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ON THE RANGE OF $\sum_{n=1}^{\infty} \pm c_n$

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Abstract. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. In this paper we answer when the range of $\sum_{n=1}^{\infty} \pm c_n$ is dense or equal to the complex plane. Some examples are given to explain our results. As its application, we calculate the Hausdorff dimension of the level sets of a Rademacher series with complex coefficients.

1. Introduction

Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and let

$$Y_{\{c_n\}} = \sum_{n=1}^{\infty} \pm c_n,$$

where the "+" and "−" signs are chosen independently with probability 1/2. When all $c_n = a_n$ are real numbers, it is known that $Y_{\{a_n\}}$ is a random variable if and only if $\{a_n\} \in \ell^2(\mathbf{N})$, i.e., $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ [8]. In this case, the distribution function of $Y_{\{a_n\}}$ is called the *infinite Bernoulli convolution*, which has been studied extensively from 1930's (see [4, 11] and the references given there). It is clear that the support of the distribution function is the whole real line if and only if $\{a_n\} \notin \ell^1(\mathbf{N})$. When all $\{c_n\} \in \ell^2(\mathbf{N})$ are complex numbers, $Y_{\{c_n\}}$ is also a random variable. Clearly $\{c_n\} \notin \ell^1(\mathbf{N})$ does not guarantee that

$$R(\{c_n\}) := \left\{ \sum_{n=1}^{\infty} \pm c_n \right\} = \mathbf{C}. \tag{1.1}$$

Motivating by this, in this paper we want to find rational conditions such that (1.1) holds.

Another motivation for this issue is the Rademacher series, see [1, 5, 7, 10, 13, 14]. A *complex Rademacher series* associated to $\{c_n\}_{n=1}^{\infty}$ is defined by $\sum_{n=1}^{\infty} c_n R(2^{n-1}x)$, where $R(x)$ is a periodic function with period 1 and $R(x) = \pm 1$ according to $x \in [0, 1/2)$ or $[1/2, 1)$, respectively. Clearly we have

$$R(\{c_n\}) = \left\{ \sum_{n=1}^{\infty} \pm c_n \right\} = \left\{ \sum_{n=1}^{\infty} c_n R(2^{n-1}x) \colon x \in [0,1) \right\}. \tag{1.2}$$

We cannot give a sufficient and necessary condition for the question (1.1). In stead of it, we obtain a criterion for $R(\{c_n\})$ being dense in the complex plane.

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Let $c_n = a_n + ib_n \in \mathbf{C}$ for $n \geq 1$ with $\{c_n\} = o(1)$, which means that $\lim_{n\to\infty} c_n = 0$. If $\{\alpha a_n + \beta b_n\}_{n=1}^{\infty} \notin \ell^1$ for any $\alpha, \beta \in \mathbf{R}$ with $\alpha + i\beta \neq 0$, we call the sequence $\{c_n\}_{n=1}^{\infty}$ a *linearly non-summable sequence.*

Theorem 1.1. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $\{c_n\} = o(1)$. Then $R(\{c_n\})$ is dense in the complex plane $\mathbf{C}$ if and only if $\{c_n\}_{n=1}^{\infty}$ is linearly non-summable.*

We are surprised that there are some examples which satisfy $\overline{R(\{c_n\})} = \mathbf{C}$ but $\overline{R(\{c_n\})} \neq \mathbf{C}$ (Example 4.4). At the same time, there are some examples with $\overline{R(\{c_n\})} = \mathbf{C}$ but we do not know whether they are equal to $\mathbf{C}$, an example with this property is $c_n = \frac{1}{n \ln n+1} + \frac{i}{n}$ for $n \geq 1$. The key step of the proof of Theorem 1.1 is the combination lemma (Lemma 2.3).

To give a sufficient condition for $R(\{c_n\}) = \mathbf{C}$ we begin with a notation.

Definition 1.2. Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ be a complex sequence with $\{c_n\} = o(1)$. A number t is called a *ratio* of $\{c_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{c_{n_k}\}_{k=1}^{\infty} \notin \ell^1$ such that $a_{n_k}/b_{n_k} \to t$ as $k \to \infty$, where the number t may be infinity.

It is easy to check that a complex sequence $\{c_n\}_{n=1}^{\infty}$ is linearly non-summable if it has two distinct ratios.

Theorem 1.3. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Then $R(\{c_n\})$ is the complex space if $\{c_n\}_{n=1}^{\infty}$ has two different ratios.*

The difficult part of the proof of Theorem 1.3 is how to show that $R(\{c_n\})$ contains a nonempty interior. We will use Moran function systems (Proposition 3.1) to overcome it.

The other one interesting problem on this issue is to study the level set of Rademacher series. As far back as 1930, Kaczmarz and Steinhaus [9] showed that, for any $a \in \mathbf{R}$, the level set

$$E_a := \left\{ x \in [0,1) \colon \sum_{n=1}^{\infty} a_n R(2^{n-1}x) = a \right\}$$

has continuous cardinality if $\{a_n\} \notin \ell^1$ and $\{a_n\} = o(1)$. In 1962, Beyer [2] proved

$$\dim_H E_a = 1$$

under the assumption $\{a_n\} \in \ell^2 \setminus \ell^1$. Wu [12] showed the same result under the conditions $\{a_n\} \notin \ell^1, \{a_n\} = o(1)$ and another man-made condition $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$. Finally, Xi [14] obtained the result without the man-made condition.

In the complex case, we define the level set by $E_c = \{x \in [0,1) \colon \sum_{n=1}^{\infty} c_n R(2^{n-1}x) = c\}$ for any $c \in \mathbf{C}$. we show that

Theorem 1.4. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $\{c_n\} = o(1)$.*

(1) *If $\{c_n\}$ has two distinct ratios, then $\dim_H E_c = 1$ for any $c \in \mathbf{C}$.*
(2) *If $\{c_n\}$ is linearly non-summable with one ratio, then*

$$\dim_H \left\{ x \in [0,1) \colon \sum_{n=1}^{\infty} c_n R(2^{n-1}x) \in B(c,\delta) \right\} = 1$$

for any $\delta > 0$, where $B(c,\delta)$ is the ball with center c and radius δ.

2. The combination lemma and proof of Theorem 1.1

Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ be a complex sequence with $\{c_n\} = o(1)$ and let $\{-1,1\}^{\mathbf{N}}$ be the set of all sequences $\{x_n\}_{n\in\mathbf{N}}$ satisfying $x_n \in \{-1,1\}$ for $n \geq 1$. We begin with the existence of ratios. Note that, in the definition of a ratio, we demand that the subsequence is not in ℓ^1.

