Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 40, 2015, 361–373

AN INTEGRAL OPERATOR PRESERVING s-CARLESON MEASURE ON THE UNIT BALL

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Abstract. We establish an integral operator which preserves s-Carleson measure on the unit ball. As an application, we characterize the distance from Bloch-type functions to the analytic function space $F(p,q,s)$ on the ball.

1. Introduction

Let $\mathbf{B}_n$ be the unit ball of $\mathbf{C}^n$ with boundary $\mathbf{S}_n$ and $H(\mathbf{B}_n)$ the space of holomorphic functions on $\mathbf{B}_n$. When $n=1$, we have the unit disc $\mathbf{D}$.

If $\zeta \in \mathbf{S}_n$ and $r>0$, let $B(\zeta,r)=\{z\in\mathbf{B}_n : |1-\langle z,\zeta\rangle|<r\}$. For a constant $s>0$ and a positive Borel measure μ on $\mathbf{B}_n$, we call μ an s-Carleson measure if

$$\|\mu\|_{\mathcal{CM}_s}=\sup\left\{\frac{\mu(B(\zeta,r))}{r^{ns}} : \zeta\in\mathbf{S}_n,\ r>0\right\}<\infty.$$

We write $\mathcal{CM}_s$ for the class of all s-Carleson measures. When $s=1$, the s-Carleson measure becomes the classical Carleson measure on the ball. See [15] for more details. The Carleson measure plays a crucial role in lots of theories.

Motivated by Lemma 3.1.2 in [7] and Theorem 2.5 in [5], we investigate an integral operator which preserves s-Carleson measures on the unit ball. For $t,\lambda>0$, we define formally a linear operator $T_{t,\lambda}$ as

$$T_{t,\lambda}f(z)=\int_{\mathbf{B}_n}\frac{(1-|w|^2)^\lambda}{|1-\langle z,w\rangle|^{t+\lambda}}f(w)\,\mathrm{d}v(w),\quad z\in\mathbf{B}_n,$$

where $\mathrm{d}v$ is the volume measure on $\mathbf{B}_n$ normalized with $v(\mathbf{B}_n)=1$ and $f\in H(\mathbf{B}_n)$. The main result of this manuscript shows that $\mathcal{CM}_s$ is invariant under $T_{t,\lambda}$, which is stated as following:

Theorem 1. *Assume $0<s\le 1$, $1\le p<\infty$, and $\alpha>-1$. Let $\lambda>(\alpha+1-p)/p$, $t>n+1-(\alpha+1)/p$ and f be Lebesgue measurable on $\mathbf{B}_n$. If $|f(z)|^p(1-|z|^2)^\alpha\,\mathrm{d}v(z)$ belongs to $\mathcal{CM}_s$, then $|T_{t,\lambda}f(z)|^p(1-|z|^2)^{p(t-n-1)+\alpha}\,\mathrm{d}v(z)$ also belongs to $\mathcal{CM}_s$.*

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doi:10.5186/aasfm.2015.4017
2010 Mathematics Subject Classification: Primary 30H25, 30H30, 45P05.
Key words: Carleson measures, analytic $F(p,q,s)$ spaces, Bloch-type spaces.

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Cezhong Tong is supported by the National Natural Science Foundation of China (Grant Nos. 11301132 and 11171087) and Natural Science Foundation of Hebei Province (Grant No. A2013202265); Cheng Yuan is supported by NSFC 11226086 of China and Tianjin Advanced Education Development Fund 20111005.

For $f \in H(\mathbf{B}_n)$ with homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

the radial derivative of f is defined as

$$Rf(z) = \sum_{k=1}^{\infty} k f_k(z).$$

It is easy to see that $Rf \in H(\mathbf{B}_n)$ with

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t}\,\mathrm{d}t \tag{1}$$

For $0 < \alpha < \infty$, the Bloch-type space on $\mathbf{B}_n$, denoted by $\mathcal{B}_\alpha$, is the space of analytic functions on $\mathbf{B}_n$ satisfying

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in \mathbf{B}_n} (1-|z|^2)^\alpha |Rf(z)| < \infty.$$

It is well known that $\mathcal{B}_\alpha$ is a Banach space under the norm

$$\|f\|^*_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_{\mathcal{B}_\alpha}.$$

In particular, $\mathcal{B}_1$ becomes the classic Bloch space $\mathcal{B}$, which is the maximal Möbius invariant Banach space.

For any point $a \in \mathbf{B}_n \setminus \{0\}$ we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbf{B}_n,$$

where $s_a = \sqrt{1-|a|^2}$, $P_a(z) = \langle z, a\rangle a/|a|^2$ and $Q_a(a) = z - P_a(z)$. When $a = 0$, we simply define $\varphi_a(z) = -z$. It is easy to check that $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a(\varphi_a(z)) = z$ and $1 - |\varphi_a(z)|^2 = (1-|a|^2)(1-|z|^2)/|1-\langle z, a\rangle|^2$. All these basic facts can be found in [15].

Let $0 < p < \infty$, $0 \le s < \infty$, $-1 < q + s < \infty$, $-1 < q + n < \infty$. The space $F(p, q, s)$, known as the *general family of function spaces*, is defined as the set of $f \in H(\mathbf{B}_n)$ for which

$$\|f\|^p_{F(p,q,s)} = \sup_{a \in \mathbf{B}_n} \int_{\mathbf{B}_n} |Rf(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s \,\mathrm{d}v(z) < \infty.$$

The spaces $F(p, q, s)$ were first introduced by Zhao on $\mathbf{D}$ in [12]. Recently, Zhang, He and Cao characterized several equivalent norms of $F(p, q, s)$ on $\mathbf{B}_n$ in [11].

