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CORRECTION TO THE PAPER "ON THE SECOND MAIN THEOREM OF CARTAN"

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Unfortunately, the proof of Theorem 2 in [1] contains a mistake: the exponents s_k in (20) can be complex, and this affects most of the arguments that follow. Below is the modified proof of Theorem 2.

To prove Theorem 2, we use the following two facts about the class $\mathfrak{F}$:

1. $\mathfrak{F}$ is a differential ring [2]. This means that $\mathfrak{F}$ is closed under addition, multiplication and differentiation.
2. All functions $y \in \mathfrak{F}$ are entire functions of completely regular growth in the sense of Levin–Pflüger [4], with piecewise-trigonometric indicators, the notions which we recall now.

Let f be a holomorphic function in an angular sector $S = \{re^{i\theta}\colon |\theta - \theta_0| < \epsilon,\ r > 0\}$. We say that f has *completely regular growth* with respect to order $\rho > 0$ if the following finite limit exists

$$\lim_{r\to\infty,\ re^{i\theta}\notin E} \frac{\log|f(re^{i\theta})|}{|r|^\rho} =: h_f(\rho,\theta), \tag{1}$$

uniformly with respect to θ, for $|\theta - \theta_0| < \epsilon$. Here $E \subset S$ is an exceptional set which can be covered by discs centered at z_j of radii r_j, such that

$$\sum_{j\colon |z_j|<r} r_j = o(r), \quad r \to \infty.$$

Such sets E are called C_0-sets in [4].

The limit $h_f(\rho,\theta)$ is called the indicator. It is always continuous as a function of $\theta \in (-\epsilon,\epsilon)$. Notice that if f has completely regular growth with respect to order ρ, then it has completely regular growth with respect to any larger order, and the indicator with respect to the larger order is zero.

An entire function f is said to be of completely regular growth, if it has completely regular growth in any sector with respect to its order $\rho = \rho(f)$.

If f_1 and f_2 are two functions of completely regular growth with respect to the same order ρ then evidently

$$h_{f_1+f_2}(\rho,\theta) \le \max\{h_{f_1}(\rho,\theta), h_{f_2}(\rho,\theta)\},$$

and equality holds if $h_{f_1}(\rho,\theta) \ne h_{f_2}(\rho,\theta)$.

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Petrenko [5, Sect. 4.3] proved that all entire functions in $\mathfrak{F}$ have completely regular growth with piecewise-trigonometric indicators. We say that h is piecewise-trigonometric if the interval $[0, 2\pi]$ can be partitioned into finitely many intervals such that $h(\theta) = c_k \sin \rho(\theta - \theta_k)$ on each interval.

Let $V \subset \mathfrak{F}$ be a vector space of finite dimension $n + 1$. Let ρ be the maximal order of elements of V. From now on, all indicators will be considered with respect to this order ρ, and we suppress it from notation.

For each V there exist finitely many rays such that for any sector S complementary to these rays all possible indicators of elements of V are strictly ordered:

$$h_1(\theta) < h_2(\theta) < \ldots < h_m(\theta), \quad e^{i\theta} \in S. \tag{2}$$

Here $m \geq 1$ is the number of distinct indicators in S. Such sectors will be called *admissible* for a vector space V.

We fix an admissible sector S of our vector space V, and construct a *special basis* in V. Let h_j be the indicator of some element of V. Then we define $V_j \subset V$ be the subspace consisting of functions whose indicator at most h_j. If all possible indicators are ordered as in (2), then

$$V_1 \subset V_2 \subset \ldots \subset V_m = V.$$

We choose $\dim V_1$ linearly independent functions in V_1, then $\dim V_2 - \dim V_1$ functions in V_2 which represent linearly independent elements of the factor space V_2/V_1, and so on. So that the basis elements chosen from $V_j \backslash V_{j-1}$ are linearly independent as elements of V_j/V_{j-1}.

Let $w_0, w_1, \ldots, w_n$ be this basis, ordered in such a way that the indicators increase,

$$h_{w_0}(\theta) \leq h_{w_1}(\theta) \leq \ldots \leq h_{w_n}(\theta), \quad e^{i\theta} \in S. \tag{3}$$

Notice that, the indicator of any linear combination of the form

$$c_0 w_0 + \ldots + c_{n-1} w_{n-1} + w_n \tag{4}$$

is the same as the indicator of w_n. This sequence (w_j) is called a special basis of V corresponding to the sector S.

Lemma 2. *Outside of a C_0 exceptional set E as in (1), the special basis satisfies*

$$\log |W(w_0, \ldots, w_n)| = \sum_{j=0}^{n} \log |w_j| + o(r^\rho)$$

in the sector S.

Proof. If f_1 and f_2 are two functions of completely regular growth in S, then the limit in (1) also exists for their ratio $f = f_1/f_2$ and this limit is equal to $h_{f_1}(\theta) - h_{f_2}(\theta)$. Let

$$\mathcal{L}(w_0, \ldots, w_n) = \frac{W(w_0, \ldots, w_n)}{w_0, \ldots, w_n}.$$

The statement of the Lemma is equivalent to $h_{\mathcal{L}}(\theta) \equiv 0$.

