Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 40, 2015, 669–682

ON ASYMMETRIC p-HYPERELLIPTIC RIEMANN SURFACES

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Abstract. In this paper we study asymmetric and pseudo-symmetric Riemann surfaces of genus $g \geq 2$ with full automorphism group $\mathbf{Z}_{4n}$ for $n = 1$ or prime. We give necessary and sufficient conditions for the existence of such a surface and we find all values of p and q for which the surface is p-hyperelliptic and (q, n)-gonal.

1. Introduction

In this work we focus our attention on asymmetric Riemann surfaces with a cyclic group of automorphisms. Recall that a Riemann surface is called *asymmetric* (or *pseudo-real*, see for example [5]), if it admits an anticonformal automorphism, but no anticonformal involution. Also, if such an automorphism has order 4, then it is called a *pseudo-symmetry* and the surface itself is called *pseudo-symmetric*. Note that being pseudo-symmetric does not automatically make the surface asymmetric, hence we specify that in this paper we focus only on the asymmetric case. Observe also that asymmetric Riemann surfaces appear naturally when we consider the fixed point set $\mathrm{Fix}(\imath)$ of the involution $\imath\colon \mathcal{M}_g \to \mathcal{M}_g$ on the moduli space of Riemann surfaces of genus g, which maps a Riemann surface onto its complex conjugate. The set $\mathrm{Fix}(\imath)$ consists of two parts: symmetric Riemann surfaces, which are those that admit an anticonformal involution, and asymmetric surfaces, which are not symmetric but are isomorphic to their conjugate.

The starting point for this paper are works of Bagiński and Gromadzki [1] and Etayo Gordejuela [9] and Bujalance and Turbek [7] and Bujalance, Conder and Costa [5]. The minimal genus problem for cyclic actions $\mathbf{Z}_n$ for arbitrary n on pseudo-real surfaces was completely solved in [1]. In [9] asymmetric surfaces with cyclic automorphism groups are studied, and the work [5] gives many general properties of asymmetric surfaces, such as the existence of an asymmetric surface of any genus $g \geq 2$, and a description of such surfaces of genera 2 and 3 and a sharp bound on the order of the automorphism group. The case of hyperelliptic asymmetric surfaces was studied in [14], while in [7] the authors determine defining equations for such surfaces and also treat the special case of hyperelliptic asymmetric pseudo-symmetric surfaces.

doi:10.5186/aasfm.2015.4040

2010 Mathematics Subject Classification: Primary 30F99; Secondary 14H37.

Key words: Riemann surface, symmetry of Riemann surface, asymmetric Riemann surface, pseudo-symmetric Riemann surface, Fuchsian groups, NEC groups.

First author supported by the Research Grant NCN 2012/05/B/ST1/02171.

A compact Riemann surface X of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a *p-hyperelliptic involution*, for which X/ρ is an orbifold of genus p. In the particular cases $p = 0$ and 1, we obtain *hyperelliptic* and *elliptic-hyperelliptic* Riemann surfaces respectively. The notion of p-hyperellipticity arises as a special case of a so-called cyclic (q, n)-gonal surface, which is defined as one that admits a conformal automorphism δ of prime order n, such that X/δ has genus q. Farkas and Kra proved in [10] that for $g > 4p + 1$, the p-hyperelliptic involution is unique and hence central in the group of all automorphisms of X. This enabled the groups of automorphisms for hyperelliptic surfaces to be determined later in [6], for elliptic-hyperelliptic surfaces in [15], and for 2-hyperelliptic surfaces in [16]. The groups of automorphisms of cyclic trigonal Riemann surfaces and cyclic p-gonal Riemann surfaces were studied in [3] and [17] respectively.

We start our paper by considering those Riemann surfaces of given genus g, which admit an anticonformal automorphism δ of order 4 such that δ^2 is a p-hyperelliptic involution. For given g, we give sharp upper and lower bounds on p in terms of g, and find all the values of p which are realized. In the second part of the paper we continue this theme by looking at those surfaces for which the full group of automorphisms has order $4n$ for some prime integer n. We give necessary and sufficient conditions for the existence of the surface in question, and also determine the degree of its hyperellipticity. Furthermore, for a given integer $g \geq 2$ and prime $n > 2$, we find integers q for which there exists an asymmetric (q, n)-gonal Riemann surface of genus g with automorphism group $\mathbf{Z}_{4n}$.