As the sequel of [10], this manuscript aims to characterize the distance from $f \in \mathcal{B}_\alpha$ to $F(p, q, s)$ on $\mathbf{B}_n$ as an application of Theorem 1. Let $X \subset \mathcal{B}_\alpha$ be an analytic function space. The distance from a Bloch-type function f to X is defined by

$$\mathrm{dist}_{\mathcal{B}_\alpha}(f, X) = \inf_{g \in X} \|f - g\|_{\mathcal{B}_\alpha}.$$

The second result of this paper is motivated by [1, 2, 6, 9, 13], which states as following:

Theorem 2. *Suppose* $1 \le p < \infty$, $0 < s \le n$, $-1 < q + s < \infty$ *and* $f \in \mathcal{B}_{\frac{n+1+q}{p}}$. *Then*

$$\operatorname{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F(p,q,s)) \approx \inf\Big\{\varepsilon > 0\colon \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\,\mathrm{d}v(z)}{(1-|z|^2)^{n+1-s}} \in \mathcal{CM}_{\frac{s}{n}}\Big\}$$

where $\widetilde{\Omega}_\varepsilon(f) = \big\{z \in \mathbf{B}_n\colon (1-|z|^2)^{\frac{n+1+q}{p}}|Rf(z)| \ge \varepsilon\big\}$ *and* $\chi_{\widetilde{\Omega}_\varepsilon(f)}$ *is the characteristic function of the set* $\widetilde{\Omega}_\varepsilon(f)$.

The argument in our proof of Theorem 2 is a generalization of [10], which follows from Theorem 3.1.3 in [7]. The distance from a $\mathcal{B}_\alpha$ function to Campanato–Morrey space on $\mathbf{D}$ was given in [8] with the similar idea.

Notation. Throughout this paper, we only write $U \lesssim V$ (or $V \gtrsim U$) for $U \le cV$ for a positive constant c, and moreover $U \approx V$ for both $U \lesssim V$ and $V \lesssim U$.

2. Preliminaries

The following result is well-known, for example, see Theorem 50 in [14] for a proof.

Lemma 3. *Let* $s, \gamma \in (0,\infty)$ *and* μ *be a nonnegative Borel measure on* $\mathbf{B}_n$. *Then* $\mu \in \mathcal{CM}_s$ *if and only if*

$$\|\mu\|_{\mathcal{CM}_s,\gamma} = \sup_{w\in\mathbf{B}_n}\int_{\mathbf{B}_n}\frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{\gamma+ns}}\,\mathrm{d}\mu(z) < \infty. \tag{2}$$

It is easy to check that if (2) holds for some $\gamma > 0$, it holds for all $\gamma > 0$. According to Lemma 3, the following corollary can be easily obtained.

Corollary 4. *Let* f *be an analytic function on* $\mathbf{B}_n$. *Then* $f \in F(p,q,s)$ *if and only if* $|Rf(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}v(z)$ *is an* s/n*-Carleson measure if and only if there exists an* $\gamma > 0$ *such that*

$$\|f\|^p_{F(p,q,s),\gamma} = \sup_{w\in\mathbf{B}_n}\int_{\mathbf{B}_n}\frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{\gamma+s}}|Rf(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}v(z) < \infty.$$

We also need the following standard result from [15].

Lemma 5. *Suppose* $t > -1$ *and* $c > 0$. *Then*

$$\int_{\mathbf{B}_n}\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\,\mathrm{d}v(w) \approx \frac{1}{(1-|z|^2)^c}$$

for all $z \in \mathbf{B}_n$.

The following lemma is quoted from [3], which is Lemma 2.5 there.

Lemma 6. *Suppose* $s > -1$ *and* $r, t > 0$. *If* $t < s+n+1 < r$, *then*

$$\int_{\mathbf{B}_n}\frac{(1-|w|^2)^s\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^r|1-\langle\eta,w\rangle|^t} \lesssim \frac{1}{(1-|z|^2)^{r-s-n-1}|1-\langle\eta,z\rangle|^t}.$$

Next we show that $F(p,q,s)$ is contained in $\mathcal{B}_{\frac{n+1+q}{p}}$. Similar result on the disk can be found in [12].

Lemma 7. *Suppose* $1 \le p < \infty$, $0 \le s < \infty$ *and* $\max\{-n-1, -s-1\} < q < \infty$, *then* $F(p,q,s) \subset \mathcal{B}_{\frac{n+1+q}{p}}$. *Moreover, if* $s > n$, *then* $F(p,q,s) = \mathcal{B}_{\frac{n+1+q}{p}}$.

Proof. By using the reproducing formula on Rf we can get that

$$Rf(z) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)} \int_{\mathbf{B}_n} \frac{(1-|w|^2)^\alpha Rf(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,\mathrm{d}v(w) \tag{3}$$

for all $\alpha > -1$. In this proof we take $\alpha > \frac{q+s}{p}$ and $0 < \gamma < n+1+q$.

