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INTERPOLATION OF APPROXIMATION NUMBERS BETWEEN HILBERT SPACES

Radosław Szwedek

A. Mickiewicz University, Faculty of Mathematics and Computer Science
Umultowska 87, 61-614 Poznań, Poland; szwedek@amu.edu.pl

Abstract. We investigate whether the approximation numbers of operators behave well under the two-sided complex interpolation of Hilbert spaces. We study geometric interpolation of the approximation numbers as well as the entropy moduli. We also study geometric properties of the entropy and approximation numbers of operators between Hilbert spaces. In particular, we provide the quantitative estimates of approximation numbers as well as the interpolation results on normal operators.

1. Introduction

The theory of s-numbers plays a fundamental role in the study of operators and the local theory of Banach spaces. The axiomatic approach to s-numbers was developed by Pietsch in [19]. Particularly important s-numbers of an operator $T \in L(E,F)$ are the following:

– *approximation numbers* $a_n(T) := \inf\{\|T - S\| \colon S \in L(E,F),\ \operatorname{rank}(S) < n\}$,
– *Gelfand numbers* $c_n(T) := \inf\{\|T|_G\| \colon G \subset E,\ \operatorname{codim}(G) < n\}$,
– *Kolmogorov numbers* $d_n(T) := \inf\{\varepsilon > 0 \colon G \subset F,\ \dim(G) < n,$ $T(U_E) \subset G + \varepsilon U_F\}$,
– *Weyl numbers* $x_n(T) := \sup\{a_n(TS) \colon S \in L(\ell_2, X),\ \|S\| \leqslant 1\}$.

If we denote by T' the dual operator of T, then $c_n(T) = d_n(T')$ for an arbitrary operator, while the analogous equalities $a_n(T) = d_n(T')$, $d_n(T) = c_n(T')$ are true, in general, for compact operators only. In the context of eigenvalues a central role is played by the Weyl numbers, which were introduced also by Pietsch. For operators acting between Hilbert spaces the various s-numbers are known to coincide.

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples and $\theta \in (0,1)$. In the case where a Banach space X belongs to the class $\mathcal{C}_K(\theta; \vec{A})$ and $B := B_0 = B_1$ or $A := A_0 = A_1$ and a Banach space Y belongs to the class $\mathcal{C}_J(\theta; \vec{B})$, the following inequalities (see, e.g. [21, 6.6.5.3]) hold

$$d_{n+m-1}(T \colon X \to B) \leqslant C\, d_n(T \colon A_0 \to B)^{1-\theta}\, d_m(T \colon A_1 \to B)^{\theta},$$
$$c_{n+m-1}(T \colon A \to Y) \leqslant C\, c_n(T \colon A \to B_0)^{1-\theta}\, c_m(T \colon A \to B_1)^{\theta}.$$

We refer to these as "one-sided" interpolation results. It has long been known that these s-numbers (as well as the others), in general, do not behave well under complex

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interpolation even for identity maps between finite dimensional spaces, as the following observation of Carl shows (see, e.g. [8]). Consider $A_0 = A_1 = B_0 := \ell_1^{3n}$, $B_1 := \ell_\infty^{3n}$ and $\theta = 1/2$. Then $X := [A_0, A_1]_\theta \cong \ell_1^{3n}$ and $Y := [B_0, B_1]_\theta \cong \ell_2^{3n}$. Because of the duality relation $c_n(T) = d_n(T')$, it is enough to consider the Kolmogorov numbers only. Therefore

$$d_n(I\colon A_0 \to B_0)^{1-\theta}\, d_n(I\colon A_1 \to B_1)^{\theta} \asymp n^{-1/4} \quad \text{and}$$
$$d_{2n-1}(I\colon X \to Y) \asymp 3^{-1/2}.$$

For an extensive survey of results concerning the asymptotic behaviour of s-numbers of such operators we refer to [15].

The problem on the "two-sided" interpolation of s-numbers is the problem of finding conditions on the Banach couples $\vec{A}$, $\vec{B}$ and the interpolation functor $\mathcal{F}$ of exponent $\theta \in (0,1)$ under which there exists a constant $C > 0$ such that for every operator $T\colon \vec{A} \to \vec{B}$ and each $n, m \in \mathbf{N}$ the following inequality

$$w_{n+m-1}(T\colon X \to Y) \leqslant C\, w_n(T\colon A_0 \to B_0)^{1-\theta}\, w_m(T\colon A_1 \to B_1)^{\theta} \tag{1.1}$$

is valid, where w is one of s-numbers, $X := \mathcal{F}(\vec{A})$ and $Y := \mathcal{F}(\vec{B})$.

It is still not completely clear under which conditions this "two-sided" interpolation problem has a positive answer. In connection with this, it seems important to investigate concrete nontrivial cases. In this paper we deliver a variant of (1.1) in the case of Hilbert spaces. We show that the s-numbers of an operator behave well under complex interpolation between Hilbert spaces. We actually prove that there exists a constant $C > 0$, such that for all Hilbert couples $\vec{H} = (H_0, H_1)$, $\vec{K} = (K_0, K_1)$ and every operator $T\colon \vec{H} \to \vec{K}$, the estimate

$$A_n\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant C\, A_n\big(T\colon H_0 \to K_0\big)^{1-\theta} A_n\big(T\colon H_1 \to K_1\big)^{\theta}$$

holds for all $\theta \in (0,1)$ and $n \in \mathbf{N}$, where $A_n(T)$ denotes $\big(\prod_{i=1}^n a_i(T)\big)^{1/n}$.

Let us remark that the "spectral" heart of our proof is inspired by the elegant idea of Halmos [10] and McCarthy [6]. We also stress that these results are obtained using the "geometric interpolation" methods, employing related ideas from [6, 7, 10]. It should be mentioned, too, that there is a lack of positive answers of this "two-sided" interpolation problem in the available literature and our result may be a unique finding.

Let us mention that it was Allakhverdiev [1] who discovered the coincidence of the approximation numbers $a_n(T)$ with the eigenvalues $\lambda_n(|T|)$, which are commonly referred to as the singular numbers $s_n(T)$. It is well known that in the case where T is a self-adjoint operator on a Hilbert space, we also have $a_n(T) = |\lambda_n(T)|$ (see, e.g. [4]). In this paper we extend this result to normal operators.

We also deliver estimates of inner entropy numbers $\varphi_n(T)$, entropy numbers $\varepsilon_n(T)$ and entropy moduli $g_n(T)$ of operators between Hilbert spaces, inspired by the celebrated Gordon, König and Schütt [9] inequality, namely

$$\varphi_n(T) \leqslant 4 \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)\,, \quad \varepsilon_n(T) \leqslant 6 \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T) \quad \text{and} \quad g_n(T) \leqslant 6\, A_n(T)$$

as well as

$$\varphi_{n^{2^k}}(T) \leqslant 2^{1/2^{k-1}} \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)$$

and

$$\varepsilon_{n^{2^k}}(T) \leqslant 2^{1+1/2^{k-1}} \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)\,,$$

for all $k, n \in \mathbf{N}$. The asymptotic behaviour of $\varepsilon_n(T) \asymp \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)$ and $g_n(T) \asymp A_n\big(T\big)$ is already known (see, [4, Theorems 3.4.1 and 3.4.2]). We improve the equivalence constants.

We prove an interpolation theorem on normal operators, which seems to be of independent interest. It turns out that whenever T on a regular Hilbert couple $\vec{H}$ is normal on both "endpoints" H_0 and H_1, the approximation numbers $a_k\big(T\colon [\vec{H}]_\theta \to [\vec{H}]_\theta\big)$ coincide for all $\theta \in [0,1]$, namely

$$a_k(T\colon H_0 \to H_0) = a_k\big(T\colon [\vec{H}]_\theta \to [\vec{H}]_\theta\big) = a_k(T\colon H_1 \to H_1)\,, \quad k \in \mathbf{N}.$$

2. Preliminaries

We recall some basic concepts and results from the spectral theory of operators we will use later on. Given an operator $T \in L(X)$ on a complex Banach space X, T is said to be a Fredholm operator provided that its kernel, $N(T)$, is finite dimensional and the $R(T)$ has a finite codimension. This last condition implies that $R(T)$ is closed. It is well known that T is a Fredholm operator if and only if its equivalence class is invertible in the Calkin algebra $L(X)/K(X)$.

Let $\sigma(T)$ denote the spectrum of T. The essential spectrum $\sigma_{ess}(T)$ is the set of all $\lambda \in \mathbf{C}$ such that $\lambda I_X - T$ is not Fredholm. The essential spectral radius is given by

$$r_{ess}(T) := \sup\{|\lambda|;\ \lambda \in \sigma_{ess}(T)\}.$$

An operator $T \in L(X)$ with $r_{ess}(T) = 0$ is called a Riesz operator. Examples of Riesz operators are power compact operators.