2. Preliminaries

The main tool in our work is combinatorial group theory, and more specifically, the theory of non-euclidean crystallographic groups (or simply *NEC groups*), which are the discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is described by the so-called *signature*:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{1}$$

Here the brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i are called the *proper periods*, and h is the *orbit genus* of Λ.

A presentation for the group Λ with signature (1) is given by the following generators, called *canonical generators*: $x_1, \ldots, x_r$, and e_i and c_{ij} for $1 \leq i \leq k$ and $0 \leq j \leq s_i$, and $a_1, b_1, \ldots, a_h, b_h$ if the sign is $+$, or $d_1, \ldots, d_h$ otherwise, while the defining relations are given by: $x_i^{m_i} = 1$ for $1 \leq i \leq r$, $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$, $e_i^{-1}c_{is_i}e_i = c_{i0}$ for $1 \leq i \leq k$, $1 \leq j \leq s_i$ and $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} = 1$ or $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1$, according to whether the sign is $+$ or $-$. The last relation will be called the long relation. The elements x_i are elliptic transformations, the a_i and b_i are hyperbolic translations, the d_i are glide reflections, and the c_{ij} are hyperbolic reflections.

It is well known that an abstract group with such a presentation can be realized as a NEC group Λ if and only if the value

$$\mu(\Lambda) = \eta h + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right),$$

is positive, where $\eta = 2$ or 1 according to whether the sign is + or −. The value $\mu(\Lambda)$ is the hyperbolic area of a fundamental region for Λ. The Riemann–Hurwitz formula says that

$$[\Lambda : \Lambda'] = \frac{\mu(\Lambda')}{\mu(\Lambda)},$$

for every subgroup Λ' of finite index in the NEC group Λ.

Let us also note that the NEC groups having no orientation reversing elements are just the classical Fuchsian groups, which have signatures of the form

$$(h; +; [m_1, \ldots, m_r]; \{-\}),$$

often simplified to $(h; m_1, \ldots, m_r)$. Particularly important here are the *Fuchsian surface groups*, which are torsion-free. A Fuchsian surface group Γ has signature of the form $(g; -)$, and then $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g. Conversely, every compact Riemann surface X of genus $g \geq 2$ can be represented as such an orbit space for some Fuchsian surface group Γ. A finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ, where Γ is a normal subgroup of Λ.

For a given NEC group Λ, the subgroup Λ^+ consisting of all orientation preserving elements of Λ is called the *canonical Fuchsian subgroup of* Λ. Also if the signature of Λ is as given in (1), then from [13] we know that Λ^+ has signature

$$(\eta h + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{2}$$

Next, a group G is called *abstractly oriented* if there exists an epimorphism $\alpha\colon G \to Z_2 = \{\pm 1\}$, called an *abstract orientation*. Here an element $g \in G$ is said to be *orientation preserving* if and only if $\alpha(g) = +1$.

A Riemann surface is called *asymmetric* if it admits orientation reversing automorphisms, but none of those is an involution. It is well known, see for example [5], that the order of the full automorphism group of such a surface is divisible by 4. A special case occurs when an orientation reversing element has order 4; we call such an element a *pseudo-symmetry*, following the terminology from [14] and [7]. Observe also that an asymmetric surface can be seen as the orbit space $\mathcal{H}/\Gamma$, where Γ is a Fuchsian surface group such that $\Gamma = \ker\theta$ for some smooth epimorphism $\theta\colon \Lambda \to G = \Lambda/\Gamma$, where Λ has sign '−' and no canonical reflections (see [5]). In this case, Λ has signature of the form

$$(h; -; [m_1, \ldots, m_r]; \{-\})$$

and its canonical Fuchsian subgroup has signature

$$(h - 1; m_1, m_1, \ldots, m_r, m_r).$$

3. Degree of hyperellipticity

In this section we study p-hyperellipticity of asymmetric Riemann surfaces with a cyclic group of automorphisms. As we mentioned in the Introduction, the hyperelliptic asymmetric surfaces were treated in [14] and [7].

Let X be an asymmetric Riemann surface, and let δ be an anticonformal automorphism of X, say of order m. Then 4 divides m, since if m is odd then δ is a power of δ^2 and so would be orientation preserving, while if $m \equiv 2 \mod 4$ then $\delta^{m/2}$ would be a symmetry of X. Also if $m/4$ is odd, then δ^m is a pseudo-symmetry of X and $\delta^{m/2}$ is a p-hyperelliptic involution for X.