When $p = 1$, it is easy to check that

$$\begin{aligned}
(1-|z|^2)^{n+1+q}|Rf(z)| &\lesssim \int_{\mathbf{B}_n} \frac{(1-|z|^2)^{n+1+q}(1-|w|^2)^\alpha |Rf(w)|}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,\mathrm{d}v(w)\\
&= \int_{\mathbf{B}_n} \frac{(1-|w|^2)^{q+s}(1-|z|^2)^\gamma |Rf(w)|}{|1-\langle z,w\rangle|^{\gamma+s}} \frac{(1-|z|^2)^{n+1+q-\gamma}(1-|w|^2)^{\alpha-q-s}}{|1-\langle z,w\rangle|^{n+1+\alpha-\gamma-s}}\,\mathrm{d}v(w)\\
&\le \int_{\mathbf{B}_n} \frac{(1-|z|^2)^\gamma}{|1-\langle z,w\rangle|^{\gamma+s}} |Rf(w)|(1-|w|^2)^{q+s}\,\mathrm{d}v(w)\\
&\quad \cdot \sup_{w\in\mathbf{B}_n} \frac{(1-|z|^2)^{n+1+q-\gamma}(1-|w|^2)^{\alpha-q-s}}{|1-\langle z,w\rangle|^{n+1+\alpha-\gamma-s}}\\
&\le \|f\|_{F(p,q,s),\gamma} \sup_{w\in\mathbf{B}_n} \frac{(1-|z|^2)^{n+1+q-\gamma}(1-|w|^2)^{\alpha-q-s}}{|1-\langle z,w\rangle|^{n+1+\alpha-\gamma-s}}.
\end{aligned}$$

Since $n+1+q-\gamma > 0$ and $\alpha - q - s > 0$, it follows that

$$\sup_{w\in\mathbf{B}_n} \frac{(1-|z|^2)^{n+1+q-\gamma}(1-|w|^2)^{\alpha-q-s}}{|1-\langle z,w\rangle|^{n+1+\alpha-\gamma-s}} \lesssim 1.$$

Thus $F(p,q,s) \subset \mathcal{B}_{\frac{n+1+q}{p}}$ when $p = 1$.

When $p > 1$, take $p' = p/(p-1)$. Then it follows from the Hölder's inequality that

$$\begin{aligned}
|Rf(z)| &\lesssim \int_{\mathbf{B}_n} \frac{(1-|w|^2)^{\frac{q+s}{p}}(1-|z|^2)^{\frac{\gamma}{p}}|Rf(w)|}{|1-\langle z,w\rangle|^{\frac{s+\gamma}{p}}} \frac{(1-|z|^2)^{-\frac{\gamma}{p}}(1-|w|^2)^{\alpha-\frac{q+s}{p}}}{|1-\langle z,w\rangle|^{n+1+\alpha-\frac{s+\gamma}{p}}}\,\mathrm{d}v(w)\\
&\le \left(\int_{\mathbf{B}_n} \frac{(1-|z|^2)^\gamma}{|1-\langle z,w\rangle|^{s+\gamma}} |Rf(w)|^p (1-|w|^2)^{q+s}\,\mathrm{d}v(w)\right)^{\frac{1}{p}}\\
&\quad \cdot \frac{1}{(1-|z|^2)^{\frac{\gamma}{p}}} \left(\int_{\mathbf{B}_n} \frac{(1-|w|^2)^{p'(\alpha-\frac{q+s}{p})}}{|1-\langle z,w\rangle|^{p'(n+1+\alpha-\frac{s+\gamma}{p})}}\,\mathrm{d}v(w)\right)^{\frac{1}{p'}}\\
&\le \frac{\|f\|_{F(p,q,s),\gamma}}{(1-|z|^2)^{\frac{\gamma}{p}}} \left(\int_{\mathbf{B}_n} \frac{(1-|w|^2)^{p'(\alpha-\frac{q+s}{p})}}{|1-\langle z,w\rangle|^{p'(n+1+\alpha-\frac{s+\gamma}{p})}}\,\mathrm{d}v(w)\right)^{\frac{1}{p'}}\\
&\lesssim \|f\|_{F(p,q,s),\gamma} \frac{1}{(1-|z|^2)^{\frac{\gamma}{p}}} \left(\frac{1}{(1-|z|^2)^{\frac{n+1+q-\gamma}{p-1}}}\right)^{\frac{1}{p'}}\\
&= \|f\|_{F(p,q,s),\gamma} \frac{1}{(1-|z|^2)^{\frac{n+1+q}{p}}}.
\end{aligned}$$

Apparently, Lemma 5 is applied in the last inequality. This gives that $F(p,q,s) \subset \mathcal{B}_{\frac{n+1+q}{p}}$ when $1 < p < \infty$.

Now, suppose $s > n$, let $f \in \mathcal{B}_{\frac{n+1+q}{p}}$, then

$$|Rf(z)|(1-|z|^2)^{\frac{n+1+q}{p}} \le \|f\|_{\mathcal{B}_{\frac{n+1+q}{p}}} < \infty$$

for all $z \in \mathbf{B}_n$. It follows that

$$\begin{aligned}
\|f\|^p_{F(p,q,s)} &= \sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} |Rf(z)|^p (1-|z|^2)^{q+s} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^s \mathrm{d}v(z) \\
&= \sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} |Rf(z)|^p (1-|z|^2)^{q+n+1}(1-|z|^2)^{s-n-1} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^s \mathrm{d}v(z) \\
&\le \|f\|^p_{\mathcal{B}_{\frac{n+1+q}{p}}} \sup_{a\in\mathbf{B}_n} (1-|a|^2)^s \int_{\mathbf{B}_n} \frac{(1-|z|^2)^{s-n-1}}{|1-\langle z,a\rangle|^{2s}} \mathrm{d}v(z) \approx \|f\|^p_{\mathcal{B}_{\frac{n+1+q}{p}}}.
\end{aligned}$$

