré space, and that μ is doubling. Moreover, we assume that $\Omega \subset X$ is nonempty, open, and such that $C_p(X \setminus \Omega) > 0$.

When μ is doubling, it is true that X is proper if and only if X is complete, and also that X supports a (p,p)-Poincaré inequality if and only if X supports a p-Poincaré inequality (the necessity follows from Hölder's inequality, and the sufficiency was proved in Hajłasz–Koskela [19]; see also Corollary 4.24 in Björn–Björn [4]). Thus, the difference between the standing assumptions of this section and of the previous is that here we make the additional assumption that μ is doubling.

Note that under these assumptions, Poincaré inequalities are self-improving in the sense that X supports a q-Poincaré inequality for some $q < p$ (this was proved by Keith–Zhong [23]). Hence, in this section, we make the same assumptions as Kinnunen–Martio [25], and we can therefore use Theorems 5.1 and 5.5 from [25].

Theorem 4.1. *If $\psi\colon \Omega \to \overline{\mathbf{R}}$ and $f \in D^p(\Omega)$ are such that $\mathcal{K}_{\psi,f} \neq \varnothing$, then there exists a unique lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem.*

The *lsc-regularization* of a function u is the (lower semicontinuous) function u^* defined by

$$u^*(x) := \operatorname*{ess\,lim\,inf}_{y \to x} u(y) := \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u.$$

Proof. Theorem 3.4 asserts that there exists a solution u of the $\mathcal{K}_{\psi,f}$-obstacle problem and that all solutions are equal to u q.e. in Ω. By Proposition 3.9, u is a superminimizer in Ω, and hence by Theorem 5.1 in Kinnunen–Martio [25], we have $u^* = u$ q.e. in Ω. Thus, u^* is the unique lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. $\square$

The following comparison principle improves upon Lemma 3.6.

Lemma 4.2. *Let $\psi_j\colon \Omega \to \overline{\mathbf{R}}$ and $f_j \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi_j,f_j} \neq \varnothing$, and let u_j be the lsc-regularized solution of the $\mathcal{K}_{\psi_j,f_j}$-obstacle problem for $j = 1, 2$. If $\psi_1 \leq \psi_2$ q.e. in Ω and $(f_1 - f_2)_+ \in D_0^p(\Omega)$, then $u_1 \leq u_2$ in Ω.*

Proof. Lemma 3.6 asserts that $u_1 \le u_2$ q.e. in Ω, and since both u_1 and u_2 are lsc-regularized, it follows that for all $x \in \Omega$, we have

$$u_1(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u_1(y) \le \operatorname*{ess\,lim\,inf}_{y \to x} u_2(y) = u_2(x). \qquad \square$$

Definition 4.3. Let $V \subset X$ be nonempty and open. We say that $u \in N^{1,p}_{\mathrm{loc}}(V)$ is *p-harmonic* in V if it is a continuous minimizer in V.

Kinnunen–Martio [25] proved that the solution u of their obstacle problem is continuous in Ω and is a minimizer in the open set $\{x \in \Omega\colon u(x) > \psi(x)\}$ whenever the obstacle ψ is continuous in Ω (Theorem 5.5 in [25]). This is true also for the $\mathcal{K}^{\mathrm{B}}_{\psi,f}(\Omega)$-obstacle problem (see, e.g., Theorem 8.28 in Björn–Björn [4]), and also, as we shall see, for our obstacle problem (that allows for unbounded sets).

Theorem 4.4. *Let $\psi\colon \Omega \to [-\infty, \infty)$ be continuous and let $f \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi,f} \ne \varnothing$. Then the lsc-regularized solution u of the $\mathcal{K}_{\psi,f}$-obstacle problem is continuous in Ω and p-harmonic in the open set $A = \{x \in \Omega\colon u(x) > \psi(x)\}$.*

We also have the following corresponding pointwise result.

Theorem 4.5. *Let $\psi\colon \Omega \to [-\infty, \infty)$ and $f \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi,f} \ne \varnothing$. Then the lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem is continuous at $x \in \Omega$ if ψ is continuous at x.*

Proof. Let $x \in \Omega$ be a given point where ψ is continuous. Let $\Omega' \Subset \Omega$ be open and containing x, and let u be the lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. Proposition 3.7 asserts that u is a solution of the $\mathcal{K}^{\mathrm{B}}_{\psi,u}(\Omega')$-obstacle problem. By Theorem 8.29 in Björn–Björn [4] (which is a special case of Corollary 3.4 in Farnana [16]), it follows that u is continuous at x. $\square$

Proof of Theorem 4.4. The first part follows directly from Theorem 4.5. Now we prove that u is a minimizer in A. The set A is open since ψ and u are continuous. Choose a ball $B \subset A$ and let

$$A_n := \Big\{x \in nB \cap A\colon \inf_y d(x,y) > \delta/n\Big\}, \quad n = 1, 2, \dots,$$

where the infimum is taken over all $y \in \partial A$ and $\delta > 0$ is chosen small enough so that $A_1 \ne \varnothing$. Clearly, $A_1 \Subset A_2 \Subset \cdots \Subset A = \bigcup_{n=1}^{\infty} A_n$. For each $n = 1, 2, \dots$, Proposition 3.7 asserts that u is a solution of the $\mathcal{K}^{\mathrm{B}}_{\psi,u}(A_n)$-obstacle problem, and therefore u is p-harmonic in A_n according to Theorem 5.5 in Kinnunen–Martio [25]. It follows that u is p-harmonic in A (see, e.g., Theorem 9.36 in Björn–Björn [4]). $\square$

Due to Theorem 4.4, the following definition makes sense.

Definition 4.6. The *p-harmonic extension* $H_\Omega f$ of a function $f \in D^p(\Omega)$ to Ω is the continuous solution of the $\mathcal{K}_{-\infty,f}(\Omega)$-obstacle problem.

Note that Definition 4.6 is a generalization of Definition 8.31 in Björn–Björn [4] to Dirichlet functions and to unbounded sets (see Remark 3.3).

Since $H_\Omega f$ is the unique p-harmonic function in Ω such that $f - H_\Omega f \in D^p_0(\Omega)$, we have solved the Dirichlet problem for p-harmonic functions in open sets with boundary values in $D^p(\Omega)$ taken in Sobolev sense. We conclude this paper with a short proof of the following comparision principle.

Lemma 4.7. *If $f_1, f_2 \in D^p(\Omega)$ and $(f_1 - f_2)_+ \in D^p_0(\Omega)$, then $H_\Omega f_1 \le H_\Omega f_2$ in Ω.*

The same conclusion holds if $f_1, f_2 \in D^p(\overline{\Omega})$ and $f_1 \le f_2$ q.e. on $\partial\Omega$.

The first part is just a special case of Lemma 4.2.

Proof. We prove the second part. Clearly, $(f_1 - f_2)_+ \in D^p(\overline{\Omega})$. Since $f_1 \le f_2$ q.e. on $\partial\Omega$, we have $(f_1 - f_2)_+ = 0$ q.e. on $\overline{\Omega} \setminus \Omega$, and hence $(f_1 - f_2)_+ \in D^p_0(\Omega; \overline{\Omega})$. Since $D^p_0(\Omega) = D^p_0(\Omega; \overline{\Omega})$ according to Proposition 2.9, the result follows from the first part. $\square$

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