Theorem 3.1. *A pseudo-symmetric asymmetric Riemann surface of genus $g \geq 2$ is p-hyperelliptic for some integer p in the range $0 \leq p \leq (g+1)/2$, such that p has the same parity as g. Moreover, there exists at least one p-hyperelliptic pseudo-symmetric (and asymmetric) Riemann surface, for every such p.*

Proof. Assume that an asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ admits an anticonformal automorphism δ of order 4. Then $G = \langle\delta\rangle = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup of index 4. The proper periods in the signature of Λ are all equal to 2, because δ^2 is the only nontrivial orientation preserving element in G. Thus Λ has signature

$$s(\Lambda) = (p+1; -; [2, \overset{r}{\ldots}, 2]; \{-\})$$

for some nonnegative integers p and r. By (2), the canonical Fuchsian subgroup Λ^+ of Λ has signature $s(\Lambda^+) = (p; 2, \overset{2r}{\ldots}, 2)$, and Γ is a subgroup of Λ^+ of index 2. By the Riemann–Hurwitz formula for (Λ^+, Γ), $r = g + 1 - 2p$ and so

$$s(\Lambda) = (p+1; -; [2, \overset{g+1-2p}{\ldots}, 2]; \{-\}). \tag{3}$$

The element δ^2 generating the group $\Lambda^+/\Gamma \cong \mathbf{Z}_2$ is p-hyperelliptic involution for X.

It remains to check for which p in the range $0 \leq p \leq (g+1)/2$ there exists a smooth epimorphism $\theta\colon \Lambda \to \mathbf{Z}_4 = \langle\delta\rangle$. Since such an epimorphism maps the elliptic generators of Λ to δ^2 and the remaining generators to δ or δ^{-1}, it follows that the long relation $x_1 \ldots x_r d_1^2 \ldots d_h^2 = 1$ is preserved if and only if $p+1$ and r have the same parity. Thus θ exists for any p in the range $0 \leq p \leq (g+1)/2$ having the same parity as g. Then the orbit space $X = \mathcal{H}/\operatorname{Ker}\theta$ is a p-hyperelliptic pseudo-symmetric (and asymmetric) Riemann surface. □

Now we shall make the bounds more precise, depending on the structure of g.

Corollary 3.2. *Let X be a p-hyperelliptic pseudo-symmetric and asymmetric Riemann surface of genus $g \geq 2$. Then $0 \leq p \leq g/2$ for $g \equiv 0 \mod 4$, and $1 \leq p \leq (g+1)/2$ for $g \equiv 1 \mod 4$, and $0 \leq p \leq (g-2)/2$ for $g \equiv 2 \mod 4$, and $1 \leq p \leq (g-1)/2$ for $g \equiv 3 \mod 4$.*

Proof. By Theorem 3.1, the degree of hyperellipticity of a pseudo-symmetric Riemann surface has the same parity as the genus of the surface. Moreover, for any integer p in the range $0 \leq p \leq (g+1)/2$ having the same parity as a given $g \geq 2$, there exists at least one p-hyperelliptic pseudo-symmetric Riemann surface of genus g. Thus the lower bound on p is 1 or 0, depending on whether g is odd or even. In particular, there is no hyperelliptic asymmetric and pseudo-symmetric surface of odd genus - as was proved in [7].

The greatest integer having the same parity as g and not exceeding $(g+1)/2$ depends on g in the manner specified in Corollary. □

Now we proceed to a slightly more complicated case, where the order of the automorphism group is $4n$ for some prime integer $n \geq 2$. For this we will need the following well-known theorem of Macbeath (from [11]), in which $N_G(\langle g\rangle)$ means the normalizer in G of the subgroup generated by g.

Theorem 3.3. *Let $G = \Delta/\Gamma$ be the group of orientation preserving automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$, and let $x_1, x_2, \ldots, x_r$ be the set of canonical elliptic generators of Δ, with periods $m_1, \ldots, m_r$ respectively. Let $\theta\colon \Delta \to G$ be the canonical epimorphism. Then the number m of points of X fixed by $g \in G$ is given* *by the formula*

$$m = |N_G(\langle g \rangle)| \sum 1/m_i,$$

where the sum is taken over those i for which g is conjugate to a power of $\theta(x_i)$. $\square$

A conformal automorphism ρ of a Riemann surface X such that ρ has prime order n, and $X/\langle\rho\rangle$ has genus q, is called a (q,n)-*gonal* automorphism of X.