This completes the proof. □

3. Proof of Theorem 1

Proof. When $p = 1$, according to Lemma 3, it is sufficient to show that

$$\sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} \frac{(1-|a|^2)^\gamma}{|1-\langle z,a\rangle|^{\gamma+ns}} |T_{t,\lambda} f(z)|(1-|z|^2)^{t-n-1+\alpha} \mathrm{d}v(z) < \infty$$

for some $\gamma > 0$. That is to show

$$\sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} \frac{(1-|a|^2)^\gamma (1-|z|^2)^{t-n-1+\alpha}}{|1-\langle z,a\rangle|^{\gamma+ns}} \left| \int_{\mathbf{B}_n} \frac{(1-|w|^2)^\lambda f(w)}{|1-\langle z,w\rangle|^{t+\lambda}} \mathrm{d}v(w) \right| \mathrm{d}v(z)$$

is finite. By Fubini's theorem, we need to verify that

$$\sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} \frac{(1-|a|^2)^\gamma |f(w)|}{(1-|w|^2)^{-\lambda}} \int_{\mathbf{B}_n} \frac{(1-|z|^2)^{t-n-1+\alpha} \mathrm{d}v(z)}{|1-\langle z,w\rangle|^{t+\lambda} |1-\langle z,a\rangle|^{\gamma+ns}} \mathrm{d}v(w)$$

is finite.

Choose γ such that $\gamma + ns < t + \alpha$. Notice that $t - n - 1 + \alpha > -1$ and $\lambda > \alpha$ in this case. Then by Lemma 6 the last integral can be controlled by

$$\sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} \frac{(1-|a|^2)^\gamma}{|1-\langle w,a\rangle|^{\gamma+ns}} |f(w)|(1-|w|^2)^\alpha \mathrm{d}A(w).$$

The desired result follows from Lemma 3, since $|f(z)|(1-|z|^2)^\alpha \mathrm{d}v(z)$ is an s-Carleson measure.

When $1 < p < \infty$, it is sufficient to show that

$$\int_{B(\zeta,r)} |T_{t,\lambda} f(z)|^p (1-|z|^2)^{p(t-n-1)+\alpha} \mathrm{d}v(z) \lesssim r^{ns}$$

holds for all $\zeta \in \mathbf{S}_n$ and $r > 0$.

For each fixed $r>0$, there exists a smallest $N_r \in \mathbf{N}$ such that $2^{N_r} r \geq 2$, which means that $B(\zeta, 2^{N_r} r) = \mathbf{B}_n$. So we can make the following estimates:

$$\begin{aligned}
&\int_{B(\zeta,r)} |T_{t,\lambda} f(z)|^p (1-|z|^2)^{p(t-n-1)+\alpha}\, dv(z)\\
&= \int_{B(\zeta,r)} \left| \int_{\mathbf{B}_n} \frac{(1-|w|^2)^\lambda}{|1-\langle z,w\rangle|^{t+\lambda}} f(w)\, dv(w) \right|^p (1-|z|^2)^{p(t-n-1)+\alpha}\, dv(z)\\
&= \int_{B(\zeta,r)} \left| \left(\int_{B(\zeta,2r)} + \int_{\mathbf{B}_n \setminus B(\zeta,2r)} \right) \frac{(1-|w|^2)^\lambda}{|1-\langle z,w\rangle|^{t+\lambda}} f(w)\, dv(w) \right|^p (1-|z|^2)^{p(t-n-1)+\alpha}\, dv(z)\\
&\lesssim \int_{B(\zeta,r)} \left(\int_{B(\zeta,2r)} \frac{(1-|w|^2)^\lambda |f(w)|}{|1-\langle z,w\rangle|^{t+\lambda}}\, dv(w) \right)^p (1-|z|^2)^{p(t-n-1)+\alpha}\, dv(z)\\
&\quad + \int_{B(\zeta,r)} \left(\int_{\mathbf{B}_n \setminus B(\zeta,2r)} \frac{(1-|w|^2)^\lambda |f(w)|}{|1-\langle z,w\rangle|^{t+\lambda}}\, dv(w) \right)^p \frac{dv(z)}{(1-|z|^2)^{p(n+1-t)-\alpha}} = \mathrm{Int}_1 + \mathrm{Int}_2.
\end{aligned}$$

For Int_1, consider the linear operator $T\colon L^p(\mathbf{B}_n, dv) \to L^p(\mathbf{B}_n, dv)$ defined by

$$(Tf)(z) = \int_{\mathbf{B}_n} K(z,w) f(w)\, dv(w),$$

where the kernel is given by

$$K(z,w) = \frac{(1-|w|^2)^{\lambda-\alpha/p}(1-|z|^2)^{t-n-1+\alpha/p}}{|1-\langle z,w\rangle|^{t+\lambda}}.$$

We can apply Schur's test (see e.g. [16]) to verify that T is a bounded operator on $L^p(\mathbf{B}_n, dv)$. Indeed, if we take $p' = p/(p-1)$ again and let $h(z) = (1-|z|^2)^{-\frac{1}{pp'}}$, then it follows from Lemma 5 that

$$\int_{\mathbf{B}_n} K(z,w) h^p(z)\, dv(z) \lesssim h^p(w)$$

and

$$\int_{\mathbf{B}_n} K(z,w) h^{p'}(w)\, dv(w) \lesssim h^{p'}(z).$$

Accordingly, the integral operator T is bounded from $L^p(\mathbf{B}_n, dv)$ to $L^p(\mathbf{B}_n, dv)$.