Proposition 2.1. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ be a complex sequence with $\{c_n\} = o(1)$. Then there exists at least one ratio.*

Proof. Let $\Lambda_1 = \{n\colon |\frac{a_n}{b_n}| \leq 1\}$ and $\Lambda_2 = \{n\colon |\frac{a_n}{b_n}| \geq 1\}$. Then, by the symmetry of a_n and b_n, without loss of generality we assume that $\sum_{n\in\Lambda_1} |c_n| = \infty$.

Let $\mathcal{A}_{j,k} = \{n\colon a_n/b_n \in [j/2^k, (j+1)/2^k]\}$ for $k \geq 0$ and $-2^k \leq j < 2^k$. Then, for each k, there exists j_k such that $\{[j_k/2^k, (j_k+1)/2^k]\}_{k=0}^{\infty}$ is a decreasing sequence of sets and $\sum_{n\in\mathcal{A}_{j_k,k}} |c_n| = \infty$ for each $k \geq 0$.

We claim that t_0 is a ratio of $\{c_n\}_{n=1}^{\infty}$, where $t_0 = \lim_{k\to\infty} \frac{j_k}{2^k}$. We prove the claim as follows: Choose a finite set $\mathcal{B}_1$ from $\mathcal{A}_{j_1,1}$ so that $\sum_{n\in\mathcal{B}_1} |c_n| \geq 1$. Then choose $\mathcal{B}_2$ from $\mathcal{A}_{j_2,2}$ so that $\min\{n\colon n \in \mathcal{B}_2\} > \max\{n\colon n \in \mathcal{B}_1\}$ and $\sum_{n\in\mathcal{B}_2} |c_n| \geq 1$. As so on we can choose $\{\mathcal{B}_k\}_{k=1}^{\infty}$ satisfying $\min\{n\colon n \in \mathcal{B}_{k+1}\} > \max\{n\colon n \in \mathcal{B}_k\}$ and $\sum_{n\in\mathcal{B}_k} |c_n| \geq 1$ for each $k \geq 1$. Then the sequence $\{c_n : n \in \cup_{k=1}^{\infty}\mathcal{B}_k\}$ has the ratio t_0. □

The following result says that any two different ratios can be changed arbitrarily without loss anything.

Proposition 2.2. *Let $\{c_n = a_n+ib_n\}_{n=1}^{\infty}$ be a complex sequence and let $\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}$ be a non-singular matrix. Then $R(\{a_n + ib_n\})$ is dense in (equal to) $\mathbf{C}$ if and only if $R(\{(\alpha_1 a_n + \beta_1 b_n) + i(\alpha_2 a_n + \beta_2 b_n)\})$ is dense in (equal to, resp.) $\mathbf{C}$.*

Proof. The assertion follows from the identity

$$R(\{a_n + ib_n\}) \begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix} = R(\{(\alpha_1 a_n + \beta_1 b_n) + i(\alpha_2 a_n + \beta_2 b_n)\}).$$ □

From now on, for any $c = a + ib \in \mathbf{C}$, we use the norm $\|c\| = \max\{|a|, |b|\}$ throughout this paper. For any complex sequence $\{c_n\}_{n\in I}$ where $I \subseteq \mathbf{N}$, we denote that

$$\|\{c_n\}_{n\in I}\| = \sup_{n\in I} \|c_n\|.$$

The following fact will be used in the proof of Lemma 2.3: Let $\|c_1 = a_1+ib_1\| \leq 1$ and $\|c_2 = a_2 + ib_2\| \leq 1$. Then it is easy to check that: $\|c_1 \pm c_2\| > 1$ *is equivalent to that* $|a_1| + |a_2| > 1$, $|b_1| + |b_2| > 1$ *and* $a_1 a_2 b_1 b_2 < 0$. The following combination lemma plays a key role in the proof of Theorem 1.1.

Lemma 2.3. (Combination lemma) *Let $\{c_n\}_{n=1}^{5}$ be complex numbers satisfying $\|c_n\| \leq 1$ for $1 \leq n \leq 5$ and $\|c_n \pm c_{n+1}\| > 1$ for $1 \leq n \leq 4$. Then there exists $\{x_n\}_{n=1}^{5} \in \{-1,1\}^5$ such that*

$$\left\|\sum_{n=1}^{5} x_n c_n\right\| \leq 2.$$

Proof. Let $\mathrm{sign}(x)$ be the sign function, that is, $\mathrm{sign}(x) = -1$, 0 and 1 according to $x < 0$, $x = 0$ and $x > 0$ respectively. Denote $u = c_1\mathrm{sign}(a_1) - c_2\mathrm{sign}(a_2) -$

$c_3\mathrm{sign}(a_3)+c_4\mathrm{sign}(a_4)$. By the above fact and the hypotheses, we have the imaginary part of u satisfying $|\mathrm{Im}(u)| = \big||b_1| + |b_2| - |b_3| - |b_4|\big| \le 1$. Similarly, write $v = c_2\mathrm{sign}(a_2) - c_3\mathrm{sign}(a_3) - c_4\mathrm{sign}(a_4) + c_5\mathrm{sign}(a_5)$, we have $|\mathrm{Im}(v)| \le 1$.

We claim that, if the real part of u satisfies $|\mathrm{Re}(u)| > 1$, then $|\mathrm{Re}(v)| \le 1$. Since $\mathrm{Re}(u) = |a_1|-|a_2|-|a_3|+|a_4|$ and $|a_2|+|a_3| > 1$, the condition $|\mathrm{Re}(u)| > 1$ implies that $|a_1|-|a_2|-|a_3|+|a_4| < -1$. Similarly, if $|\mathrm{Re}(v)| > 1$, then $|a_2|-|a_3|-|a_4|+|a_5| < -1$. Consequently, $|a_1|-2|a_3|+|a_5| < -2$, which leads to $2 < 2|a_3| \le 2$ and it is impossible. Hence the claim follows.