Lemma 3.4. *A (q,n)-gonal automorphism of a Riemann surface of genus $g \geq 2$ has $r = 2 + (2g - 2nq)/(n-1)$ fixed points on X.*

Proof. Suppose ρ is a (q,n)-gonal automorphism of a Riemann surface X of genus $g \geq 2$. Then X can be represented as the orbit space $\mathcal{H}/\Gamma$ for some surface Fuchsian group Γ with signature $(g;-)$, and $G = \langle\rho\rangle = \Lambda/\Gamma$ for some Fuchsian group Λ containing Γ as a normal subgroup of index n. Since $|G| = n$ is prime and Γ is torsion free, it follows that the elliptic generators x_i of Λ all have order n. Moreover, $X/\langle\rho\rangle \simeq \mathcal{H}/\Lambda$ has genus q and so Λ has signature $(q;+;[n,\overset{r}{\ldots},n];\{-\})$ for some integer r, which by the Riemann–Hurwitz formula is equal to $2 + (2g-2nq)/(n-1)$. Thus according to Theorem 3.3, ρ has r fixed points on X. $\square$

We shall study anticonformal automorphisms of order $4n$ of an asymmetric Riemann surface. We will consider the case $n > 2$ in the next section, and the case $n = 2$ in the final section.

3.1. Cyclic groups of automorphisms $\mathbf{Z}_{4n}$ for $n > 2$. Throughout this chapter we shall assume that n is a prime odd integer.

Lemma 3.5. *Suppose that an asymmetric Riemann surface X of genus $g \geq 2$ admits an anticonformal automorphism δ of order $4n$. Then there exist nonnegative integers q, γ of the same parity as g such that $g - nq \equiv 0\ (n-1)$ and $a = (g - nq)/(n-1) \geq -1$, and $a \neq -1$ for $\gamma = 0$. Furthermore, the group $G = \langle\delta\rangle$ acts with signature*

$$(\gamma+1;-;[2,\overset{q-2\gamma-k}{\ldots},2,n,\overset{(a-k)/2}{\ldots},n,2n,\overset{k+1}{\ldots},2n];\{-\}), \tag{4}$$

where k is an integer in the range $-1 \leq k \leq \min(a, q-2\gamma)$ having the same parity as a.

Proof. If $X = \mathcal{H}/\Gamma$ has an anticonformal automorphism δ of order $4n$, then there exists a NEC group Λ such that $\Lambda/\Gamma = \langle\delta\rangle$. Since nontrivial cyclic subgroups of $\mathbf{Z}_{4n}$ generated by conformal automorphisms have orders $2n, n$ or 2, it follows that Λ has signature

$$s(\Lambda) = (\gamma+1;-;[2,\overset{k_1}{\ldots},2,n,\overset{k_2}{\ldots},n,2n,\overset{k_3}{\ldots},2n];\{-\}),$$

for some nonnegative integers γ, k_1, k_2 and k_3. By (2), the canonical Fuchsian subgroup Λ^+ of Λ has signature

$$s(\Lambda^+) = (\gamma;2,\overset{2k_1}{\ldots},2,n,\overset{2k_2}{\ldots},n,2n,\overset{2k_3}{\ldots},2n),$$

and applying the Riemann–Hurwitz formula for (Λ^+, Γ), we get

$$g - 1 = 2n\gamma - 2n + nk_1 + 2k_2(n-1) + k_3(2n-1). \tag{5}$$

The elements δ^4 and δ^{2n} are respectively a (q,n)-gonal automorphism and a p-hyperelliptic involution, with $r = 2 + (2g-2nq)/(n-1)$ and $s = 2g + 2 - 4p$ fixed points, for some nonnegative integers p and q.

The cyclic group $\mathbf{Z}_{2n}$ generated by δ^4 and δ^{2n} acts with signature $s(\Lambda^+)$. Thus by Theorem 3.3, $s = 2nk_1 + 2k_3$ and $r = 4k_2 + 2k_3$. Let $a = r/2 - 1$ and $k = k_3 - 1$. Then $nk_1 = s/2 - k_3 = g - 2p - k$ and $2k_2 = r/2 - k_3 = a - k$. Substituting the last equalities to (5), we get $p = n\gamma + (a + k)(n - 1)/2$. Thus

$$\begin{aligned} k_1 &= (g - 2p - k)/n = \{[nq + a(n-1)] - [2n\gamma + (a+k)(n-1)] - k\}/n \\ &= q - 2\gamma - k. \end{aligned}$$

For any integer k in the range $-1 \leq k \leq \min(a, q - 2\gamma)$ having the same parity as a, the parameters $k_1 = q - 2\gamma - k$, $k_2 = (a - k)/2$ and $k_3 = k + 1$ are nonnegative integers.