Now we rewrite Int_1 as

$$\mathrm{Int}_1 = \int_{B(\zeta,r)} \left(\int_{B(\zeta,2r)} K(z,w)|f(w)|(1-|w|^2)^{\alpha/p}\, dv(w) \right)^p dv(z),$$

and let

$$g(w) = |f(w)|(1-|w|^2)^{\alpha/p} \chi_{B(\zeta,2r)}(w),$$

where χ_E stands for the characteristic function of E. Recall that $|f(w)|^p(1-|w|^2)^\alpha\, dv(w)$ is an s-Carleson measure, we have

$$\|g\|_{L^p}^p = \int_{B(\zeta,2r)} |f(w)|^p (1-|w|^2)^\alpha\, dv(w) \lesssim (2r)^{ns} \lesssim r^{ns}.$$

Thus, we get

$$\mathrm{Int}_1 \lesssim \int_{\mathbf{B}_n} \left| \int_{\mathbf{B}_n} K(z,w) g(w)\, \mathrm{d}v(w) \right|^p \mathrm{d}v(z) = \|Tg\|_{L^p}^p \lesssim \|g\|_{L^p}^p \lesssim r^{ns}$$

as desired.

To handle Int_2, we note first that for $k = 2, 3, \cdots, N_r$, the inequality $|1 - \langle z, w\rangle| \gtrsim 2^k r$ holds for $z \in B(\zeta, r)$ and $w \in B(\zeta, 2^k r) \setminus B(\zeta, 2^{k-1} r)$. For fixed $c > -1$, if we write $Q(\zeta, r) = \{\xi \in \mathbf{S}_n \colon |1 - \langle \zeta, \xi\rangle| < r\}$ and denote σ the normalized surface measure on $\mathbf{S}_n$, then a straightforward computation shows that

$$\int_{B(\zeta, 2^k r)} (1 - |z|^2)^c\, \mathrm{d}v(z) \lesssim \int_{Q(\zeta, 2^k r)} \mathrm{d}\sigma \int_{1 - 2^k r}^1 2n t^{2n-1} (1 - t^2)^c\, \mathrm{d}t \lesssim (2^k r)^{n+1+c}.$$

Notice that

$$\mathbf{B}_n \setminus B(\zeta, 2r) = \bigcup_{k=1}^{N_r - 1} B(\zeta, 2^{k+1} r) \setminus B(\zeta, 2^k r).$$

Since $p(t - n - 1) + \alpha > -1$, we have

$$\begin{aligned} \mathrm{Int}_2 &\lesssim \int_{B(\zeta, r)} \left(\sum_{k=1}^{N_r - 1} \int_{B(\zeta, 2^{k+1} r) \setminus B(\zeta, 2^k r)} \frac{(1 - |w|^2)^\lambda |f(w)|}{|1 - \langle z, w\rangle|^{t+\lambda}}\, \mathrm{d}v(w) \right)^p \frac{\mathrm{d}v(z)}{(1 - |z|^2)^{p(n+1-t) - \alpha}} \\ &\lesssim \int_{B(\zeta, r)} \left(\sum_{k=1}^{N_r - 1} \int_{B(\zeta, 2^{k+1} r) \setminus B(\zeta, 2^k r)} \frac{(1 - |w|^2)^\lambda |f(w)|}{(2^k r)^{t+\lambda}}\, \mathrm{d}v(w) \right)^p \frac{\mathrm{d}v(z)}{(1 - |z|^2)^{p(n+1-t) - \alpha}} \\ &\lesssim r^{n+1+p(t-n-1)+\alpha} \left(\sum_{k=1}^{N_r - 1} \int_{B(\zeta, 2^{k+1} r)} \frac{(1 - |w|^2)^\lambda |f(w)|}{(2^k r)^{t+\lambda}}\, \mathrm{d}v(w) \right)^p. \end{aligned}$$

Keep in mind that $|f(w)|^p (1 - |w|^2)^\alpha\, \mathrm{d}v(w)$ is an s-Carleson measure and $\lambda > (1 + \alpha - p)/p$, we can use the Hölder's inequality to get that

$$\begin{aligned} &\int_{B(\zeta, 2^{k+1} r)} (1 - |w|^2)^\lambda |f(w)|\, \mathrm{d}v(w) \\ &\le \left(\int_{B(\zeta, 2^{k+1} r)} (1 - |w|^2)^\alpha |f(w)|^p\, \mathrm{d}v(w) \right)^{\frac{1}{p}} \left(\int_{B(\zeta, 2^{k+1} r)} (1 - |w|^2)^{(\lambda - \frac{\alpha}{p}) p'}\, \mathrm{d}v(w) \right)^{\frac{1}{p'}} \\ &\lesssim \left(2^{k+1} r\right)^{\frac{ns}{p}} \times \left(2^{k+1} r\right)^{(\lambda p' - \frac{p'}{p}\alpha + n + 1)\frac{1}{p'}}. \end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned} \mathrm{Int}_2 &\lesssim r^{n+1+p(t-n-1)+\alpha} \left(\sum_{k=1}^{\infty} \frac{\left(2^{k+1} r\right)^{\frac{ns}{p}} \times \left(2^{k+1} r\right)^{(\lambda p' - \frac{p'}{p}\alpha + n + 1)\frac{1}{p'}}}{(2^k r)^{t+\lambda}} \right)^p \\ &\lesssim r^{ns} \left(\sum_{k=1}^{\infty} 2^{k\left(\frac{ns - \alpha + (p-1)(n+1)}{p} - t\right)} \right)^p. \end{aligned}$$

The assumptions $t > n + 1 - \frac{\alpha+1}{p}$ and $0 < s \le 1$ imply that $t > \frac{ns - \alpha + (p-1)(n+1)}{p}$. This completes the proof. □

4. Proof of Theorem 2

Proof. Firstly, we prove

$$\operatorname{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F(p,q,s)) \lesssim \inf\left\{\varepsilon > 0\colon \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\,\mathrm{d}v(z)}{(1-|z|^2)^{n+1-s}} \in \mathcal{CM}_{\frac{s}{n}}\right\}. \tag{4}$$