The result follows by choosing $\{x_n\}_{n=1}^5 \in \{-1,1\}^5$ such that $\sum_{n=1}^5 x_n c_n = u + c_5$ when $|\mathrm{Re}(u)| \le 1$ or $\sum_{n=1}^5 x_n c_n = c_1 + v$ when $|\mathrm{Re}(u)| > 1$. □

Lemma 2.4. *Let $\{c_n\}_{n=1}^N$ be complex numbers with all $\|c_n\| \le 1$. Then there exists $\{x_n\}_{n=1}^N \in \{-1,1\}^N$ such that*

$$\left\|\sum_{n=1}^k x_n c_n\right\| \le 5, \quad \text{for all } 1 \le k \le N.$$

Proof. We prove it by induction. Assume that the result holds for N and $N \ge 5$. For $N+1$, we show it by two cases:

Case 1. There exists j, $1 \le j \le 4$, such that either $\|c_j+c_{j+1}\| \le 1$ or $\|c_j-c_{j+1}\| \le 1$. Without loss of generality we say that $\|c_1+c_2\| \le 1$. By induction there exists $\{x_i\}_{i=1}^N \in \{-1,1\}^N$ such that

$$\left\|x_1(c_1+c_2) + \sum_{i=2}^k x_i c_{i+1}\right\| \le 5$$

for $2 \le k \le N$, this implies the assertion for $N+1$;

Case 2. The assumption in Case 1 fails. We replace $\{c_1, c_2, c_3, c_4\}$ by u if $\|u\| \le 1$ or $\{c_2, c_3, c_4, c_5\}$ by v if $\|u\| > 1$, where u and v are given in the proof of Lemma 2.3. Then the assertion follows by Lemma 2.3 and the same idea of Case 1. □

Now we can give a result on the controlling problem.

Proposition 2.5. *Let $\{c_n\}_{n=1}^\infty$ be a sequence of complex numbers with $\{c_n\} = o(1)$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \in \{-1,1\}^{\mathbf{N}}$ such that*

$$\left\|\sum_{n=1}^\infty x_n c_n\right\| \le 5\|\{c_n\}_{n=1}^\infty\|.$$

Proof. Since $\{c_n\} = o(1)$ as $n \to \infty$, there exists an increasing natural number sequence $\{N_k\}_{k=0}^\infty$ such that $N_0 = 1$ and

$$\left\|\{c_n\}_{n=N_{k-1}}^{N_k-1}\right\| \le 2^{-k}\|\{c_n\}_{n=1}^\infty\|$$

for all $k \ge 1$. Clearly the result follows by using Lemma 2.4 for each subsequence $\{c_n\}_{n=N_{k-1}}^{N_k-1}$. □

Lemma 2.6. *Let $\{c_n = a_n+ib_n\}_{n=1}^\infty$ with $\{c_n\} = o(1)$ be linearly non-summable. If there exists a non-summable subsequence of $\{c_n\}_{n=1}^\infty$ such that its real or imaginary part is summable, then $R(\{c_n\})$ is dense in $\mathbf{C}$.*

Proof. Let $\{c_{n_k}\}_{k=1}^{\infty}$ be a non-summable subsequence satisfying that its real or imaginary part is summable. Without loss of generality we assume that its real part $\sum_{k=1}^{\infty} |a_{n_k}|$ converges and $n_1 > 1$.

Note that $\{a_n\}_{n=1}^{\infty}$ is not in ℓ^1 with $a_n \to 0$. For any $a + bi \in \mathbf{C}$, there exists a sequence $\{x_n\} \in \{-1, 1\}^{\mathbf{N}}$ such that $a = \sum_{n=1}^{\infty} a_n x_n$. We denote $B_k = \sum_{n=1}^{n_k-1} b_n x_n$ for $k \geq 1$, where $\{n_k\}$ is given in the above subsequence. Let $\Lambda_k = \{n_k, n_{k+1}, n_{k+2}, \ldots\}$ and $\Lambda_k^c = \mathbf{N} \setminus (\Lambda_k \cup \{1, 2, \ldots, n_k - 1\})$.

Since $\{b_{n\in\Lambda_k}\}$ is not summable, there exist $y_n \in \{-1, 1\}$ for $n \geq n_k$ dependent on k such that $\sum_{n\in\Lambda_k} b_n y_n = b - B_k$ and $\| \sum_{n\in\Lambda_k^c} c_n y_n \| \leq 5 \| \{c_n\}_{n\in\Lambda_k^c} \|$. Hence we have

$$\begin{aligned}\sum_{n=1}^{n_k-1} c_n x_n + \sum_{n=n_k}^{\infty} c_n y_n &= a - \sum_{n=n_k}^{\infty} a_n x_n + iB_k + \sum_{n\in\Lambda_k} a_n y_n + i(b - B_k) + \sum_{n\in\Lambda_k^c} c_n y_n \\ &= a + bi - \sum_{n=n_k}^{\infty} a_n x_n + \sum_{n\in\Lambda_k} a_n y_n + \sum_{n\in\Lambda_k^c} c_n y_n.\end{aligned}$$

Note that

$$\left| - \sum_{n=n_k}^{\infty} a_n x_n + \sum_{n\in\Lambda_k} a_n y_n + \sum_{n\in\Lambda_k^c} c_n y_n \right| \leq \left| \sum_{n=n_k}^{\infty} a_n x_n \right| + \sum_{n\in\Lambda_k} |a_n| + 5 \| \{c_n\}_{n\in\Lambda_k^c} \|,$$

which tends to zero when k tends to infinity. Then the proof is complete. □

Proof of Theorem 1.1. We first prove the sufficiency. Suppose that $\{c_n\}_{n=1}^{\infty}$ is linearly non-summable. By Lemma 2.1 and Proposition 2.2 we may assume that there exists a subsequence $\{c_{n_k}\}_{k=1}^{\infty} \notin \ell^1$ such that a_{n_k}/b_{n_k} tends to 0 when k tends to infinity. In this case we still have $\sum_{n=1}^{\infty} |a_n| = \infty$ by the linear non-summation. When $\sum_{k=1}^{\infty} |a_{n_k}|$ converges, the sufficient condition follows by Lemma 2.6; When $\sum_{k=1}^{\infty} |a_{n_k}|$ diverges, we construct a subsequence $\{l_k\}$ of $\{n_k\}$ such that $\sum_{k=1}^{\infty} |a_{l_k}| < \infty$ and $\sum_{k=1}^{\infty} |b_{l_k}| = \infty$. This implies the sufficiency according to Lemma 2.6 again.

Now we construct a desired subsequence of $\{c_{n_k}\}_{k=1}^{\infty}$ if $\sum_{k=1}^{\infty} |a_{n_k}|$ diverges. Note that in this case $\sum_{k=1}^{\infty} |b_{n_k}|$ diverges also. Denote $\Lambda_m = \{k \colon |a_{n_k}/b_{n_k}| < 2^{-m}\}$ for $m \geq 1$. Then k belongs to Λ_m for sufficiently large k and thus $\sum_{k\in\Lambda_m} |b_k| = \infty$ for each m. We can choose finite sets $\Gamma_k \subset \Lambda_k$ such that $\sum_{n\in\Gamma_k} |b_n| \in (1, 2)$ and $\max \Gamma_k < \min \Gamma_{k+1}$ for $k \geq 1$. We claim that $\Gamma := \cup_{k=1}^{\infty} \Gamma_k$ is the index set of a desired subsequence. In fact,

$$\sum_{n\in\Gamma} |a_n| = \sum_{k=1}^{\infty} \sum_{n\in\Gamma_k} |a_n| \leq \sum_{k=1}^{\infty} 2^{-k} \sum_{n\in\Gamma_k} |b_n| \leq 2$$

and

$$\sum_{n\in\Gamma} |b_n| = \sum_{k=1}^{\infty} \sum_{n\in\Gamma_k} |b_n| \geq \sum_{k=1}^{\infty} 1 = \infty.$$

Now we prove the necessity. Suppose that $R(\{c_n\})$ is dense in $\mathbf{C}$, then both the real and imaginary parts of $\{c_n\}$ are not in ℓ^1. If there exist α and β such that $\{\alpha a_n + \beta b_n\}_{n=1}^{\infty} \in \ell^1$, this implies a contradiction by Proposition 2.2. □

3. Moran function systems and proof of Theorem 1.3

In the proof of Theorem 1.3, the difficult point is to show that $R(\{c_n\})$ contains a nonempty interior. The following proposition will help us to show it [6]. We begin with some notations.