Now let $\theta : \Lambda \to \mathbf{Z}_{4n} = \langle \delta \rangle$ be an epimorphism with torsion-free kernel for a NEC group Λ with signature (4). Then

$$\begin{array}{lll} & \theta(x_i) = \delta^{2n}, & i = 1, \ldots, k_1 = q - 2\gamma - k, \\ & \theta(x_{k_1+i}) = \delta^{4p_i}, & i = 1, \ldots, k_2 = (a-k)/2, \\ (6) & \theta(x_{k_1+k_2+i}) = \delta^{2q_i}, & i = 1, \ldots, k_3 = k + 1, \\ & \theta(d_j) = \delta^{t_j}, & j = 1, \ldots, \gamma + 1, \end{array}$$

for some nonnegative integers p_i, q_i and t_j such that all p_i are co-prime with n, all q_i are co-prime with $2n$, and all t_j are odd. Thus the product of all elliptic generators is mapped to δ^{2t}, for $t = [n(q - k) + 2\sum_{i=1}^{(a-k)/2} p_i + \sum_{i=1}^{k+1} q_i]$. If q and k have the same parity, then t has the same parity as $\sum_{i=1}^{k+1} q_i$ and so its parity is opposite to k. For k and q having different parity, t has the same parity as k. In both cases, t has parity opposite to q. Thus the long relation is preserved if and only if γ has the same parity as q. The last condition is sufficient for the existence of θ. Indeed, if q and γ are odd, then θ can be defined by (6), where $p_i = 1$ for $1 \leq i \leq k_2$; $q_i = 1$ for $1 \leq i \leq k_3$; $t_1, \ldots, t_{\gamma-1} = 1, -1, \ldots, 1, -1$; $t_\gamma = -t + 1$ and $t_{\gamma+1} = -1$.

If γ and q are even, then θ can be defined by (6), where all p_i and q_i are equal to 1; $t_1, \ldots, t_\gamma = 1, -1, \ldots, 1, -1$ and $t_{\gamma+1} = -t$.

The case where $\gamma = 0$ needs more clarification. Since $a \geq k$ and $k \geq -1$, it follows that $k = -1$ for $a = -1$. In this case, all periods are equal to 2 in the signature of Λ^+, and therefore there is no epimorphism $\theta : \Lambda \to \mathbf{Z}_{4n}$ if $\gamma = 0$. If $a \neq -1$, then Λ has at least one elliptic generator x_i such that $\theta(x_i) = \delta^4$ or δ^2. Thus for $\gamma = 0$, the elements $\theta(x_i)$ and $\theta(d_1) = \delta^{-t}$ generate $\mathbf{Z}_{4n}$, as t is odd. □

By the proof of Lemma 3.5, we get the following

Proposition 3.6. *Let δ be an anticonformal automorphism of order $4n$ of an asymmetric Riemann surface, and let (γ, q, a, k) be a sequence of integers satisfying the conditions of Lemma 3.5. Then δ^{2n} is a p-hyperelliptic involution, for $p = n\gamma + (a + k)(n - 1)/2$.*

The value of p is given in Table 1 for $q \leq 2$ and in Table 2 for $a \leq 2$.

To answer the question whether a cyclic group of order $4n$ is the full automorphism group of an asymmetric Riemann surface, we will need the theory of maximal NEC groups and maximal signatures.

A NEC group Λ is called *maximal* if there does not exist another NEC group containing it properly.

q	γ	k	p	Conditions
0	0	0	$a(n-1)/2$	$a \geq 0$ even
	0	-1	$(a-1)(n-1)/2$	$a \geq 1$ odd
1	1	-1	$(a+1)(n-1)/2$	$a \geq 1$ odd
2	0	0	$a(n-1)/2$	$a \geq 0$ even
	0	-1	$(a-1)(n-1)/2$	$a \geq 1$ odd
	0	1	$(a+1)(n-1)/2$	$a \geq 1$ odd
	0	2	$(a+2)(n-1)/2$	$a \geq 0$ even