When $\alpha > -1$, for $f \in \mathcal{B}_{\frac{n+1+q}{p}}$, $Rf(z)$ can be rewritten as

$$\int_{\mathbf{B}_n} \frac{Rf(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}},$$

where

$$\mathrm{d}v_\alpha(z) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)}(1-|z|^2)^\alpha\,\mathrm{d}v(z).$$

Similarly as [15] and [9], it follows from $Rf(0) = 0$ that

$$Rf(z) = \int_{\mathbf{B}_n} Rf(w)\left(\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} - 1\right)\mathrm{d}v_\alpha(w)$$

for all $z \in \mathbf{B}_n$. According to (1),

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t}\,\mathrm{d}t = \int_{\mathbf{B}_n} Rf(w)L(w,z)\,\mathrm{d}v_\alpha(w),$$

where the kernel

$$L(z,w) = \int_0^1 \left(\frac{1}{(1-t\langle z,w\rangle)^{n+1+\alpha}} - 1\right)\frac{\mathrm{d}t}{t}.$$

Define

$$f_1(z) = f(0) + \int_{\widetilde{\Omega}_\varepsilon(f)} Rf(w)L(z,w)\,\mathrm{d}v_\alpha(w)$$

and

$$f_2(z) = \int_{\mathbf{B}_n\setminus\widetilde{\Omega}_\varepsilon(f)} Rf(w)L(z,w)\,\mathrm{d}v_\alpha(w).$$

Then

$$f(z) = f_1(z) + f_2(z).$$

We can just verify that $\frac{\chi_{\widetilde{\Omega}_\epsilon(f)}(z)\,\mathrm{d}v(z)}{(1-|z|^2)^{n+1-s}} \in \mathcal{CM}_{\frac{s}{n}}$ implies $f_1 \in F(p,q,s)$ and $f_2 \in \mathcal{B}_{\frac{n+1+q}{p}}$ with $\|f_2\|_{\mathcal{B}_{\frac{n+1+q}{p}}} \lesssim \varepsilon$.

When w is fixed, $L(z,w)$ becomes a holomorphic function in z. And it is easy to check that

$$RL(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} - 1,$$

and

$$|RL(z,w)| \lesssim \frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}}.$$

We choose $\alpha = \frac{n+1+q}{p}$, then

$$\begin{aligned}|Rf_1(z)| &= \left|\int_{\widetilde{\Omega}_\varepsilon(f)} Rf(w)RL(z,w)\,\mathrm{d}v_\alpha(w)\right|\\ &\lesssim \int_{\widetilde{\Omega}_\varepsilon(f)} |Rf(w)|(1-|w|^2)^{\frac{n+1+q}{p}}|RL(z,w)|\,\mathrm{d}v(w)\\ &\lesssim \|f\|_{\mathcal{B}_{\frac{n+1+q}{p}}}\int_{\mathbf{B}_n}\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)|RL(z,w)|\,\mathrm{d}v(w)\\ &\lesssim \|f\|_{\mathcal{B}_{\frac{n+1+q}{p}}}\int_{\mathbf{B}_n}\frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{|1-\langle z,w\rangle|^{n+1+\frac{n+1+q}{p}}}\,\mathrm{d}v(w)\\ &= \|f\|_{\mathcal{B}_{\frac{n+1+q}{p}}}\int_{\mathbf{B}_n}\frac{(1-|w|^2)^{\frac{n+1}{p}}}{|1-\langle z,w\rangle|^{n+1+\frac{n+1+q}{p}}}\frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{(1-|w|^2)^{\frac{n+1}{p}}}\,\mathrm{d}v(w).\end{aligned}$$

If we write

$$g(w) = \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(w)}{(1-|w|^2)^{\frac{n+1}{p}}},$$

then

$$|g(w)|^p(1-|w|^2)^s\,\mathrm{d}v(w) = \chi_{\widetilde{\Omega}_\varepsilon(f)}(w)(1-|w|^2)^{s-n-1}\,\mathrm{d}v(w).$$

So, if

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)(1-|z|^2)^{s-n-1}\,\mathrm{d}v(z)$$

is in $\mathcal{CM}_{\frac{s}{n}}$, Theorem 1 with $\lambda = \frac{n+1}{p}$ and $t = n+1+\frac{q}{p}$ implies that

$$|Rf_1(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}v(z)$$

belongs to $\mathcal{CM}_{\frac{s}{n}}$. This means $f_1 \in F(p,q,s)$. Meanwhile, we have

$$|Rf_2(z)| \lesssim \varepsilon\int_{\mathbf{B}_n}\frac{\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\frac{n+1+q}{p}}} \approx \frac{\varepsilon}{(1-|z|^2)^{\frac{n+1+q}{p}}}.$$

This gives that $f_2 \in \mathcal{B}_{\frac{n+1+q}{p}}$ with $\|f_2\|_{\mathcal{B}_{\frac{n+1+q}{p}}} \lesssim \varepsilon$. Thus we verified (4).