The classical Fredholm theory gives that the set

$$\Lambda(T) = \{\lambda \in \sigma(T)\colon\ |\lambda| > r_{ess}(T)\}\,.$$

is at most countable and consists of isolated eigenvalues of finite algebraic multiplicity. Following the Riesz theory of operators (see [4] and [22] for more details), for an operator $T \in L(X)$ acting on a complex Banach space X, we can assign an eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ from the elements of the set $\Lambda(T) \cup \{r_{ess}(T)\}$ as follows: The eigenvalues are arranged in an order of non-increasing absolute values and each eigenvalue is counted according to its algebraic multiplicity. If T possesses less than n eigenvalues λ with $|\lambda| > r_{ess}(T)$, we let $\lambda_n(T) = \lambda_{n+1}(T) = \ldots = r_{ess}(T)$. The order could be non-uniquely determined; we choose a fixed order of this form.

Let us recall that in this framework we also have the following equalites

$$\lambda_n(T^m) = \lambda_n(T)^m\,, \quad m, n \in \mathbf{N} \tag{2.1}$$

and

$$\lambda_n(RS) = \lambda_n(SR)\,, \quad n \in \mathbf{N}, \tag{2.2}$$

where $R \in L(E,F)$, $S \in L(F,E)$ are operators acting between complex Banach spaces.

3. Geometric properties of entropy moduli and entropy numbers

In this section we prove the main results of the paper. First, we recall some important definitions. Let $T\colon E \to F$ be an operator between Banach spaces and let $n \in \mathbf{N}$. The n-th *entropy number* $\varepsilon_n(T) = \varepsilon_n(T\colon E \to F)$ is defined to be the infimum of all $\varepsilon > 0$ such that there exist $y_1, \dots, y_n \in Y$ for which

$$T(U_E) \subset \bigcup_{j=1}^{n} \{y_j + \varepsilon U_F\}\,,$$

where U_E denotes the closed unit ball of E. We notice here that these numbers and their speed of convergence provide a quantitative way to measure the "degree of compactness" of an operator between Banach spaces. The measure of non-compactness $\beta(T)$ is defined by $\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$.

The entropy numbers of operators are useful in the analysis of the asymptotic behaviour of eigenvalues. The celebrated inequality due to Carl and Triebel [5] gives an estimate of eigenvalues of $T \in L(X)$ acting on a complex Banach space X, by single entropy numbers

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbf{N}} k^{1/2n}\varepsilon_k(T)\,, \quad n \in \mathbf{N},$$

where $\{\lambda_n(T)\}$ is an eigenvalue sequence of T.

This motivated the following notion; given an operator $T \in L(E,F)$ between Banach spaces, we define the n-th entropy modulus $g_n(T) = g_n(T\colon E \to F)$ by

$$g_n(T) := \inf_{k\in\mathbf{N}} k^{1/2n}\varepsilon_k(T)\,, \quad n \in \mathbf{N}.$$

It is also well known that for every operator $T\colon X \to X$ on a complex Banach space X,

$$\mathcal{G}_n(T\colon X \to X) := \lim_{m\to\infty} g_n(T^m)^{1/m} = \left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n}, \quad n \in \mathbf{N}. \tag{3.1}$$

This formula was proved by Makai–Zemánek (see, e.g. [4, 17]). By the Carl–Triebel inequality, $\mathcal{G}_n(T) \leqslant g_n(T)$ for each $n \in \mathbf{N}$.

Until now, we considered entropy moduli $g_n(T)$ as a function of any operator T acting between arbitrary Banach spaces. Here and subsequently, we will sometimes drop the assumption that these spaces are complete.

Proposition 3.1. *Let E and F be arbitrary Banach spaces, and $T \in L(E,F)$. Assume that there exist subspaces $E^0 \subset E$ and $F^0 \subset F$ which are dense in E and F, respectively such that $T\big(E^0\big) \subset F^0$. Then*

$$g_n\big(T\colon E \to F\big) = g_n\big(T\colon E^0 \to F^0\big), \quad n \in \mathbf{N}.$$

Proof. Fix $k \in \mathbf{N}$. It suffices to show that $\varepsilon_k\big(T\colon E \to F\big) = \varepsilon_k\big(T\colon E^0 \to F^0\big)$. Indeed, given $\varepsilon > \varepsilon_k\big(T\colon E^0 \to F^0\big)$, we can find $y_i^0 \in F^0$, $1 \leqslant i \leqslant k$, such that

$$T\left(U_{E^0}\right) \subset \bigcup_{i=1}^{k} \left\{y_i^0 + \varepsilon U_{F^0}\right\}.$$

This implies

$$T\left(U_E\right) \subset \overline{T\left(U_{E^0}\right)} \subset \bigcup_{i=1}^{k} \left\{y_i^0 + \varepsilon U_F\right\}$$

and so $\varepsilon_k\big(T\colon E \to F\big) \leqslant \varepsilon_k\big(T\colon E^0 \to F^0\big)$. For the opposite inequality, suppose that $\varepsilon > \varepsilon_k\big(T\colon E \to F\big)$ and $\delta > 0$. Likewise, there exists a covering such that

$$T\left(U_E\right) \subset \bigcup_{i=1}^{k} \left\{y_i + \varepsilon U_F\right\}, \quad \text{where } y_i \in F,\ 1 \leqslant i \leqslant k.$$

Choose $y_i^0 \in F^0$ which satisfy $\|y_i^0 - y_i\|_F < \delta$, $1 \leqslant i \leqslant k$. Hence

$$T\left(U_{E^0}\right) \subset \bigcup_{i=1}^{k} \left\{y_i + \varepsilon U_F\right\} \cap F^0 \subset \bigcup_{i=1}^{k} \left\{y_i^0 + (\varepsilon + \delta)\, U_{F^0}\right\}.$$

This gives $\varepsilon_k\big(T\colon E^0 \to F^0\big) \leqslant \varepsilon_k\big(T\colon E \to F\big)$. □

The next two results come from [18, Theorem 3.2 and Proposition 3.4].

Theorem 3.2. *Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then*

$$\lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m} = \sup_{n\in\mathbf{N}} k^{-1/2n} \left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n}, \quad k \in \mathbf{N}.$$

Following [18], we define for every operator T on a complex Banach space X the n-th *spectral entropy number*

$$\mathcal{E}_n(T) := \lim_{m\to\infty} \varepsilon_{n^m}(T^m)^{1/m}.$$

Clearly, $\mathcal{E}_n(T) \leqslant \varepsilon_n(T)$ for each $n \in \mathbf{N}$.

Proposition 3.3. *Let X be a complex Banach space and $T \in L(X)$. Then*

$$r_{ess}(T) \leqslant \ldots \leqslant \mathcal{E}_2(T) \leqslant \mathcal{E}_1(T) = r(T) \quad \textit{and} \quad \lim_{n\to\infty} \mathcal{E}_n(T) = r_{ess}(T).$$

Let us recall that the approximation numbers $a_n(T)$ admit the geometrical representation (see [20], as well as [4, Propositions 2.4.2 and 2.4.5])

$$a_n(T\colon H \to K) = \inf\{\|T - TP\| : P \in L(H) \text{ is an orthogonal projection with } \operatorname{rank} P < n\}$$

where H, K are arbitrary Hilbert spaces and $T \in L(H, K)$. In what follows $A_n(T\colon\ H \to K)$ denotes $\big(\prod_{i=1}^n a_i(T\colon H \to K)\big)^{1/n}$. Since

$$\|T(I-P)\|^2 = \|(I-P)T^*T(I-P)\| \leqslant \|T^*T - T^*TP\|,$$

we also have

$$a_n(T\colon H \to K)^2 \leqslant a_n(T^*T\colon H \to H). \tag{3.2}$$

We also recall that

$$a_n(T\colon H \to K) = a_n(|T| : H \to H) = a_n(T^*\colon K \to H), \quad n \in \mathbf{N},$$

by the polar decomposition of T where $|T| := (T^*T)^{1/2}$, $T = U\,|T|$, $|T| = U^*T$ and $U\colon H \to K$ is a partial isometry with $\|U\| = 1$. Similarly, for each $n \in \mathbf{N}$, we have

$$\begin{aligned} \varphi_n(T\colon H \to K) &= \varphi_n(|T| : H \to H) = \varphi_n\big(T^*\colon K \to H\big), \\ \varepsilon_n(T\colon H \to K) &= \varepsilon_n(|T| : H \to H) = \varepsilon_n\big(T^*\colon K \to H\big). \end{aligned} \tag{3.3}$$

Let us remark that Allakhverdiev [1] proved the following formula

$$a_n(T\colon H \to K) = \lambda_n(|T| : H \to H)\,, \quad n \in \mathbf{N}.$$

It is well known that in the case where $T \in L(H)$ is a self-adjoint operator acting in a Hilbert space H, we have the following equality (see [4, Proposition 4.4.1])

$$a_n(T) = |\lambda_n(T)|\,, \quad n \in \mathbf{N}.$$

Now we can strengthen this result. In the proof we use König's result (see [11, 12] or [4, Theorem 4.3.1]) which states that for every operator $T \in L(X)$ on a complex Banach space the following formula holds:

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n(T^m)^{1/m}\,, \quad n \in \mathbf{N}.$$

Proposition 3.4. *Let H be a complex Hilbert space and $T \in L(H)$ be a normal operator. Then*

$$a_n(T) = |\lambda_n\,(T)|\,, \quad n \in \mathbf{N}.$$

Proof. Let $n \in \mathbf{N}$. Let $P \in L(H)$ be an orthogonal projection with $\operatorname{rank} P < n$. Since $T^*T = TT^*$,

$$\begin{aligned} \big\|T^2(I-P)\big\|^2 &= \big\|(I-P)(T^2)^*(T^2)(I-P)\big\| \\ &= \|(I-P)T^*TT^*T(I-P)\| = \|(T^*T)(I-P)\|^2 \end{aligned}$$

shows that $a_n(T^2) = a_n(T^*T)$. We check by induction that

$$a_n\big(T^{2^m}\big) = a_n\big((T^*T)^{2^{m-1}}\big)$$

for each $m \in \mathbf{N}$. Since T^*T is self-adjoint, [4, Proposition 4.4.1]) shows that

$$a_n\big((T^*T)^{2^{m-1}}\big) = \lambda_n\big((T^*T)^{2^{m-1}}\big) = \lambda_n(|T|)^{2^m} = a_n(|T|)^{2^m}\,.$$

Therefore

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n\big(T^{2^m}\big)^{1/2^m} = a_n(T)\,,$$

by the mentioned König formula. □

Theorem 3.2 and Proposition 3.4 now yields

$$\mathcal{G}_n(T) = A_n(T) \quad \text{as well as} \quad \mathcal{E}_n(T) = \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)\,, \quad n \in \mathbf{N}, \tag{3.4}$$

where $T \in L(H)$ is a normal operator on a complex Hilbert space H.

Given $n \in \mathbf{N}$, the n-th inner entropy number $\varphi_n(T) = \varphi_n(T\colon E \to F)$ is defined to be the supremum of all $\rho > 0$ such that there exist $x_1, \ldots, x_p \in U_E$, $p > n$ such that

$$\|T(x_i - x_j)\| > 2\rho, \quad 1 \leqslant i < j \leqslant p.$$

There is a close relation between entropy and inner entropy numbers, namely $\varphi_n(T) \leqslant \varepsilon_n(T) \leqslant 2\,\varphi_n(T)$. The inner entropy numbers $\varphi_n(T)$ are additive with a constant equals 2 because their additivity results from the additivity of the entropy numbers $\varepsilon_n(T)$ (see, e.g., [4]). However, the following variant of this property is also true:

Proposition 3.5. *Let E, F be arbitrary Banach spaces and $T_1, T_2 \in L(E,F)$. Then*

$$\varphi_n(T_1 + T_2) \leqslant \|T_1\| + \varphi_n(T_2)\,, \quad n \in \mathbf{N}.$$

Proof. Fix $\rho < \varphi_n(T_1 + T_2)$. There exist elements $x_1, \ldots, x_p \in U_E$, $p > n$ such that

$$2\rho < \|(T_1 + T_2)(x_i - x_j)\| \leqslant 2\,\|T_1\| + \|T_2(x_i - x_j)\|\,, \quad 1 \leqslant i < j \leqslant p.$$

It is easily seen that

$$\min_{1 \leqslant i < j \leqslant p} \|T_2(x_i - x_j)\| \leqslant 2\sigma, \quad \sigma > \varphi_n(T_2)\,.$$

Therefore $\rho < \|T_1\| + \sigma$, and $\varphi_n(T_1 + T_2) \leqslant \|T_1\| + \varphi_n(T_2)$ as required. □

We next state two auxiliary results, all of which have geometrical character. First, we prove an analogue of (3.2) in terms of the inner entropy numbers:

Proposition 3.6. *Let H, K be Hilbert spaces and $T \in L(H,K)$. Then*

$$\varphi_n(T\colon H \to K)^2 \leqslant \varphi_n(T^*T\colon H \to H)\,, \quad n \in \mathbf{N}.$$

Proof. Let $\rho < \varphi_n(T)$. There exist elements $x_1, \ldots, x_p \in U_E$, $p > n$ such that

$$\begin{aligned}(2\rho)^2 &< \|T(x_i - x_j)\|^2 = \langle T(x_i - x_j), T(x_i - x_j)\rangle \\ &= \langle T^*T(x_i - x_j), x_i - x_j\rangle \leqslant 2\,\|T^*T(x_i - x_j)\|\,, \quad 1 \leqslant i < j \leqslant p,\end{aligned}$$

the last inequality being a consequence of the Cauchy–Schwarz inequality. This gives $\rho^2 < \varphi_n(T^*T)$, and $\varphi_n(T)^2 \leqslant \varphi_n(T^*T)$ as claimed. □

Let E, F be arbitrary Banach spaces. It is known that the entropy moduli $\{g_n(T)\}_{n\in\mathbf{N}}$ are injective in a weaker sense, namely $g_n(T) \leqslant 2g_n(JT)$ for any $T \in L(E,F)$ and any metric injection $J\colon F \to \widetilde{F}$, where $\widetilde{F}$ is a Banach space. The factor 2 cannot be reduced in general (see, e.g., [4, (3.5.17)]). Nevertheless, in the case of Hilbert spaces, we have the following equality:

Proposition 3.7. *Let E be an arbitrary Banach space. Assume that H is a Hilbert space such that K is a closed subspace of H and let $J\colon K \to H$ denote the isometric embedding of K into H. If $T \in L(E,K)$, then*

$$g_n(T\colon E \to K) = g_n(JT\colon E \to H)\,, \quad n \in \mathbf{N},$$

i.e., the entropy numbers $\{\varepsilon_n(T)\}_{n\in\mathbf{N}}$ are injective.

Proof. We start with the observation that $H = K \oplus K^\perp$, so there exists an orthogonal projection $P\colon H \to H$ of H onto K. Let $\widetilde{P}\colon H \to K$ denote the operator of H onto K induced by P. We obviously have $P = J\widetilde{P}$ and $\widetilde{P}J = I$, where $I\colon K \to K$ stands for the identity operator, thus

$$g_n(T) = g_n\bigl(\widetilde{P}JT\bigr) \leqslant \|\widetilde{P}\| g_n(JT) \leqslant \|J\|\, g_n(T)\,, \quad n \in \mathbf{N},$$

and this completes the proof. □

We now state and prove the following result which is inspired by the celebrated inequality due to Gordon, König and Schütt [9] (see also [4, Theorems 1.3.2 and 1.4.1]).

Theorem 3.8. *Let H be a complex Hilbert space and $T \in L(H)$ be a normal operator. Then*

$$\varphi_n(T) \leqslant 4\,\mathcal{E}_n(T)\,, \quad \varepsilon_n(T) \leqslant 6\,\mathcal{E}_n(T) \quad \text{and} \quad g_n(T) \leqslant 6\,\mathcal{G}_n(T)\,, \quad n \in \mathbf{N}. \tag{3.5}$$

In particular,

$$\varphi_{n^{2^k}}(T) \leqslant 2^{1/2^{k-1}}\,\mathcal{E}_n(T) \quad \text{and} \quad \varepsilon_{n^{2^k}}(T) \leqslant 2^{1+1/2^{k-1}}\,\mathcal{E}_n(T)\,, \quad k, n \in \mathbf{N}. \tag{3.6}$$

Proof. Suppose for the moment that T is a positive operator. Clearly T is self-adjoint with nonnegative eigenvalues. Fix $n \in \mathbf{N}$. Let $\{\lambda_m(T)\}_{m=1}^{\infty}$ be an eigenvalue sequence of T. By Proposition 3.3, there exists an index $r \in \mathbf{N}$ with $|\lambda_r(T)| \leqslant 2\,\mathcal{E}_n(T)$. In order to prove the first two inequalities, we need consider two cases:

(i) $\lambda_1(T) \leqslant 2\,\mathcal{E}_n(T)$.

(ii) There exists $k \in \mathbf{N}$ such that $\lambda_{k+1}(T) \leqslant 2\,\mathcal{E}_n(T) < \lambda_k(T)$.