Table 1. The values of p for $q \in \{0, 1, 2\}$.

a	k	p	Conditions
-1	-1	$n(\gamma-1)n+1$	$1 \leq \gamma \leq (q+1)/2,$
0	0	$n\gamma$	$0 \leq \gamma \leq q/2$
1	-1	$n\gamma$	$0 \leq \gamma \leq (q+1)/2$
1	1	$n(\gamma+1)-1$	$0 \leq \gamma \leq (q-1)/2$
2	0	$n(\gamma+1)-1$	$0 \leq \gamma \leq q/2$
2	2	$n\gamma+2(n-1)$	$0 \leq \gamma \leq (q-2)/2$

Table 2. The values of p for $a \in \{-1, 0, 1, 2\}$.

A signature σ is said to be *maximal*, if for every NEC group Λ' with signature σ' containing a NEC group Λ with signature σ such that the Teichmüller spaces of Λ and Λ' have the same dimensions, the equality $\Lambda = \Lambda'$ holds. Otherwise, the pair (σ, σ') is called a *normal* or *non-normal pair* according to whether Λ is a normal subgroup of Λ' or not. The complete lists of normal and non-normal pairs were given in [2] and [8] respectively (see also [12] and [4]). If σ is a maximal signature, then there exists a maximal NEC group with signature σ.

If an automorphism group $G = \Lambda/\Gamma$ is not the full automorphism group of a Riemann surface $X = \mathcal{H}/\Gamma$, then Λ is properly contained with finite index in another NEC group Λ' normalizing Γ.

Theorem 3.7. *The cyclic group $G = \mathbf{Z}_{4n}$ is the full automorphism group of an asymmetric Riemann surface of genus $g \geq 2$ if and only if there is a sequence of integers $(\gamma, q, a, k) \neq (1, 1, 1, -1)$, $(0, 0, 1, -1)$ and $(0, 0, 2, 0)$ such that: $\gamma, q \geq 0$; $a, k \geq -1$; $g = qn + a(n-1)$; a and k have the same parity; q and γ have the same parity and $k \leq \min(a, q-2\gamma)$. In particular, g is different from $2n-1, n-1$ and $2n-2$.*

Proof. By Lemma 3.5, the cyclic group $\mathbf{Z}_{4n}$ generated by an anticonformal automorphism acts on an asymmetric Riemann surface of genus g with signature (4). Since the signature has no period cycles, it must be one of the following from the lists in [2] and [8]: $(3; -; [-]; \{-\})$, $(2; -; [t]; \{-\})$, $(1; -; [t, t]; \{-\})$ and $(1; -; [t, u]; \{-\})$, where $t \geq 3$ in the third signature, and $\max(t, u) \geq 3$ in the fourth. The only sequences (γ, q, a, k) of integers satisfying the conditions of Lemma 3.5, for which (4) is one of signatures listed above, are: $(1, 1, 1, -1)$, $(0, 0, 1, -1)$ and $(0, 0, 2, 0)$. They provide the following non-maximal signatures:

$$\sigma_1 = (2; -; [n]; \{-\}),\ \sigma_2 = (1; -; [2, n]; \{-\}),\ \sigma_3 = (1; -; [n, 2n]; \{-\}).$$

For a NEC group Λ with signature σ_i, for $i = 1, 2, 3$, there exists a NEC group Λ' with signature σ_i', which contains Λ as a subgroup of index 2, where

$$\sigma_1' = (0; +; [2, 2]; \{(n)\}),\ \sigma_2' = (0; +; [2]; \{(2, n)\}),\ \sigma_3' = (0; +; [2]; \{(n, 2n)\}).$$

We shall prove that for any epimorphism $\theta\colon \Lambda \to G = \mathbf{Z}_{4n}$ whose kernel Γ has signature $(g; -)$, there exist an epimorphism $\theta'\colon \Lambda' \to G' = \mathbf{D}_{4n}$, and group embeddings $i\colon \Lambda \to \Lambda'$ and $j\colon G \to G'$, such that $\theta' \cdot i = j \cdot \theta$. This means that $G = \Lambda/\Gamma \subset \Lambda'/\Gamma = G' \subseteq \mathrm{Aut}(X)$, for $X = \mathcal{H}/\Gamma$. Since σ_i' contains period cycles, it follows that G' contains symmetries, and therefore X is not an asymmetric Riemann surface. Consequently, the cyclic group $\mathbf{Z}_{4n}$ acting with signature σ_i is not an automorphism group of an asymmetric Riemann surface.