Given a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers with all $n_k \geq 2$ and a sequence $\{f_{k,i}(x)\colon k \geq 1, i = 1, 2, \ldots, n_k\}$ of functions from $\mathbf{R}^n$ to itself, which satisfy that

$$\|f_{k,i}(x) - f_{k,i}(y)\| \leq r\|x - y\|$$

for all $k \geq 1$ and $1 \leq i \leq n_k$, where $0 < r < 1$. We say that the sequence is a *Moran function system with contraction ratio not exceeding* r. Define $\prod_{k=1}^{m}\{1, 2, \ldots, n_k\} = \{\sigma = \sigma_1\sigma_2\cdots\sigma_m \colon \sigma_k \in \{1, 2, \ldots, n_k\},\ 1 \leq k \leq m\}$ for $m \geq 1$ and $\prod_{k=1}^{\infty}\{1, 2, \ldots, n_k\} = \{\sigma = \sigma_1\sigma_2\cdots\cdots \colon \text{each } \sigma_k \in \{1, 2, \ldots, n_k\}\}$. For any $\sigma = \sigma_1\cdots\sigma_m$, we define

$$f_\sigma(x) = f_{1,\sigma_1} \circ f_{2,\sigma_2} \circ \cdots \circ f_{m,\sigma_m}(x),$$

which is the composing function of $f_{i,\sigma_i}, i = 1, 2, \ldots, m$.

Proposition 3.1. *Let $F = \{f_{k,i}(x)\colon i = 1, 2, \ldots, n_k, k \geq 1\}$ be a Moran function system with contraction ratio not exceeding r. Suppose that the set $\{f_{k,i}(0)\colon i = 1, 2, \ldots, n_k, k \geq 1\}$ is bounded with bound M, then*

(1) *For any $\sigma = \sigma_1\sigma_2\cdots \in \prod_{k=1}^{\infty}\{1, 2, \ldots, n_k\}$, the limit*

$$\lim_{k\to\infty} f_{\sigma_1\cdots\sigma_k}(0)$$

exists, we denote the value as $f_\sigma(0)$;

(2) *The set K_F, $K_F := \{f_\sigma(0)\colon \sigma \in \prod_{k=1}^{\infty}\{1, 2, \ldots, n_k\}\}$, is a nonempty compact set;*

(3) *Let Q be a compact set so that $Q \subseteq \bigcup_{i=1}^{n_k} f_{k,i}(Q)$ for all $k \geq 1$, then $Q \subseteq K_F$.*

Proof. Let $B(0, R)$ be the closure ball with center 0 and radius R. Then, for each k and i with $1 \leq i \leq n_k$, we have

$$f_{k,i}(B(0, R)) \subseteq B(f_{k,i}(0), rR) \subseteq B(0, M + rR) \subseteq B(0, R)$$

if $R > M/(1-r)$. This implies that $\{\bigcup_{\sigma\in\prod_{i=1}^{m}\{1,2,\ldots,n_i\}} f_\sigma(B(0, R))\}_{m=1}^{\infty}$ is a decreasing sequence of compact sets. Hence,

$$\bigcap_{m=1}^{\infty} \bigcup_{\sigma\in\prod_{i=1}^{m}\{1,2,\ldots,n_i\}} f_\sigma(B(0, R))$$

is a nonempty compact set, which is independent of large R. For any $\sigma = \sigma_1\sigma_2\cdots \in \prod_{k=1}^{\infty}\{1, 2, \ldots, n_k\}$, we have

$$\bigcap_{k=1}^{\infty} f_{\sigma_1\cdots\sigma_k}(B(0, R)) = \Big\{\lim_{k\to\infty} f_{\sigma_1\cdots\sigma_k}(0) = f_\sigma(0)\Big\}.$$

Hence,

$$K_F = \bigcap_{m=1}^{\infty} \bigcup_{\sigma\in\prod_{i=1}^{m}\{1,2,\ldots,n_i\}} f_\sigma(B(0, R)). \tag{3.1}$$

This deduces (1) and (2).

By the hypothesis in (3) we have

$$Q \subseteq \bigcup_{\sigma \in \prod_{k=1}^{m}\{1,2,\dots,n_k\}} f_\sigma(Q)$$

for $m \geq 1$. Choosing R so that $Q \subseteq B(0,R)$, we obtain $Q \subseteq K_F$ by the above and (3.1). □

Next we construct a Moran function system F and a cube $Q = [-5,5] \times [-5,5]$ such that $Q \subseteq K_F \subseteq R(\{c_n\})$ if $\{c_n\}_{n=1}^{\infty}$ has two distinct ratios.

The next lemma says that we can construct a subsequence from $\{c_n\}$ such that its real part and imaginary part are comparable to the sequence $\{\delta^n\}_{n=1}^{\infty}$, where δ is given arbitrarily in $(0,1)$.