If $\lambda_1(T) \leqslant 2\,\mathcal{E}_n(T)$, then $\varphi_n(T) \leqslant \varepsilon_n(T) \leqslant \|T\| = \lambda_1(T) \leqslant 2\,\mathcal{E}_n(T)$. In the latter case, there exists $k \in \mathbf{N}$ such that $\lambda_{k+1}(T) \leqslant 2\,\mathcal{E}_n(T) < \lambda_k(T)$ where $\lambda_1(T), \ldots, \lambda_k(T) \in \Lambda(T)$. Therefore, there exists a k-dimensional subspace K of H (see, e.g., [4, Lemma 4.2.1]), invariant under T such that T_K possesses $\lambda_1(T), \ldots, \lambda_k(T)$ as its eigenvalues, where $T_K \colon K \to K$ denotes the restriction of T to K. By the spectral theorem for normal operators, there also exists an orthonormal basis $\{f_i\}_{i=1}^{k}$ of K consisting only of eigenvectors of T_K, ordered the same way as eigenvalues (i.e., $Tf_i = \lambda_i(T)\,f_i$, $1 \leqslant i \leqslant k$). Hence, $S \colon K \to \ell_2^k$ and $R \colon \ell_2^k \to K$ given by

$$S\Big(\sum_{i=1}^{k} \alpha_i f_i\Big) = \sum_{i=1}^{k} \alpha_i e_i \quad \text{and} \quad R\Big(\sum_{i=1}^{k} \alpha_i e_i\Big) = \sum_{i=1}^{k} \alpha_i f_i, \quad \alpha_i \in \mathbf{C},\ 1 \leqslant i \leqslant k,$$

are isometries. Since $H = K \oplus K^{\perp}$, there exists an orthogonal projection P of H onto K, which commutes with T. Denote by $\widetilde{P} \colon H \to K$ the operator of H onto K induced by P and let $J \colon K \to H$ be the isometric embedding of K into H. We have $T_K = \widetilde{P}TPJ$ and $TP = JT_K\widetilde{P}$, because $T_K = \widetilde{P}TJ$, $PJ = J$ and $P = J\widetilde{P}$. Hence

$$\varphi_m(TP) = \varphi_m\big(JT_K\widetilde{P}\big) \leqslant \varphi_m(T_K) = \varphi_m\big(\widetilde{P}TPJ\big) \leqslant \varphi_m(TP)$$

and $\varepsilon_m(TP) = \varepsilon_m(T_K)$. Therefore $\varepsilon_m(TP) = \varepsilon_m(D)$ and $\varphi_m(TP) = \varphi_m(D)$, where D denotes the operator ST_KR from ℓ_2^k into itself and $m \in \mathbf{N}$. D turns out to be a diagonal operator generated by the sequence $\lambda_1(T), \ldots, \lambda_k(T)$, namely

$$D\,(\alpha_1, \ldots, \alpha_k) = (\lambda_1(T)\,\alpha_1, \ldots, \lambda_k(T)\,\alpha_k)\,, \quad \alpha_i \in \mathbf{C},\ 1 \leqslant i \leqslant k.$$

Since the image $D\,(U)$ of the closed unit ball U of ℓ_2^k is compact, $\lim_{m\to\infty} \varphi_m(D) = 0$. Suppose that there exists a set $\{y_1, \ldots, y_N\}$ of elements in $D\,(U)$ with

$$\|y_i - y_j\| > 4\,\mathcal{E}_n(T)\,, \quad 1 \leqslant i < j \leqslant N,$$

and that N is maximal in this respect. Therefore $\varphi_N(D) \leqslant 2\,\mathcal{E}_n(T)$ and $\varepsilon_N(D) \leqslant 4\,\mathcal{E}_n(T)$.

We now give an estimate for N. Since the sets $\{y_i + 2\,\mathcal{E}_n(T)\,U\}$, $1 \leqslant i \leqslant N$ are pairwise disjoint and $2\,\mathcal{E}_n(T) < \lambda_k(T) \leqslant \cdots \leqslant \lambda_1(T)$, it follows that

$$\bigcup_{j=1}^{N} \{y_j + 2\,\mathcal{E}_n(T)\,U\} \subseteq D\,(U) + 2\,\mathcal{E}_n(T)\,U \subseteq 2D\,(U)\,.$$

This yields

$$N\,(2\,\mathcal{E}_n(T))^{2k} \leqslant 2^{2k} \prod_{i=1}^{k} \lambda_i(T)^2\,,$$

which follows by the fact that the comparison of volumes takes place in a real euclidean space of dimension $2k$. Thus

$$N \leqslant \mathcal{E}_n(T)^{-2k} \prod_{i=1}^{k} \lambda_i(T)^2 \leqslant n,$$

the last inequality being a consequence of the definition of $\mathcal{E}_n(T)$.

It remains to estimate $\varphi_n(T)$ and $\varepsilon_n(T)$. Proposition 3.5 now leads to

$$\varphi_n(T) \leqslant \varphi_N(T) \leqslant \|T(I-P)\| + \varphi_N(TP) = \lambda_{k+1}(T) + \varphi_n(D) \leqslant 4\,\mathcal{E}_n(T)\,.$$

In a similar fashion, we obtain $\varepsilon_n(T) \leqslant 6\,\mathcal{E}_n(T)$.

We now prove the third inequality. Fix $k \in \mathbf{N}$. If $\lambda_k(T) = 0$ then $\operatorname{rank} T < k$ (see e.g., Proposition 3.4) and hence $g_k(T) = 0$. Suppose that $\lambda_k(T) > 0$. We can now proceed analogously to the first part of the proof. Since $\lambda_k(T) \leqslant \|D\|$, there exists a maximal set $\{y_1, \ldots, y_N\}$ of elements in $D\,(U)$ with

$$\|y_i - y_j\| > 2\,\lambda_k(T)\,, \quad 1 \leqslant i < j \leqslant N.$$

This clearly forces $\varepsilon_N(TP) = \varepsilon_N(D) \leqslant 2\,\lambda_k(T)$ and thus $\varepsilon_N(T) \leqslant \|T(I-P)\| + \varepsilon_N(TP) \leqslant 3\,\lambda_k(T)$. To estimate N, we note that the sets $\{y_i + \lambda_k(T)\,U\}$, $1 \leqslant i \leqslant N$ are pairwise disjoint and therefore

$$\bigcup_{j=1}^{N} \{y_j + \lambda_k(T)\,U\} \subseteq D\,(U) + \lambda_k(T)\,U \subseteq 2D\,(U)\,.$$

Now we carry out a comparison of volumes and conclude that

$$N\,\lambda_k(T)^{2k} \leqslant 2^{2k} \prod_{i=1}^{k} \lambda_i(T)^2\,.$$

Hence

$$N^{1/2k} \leqslant 2\,\lambda_k(T)^{-1} \left(\prod_{i=1}^{k} \lambda_i(T)\right)^{1/k}$$

and so

$$g_k(T) \leqslant N^{1/2k} \varepsilon_N(T) \leqslant 6 \left(\prod_{i=1}^{k} \lambda_i(T)\right)^{1/k} = 6\,\mathcal{G}_k(T)\,.$$

We are now in a position to show the last two estimates. Fix $k \in \mathbf{N}$. Repeated application of Proposition 3.6 enables us to write

$$\varphi_{n^{2^k}}(T)^{2^k} \leqslant \ldots \leqslant \varphi_{n^{2^k}}\left(T^{2^k}\right) \leqslant 4\,\mathcal{E}_{n^{2^k}}\left(T^{2^k}\right) = 4\,\mathcal{E}_n(T)^{2^k},$$

where the last equality is a consequence of Proposition 3.3. By the above,

$$\varepsilon_{n^{2^k}}(T)^{2^k} \leqslant 2^{2^k}\,\varphi_{n^{2^k}}(T)^{2^k} \leqslant 4 \cdot 2^{2^k}\,\mathcal{E}_n(T)^{2^k}.$$

The proof is completed by showing that the result will remain unaffected if we assume merely that T is normal. Using Proposition 3.4, Theorem 3.2 and (3.3) we obtain

$$\varphi_n(T) = \varphi_n(|T|) \leqslant 4\,\mathcal{E}_n(|T|) = 4 \sup_{n \in \mathbf{N}} k^{-1/2n} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = 4\,\mathcal{E}_n(T).$$

The same conclusion can be drawn for the remaining estimates. □

Let H be Hilbert space and $T \in L(H)$ be a normal operator. Note that by Propositions 3.3 and 3.4 we also have $\varepsilon_1(T) = \mathcal{E}_1(T)$ and $\beta(T) = r_{ess}(T)$. Nevertheless, one may check that $\varepsilon_n(T) \leqslant \mathcal{E}_n(T)$, $n \geqslant 2$ does not hold even for self-adjoint operators T acting in a finite dimensional Hilbert space. Unfortunately, we do not know whether or not the constant appearing in $\varepsilon_n(T) \leqslant 6\,\mathcal{E}_n(T)$, $n \in \mathbf{N}$ is optimal. The corresponding part of our proof of Theorem 3.8 is based upon ideas found in [9], [4, Theorem 1.3.2]. We also note that the first (resp., the second) formula of (3.6) which is a generalization of the first inequality of (3.5), shows that at the expense of replacing n by n^{2^k} on the left hand side of (3.5) we may replace the factor 4 (resp., 6) on the right hand side of (3.5) by $2^{1/2^{k-1}}$ (resp., $2^{1+1/2^{k-1}}$), $k \in \mathbf{N}$.