Suppose that δ is a generator of $G = \mathbf{Z}_{4n}$, and $G' = \mathbf{D}_{4n}$ is a dihedral group generated by two involutions τ and a such that τ reverses orientation, a preserves orientation and $(\tau a)^{4n} = 1$. Then there is an embedding $j\colon G \to G'$ given by $j(\delta) = \tau a$.

First, assume that G and G' act with signatures σ_1 and σ_1' respectively. Then without lost of generality, we can define $i\colon \Lambda \to \Lambda'$ by

$$i(d_1) = e_1'c_{10}'x_1',\ i(d_2) = x_1'c_{10}' \text{ and } i(x_1) = c_{10}'c_{11}',$$

where we use prime for generators of Λ'. Any epimorphism $\theta\colon \Lambda \to G$ maps x_1, d_1 and d_2 to δ^{4p}, δ^{t_1} and δ^{t_2} respectively, for some integers t_1, t_2 and p satisfying the congruence $2p + t_1 + t_2 \equiv 0 \mod 2n$, such that t_1, t_2 are odd and p is co-prime with n. It is easy to check that $\theta' \cdot i = j \cdot \theta$ for an epimorphism $\theta'\colon \Lambda' \to G'$ defined by the assignments:

$$c_{10}' \mapsto \tau,\ c_{11}' \mapsto \tau(\tau a)^{4p},\ x_1' \mapsto \tau(a\tau)^{t_2} \text{ and } e_1' \mapsto (\tau a)^{t_1+t_2}.$$

If Λ and Λ' have signatures σ_2 and σ_2', then the embedding $i\colon \Lambda \to \Lambda'$ can be defined by

$$i(x_1) = c_{10}'c_{11}',\ i(x_2) = c_{11}'c_{12}' \text{ and } i(d_1) = e_1'c_{10}'.$$

Any epimorphism $\theta\colon \Lambda \to G$ maps x_1, x_2 and d_1 to δ^{2n}, δ^{4p} and δ^{t_1} respectively, for some integers p and t_1 satisfying the congruence $n + 2p + t_1 \equiv 0 \mod 2n$ with $(p, n) = 1$ and t_1 odd. Now θ can be extended to an epimorphism $\theta'\colon \Lambda' \to G'$ by the assignments

$$c_{10}' \mapsto \tau,\ c_{11}' \mapsto \tau(\tau a)^{2n},\ c_{12}' \mapsto \tau(\tau a)^{4p+2n} \text{ and } x_1' = e_1' \mapsto (\tau a)^{t_1}\tau,$$

and therefore X is symmetric again.

Finally, if Λ and Λ' have signatures σ_3 and σ_3', then the inclusion $\Lambda \subset \Lambda'$ can be defined as for $i = 2$. Now $\theta(x_1) = \delta^{4p}$, $\theta(x_2) = \delta^{2q}$ and $\theta(d_1) = \delta^{t_1}$, for some integers p, q and t_i such that p is co-prime with n and q is co-prime with $2n$ and t_1 is odd and $2p + q + t_1 \equiv 0 \mod 2n$. The epimorphism θ can be extended to $\theta'\colon \Lambda' \to G'$ by

$$c_{10}' \mapsto \tau, c_{11}' \mapsto \tau(\tau a)^{4p},\ c_{12}' \mapsto \tau(\tau a)^{4p+2q} \text{ and } x_1' = e_1' \mapsto (\tau a)^{t_1}\tau. \qquad \square$$

Corollary 3.8. *The cyclic group $\mathbf{Z}_{4n}$ does not act on an asymmetric Riemann surface of genus $g = 2n - 1$, $n - 1$ or $2n - 2$.*

Next, we ask which integers can be the genus of an asymmetric Riemann surface with the automorphism group $\mathbf{Z}_{4n}$, and for which q the surface is (q, n)-gonal.