Lemma 3.2. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ be a complex sequence with $\{c_n\} = o(1)$ and $\lim_{n\to\infty} a_n/b_n = t$, where $0 < t < \infty$. Then, for any $0 < \delta < 1$ and a positive number sequence $\{\eta_k\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} \eta_k < \infty$, there exists a sequence $\{\Lambda_k\}_{k=1}^{\infty}$ of finite sets with $\min \Lambda_{k+1} > \max \Lambda_k$ for all $k \geq 1$ such that*

$$\Big|\sum_{n\in\Lambda_k} |b_n| - \delta^k\Big| \leq \eta_k, \quad \left|\frac{\sum_{n\in\Lambda_k} a_n \operatorname{sign}(b_n)}{\sum_{n\in\Lambda_k} |b_n|} - t\right| < \eta_k$$

for all $k \geq 1$ and

$$\sum_{k=1}^{\infty} \sum_{n\in\Lambda_k} \|c_n\| < \infty.$$

Proof. We define $\Gamma_k = \{n : |a_n/b_n - t| + \|c_n\| < \eta_k\}$ for $k \geq 1$. Then by hypotheses we have

$$\sum_{n\in\Gamma_k} |b_n| = \infty.$$

Now we choose finite sets Λ_k from Γ_k by induction. Note that b_n tends to zero when n tends to infinity, we can choose a finite set Λ_1 from Γ_1 such that $\big|\sum_{n\in\Lambda_1} |b_n| - \delta\big| \leq \eta_1$, and then choose a finite set Λ_2 from Γ_2 such that $\min \Lambda_2 > \max \Lambda_1$ and $\big|\sum_{n\in\Lambda_2} |b_n| - \delta^2\big| \leq \eta_2$. In general we obtain a sequence $\{\Lambda_k\}_{k=1}^{\infty}$ of finite sets with $\min \Lambda_{k+1} > \max \Lambda_k$ for all $k \geq 1$ and

$$\big|\sum_{n\in\Lambda_k} |b_n| - \delta^k\big| \leq \eta_k, \quad \text{for } k \geq 1.$$

Now we show the second inequality. Write $t_n = a_n/b_n$, then $|t_n - t| < \eta_k$ for $n \in \Gamma_k$. Hence, the second assertion follows from that

$$\frac{\sum_{n\in\Lambda_k} a_n \operatorname{sign}(b_n)}{\sum_{n\in\Lambda_k} |b_n|} - t = \sum_{n\in\Lambda_k} (t_n - t) \frac{|b_n|}{\sum_{m\in\Lambda_k} |b_m|}.$$

Note that $|a_n| \leq (|t| + \eta_k)|b_n|$ for $n \in \Gamma_k$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{n\in\Lambda_k} \|c_n\| &\leq \sum_{k=1}^{\infty} \sum_{n\in\Lambda_k} (|a_n| + |b_n|) \leq \sum_{k=1}^{\infty} (1 + |t| + \eta_k) \sum_{n\in\Lambda_k} |b_n| \\ &\leq \sum_{k=1}^{\infty} (1 + |t| + \eta_k)(\delta^k + \eta_k) \leq (1 + |t| + \sum_{k=1}^{\infty} \eta_k) \sum_{k=1}^{\infty} (\delta^k + \eta_k) < \infty. \quad \square\end{aligned}$$

Applying Lemma 3.2 to a special case, we have the following lemma.

Lemma 3.3. *Let* $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ *and* $\{\gamma_n = \alpha_n + i\beta_n\}_{n=1}^{\infty} \notin \ell^1$ *with* $\{c_n\}, \{\gamma_n\} = o(1)$. *Suppose that* $\lim_{n\to\infty} a_n/b_n = 2$ *and* $\lim_{n\to\infty} \beta_n/\alpha_n = 3$, *then, for any* $\delta \in (0,1)$, *there exist two sequences* $\{\Lambda_k\}_{k=1}^{\infty}, \{\Gamma_k\}_{k=1}^{\infty}$ *of finite sets with* $\min \Lambda_{k+1} > \max \Lambda_k$ *and* $\min \Gamma_{k+1} > \max \Gamma_k$ *for all* $k \geq 1$ *such that*

$$\frac{105}{64}\delta^k \leq \sum_{n\in\Lambda_k} a_n \mathrm{sign}(b_n) \leq \frac{153}{64}\delta^k, \quad \frac{7}{8}\delta^k \leq \sum_{n\in\Lambda_k} |b_n| \leq \frac{9}{8}\delta^k$$

and

$$\frac{7}{8}\delta^k \leq \sum_{n\in\Gamma_k} |\alpha_n| \leq \frac{9}{8}\delta^k, \quad \frac{161}{64}\delta^k \leq \sum_{n\in\Gamma_k} \beta_n \mathrm{sign}(\alpha_n) \leq \frac{225}{64}\delta^k.$$

Proof. Using Lemma 3.2 for $\{c_n = a_n + ib_n\}_{n=1}^{\infty}$ and taking $\eta_k = \frac{1}{8}\delta^k$, we have

$$\frac{7}{8}\delta^k \leq \sum_{n\in\Lambda_k} |b_n| \leq \frac{9}{8}\delta^k$$

and

$$(2-\frac{1}{8})\frac{7}{8}\delta^k \leq (2-\frac{1}{8}\delta^k)\frac{7}{8}\delta^k \leq \sum_{n\in\Lambda_k} a_n \mathrm{sign}(b_n) \leq (2+\frac{1}{8}\delta^k)\frac{9}{8}\delta^k \leq (2+\frac{1}{8})\frac{9}{8}\delta^k.$$

Similarly, using Lemma 3.2 for $\{\beta_n + i\alpha_n\}_{n=1}^{\infty}$ and taking $\eta_k = \frac{1}{8}\delta^k$, we have

$$\frac{7}{8}\delta^k \leq \sum_{n\in\Gamma_k} |\alpha_n| \leq \frac{9}{8}\delta^k$$

and

$$(3-\frac{1}{8})\frac{7}{8}\delta^k \leq \sum_{n\in\Gamma_k} \beta_n \mathrm{sign}(\alpha_n) \leq (3+\frac{1}{8})\frac{9}{8}\delta^k. \qquad \square$$

Now we construct the desired Moran function system from Lemma 3.3. Let $\{\Lambda_k\}$ and $\{\Gamma_k\}$ be given in Lemma 3.3. We define

$$a_k^1 = \sum_{n\in\Lambda_k} a_n \mathrm{sign}(b_n), \quad b_k^1 = \sum_{n\in\Lambda_k} |b_n|$$

and

$$\alpha_k^1 = \sum_{n\in\Gamma_k} |\alpha_n|, \quad \beta_k^1 = \sum_{n\in\Gamma_k} \beta_n \mathrm{sign}(\alpha_n)$$

for $k \geq 1$. Write

$$\mathcal{D}_k = \{\pm[(a_k^1 + ib_k^1) \pm (\alpha_k^1 + i\beta_k^1)]\} = \{d_{k,1}, d_{k,2}, d_{k,3}, d_{k,4}\}$$

and then define

$$f_{k,d}(z) = \delta z + \delta^{1-k} d, \quad d \in \mathcal{D}_k.$$

We will show that the above sequence of functions is the desired one. Let $Q = [-5,5]\times[-5,5]$. It is not difficult to see that, for all $k \geq 1$ and δ closed to 1, for example $\delta = 0.99$, we have

$$Q \subseteq \bigcup_{d\in\mathcal{D}_k} f_{k,d}(Q). \tag{3.2}$$

In fact, by Lemma 3.3 we have $(a_k^1 + \alpha_k^1, b_k^1 + \beta_k^1) \subset [\delta^k, 4.7\delta^k] \times [\delta^k, 4.7\delta^k]$ and $(a_k^1 - \alpha_k^1, b_k^1 - \beta_k^1) \subset [0.4\delta^k, 2\delta^k] \times [-3\delta^k, -\delta^k]$. Then, when δ is chosen closed to 1 enough, we have $[0,5]\times[0,5] \subseteq f_{k,d}(Q)$ if $d = (a_k^1+ib_k^1)+(\alpha_k^1+i\beta_k^1)$ and $[0,5]\times[-5,0] \subseteq f_{k,d}(Q)$ if $d = (a_k^1 + ib_k^1) - (\alpha_k^1 + i\beta_k^1)$. This implies (3.2) by the symmetry property of all $\mathcal{D}_k$.