Let us remark, that with Theorem 3.8 at hand, we can obtain the following estimates, which are interesting results in their own right (cf. Proposition 3.6):

Corollary 3.9. *Let H, K be complex Hilbert spaces and $T \in L(H, K)$. Then*

$$\varphi_n(T\colon H \to K) \leqslant 4\,\varepsilon_{n^2}(T^*T\colon H \to H)^{1/2} \quad \text{and}$$
$$\varepsilon_n(T\colon H \to K) \leqslant 6\,\varepsilon_{n^2}(T^*T\colon H \to H)^{1/2} \quad \text{and}$$
$$g_n(T\colon H \to K) \leqslant 6\,g_n(T^*T\colon H \to H)^{1/2}, \quad n \in \mathbf{N}.$$

Proof. Let $n \in \mathbf{N}$. By (3.3) and Theorem 3.8 we have

$$\varphi_n(T) = \varphi_n(|T|) \leqslant 4\,\mathcal{E}_n(|T|) = 4\,\mathcal{E}_{n^2}\left(|T|^2\right)^{1/2} \leqslant 4\,\varepsilon_{n^2}\left(T^*T\right)^{1/2},$$

and the first estimate follow. The remaining assertion can be verified in a similar way. □

It is easy to check, that the following inequality stated in Theorem 3.8, $\varepsilon_n(T) \leqslant 6\,\mathcal{E}_n(T)$, (resp., $g_n(T) \leqslant 6\,\mathcal{G}_n(T)$), $n \in \mathbf{N}$, can be recovered using Corollary 3.9 and Theorem 3.2 (resp., the Makai–Zemánek formula (3.1). Note that the last inequality stated in Corollary 3.9 can also be regarded as a variant of (3.2).

The next result is clearly motivated by [4, Theorems 3.4.1 and 3.4.2], where the constant appearing at the right-hand side of the inequality is equal to 14 and 10, respectively. The proof is analogous in spirit to that of Corollary 3.9, thus it will only be indicated briefly.

Theorem 3.10. *Let H, K be complex Hilbert spaces and $T \in L(H,K)$. Then, for each $n \in \mathbf{N}$, we have*

$$A_n(T) \leqslant g_n(T) \leqslant 6\, A_n(T)\,,$$
$$\sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T) \leqslant \varepsilon_n(T) \leqslant 6 \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)\,,$$
$$1/2 \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T) \leqslant \varphi_n(T) \leqslant 4 \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T)\,.$$

In particular,

$$\varphi_{n^{2^k}}(T) \leqslant 2^{1/2^{k-1}} \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T) \quad \text{and}$$
$$\varepsilon_{n^{2^k}}(T) \leqslant 2^{1+1/2^{k-1}} \sup_{m\in\mathbf{N}} n^{-1/2m} A_m(T), \quad k,n \in \mathbf{N}.$$

Proof. We only show the first three inequalities. Fix $n \in \mathbf{N}$. That $\mathcal{E}_n(|T|) \leqslant \varepsilon_n(|T|)$ follows from Theorem 3.2. Thus $g_n(T\colon H \to K) = g_n(|T| : H \to H)$. By the Carl–Triebel inequality (cf. [4, 5]), $\mathcal{G}_n(|T|) \leqslant g_n(|T|)$. Theorem 3.8 now shows that $\varphi_n(|T|) \leqslant 4\,\mathcal{E}_n(|T|)$, $\varepsilon_n(|T|) \leqslant 6\,\mathcal{E}_n(|T|)$ and $g_n(|T|) \leqslant 4\,\mathcal{G}_n(|T|)$, and the proof is completed by (3.3) and (3.4). □

In the sequel we use the following lemma.

Lemma 3.11. *Let H, K be arbitrary Hilbert spaces and $T \in L(H,K)$. Suppose that there exist operators $\{P_n\}_{n\in\mathbf{N}}$ and $\{Q_n\}_{n\in\mathbf{N}}$ which which have norm less or equal to* 1 *and approximate identity on finite subsets of H and K, respectively. If*

$$P_n = P_{n+1}P_n \quad \text{and} \quad Q_n = Q_nQ_{n+1}, \quad n \in \mathbf{N},$$

then

$$g_k(T\colon H \to K) \leqslant 6 \lim_{n\to\infty} g_k(Q_nTP_n\colon H \to K)$$

and

$$A_k(T\colon H \to K) \leqslant 12 \lim_{n\to\infty} A_k(Q_nTP_n\colon H \to K)\,, \quad k \in \mathbf{N}.$$

Proof. We first show that

$$\varphi_k(T\colon H \to K) = \lim_{n\to\infty} \varphi_k(Q_nTP_n\colon H \to K)\,, \quad k \in \mathbf{N}. \tag{3.7}$$

Note that $\limsup_{n\to\infty} \varphi_k(Q_nTP_n\colon H \to K) \leqslant \varphi_k(T\colon H \to K)$ holds trivially. Since the operators P_n and Q_n approximate identity on finite subsets of H and K, respectively, it follows that

$$\begin{aligned} \|(T - Q_nTP_n)x\|_K &\leqslant \|(I-Q_n)Tx\|_K + \|Q_nT(I-P_n)x\|_K \\ &\leqslant \|(I-Q_n)Tx\|_K + \|T\|_{H\to K}\,\|(I-P_n)x\| \to 0 \end{aligned} \tag{3.8}$$

as $n \to \infty$ and hence $\|Tx\|_K = \lim_{n\to\infty} \|Q_nTP_nx\|_K$ for all $x \in H$. Fix $\varepsilon > 0$. By the definition of $\varphi_k(T)$, there exists a set of elements $x_1, \dots, x_{n+1} \in U_H$ such that

$$2\,\varphi_k(T\colon H \to K) - \varepsilon < \|T(x_i - x_j)\|_K\,, \quad 1 \leqslant i < j \leqslant n+1,$$

and an integer N such that

$$\|T(x_i - x_j)\|_K - \varepsilon < \|Q_nTP_n(x_i - x_j)\|_K\,, \quad n > N, \quad 1 \leqslant i < j \leqslant n+1.$$

Hence, by definition again, we have

$$\min_{1\leqslant i<j\leqslant n+1} \|Q_nTP_n(x_i - x_j)\|_K < 2\,\varphi_k(Q_nTP_n) + \varepsilon, \quad n > N,$$

and thus the inequality $\varphi_k(T\colon H \to K) \leqslant \liminf_{n\to\infty} \varphi_k(Q_nTP_n\colon H \to K)$ follows.

Let us observe that $P_nU_H \subset U_H$ and $P_nU_H \subset P_{n+1}U_H$. We claim that

$$g_k(Q_nTP_n) \leqslant g_k(Q_{n+1}TP_{n+1})\,. \tag{3.9}$$

Indeed, $Q_nTP_nU_H \subset Q_nTP_{n+1}U_H$ and $g_k(Q_nQ_{n+1}TP_{n+1}) \leqslant g_k(Q_{n+1}TP_{n+1})$. Since $g_k(Q_nTP_n) \leqslant \|T\|$, the sequence $\{g_k(Q_nTP_n)\}_{n\in\mathbf{N}}$ converges. In this way one can also check that $\{A_k(Q_nTP_n)\}_{n\in\mathbf{N}}$ converges.

Without loss of generality we can assume that $g_k(T) > 0$. Hence, $\operatorname{rank} T \geqslant k$. Fix $\varepsilon > 0$. There exists a sequence $\{m_n\}_{n\in\mathbf{N}}$ such that

$$m_n^{1/2k}\varepsilon_{m_n}(Q_nTP_n) < g_k(Q_nTP_n) + \varepsilon/2. \tag{3.10}$$

In order to prove the remaining inequality, we need consider two cases:

(i) There exists a constant subsequence $\{N_n\}_{n\in\mathbf{N}}$ of $\{m_n\}_{n\in\mathbf{N}}$.

(ii) There exists a strictly increasing subsequence $\{N_n\}_{n\in\mathbf{N}}$ of $\{m_n\}_{n\in\mathbf{N}}$.

Assume that $N_n = N$, $n \in \mathbf{N}$. Since $g_k(T) \leqslant 2\,N^{1/2k}\varphi_N(T)$, (3.7) shows that

$$A_k(T) \leqslant g_k(T) \leqslant 2 \lim_{n\to\infty} g_k(Q_nTP_n) + \varepsilon \leqslant 12 \lim_{n\to\infty} A_k(Q_nTP_n) + \varepsilon,$$

by Theorem 3.10, (3.3) and (3.4). Now we turn to case (ii). Fix $m \in \mathbf{N}$. We conclude from (3.9) and (3.10) that there exists C such that

$$\varepsilon_{N_n}(Q_mTP_m) \leqslant C\,N_n^{-1/2k}, \quad n \in \mathbf{N},$$

hence that $\operatorname{rank} Q_mTP_m \leqslant k$ (see, e.g. [4, Lemma 1.3.1 and (1.3.14)']). We next prove that $\operatorname{rank} T = k$. Conversely, suppose that $\operatorname{rank} T > k$. Then we find d-dimensional subspaces $H_d \subset H$ and $K_d \subset K$ with $d > k$, which satisfy $T(H_d) = K_d$. We will denote by $S\colon H_d \to K_d$ the operator of H_d onto K_d induced by T. There exists an orthogonal projection $P\colon H \to H$ of H onto H_d. Here $\widetilde{P}\colon H \to H_d$ denote the operator of H onto H_d induced by P. The isometric embedding of K_d into K will be denoted by J. We conclude from (3.8) that $\big\|JS\widetilde{P} - Q_nTP_n\big\|_{H\to K} \to 0$, hence that $a_d\big(JS\widetilde{P}\big) = 0$, and finally that $\operatorname{rank} JS\widetilde{P} < d$ (see, e.g. [4, Rank property (A4), p.42]), which is impossible. The result is $\operatorname{rank} T = k$.