In order to prove the converse inequality of (4), we assume that

$$\mathrm{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F(p,q,s)) < \inf\left\{\varepsilon > 0\colon \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)\,\mathrm{d}v(z)}{(1-|z|^2)^{n+1-s}} \in \mathcal{CM}_{\frac{s}{n}}\right\}.$$

For short, let ε_0 denote the right-hand quantity of the last inequality. We only consider the case $\varepsilon_0 > 0$. Then there exists an ε_1 such that

$$0 < \varepsilon_1 < \varepsilon_0 \quad \text{and} \quad \mathrm{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F(p,q,s)) < \varepsilon_1.$$

Hence, we can find a $h \in F(p,q,s)$ such that

$$\|f-h\|_{\mathcal{B}_{\frac{n+1+q}{p}}} < \varepsilon_1.$$

Now for any $\varepsilon \in (\varepsilon_1, \varepsilon_0)$ we have that

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)(1-|z|^2)^{s-n-1}\,\mathrm{d}v(z)$$

is not in $\mathcal{CM}_{\frac{s}{n}}$. But, $\|f-h\|_{\mathcal{B}_{\frac{n+1+q}{p}}} < \varepsilon_1$ yields

$$(1-|z|^2)^{\frac{n+1+q}{p}}|Rh(z)| > (1-|z|^2)^{\frac{n+1+q}{p}}|Rf(z)| - \varepsilon_1, \quad \forall\, z \in \mathbf{B}_n,$$

and so

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z) \le \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) \quad \forall\, z \in \mathbf{B}_n.$$

This implies that

$$\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)(1-|z|^2)^{s-n-1}\,\mathrm{d}v(z)$$

does not belong to $\mathcal{CM}_{\frac{s}{n}}$. On the other hand,

$$\begin{aligned}\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)(1-|z|^2)^{s-n-1}\,\mathrm{d}v(z) &= \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\frac{(1-|z|^2)^{q+s}}{(1-|z|^2)^{q+n+1}}\,\mathrm{d}v(z)\\ &\le \frac{|Rh(z)|^p}{(\varepsilon-\varepsilon_1)^p}(1-|z|^2)^{q+s}\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)\,\mathrm{d}v(z)\\ &\le \frac{1}{(\varepsilon-\varepsilon_1)^p}|Rh(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}v(z).\end{aligned}$$

Since $h \in F(p,q,s)$,

$$|Rh(z)|^p(1-|z|^2)^{q+s}\,\mathrm{d}v(z)$$

is in $\mathcal{CM}_{\frac{s}{n}}$, and consequently

$$\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z)(1-|z|^2)^{s-n-1}\,\mathrm{d}v(z)$$

is in $\mathcal{CM}_{\frac{s}{n}}$. Now, a contradiction occurs. Thus we must have

$$\varepsilon_0 \le \mathrm{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F(p,q,s)) \lesssim \varepsilon_0$$

as required. □

5. Further remarks

For a measurable function f on $\mathbf{B}_n$, define the projection operator

$$P_{t,\lambda}f(z) = \int_{\mathbf{B}_n} \frac{(1-|w|^2)^\lambda}{(1-\langle z,w\rangle)^{t+\lambda}} f(w)\,\mathrm{d}v(w), \quad z \in \mathbf{B}_n.$$

In particular, if $\lambda > 0$ and $t = n+1$, $P_{t,\lambda}$ is called the Bergman projection. It is shown in [15] that the Bergman projection is bounded from $L^p(\mathbf{B}_n, \mathrm{d}v_\lambda)$ onto the Bergman space A^p_λ when $1 < p < \infty$.

When $1 \le p < \infty$, $\alpha > -1$, $0 < s \le 1$, we define a class $\mathcal{G}_{p,\alpha,s}$ of measurable functions on $\mathbf{B}_n$ such that

$$|f(z)|^p(1-|z|^2)^\alpha\,\mathrm{d}v(z) \in \mathcal{CM}_s.$$

Then, $f \in F(p,q,s)$ if and only if $Rf \in \mathcal{G}_{p,q+s,\frac{s}{n}} \cap H(\mathbf{B}_n)$. The next theorem shows that the Bergman projection is bounded from $\mathcal{G}_{p,\alpha,s}$ to $\mathcal{G}_{p,\alpha,s} \cap H(\mathbf{B}_n)$.

Theorem 8. *Let $1 \le p < \infty$, $\alpha > -1$, $0 < s \le 1$. The Bergman projection $P_{n+1,\lambda}$ is a bounded linear operator from $\mathcal{G}_{p,\alpha,s}$ to $\mathcal{G}_{p,\alpha,s} \cap H(\mathbf{B}_n)$.*

Proof. It can be easily checked that for all measurable f,

$$|P_{t,\lambda}f| \le |T_{t,\lambda}f|.$$

Then Theorem 1 implies the desired result. □

Further, we have the following corollary.

Corollary 9. *Let $1 \le p < \infty$, $\alpha > -1$ and $0 < s \le 1$. Suppose $\lambda > (\alpha+1-p)/p$ and $t > n+1-(\alpha+1)/p$. Then the projection $P_{t,\lambda}$ is a bounded linear operator from $\mathcal{G}_{p,\alpha,s}$ to $\mathcal{G}_{p,p(t-n-1)+\alpha,s} \cap H(\mathbf{B}_n)$.*

For an s-Carleson measure μ on $\mathbf{B}_n$, if

$$\lim_{r\to 1} \frac{\mu(B(\zeta,r))}{r^{ns}} = 0$$

for $\zeta \in \mathbf{S}_n$ uniformly, we call μ a vanishing s-Carleson measure.

The following result is well-known. See, for example, the remark after Theorem 50 in [14].

Corollary 10. *Let $s, \gamma \in (0,\infty)$ and μ be nonnegative Borel measures on $\mathbf{B}_n$. Then μ is a vanishing s-Carleson measure if and only if*

$$\lim_{|w|\to 1} \int_{\mathbf{B}_n} \frac{(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{\gamma+ns}} \, \mathrm{d}\mu(z) = 0. \tag{5}$$

By a slight modification of the proof of Theorem 1, we can obtain the following result.