Lemma 3.4. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty} \notin \ell^1$ and $\{\gamma_n = \alpha_n + i\beta_n\}_{n=1}^{\infty} \notin \ell^1$ with $\{c_n\}, \{\gamma_n\} = o(1)$. Suppose $\lim_{n\to\infty} a_n/b_n = 2$ and $\lim_{n\to\infty} \beta_n/\alpha_n = 3$, then there exists $c = a + bi \in \mathbf{C}$ such that*

$$c + [-5,5] \times [-5,5] \subseteq R(\{c_n\}) + R(\{\gamma_n\}).$$

Proof. With the same notations we have a Moran function system $G := \{f_{k,d}(x) = \delta z + \delta^{1-k}d\colon d \in \mathcal{D}_k, k \geq 1\}$, where δ is chosen so that (3.2) holds. Note that $f_{k,d}(0) = \delta^{1-k}d \in [-5,5] \times [-5,5]$. By Proposition 3.1 we have

$$Q = [-5,5] \times [-5,5] \subseteq K_G = \left\{ f_\sigma(0)\colon \sigma \in \prod_{k=1}^{\infty}\{1,2,3,4\} \right\}.$$

Since

$$f_{\sigma_1\cdots\sigma_m}(0) = d_{1,\sigma_1} + d_{2,\sigma_2} + \cdots + d_{k,\sigma_m},$$

we have

$$K_G = \left\{ \sum_{k=1}^{\infty} d_{k,\sigma_k}\colon \text{all } \sigma_k \in \{1,2,3,4\} \right\} \subseteq R(\{c_n\}_{n\in\cup_{k=1}^{\infty}\Lambda_k}) + R(\{\gamma_n\}_{n\in\cup_{k=1}^{\infty}\Gamma_k}).$$

According to Proposition 2.5, there exist $x_n \in \{-1,1\}$ for $n \in \mathbf{N} \setminus \cup_{k=1}^{\infty}\Lambda_k$ and $y_n \in \{-1,1\}$ for $n \in \mathbf{N} \setminus \cup_{k=1}^{\infty}\Gamma_k$ such that

$$\sum_{n\in\mathbf{N}\setminus\cup_{k=1}^{\infty}\Lambda_k} x_n c_n + \sum_{n\in\mathbf{N}\setminus\cup_{k=1}^{\infty}\Gamma_k} y_n\gamma_n = a + bi.$$

Hence,

$$R(\{c_n\}) + R(\{\gamma_n\}) \supseteq R(\{c_n\}_{n\in\cup_{k=1}^{\infty}\Lambda_k}) + R(\{\gamma_n\}_{n\in\cup_{k=1}^{\infty}\Gamma_k}) + a + bi \supseteq Q + c. \quad \square$$

Proof of Theorem 1.3. We first note that the sequence $\{c_n\}$ with two ratios must be linearly non-summable. By Proposition 2.2 we can assume that both 2 and 3^{-1} are ratios. Since any sequence can be decomposed into two sequences with the same ratio, we can assume that the sequence$\{c_n\}_{n=1}^{\infty}$ is decomposed into three sequences: $\{c_n^{(1)} = a_n^{(1)} + ib_n^{(1)}\}_{n=1}^{\infty}$, $\{c_n^{(2)} = a_n^{(2)} + ib_n^{(2)}\}_{n=1}^{\infty}$ and $\{c_n^{(3)}\}_{n=1}^{\infty}$, which satisfy that

$$\lim_{n\to\infty} a_n^{(1)}/b_n^{(1)} = 2, \quad \lim_{n\to\infty} b_n^{(2)}/a_n^{(2)} = 3$$

and $\{c_n^{(3)}\}_{n=1}^{\infty}$ is linearly non-summable. Then the result follows by Lemma 3.4 and Theorem 1.1. $\square$

Definition 3.5. We call a sequence $\{c_n = a_n+ib_n\}_{n=1}^{\infty}$ with $a_n/b_n \to t$ *changeable* if there exists a partition $\{\Lambda_k\}_{k=1}^{\infty}$ of $\mathbf{N}$ and a sequence $\{x_n\}_{n=1}^{\infty} \in \{-1,1\}^{\infty}$ such that the new sequence $\{\sum_{n\in\Lambda_k} x_n c_n\}_{k=1}^{\infty} \notin \ell^1$ satisfies that $\lim_{k\longrightarrow\infty} \sum_{n\in\Lambda_k} x_n c_n = 0$ and

$$\lim_{k\to\infty} \frac{\sum_{n\in\Lambda_k} x_n a_n}{\sum_{n\in\Lambda_k} x_n b_n} = t' \neq t.$$

Theorem 3.6. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty}$ with $\{c_n\} = o(1)$ be a linearly non-summable sequence satisfying $a_n/b_n \to t$. Suppose that the sequence $\{c_n\}$ can be decomposed into two non-summable sequences such that one of them is changeable, then $R(\{c_n\}) = \mathbf{C}$.*

Proof. By Definition 3.5 we can change the sequence $\{c_n\}$ so that it has at least two distinct radios. Then the result follows by Theorem 1.3. □

4. Some examples

We have showed that any complex sequence $\{c_n\}_{n=1}^{\infty} \notin \ell^1$ with $\{c_n\} = o(1)$ has at lease one ratio (Proposition 2.1) and $R(\{c_n\}) = \mathbf{C}$ if it has two different ratios (Theorem 1.3). The following example says that there exist complex sequences with only one ratio which range is the complex space.

Example 4.1. Let $\{c_n = \frac{(-1)^n}{n\ln(n+1)} + \frac{i}{n}\}_{n=1}^{\infty}$ with one ratio 0. Then $R(\{c_n\}) = \mathbf{C}$.