In the same manner we can see that

$$\big|a_i(T) - a_i(Q_nTP_n)\big| \leqslant \big\|T - Q_nTP_n\big\|_{H\to K} \to 0, \quad 1 \leqslant i \leqslant k.$$

Theorem 3.10 now yields

$$g_k(T) \leqslant 6\,A_k(T) = 6 \lim_{n\to\infty} A_k(Q_nTP_n) \leqslant 6 \lim_{n\to\infty} g_k(Q_nTP_n)\,,$$

the last inequality being a consequence of (3.3) and (3.4). □

4. Interpolation of entropy moduli and approximation numbers

In this section we look at some specific techniques from interpolation theory which can be briefly described as "geometric interpolation" methods. We also develop tools which will be essential in geometric interpolation of the entropy moduli of operators. We start with some definitions from the interpolation theory of operators. We will generally use the same notation as in [2, 3, 14].

The Banach space X will be called an *intermediate* space between A_0 and A_1 (or with respect to a Banach couple $\vec{A} := (A_0, A_1)$) provided $A_0 \cap A_1 \subset X \subset A_0 + A_1$. A Banach couple (A_0, A_1), is called *regular* if $A_j^\circ = A_j$, where A_j° denote the closure of $A_0 \cap A_1$ in A_j for $j = 0, 1$. An couple (A_0, A_1) is *ordered* if $A_0 \subset A_1$. If $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ are Banach couples and $T\colon A_0 + A_1 \to B_0 + B_1$ is a linear map such that $T|_{A_j} \in L(A_j, B_j)$ for $j = 0, 1$, then we write $T\colon \vec{A} \to \vec{B}$. The space $L(\vec{A}, \vec{B})$ of all operators $T\colon \vec{A} \to \vec{B}$ is a Banach space equipped with the norm

$$\|T\| := \max_{j=0,1} \|T|_{A_j}\|_{L(A_j,B_j)}.$$

We recall that a mapping $\mathcal{F}$ from the category of all couples of Banach spaces into the category of all Banach spaces is said to be an *interpolation functor* (or a method of interpolation) if for any couple $\vec{A}$, $\mathcal{F}(\vec{A})$ is a Banach space intermediate with respect to $\vec{A}$, and T maps $\mathcal{F}(\vec{A})$ into $\mathcal{F}(\vec{B})$ for all $T\colon \vec{A} \to \vec{B}$. If additionally there is a constant $C > 0$ such that

$$\|T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\| \leqslant C\,\|T\|_{L(\vec{A},\vec{B})}$$

for every $T\colon \vec{A} \to \vec{B}$, then $\mathcal{F}$ is called *bounded* (and *exact* if $C = 1$).

Banach spaces X and Y are said to be *interpolation spaces* with respect to $\vec{A}$ and $\vec{B}$ if X and Y are intermediate with respect to $\vec{A}$ and $\vec{B}$, and if T maps X into Y for every $T \in L(\vec{A}, \vec{B})$. If in addition there exists $C > 0$ and $\theta \in (0, 1)$ such that

$$\|T\colon X \to Y\| \leqslant C\,\|T\colon A_0 \to B_0\|^{1-\theta}\,\|T\colon A_1 \to B_1\|^{\theta}$$

for every $T \in L(\vec{A}, \vec{B})$, then X and Y are said to be *of exponent* θ (and *exact of exponent* θ if $C = 1$). Similarly, we say that $\mathcal{F}$ is (*exact*) *of exponent* θ if $\mathcal{F}(\vec{A})$ and $\mathcal{F}(\vec{B})$ are (exact) of exponent θ. It is well known that the complex interpolation space $[\vec{A}]_\theta$ is exact of exponent θ for every $\theta \in (0, 1)$.

We now turn to geometric interpolation between Hilbert spaces. The best general reference here is Donoghue [7] and M$^{\text{C}}$Carthy [6], where more details are given. Let $\vec{H} = (H_0, H_1)$ be a regular couple of Hilbert spaces. Let H be a Hilbert space which is intermediate between H_0 and H_1, and let $\theta \in (0, 1)$. Following [6], we say that H is a *geometric interpolation space of exponent* θ between H_0 and H_1 if it satisfies the following three conditions:

- H is an interpolation space exact of exponent s with respect to $\vec{H}$.
- Given any Hilbert space K; H and K are interpolation spaces exact of exponent s with respect to $\vec{H}$ and (K, K).
- Given any Hilbert space G; G and H are interpolation spaces exact of exponent s with respect to (G, G) and $\vec{H}$.

By means of an operator approach, it is showed in [6, Theorem 1.1] that there exists a unique geometric interpolation space of exponent θ between H_0 and H_1. In the language of category theory, this result says that for each $\theta \in (0,1)$ there exists a unique functor $\mathcal{F}_\theta$ mapping the category of all regular couples of Hilbert spaces to the category of all Hilbert spaces such that $\mathcal{F}_\theta$ is exact of exponent θ.

Now we define interpolation spaces using powers of a positive operator (see [6] and also [13, 14, 16]). The inner product for H_1 is a Hermitian form on $\Delta(\vec{H}) := H_0 \cap H_1$, so there exists a densely defined (not necessarily bounded), positive injective operator A on H_0 satisfying

$$\langle \xi, \eta \rangle_{H_1} = \langle A^{1/2}\xi, A^{1/2}\eta \rangle_{H_0}, \quad \xi, \eta \in \Delta(\vec{H}).$$

$\Delta(\vec{H})$ is a subset of both $\operatorname{Dom} A^{1/2}$ and $\operatorname{Ran} A^{1/2}$. Note that A is bounded if and only if H_0 is embedded in H_1. For fixed $\theta \in (0,1)$, we define a new inner product on $\Delta(\vec{H})$ by

$$\langle \xi, \eta \rangle = \langle A^{\theta/2}\xi, A^{\theta/2}\eta \rangle_{H_0}.$$

$\Delta(\vec{H})$ is contained in both $\operatorname{Dom} A^{\theta/2}$ and $\operatorname{Ran} A^{\theta/2}$. The closure of $\Delta(\vec{H})$, with respect to the norm given by the inner product $\langle \cdot, \cdot \rangle$, we will call H_θ. We remark that H_θ is a geometric interpolating space of exponent θ. Since the complex interpolation space is also a geometric interpolation space of exponent θ, H_θ coincides with $[H_0, H_1]_\theta$.

We start with a key lemma, crucial in the sequel.

Lemma 4.1. *Let $\vec{H}$ and $\vec{K}$ be regular couples of Hilbert spaces. Assume that A and B are positive operators on H_0 and K_0 that generate the H_1 and K_1 inner product, respectively. If $T \in L(\vec{H}, \vec{K})$, then*

$$g_n(T\colon H_\theta \to K_\theta) = g_n\big(B^{\theta/2}TA^{-\theta/2}\colon H_0 \to K_0\big), \quad \theta \in [0,1], \quad n \in \mathbf{N}.$$

Proof. Let us regard $\theta \in [0,1]$ as fixed. Let us denote by $H_\theta^\Delta := \big(\Delta(\vec{H}), \|\cdot\|_{H_\theta}\big)$, $H_0^\Delta := \big(\operatorname{Ran} A^{\theta/2}, \|\cdot\|_{H_0}\big)$ and $K_\theta^\Delta := \big(\Delta(\vec{K}), \|\cdot\|_{K_\theta}\big)$, $K_0^\Delta := \big(\operatorname{Ran} B^{\theta/2}, \|\cdot\|_{K_0}\big)$. We claim that $S := B^{\theta/2}TA^{-\theta/2}\colon H_0^\Delta \to K_0^\Delta$ gives rise to a bounded operator between H_0 and K_0. To see this observe that

$$\begin{aligned} \|T\widetilde{x}\|_{K_\theta}^2 &= \langle T\widetilde{x}, T\widetilde{x}\rangle_{K_\theta} = \langle B^{\theta/2}T\widetilde{x}, B^{\theta/2}T\widetilde{x}\rangle_{K_0} \\ &= \langle B^{\theta/2}TA^{-\theta/2}x, B^{\theta/2}TA^{-\theta/2}x\rangle_{K_0} = \|Sx\|_{K_0}^2, \quad \text{and} \\ \|\widetilde{x}\|_{H_\theta}^2 &= \langle \widetilde{x}, \widetilde{x}\rangle_{H_\theta} = \langle A^{\theta/2}\widetilde{x}, A^{\theta/2}\widetilde{x}\rangle_{K_0} = \langle x, x\rangle_{K_0} = \|x\|_{H_0}^2, \end{aligned} \tag{4.1}$$

where $x = A^{\theta/2}\widetilde{x}$ and $x \in H_0^\Delta$, $\widetilde{x} \in H_\theta^\Delta$. In particular, this yield

$$\big\|T\colon H_\theta^\Delta \to K_\theta^\Delta\big\| = \big\|S\colon H_0^\Delta \to K_0^\Delta\big\| \quad \text{and} \quad \big\|A^{\theta/2}\colon H_\theta^\Delta \to H_0^\Delta\big\| = 1. \tag{4.2}$$