As $\mathcal{L}$ is a determinant consisting of the logarithmic derivatives of functions of class $\mathfrak{F}$, we have $h_{\mathcal{L}}(\theta) \leq 0$ by the Lemma on the logarithmic derivative [3]. It remains to prove that $h_{\mathcal{L}}(\theta) \geq 0$.

We prove this by induction in n. The statement is evident when $n = 0$. When $n = 1$ we set $f = w_1/w_0$. Then $\mathcal{L} = f'/f$. If $h_{\mathcal{L}}(\theta_0) < 0$, we integrate f'/f along the ray $\arg z = \theta_0$. If the exceptional set E intersects which ray, we bypass it by a curve close to the ray consisting of arcs of circles. The result is that

$$f = c + O(e^{-\delta r^\rho}).$$

This implies that

$$h_{w_1 - cw_0}(\theta_0) < h_{w_1}(\theta_0),$$

which contradicts the definition of the special basis.

Suppose now that the statement of the Lemma holds for spaces V of dimension at most $m+1$, with some $m \geq 1$. We have to prove it for $n = m+1$. Assume by contradiction that $h_{\mathcal{L}(w_0,\ldots,w_n)}(\theta_0) < 0$ for some θ_0. Define functions B_j as solutions of the following system of linear equations

$$\sum_{j=0}^{n-1} B_j w_j^{(k)} = w_n^{(k)}, \quad k = 0, \ldots, n-1.$$

By Cramer's rule,

$$B_j = \pm \frac{W_j}{W_n},$$

where W_j is the Wronskian of size n made of functions w_i with $i \neq j$. We use the formula for differentiation of the logarithm of the quotient of Wronskians [6, Part VII, Probl. 59], [3, p. 251]

$$\frac{d}{dz} \log \left(\frac{W_j}{W_n} \right) = \frac{W_{j,n} W}{W_j W_n} = \frac{\mathcal{L}_{j,n} \mathcal{L}}{\mathcal{L}_j \mathcal{L}_n}, \tag{5}$$

where $W_{j,n}$ is the Wronskian of size $n-1$ with w_j and w_n deleted, and W is our Wronskian of size $n+1$. Notation $\mathcal{L}, \mathcal{L}_j, \mathcal{L}_{j,n}$ has similar meaning. Using the induction assumption, we conclude that the right hand side of (5) has negative indicator. Integrating with respect to z along an appropriate curve near the ray $\arg z = \theta_0$, that avoids the exceptional set E, we obtain $B_j = c_j + O(e^{-\delta r^\rho})$, $0 \leq j \leq n-1$, where $c_j \neq 0$ and $\delta > 0$ are constants. So we conclude that the indicator of

$$w_n - \sum_{j=0}^{n-1} c_j w_j$$

at the point θ_0 is strictly less than $h_{w_n}(\theta_0)$. This contradicts the property (4) of the special basis. The contradiction completes the proof of Lemma 2. □

Proof of Theorem 2. Let $f \colon \mathbf{C} \to \mathbf{P}^n$ be a linearly non-degenerate holomorphic curve whose homogeneous coordinates are functions of $\mathfrak{F}$.

Let ρ be the order of our curve; it is equal to the maximal order of components f_j.

Let $V \subset \mathfrak{F}$ be the subspace spanned by the homogeneous coordinates. To such a space V we associated finitely many exceptional rays, whose complement consists of admissible sectors. Let us fix any admissible sector S, and a special basis $w_0, \ldots, w_n$ in S.

Let $w_j = (f, \alpha_j)$, $0 \leq j \leq n$, then the vectors $\{\alpha_0, \ldots, \alpha_n\}$ are linearly independent. We define subspaces

$$X_k = \{w \in \mathbf{C}^{n+1} \colon (w, \alpha_0) = \ldots = (w, \alpha_{k-1}) = 0\}, \quad 1 \leq k \leq n,$$

so that $\operatorname{codim} X_k = k$. We use the notation $u = \log \|f\|$, $u_j = \log |w_j|$. If z is outside of an exceptional set E, we have

$$u_j(z) \leq u_{j+1}(z) + o(|z|^\rho), \quad 0 \leq j \leq n-1,$$

view of (3). So

$$\begin{aligned}\log d_k(z) &\leq \log \operatorname{dist}(f(z), X_k)\\ &= \max_{0\leq j\leq k-1} \log |(f(z), \alpha_j)| - \log \|f\| = u_{k-1}(z) - u(z) + o(r^\rho).\end{aligned}$$

Then, using Lemma 2 and $u = u_n + o(r^\rho)$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} \log \frac{1}{d_k(z)} &\geq -\sum_{j=0}^{n-1} u_j(z) + nu + o(r^\rho)\\ &= -\sum_{j=0}^{n} u_j(z) + (n+1)u(z) + o(r^\rho)\\ &= -\log |W(w_0, \dots, w_n)| + (n+1)u(z) + o(r^\rho).\end{aligned}$$

Integrating this with respect to θ on the sector S, and then adding over all admissible sectors, we obtain

$$\sum_{j=1}^{n} m_k(r, f) + N_1(r, f) \geq (n+1)T(r, f) + o(r^\rho).$$

Integrals over the exceptional set E contribute $o(r^\rho)$ [4]. For curves f with components in $\mathfrak{F}$ we always have $T(r, f) = cr^\rho$, so the error term is $o(T(r, f)$.

The opposite inequality follows from Theorem 1, where exceptional set is absent because we deal with functions of finite order.

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