Lemma 11. *Assume $0 < s \le 1$, $1 \le p < \infty$, and $\alpha > -1$. Let $\lambda > (\alpha+1-p)/p$, $t > n+1-(\alpha+1)/p$ and f be Lebesgue measurable on $\mathbf{B}_n$. If $|f(z)|^p(1-|z|^2)^\alpha \, \mathrm{d}v(z)$ is a vanishing s-Carleson measure, then*

$$|T_{t,\lambda} f(z)|^p (1-|z|^2)^{p(t-n-1)+\alpha} \, \mathrm{d}v(z)$$

is also a vanishing s-Carleson measure.

For $0 < \alpha < \infty$, the little Bloch-type space on $\mathbf{B}_n$, denoted by $\mathcal{B}_\alpha^0$, is the subspace of $\mathcal{B}_\alpha$ consisting of all $f \in \mathcal{B}_\alpha$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |Rf(z)| = 0,$$

and the space $F_0(p,q,s)$, is the subspace of $F(p,q,s)$ consisting of all $f \in F(p,q,s)$ such that

$$\sup_{|a|\to 1} \int_{\mathbf{B}_n} |Rf(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s \, \mathrm{d}v(z) = 0.$$

Similar to Lemma 7, we have the following corollary.

Lemma 12. *Suppose $1 \le p < \infty$, $0 \le s < \infty$ and $\max\{-n-1, -s-1\} < q < \infty$. If $f \in H(\mathbf{B}_n)$, then $f \in F_0(p,q,s)$ if and only if $|Rf(z)|^p(1-|z|^2)^{q+s} \, \mathrm{d}v(z)$ is a vanishing $\frac{s}{n}$-Carleson measure. Further, $F_0(p,q,s) \subset \mathcal{B}^0_{\frac{n+1+q}{p}}$. When $s > n$, $F_0(p,q,s) = \mathcal{B}^0_{\frac{n+1+q}{p}}$.*

For the "little-oh" case of Theorem 2, we have following corollary.

Corollary 13. *Let $0 < s \le n$, $1 \le p < \infty$, $-1 < q+s < \infty$ and let $f \in \mathcal{B}_{\frac{n+1+q}{p}}$. Then the following quantities are equivalent:*

(1) $\mathrm{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, \mathcal{B}^0_{\frac{n+1+q}{p}})$;

(2) $\mathrm{dist}_{\mathcal{B}_{\frac{n+1+q}{p}}}(f, F_0(p,q,s))$;

(3) $\inf\{\varepsilon > 0\colon \frac{\chi_{\widetilde{\Omega}_\varepsilon(f)}(z)}{(1-|z|^2)^{n+1-s}} \, \mathrm{d}v(z)$ *is a vanishing $\frac{s}{n}$-Carleson measure*$\}$.

Remark 14. Theorem 2 characterizes the closure of $F(p,q,s)$ in the $\mathcal{B}_{\frac{n+1+q}{p}}$ norm. That is, for $f \in \mathcal{B}_{\frac{n+1+q}{p}}$, f is in the closure of $F(p,q,s)$ in the $\mathcal{B}_{\frac{n+1+q}{p}}$ norm if and only if for every $\varepsilon > 0$,

$$\int_{\widetilde{\Omega}_\varepsilon(f)\cap B(\zeta,r)} (1-|z|^2)^{s-n-1}\,\mathrm{d}v(z) \lesssim r^s$$

for all $\zeta \in \mathbf{S}_n$ and $r > 0$.

The invariant Green's function $G(z,a)$ of $\mathbf{B}_n$ is defined by $G(z,a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n}\int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1}\,\mathrm{d}t.$$

The holomorphic function spaces Q_s associated with the Green's function is introduced in [4]. For $s > 0$, Q_s is defined by

$$Q_s = \left\{ f \in H(\mathbf{B}_n) \colon \sup_{a\in\mathbf{B}_n} \int_{\mathbf{B}_n} \left|\widetilde{\nabla} f(z)\right|^2 G(z,a)^s\,\mathrm{d}\tau(z) < \infty \right\},$$

and its subspace $Q_{s,0}$ is defined by

$$Q_{s,0} = \left\{ f \in H(\mathbf{B}_n) \colon \lim_{|a|\to 1} \int_{\mathbf{B}_n} \left|\widetilde{\nabla} f(z)\right|^2 G(z,a)^s\,\mathrm{d}\tau(z) = 0 \right\},$$

where $\widetilde{\nabla} f(z) = \nabla(f\circ\varphi_z)(0)$ is the Möbius invariant gradient of f, and $\mathrm{d}\tau(z) = (1-|z|^2)^{-n-1}\,\mathrm{d}v(z)$ is the Möbius invariant measure on $\mathbf{B}_n$. It is well known that for $n > 1$ and $\frac{n-1}{n} < s \le 1$, $f \in Q_s$ if and only if $|Rf(z)|^2(1-|z|^2)^{ns+2}\,\mathrm{d}\tau(z)$ is an s-Carleson measure; $f \in Q_{s,0}$ if and only if $|Rf(z)|^2(1-|z|^2)^{ns+2}\,\mathrm{d}\tau(z)$ is a vanishing s-Carleson measure. Thus $Q_s = F(2,1-n,ns)$ and $Q_{s,0} = F_0(2,1-n,ns)$. In particular, when $s = 1$, $Q_s = BMOA = F(2,1-n,n)$ and $Q_{s,0} = VMOA = F_0(2,1-n,n)$. Thus, Theorem 2 covers Jone's formula in [1], a part of Zhao's result in [13] and Xu's result in [9].

Acknowledgements. The authors would like to thank the referee for her/his helpful comments and suggestions for improving this manuscript.

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Received 13 May 2014 • Accepted 19 September 2014