From the closed graph theorem, it suffices to show that there exists a closure of S, whose domain is the whole space H_0. Take an arbitrary sequence x_n from H_0^Δ such that $\|x_n\|_{H_0} \to 0$. Since $\|A^{-\theta/2}x_n\|_{H_\theta} \to 0$, and since moreover $\|TA^{-\theta/2}x_n\|_{K_\theta} \to 0$, it follows that $\|Sx_n\|_{K_0} \to 0$. Hence S is closable. Let us denote by $\overline{S}$ the closure of S. We now show that $D(\overline{S}) = H_0$. If $x \in H_0$, then there exists a sequence x_n from $\Delta(\vec{H})$ such that $\|x_n - x\|_{H_0} \to 0$. Since $\{x_n\}$ is Cauchy, so also is $\{Sx_n\}$, by (4.2). This gives $\|Sx_n - y\|_{H_0} \to 0$ for some $y \in K_0$, and consequently $\overline{S}x = y$. From what has already been proved, it follows that $\|T\colon H_\theta \to K_\theta\| = \|\overline{S}\colon H_0 \to K_0\|$. Therefore, we can extend S to the whole H_0 and denote by $B^{\theta/2}TA^{-\theta/2}$ its closure $\overline{S}$ on H_0. Similarly,

$$\big\|A^{\theta/2}\colon H_\theta \to H_0\big\| = 1 \quad \text{and} \quad \big\|A^{-\theta/2}\colon H_0 \to H_\theta\big\| = 1, \tag{4.3}$$

Fix $k \in \mathbf{N}$; we proceed to show that $\varepsilon_k\big(T\colon H_\theta^\Delta \to K_\theta^\Delta\big) = \varepsilon_k\big(S\colon H_0^\Delta \to K_0^\Delta\big)$. Indeed, given $\varepsilon > \varepsilon_k\big(S\colon H_0^\Delta \to K_0^\Delta\big)$, we can find $y_i \in K_0^\Delta$, $1 \leqslant i \leqslant k$ such that

$$S\big(U_{H_0^\Delta}\big) \subset \bigcup_{i=1}^{k} \Big\{ y_i + \varepsilon U_{K_0^\Delta} \Big\}.$$

In other words, $\min_{1\leqslant i\leqslant k} \|Sx - y_i\|_{H_0^\Delta} \leqslant \varepsilon$ for every $x \in U_{H_0^\Delta}$. Hence, by (4.1) again, if $\widetilde{x} \in U_{H_\theta^\Delta}$, then

$$\min_{1\leqslant i\leqslant k} \|T\widetilde{x} - \widetilde{y}_i\|_{H_\theta^\Delta} \leqslant \varepsilon, \quad \text{where} \quad y_i = B^{\theta/2}\widetilde{y}_i, \quad 1 \leqslant i \leqslant k.$$

We thus get $\varepsilon_k\big(T\colon H_\theta^\Delta \to K_\theta^\Delta\big) \leqslant \varepsilon_k\big(S\colon H_0^\Delta \to K_0^\Delta\big)$. The opposite inequality can be proved in a similar way.

Let $n \in \mathbf{N}$. By definition, $g_n\big(T\colon H_\theta^\Delta \to K_\theta^\Delta\big) = g_n\big(S\colon H_0^\Delta \to K_0^\Delta\big)$. Proposition 3.1 now shows that

$$g_n\big(T\colon H_\theta^\Delta \to K_\theta^\Delta\big) = g_n\big(T\colon H_\theta \to K_\theta\big) \text{ and } g_n\big(S\colon H_0^\Delta \to K_0^\Delta\big) = g_n\big(\overline{S}\colon H_0 \to K_0\big),$$

which completes the proof. $\square$

We can now state our main result.

Theorem 4.2. *Assume that $\vec{H} = (H_0, H_1)$ and $\vec{K} = (K_0, K_1)$ are arbitrary couples of complex Hilbert spaces. For every operator $T \in L(\vec{H}, \vec{K})$, every $\theta \in (0,1)$ and each $n \in \mathbf{N}$,*

$$g_n\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant 6^4\, g_n(T\colon H_0 \to K_0)^{1-\theta}\, g_n(T\colon H_1 \to K_1)^{\theta}$$

and

$$A_n\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant 2^6\, 3^5\, A_k(T\colon H_0 \to K_0)^{1-\theta}\, A_n(T\colon H_1 \to K_1)^{\theta}.$$

Proof. Let $k, n \in \mathbf{N}$. The proof will be divided into 2 parts. First, suppose that $\vec{H}$ and $\vec{K}$ are regular. Let A (resp., B) be the positive operator on H_0 (resp., K_0) that gives the H_1 (resp., K_1) inner product. Let $\theta \in [0,1]$. Lemma 4.1 now shows that $B^{\theta/2}TA^{-\theta/2} \in L(H_0, K_0)$. Let us introduce the temporary notation R_θ for $B^{\theta/2}TA^{-\theta/2}$. By Lemma 4.1 again, Theorem 3.10, (3.3) and (3.4)

$$6^{-1}\, A_k(T\colon H_\theta \to K_\theta) \leqslant A_k\big(R_\theta\colon H_0 \to K_0\big) \leqslant 6\, A_k(T\colon H_\theta \to K_\theta). \tag{4.4}$$

We first prove a reduced form of the theorem for the family of operators R_θ. Suppose for the moment that A^{-1} and B are bounded. From equalities (2.1), (2.2), (3.3) and (3.4) we deduce that

$$\begin{aligned}
A_k\big(R_{1/2}\colon H_0 \to K_0\big) &= \mathcal{G}_k\big(\big|R_{1/2}\big|\colon H_0 \to H_0\big) = \mathcal{G}_k\big(R_{1/2}^{*}R_{1/2}\colon H_0 \to H_0\big)^{1/2}\\
&= \mathcal{G}_k\big(A^{1/4}R_{1/2}^{*}R_{1/2}A^{-1/4}\colon H_0 \to H_0\big)^{1/2}\\
&\leqslant g_k\big(A^{1/4}R_{1/2}^{*}R_{1/2}A^{-1/4}\colon H_0 \to H_0\big)^{1/2}\\
&\leqslant g_k\big(T^{*}\colon K_0 \to H_0\big)^{1/2} g_k\big(B^{1/2}TA^{-1/2}\colon H_0 \to K_0\big)^{1/2}\\
&\leqslant g_k\big(R_0\colon K_0 \to H_0\big)^{1/2} g_k\big(R_1\colon H_0 \to K_0\big)^{1/2}.
\end{aligned}$$

Theorem 3.10 now shows that the assertion holds for $\theta = 1/2$ with a constant 6,

$$A_k\big(R_{1/2}\colon H_0 \to K_0\big) \leqslant 6\, A_k(R_0\colon H_0 \to K_0)^{1/2}\, A_k(R_1\colon H_0 \to K_0)^{1/2}\,. \tag{4.5}$$

Now interpolating between R_0 and $R_{1/2}$ or $R_{1/2}$ and R_1 gives the result for $\theta = 1/4$ or $\theta = 3/4$, respectively. The constant here is equal to $6^{3/2}$. Following the same lines we find that the theorem holds for any dyadic rational in $[0,1]$ with a common constant 6^2. Indeed, one may check

$$A_k(R_\theta\colon H_0 \to K_0) \leqslant 6^2 A_k(R_0\colon H_0 \to K_0)^{1-\theta}\, A_k(R_1\colon H_0 \to K_0)^{\theta} \tag{4.6}$$

for any dyadic rational $\theta = m/2^n \in (0,1)$ by induction on n. That (4.6) is valid for $\theta = 1/2$ is already proved in (4.5). For the inductive step, suppose that (4.6) holds for $\theta = m/2^n$, $0 < m < 2^n$ with a constant equal to 6^2. It suffices to consider $\theta = m/2^{n+1}$ where $0 < m < 2^{n+1}$ is odd. Now interpolating between R_0 and $R_{2\theta}$ or $R_{1-2\theta}$ and R_1 gives (4.6) for $\theta < 1/2$ or $\theta > 1/2$, respectively.