Proof. We decompose the sequence $\{c_n\}$ into two sequences: $\{c_{4k+1}, c_{4k+2}\}_{k=0}^{\infty}$ and $\{c_{4k+3}, c_{4k+4}\}_{k=0}^{\infty}$. Note that

$$\begin{aligned} -c_{4k+1} + c_{4k+2} &= \frac{1}{(4k+1)\ln(4k+2)} + \frac{1}{(4k+2)\ln(4k+3)} - \frac{i}{(4k+1)(4k+2)} \\ &:= \alpha_k + i\beta_k \end{aligned}$$

and

$$\lim_{k\to\infty} \frac{\alpha_k}{\beta_k} = \infty.$$

Then the sequence $\{c_{4k+1}, c_{4k+2}\}_{k=0}^{\infty}$ is changeable and thus the assertion follows by Theorem 3.6. □

Remark 4.2. Let $\{c_n = \frac{1}{n\ln(n+1)} + \frac{i}{n}\}_{n=1}^{\infty}$ with one ratio 0. It is easy to check that the sequence $\{c_n\}_{n=1}^{\infty}$ is linearly non-summable, then $R(\{c_n\})$ is dense in $\mathbf{C}$ by Theorem 1.1, but we do not know whether $R(\{c_n\}) = \mathbf{C}$.

To give an example so that the range of a sequence is dense in $\mathbf{C}$ but not equal to $\mathbf{C}$, we begin with the following lemma.

Lemma 4.3. *Let $A_n = 2^{-n}(\mathbf{Z} + [-1/4, 1/4])$ and $A = \bigcup_{n=1}^{\infty} A_n$. Then $A \neq \mathbf{R}$.*

Proof. We show that $1/3 \notin A$. If $1/3 \in A$, there exists $n \geq 1$ such that $1/3 \in A_n$, i.e., $2^n/3 \in \mathbf{Z} + [-1/4, 1/4]$. Note that $2^n/3 = k_n + r_n/3$ for some $k_n \in \mathbf{Z}$ and $r_n \in \{1, 2\}$, then $\mathrm{dist}(2^n/3, \mathbf{Z}) = 1/3$. This yields a contradiction to $2^n/3 \in \mathbf{Z} + [-1/4, 1/4]$. □

Example 4.4. Let $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ be two increasing integer sequences with $m_0 = n_0 = 0$ and $n_{k+1} \geq n_k + m_k + 3$ for $k \geq 0$. Define

$$a_j = 2^{-m_k}, \quad b_j = 2^{-m_k - n_k} \quad \text{and} \quad c_j = a_j + ib_j,$$

whenever

$$\sum_{l=0}^{k-1} 2^{m_l + n_l} \leq j < \sum_{l=0}^{k} 2^{m_l + n_l} \quad \text{and } k \geq 1.$$

Then $R(\{c_j\})$ is dense in $\mathbf{C}$ but not equal to $\mathbf{C}$.

Proof. We first show that the complex sequence $\{c_j\}$ is linearly non-summable. Note that

$$\sum_{j=1}^{\infty} a_j \geq \sum_{j=1}^{\infty} b_j = \sum_{k=1}^{\infty} \sum_{\sum_{l=0}^{k-1} 2^{m_l+n_l}}^{\sum_{l=0}^{k} 2^{m_l+n_l}-1} 2^{-m_k-n_k} = \sum_{k=1}^{\infty} 1 = \infty.$$

For any $\alpha, \beta \in \mathbf{R}$ with $\alpha + i\beta \neq 0$, clearly $\{\alpha a_j + \beta b_j\}$ does not lie in ℓ^1 when $\alpha = 0$. When $\alpha \neq 0$, we have $\lim_{j\to\infty}(\alpha a_j + \beta b_j)/a_j = \alpha$. Then $\{\alpha a_j + \beta b_j\}$ is non-summable and thus $\{c_j\}$ is linearly non-summable. By Theorem 1.1, $R(\{c_j\})$ is dense in $\mathbf{C}$.

Secondly, we show that $R(\{c_j\})$ is not equal to $\mathbf{C}$. For any $\{x_j\}_{j=1}^{\infty} \in \{-1,1\}^{\infty}$ such that $\sum_{j=1}^{\infty} x_j c_j$ converges, we have

$$\sum_{j=1}^{\infty} x_j c_j = \sum_{k=1}^{\infty} l_k 2^{-m_k} + i \sum_{k=1}^{\infty} l_k 2^{-m_k-n_k},$$

where all l_k are integers. Hence, there exists $k_0 \geq 1$ such that $|l_k 2^{-m_k}| \leq 1$ for $k > k_0$. Since

$$\sum_{k=1}^{\infty} l_k 2^{-m_k-n_k} = 2^{-m_{k_0}-n_{k_0}} \left(\sum_{k=1}^{k_0} l_k 2^{m_{k_0}-m_k+n_{k_0}-n_k} + 2^{m_{k_0}} \sum_{k=k_0+1}^{\infty} l_k 2^{-m_k} 2^{-(n_k-n_{k_0})} \right)$$

and

$$\left| 2^{m_{k_0}} \sum_{k=k_0+1}^{\infty} l_k 2^{-m_k} 2^{-(n_k-n_{k_0})} \right| \leq 2^{-(n_{k_0+1}-n_{k_0}-m_{k_0}-1)} \leq \frac{1}{4},$$

we have $\sum_{k=1}^{\infty} l_k 2^{-m_k-n_k} \in A$, where A is given in Lemma 4.3. Consequently, the imaginary part of $R(\{c_j\})$ is contained in A and thus the assertion follows by Lemma 4.3. □

5. Hausdorff dimension of the level sets

Let $\{x_n\}, \{y_n\} \in \{-1,1\}^{\mathbf{N}}$. Define

$$d(\{x_n\}, \{y_n\}) = 2^{-k},$$

where k satisfies that $x_i = y_i$ for $1 \leq i < k$ and $x_k \neq y_k$. It is well-known (easy check) that $\{-1,1\}^{\mathbf{N}}$ is a complete metric space with this metric $d(\cdot,\cdot)$. Similarly we define the Hausdorff dimension on $\{-1,1\}^{\mathbf{N}}$ by, for any $B \subseteq \{-1,1\}^{\mathbf{N}}$,

$$\dim_H B = \sup \left\{ s \colon \liminf_{\delta\to 0} \left\{ \sum_{i\in I} \operatorname{diam}(U_i)^s \colon \{U_i\}_{i\in I} \text{ is a cover of } B \right.\right.$$
$$\left.\left. \text{with } \operatorname{diam}(U_i) < \delta \right\} = \infty \right\}.$$

Recall that the level set $E_c = \{x \in [0,1) \colon \sum_{n=1}^{\infty} c_n R(2^{n-1}x) = c\}$ for $c \in \mathbf{C}$ and define $F_c = \big\{\{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n=1}^{\infty} c_n x_n = c\big\}$. It is known that $\dim_H E_c = \dim_H F_c$ by bi-Lipschitz mapping from $[0,1)$ to $\{-1,1\}^{\mathbf{N}}$ [14]. We begin with a generalization of Theorem 2 in [14].