Since the map $[0,1] \to L(H_0, K_0)\colon \alpha \mapsto R_\alpha$ is continuous, the statement remains valid for any real $\theta \in [0,1]$. Indeed,

$$\begin{aligned}
&\big|a_k(R_\alpha\colon H_0 \to K_0) - a_k(R_\beta\colon H_0 \to K_0)\big| \leqslant \|R_\alpha - R_\beta\|_{H_0\to K_0}\\
&\leqslant \big\|B^{\alpha/2}T\big(A^{-\alpha/2} - A^{-\beta/2}\big)\big\|_{H_0\to K_0} + \big\|\big(B^{\alpha/2} - B^{\beta/2}\big)TA^{-\beta/2}\big\|_{H_0\to K_0}\\
&\leqslant \big\|B_n^{\alpha/2}\big\|_{K_0\to K_0}\, \|T\|_{H_0\to K_0}\, \big\|A^{-\alpha/2} - A^{-\beta/2}\big\|_{H_0\to H_0}\\
&\quad + \big\|B^{\alpha/2} - B^{\beta/2}\big\|_{K_0\to K_0}\, \|T\|_{H_0\to K_0}\, \big\|A^{-\beta/2}\big\|_{H_0\to K_0} \to 0
\end{aligned}$$

as $\alpha \to \beta$, by the spectral theorem for normal operators and Proposition 3.5. By the above,

$$\big|A_k(R_\alpha\colon H_0 \to K_0) - A_k(R_\beta\colon H_0 \to K_0)\big| \to 0 \quad \text{as } \alpha \to \beta.$$

Lemma 4.1, (4.4) and (4.6) now leads to

$$\begin{aligned}
g_k(T\colon H_\theta \to K_\theta) &\leqslant 6^3\, g_k(T\colon H_0 \to K_0)^{1-\theta}\, g_k(T\colon H_1 \to K_1)^{\theta} \quad \text{and}\\
A_k(T\colon H_\theta \to K_\theta) &\leqslant 6^4 A_k(T\colon H_0 \to K_0)^{1-\theta}\, A_k(T\colon H_1 \to K_1)^{\theta}\,.
\end{aligned} \tag{4.7}$$

Suppose now that A^{-1} and B are not necessarily bounded. For each $n \in \mathbf{N}$ we consider the operators $P_n := \int_0^n \mathrm{d}E_{A^{-1}}(\lambda)$ and $Q_n := \int_0^n \mathrm{d}E_B(\lambda)$, where $E_{A^{-1}}$ and E_B are the corresponding spectral projections. Since A^{-1} and B commutes with P_n and Q_n on H_0 and K_0, the operators P_n and Q_n have bounded extensions on $\vec{H}$ and $\vec{K}$, which are norm 1 projections on H_θ and K_θ, respectively. Thus $Q_nTP_n \in L(\vec{H}, \vec{K})$. Using $B_n^{\theta/2}TA_n^{-\theta/2} = B^{\theta/2}Q_nTP_nA^{-\theta/2}$, where $A_n^{-1} := \int_0^n \lambda \mathrm{d}E_{A^{-1}}(\lambda)$ and $B_n := \int_0^n \lambda \mathrm{d}E_B(\lambda)$ are both bounded, and following steps analogous to those above (with A^{-1}, B and R_θ replaced by A_n^{-1}, B_n, and $B_n^{\theta/2}TA_n^{-\theta/2}$, respectively) we obtain

$$\begin{aligned}
g_k(T\colon H_\theta \to K_\theta) &\leqslant 6\, g_k(Q_nTP_n\colon H_\theta \to K_\theta)\\
&\leqslant 6^4 g_k(Q_nTP_n\colon H_0 \to K_0)^{1-\theta}\, g_k(Q_nTP_n\colon H_1 \to K_1)^{\theta}\\
&\leqslant 6^4 g_k(T\colon H_0 \to K_0)^{1-\theta}\, g_k(T\colon H_1 \to K_1)^{\theta}
\end{aligned}$$

and

$$A_k(T\colon H_\theta \to K_\theta) \leqslant 2\cdot 6^5 A_k(T\colon H_0 \to K_0)^{1-\theta}\, A_k(T\colon H_1 \to K_1)^{\theta}\,, \quad \theta \in [0,1],$$

by Lemma 3.11. This finishes the first part of the proof.

We now turn to the case where the couples $\vec{H}$ and $\vec{K}$ are not necessarily regular. Let H_j° (resp., K_j°) denote the closure of $H_0 \cap H_1$ (resp., $K_0 \cap K_1$) in H_j (resp., K_j), $j = 0, 1$. Let us observe that (H_0°, H_1°) and (K_0°, K_1°) are already regular. By [2, Theorem 4.2.2] we have

$$H_\theta^\circ = [H_0^\circ, H_1^\circ]_\theta = [H_0, H_1]_\theta \quad \text{and} \quad K_\theta^\circ = [K_0^\circ, K_1^\circ]_\theta = [K_0, K_1]_\theta.$$

Since Gelfand numbers $\{c_n(T)\}_{n\in\mathbf{N}}$ and entropy numbers are injective in the sense of Proposition 3.7 and $U_{H_j^\circ} \subset U_{H_j}$, we have

$$\begin{aligned} a_n\big(T\colon H_j^\circ \to H_j^\circ\big) &= c_n\big(T\colon H_j^\circ \to H_j^\circ\big) = c_n\big(T\colon H_j^\circ \to H_j\big) \\ &\leqslant c_n(T\colon H_j \to H_j) = a_n(T\colon H_j \to H_j) \quad \text{and} \\ g_n\big(T\colon H_j^\circ \to K_j^\circ\big) &\leqslant g_n\big(T\colon H_j^\circ \to K_j\big) \leqslant g_n(T\colon H_j \to K_j)\,, \quad n \in \mathbf{N}. \end{aligned}$$

This allows us to invoke theorem for the couples of Hilbert spaces (H_0°, H_1°) and (K_0°, K_1°) to obtain

$$\begin{aligned} g_k\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) &= g_k(T\colon H_\theta^\circ \to K_\theta^\circ) \\ &\leqslant 6^4\, g_k(T\colon H_0^\circ \to K_0^\circ)^{1-\theta}\, g_k(T\colon H_1^\circ \to K_1^\circ)^\theta \\ &\leqslant 6^4\, g_k(T\colon H_0 \to K_0)^{1-\theta}\, g_k(T\colon H_1 \to K_1)^\theta \end{aligned}$$

and

$$A_k\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant 2\cdot 6^5\, A_k(T\colon H_0 \to K_0)^{1-\theta}\, A_k(T\colon H_1 \to K_1)^\theta\,,$$

where $\theta \in [0, 1]$, and this completes the proof. □

A similar interpolation result holds for the approximation numbers of normal operators acting on Hilbert spaces.

Theorem 4.3. *Assume that $\vec{H} = (H_0, H_1)$ is a regular couple of Hilbert spaces. Let A be a positive operator on H_0 that gives the H_1 inner product and let $T \in L(\vec{H})$. If the operator T on H_0 is normal and commutes with A, then*

$$a_k(T\colon H_0 \to H_0) = a_k\big(T\colon [\vec{H}]_\theta \to [\vec{H}]_\theta\big) = a_k(T\colon H_1 \to H_1)$$

for every $\theta \in [0, 1]$ and each $k \in \mathbf{N}$.

Proof. Fix $\theta \in [0, 1]$ and $n \in \mathbf{N}$. By Lemma 4.1, $A^{\theta/2} T A^{-\theta/2} \in L(H_0, K_0)$ and

$$g_n(T\colon H_\theta \to K_\theta) = g_n\big(A^{\theta/2} T A^{-\theta/2}\colon H_0 \to K_0\big). \tag{4.8}$$

The operator T commutes with all powers of A on H_0. Then T on H_θ is unitarily equivalent to T on H_0, hence normal on H_θ. Analysis similar to that in the proof of Lemma 4.1 shows that

$$g_n\big(A^{\theta/2} T A^{-\theta/2}\colon H_0 \to K_0\big) = g_n\big(T\colon H_0 \to K_0\big). \tag{4.9}$$

From (4.8), (4.9), (3.1) and (3.4) it follows that

$$\begin{aligned} A_n\big(T\colon H_\theta \to K_\theta\big) &= \mathcal{G}_n\big(T\colon H_\theta \to K_\theta\big) = \mathcal{G}_n(T\colon H_0 \to K_0) \\ &= A_n\big(T\colon H_0 \to K_0\big), \end{aligned} \tag{4.10}$$

which completes the proof. □

We conclude the paper with the following remark that the proof of Theorem 4.3 strongly depends on the assumption that T commutes with A. We remark that

McCarthy [6, Theorem 2.1] showed that whenever $\vec{H}$ is a regular and ordered (resp., not necessarily ordered) Hilbert couple and $T \in L(\vec{H})$ is normal (resp., self-adjoint) on both H_0 and H_1, then the operator T on each H_θ, $\theta \in (0,1)$ is normal (resp., self-adjoint) and commutes with A.

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