Lemma 5.1. *Let $\{c_n\}_{n=1}^{\infty} \notin \ell^1$ be a sequence of complex numbers with $\{c_n\} = o(1)$. Then*

$$\dim_H \left\{ \{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n=1}^{\infty} x_n c_n \text{ converges} \right\} = 1.$$

Proof. The proof is essentially identical to that of [14, Theorem 2] with minor modifications. □

Definition 5.2. Let $\Lambda \subseteq \mathbf{N}$. Define the *super and lower density* of Λ by

$$\overline{D}(\Lambda) = \limsup_{k\to\infty} \frac{\#(\Lambda \cap [0,k])}{k}, \quad \underline{D}(\Lambda) = \liminf_{k\to\infty} \frac{\#(\Lambda \cap [0,k])}{k},$$

respectively, where $\#E$ is the cardinalities of the set E. If $\overline{D}(\Lambda) = \underline{D}(\Lambda)$, we say the common value the *density* of Λ and denote it by $D(\Lambda)$.

Let $\Lambda = \{n_1, n_2, \cdots\} \subseteq \mathbf{N}$. We define a map h_Λ from $\{-1,1\}^{\mathbf{N}}$ to itself by

$$h_\Lambda(\{x_n\}_{n\in\mathbf{N}}) = \{x_n\}_{n\in\mathbf{N}\setminus\Lambda}.$$

Lemma 5.3. *Let $\Lambda \subseteq \mathbf{N}$ and $0 < \epsilon < 1$. If $\overline{D}(\Lambda) < \epsilon$, then*

$$(1-\epsilon)\dim_H h_\Lambda(B) \le \dim_H B$$

for any $B \subseteq \{-1,1\}^{\mathbf{N}}$.

Proof. Denote $m_k = \#(\Lambda \cap [0,k-1])$. Then, by $\overline{D}(\Lambda) < \epsilon$, $m_k/k < \epsilon$ for $k > k_0 \ge 1$. For any $\{x_n\}, \{y_n\} \in \{-1,1\}^{\mathbf{N}}$ with $d(\{x_n\},\{y_n\}) = 2^{-k} < 2^{k_0}$, we have

$$d\big(h_\Lambda(\{x_n\}), h_\Lambda(\{y_n\})\big) \le 2^{-k+m_k} = d(\{x_n\},\{y_n\})^{1-\frac{m_k}{k}} < d(\{x_n\},\{y_n\})^{1-\epsilon}.$$

Hence, by the definition of Hausdorff dimension, it is easy to check that

$$(1-\epsilon)\dim_H h_\Lambda(B) \le \dim_H B.$$ □

Lemma 5.4. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty}$ with $\{c_n\} = o(1)$ be a linearly non-summable sequence and let ϵ so that $0 < \epsilon < 1$. Suppose that $\{c_n\}$ has a unique ratio (at least two distinct ratios), then there exists a linearly non-summable subsequence $\{c_n\}_{n\in\Lambda}$ with one ratio (two distinct ratios, resp.) so that $\overline{D}(\Lambda) < \epsilon$.*

Proof. First we show the case of one ratio. By Proposition 2.2 we can assume that the unique ratio is 1, that is, $\lim_{n\to\infty} a_n/b_n = 1$. Since $\sum_{n=1}^{\infty} |a_n - b_n| = \sum_{j=1}^{q}\sum_{k=0}^{\infty} |a_{kq+j} - b_{kq+j}| = \infty$, there exists j so that $\sum_{k=0}^{\infty} |a_{kq+j} - b_{kq+j}| = \infty$. This implies that the subsequence $\{c_{kq+j}\}_{k=0}^{\infty}$ is linearly non-summable. Denote $\Lambda_q = \{kq+j \colon k = 0,1,2,\ldots\}$, by a simple calculation we have $\overline{D}(\Lambda_q) = 1/q$. Hence the assertion follows by choosing q so that $1/q < \epsilon$. Secondly, for the case of at least two ratios, the assertion follows by the same idea used for two subsequences with distinct ratios. □

The following result is the same with Theorem 1.1.

Theorem 5.5. *Let $\{c_n = a_n + ib_n\}_{n=1}^{\infty}$ with $\{c_n\} = o(1)$ be a linearly non-summable sequence and let $c \in \mathbf{C}$.*

(1) *If $\{c_n\}$ has one ratio, then*

$$\dim_H \left\{ \{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n=1}^{\infty} x_n c_n \in B(c,\delta) \right\} = 1$$

for any $\delta > 0$;

(2) *If $\{c_n\}$ has at least two distinct ratios, then*

$$\dim_H \left\{ \{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n=1}^{\infty} x_n c_n = c \right\} = 1.$$

Proof. (1) For any $\epsilon > 0$, by Lemma 5.3 there exists a linearly non-summable subsequence $\{c_n\}_{n\in\Lambda}$ with $\overline{D}(\Lambda) < \epsilon$. According to Lemma 5.1, we have

$$\dim_H \left\{ \{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n\in\mathbf{N}} x_n c_n \text{ converges} \right\} = 1.$$

To show the assertion (1), it is sufficient, by Lemma 5.3, to show that

$$\begin{aligned} &h_\Lambda\left(\left\{ \{x_n\} \in \{-1,1\}^{\mathbf{N}} \colon \sum_{n\in\mathbf{N}} x_n c_n \in B(c,\delta) \right\}\right) \\ &\supseteq \left\{ \{x_n\}_{n\in\mathbf{N}\setminus\Lambda} \colon \sum_{n\in\mathbf{N}\setminus\Lambda} x_n c_n \text{ converges} \right\}. \end{aligned} \tag{5.1}$$

For any $\{x_n\}_{n\in\mathbf{N}\setminus\Lambda}$ so that $\sum_{n\in\mathbf{N}\setminus\Lambda} x_n c_n$ converges (to d), by Theorem 1.1 there exists $\{x_n\}_{n\in\Lambda}$ such that $\sum_{n\in\Lambda} x_n c_n \in B(c-d,\delta)$. Then we have

$$\sum_{n\in\mathbf{N}} x_n c_n = \sum_{n\in\mathbf{N}\setminus\Lambda} x_n c_n + \sum_{n\in\Lambda} x_n c_n \in B(c,\delta).$$

This implies (5.1) by the definition of h_Λ.

The proof of (2) is similar to that of (1) and we omit it. □

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