Theorem 3.9. *Let $r = g \mod n$, for an integer g different from $n-1$, $2n-1$ and $2n-2$. Then g is the genus of an asymmetric Riemann surface X with automorphism group $\mathbf{Z}_{4n}$ if and only if $g \geq (n-r)(n-1)$, or g is a multiple of n or $n-1$. The surface X is a n-sheeted covering of an orbifold of genus q ramified over t points, where in the first case, $q = [g-(n-r)(n-1)]/n - c(n-1)$ and $t = 2[n(c+1)+1-r]$ for $c \geq -1$, otherwise $q = g/n - c(n-1)$ and $t = 2(cn+1)$, or $q = c(n-1)$ and $t = 2(g/(n-1) - cn + 1)$, for $c \geq 0$, depending on whether $g \equiv 0 \mod n$ or $g \equiv 0 \mod n-1$. In all cases c is an integer for which q and t are nonnegative.*

Proof. If $g \geq 2$ is the genus of an asymmetric Riemann surface with automorphism group $\mathbf{Z}_{4n}$, then by Theorem 3.7, there exist integers $a \geq -1$ and $q \geq 0$ such that $g = nq + (n-1)a$. The parameters a and q in the above presentation of g are not unique. First, suppose that $g \geq (n-r)(n-1)$ for $r = g \mod n$. Then $g = [(g-r)/n + 1 - (n-r)]n + (n-r)(n-1) = nq_{\max} + (n-1)a_{\min}$ for $q_{\max} = [g-(n-r)(n-1)]/n$ and $a_{\min} = n-r$. For $r > 1$, $q_{\max}$ is the greatest value of q and $a_{\min}$ is the smallest value of a. If $c \geq -1$ is an integer such that $c(n-1) \leq q_{\max}$, then $g = nq + (n-1)a$, for $q = q_{\max} - c(n-1)$ and $a = cn + a_{\min}$. If $g < (n-r)(n-1)$, then g is a sum of multiples of the numbers n and $n-1$ if and only if $g \equiv 0 \mod n$ or $g \equiv 0 \mod n-1$. In the first case, let c be a nonnegative integer such that $g/n - c(n-1) \geq 0$. Then $g = nq + (n-1)a$ for $q = g/n - c(n-1)$ and $a = cn$. If $g \equiv 0 \mod n-1$, then g has the above presentation for $a = g/(n-1) - cn$ and $q = c(n-1)$, where c is a nonnegative integer for which $cn \leq g/(n-1)$.

Let (a, q) be a pair determined above for an integer g such that $g \geq (n-r)(n-1)$, or g is a multiple of n or $n-1$. Let $\gamma = q \mod 2$, and $k = 0$ or $k = -1$ depending on whether a is even or odd. If $g \neq n-1$, $2n-1$ and $2n-2$, then $(a, q) \neq (1, 0), (1, 1)$ and $(2, 0)$, so $(\gamma, q, a, k) \neq (0, 0, 1, -1), (1, 1, 1, -1)$ and $(0, 0, 2, 0)$. Thus by Theorem 3.7, the cyclic group $G = \mathbf{Z}_{4n}$ acts on an asymmetric Riemann surface X of genus g with signature (4). If δ is a generator of G, then δ^4 is a (q, n)-gonal automorphism of X with $t = 2a + 2$ fixed points. Thus X is a n-sheeted covering of an orbifold of genus q ramified over t points. □

Theorem 3.10. *Let $g = qn + a(n-1)$ for $a, q \geq 3$, and let $\beta = q \mod 4$ and $\varepsilon = a \mod 2$. Then an asymmetric Riemann surface X of genus g with automorphism group $G = \mathbf{Z}_{4n}$ is p-hyperelliptic, and the sharp bounds on p are equal to*

$$p_{\min} = \begin{cases} \frac{g+n(2-q)}{2} & \text{if } \varepsilon = 0,\ q \equiv 1 \mod 2, \\ \frac{g+1+n(1-q)}{2} & \text{if } \varepsilon = 1,\ q \equiv 1 \mod 2, \\ \frac{g-nq}{2} & \text{if } \varepsilon = 0,\ q \equiv 0 \mod 2, \\ \frac{g+1-n(1+q)}{2} & \text{if } \varepsilon = 1,\ q \equiv 0 \mod 2, \end{cases} \tag{7}$$

and

$$p_{\max} = \begin{cases} (g - \varepsilon(n-1) - \beta)/2 & \text{if } \beta \equiv 0 \mod 2, \\ (g+1)/2 & \text{if } \varepsilon \equiv 1 \mod 2,\ \beta = 1, \\ (g - \varepsilon(n-1) - (4-\beta))/2 & \text{if } \beta = 1,\ \varepsilon = 0 \text{ or } \beta = 3. \end{cases} \tag{8}$$

Proof. Let (γ, q, a, k) be a sequence of integers associated by Theorem 3.7 with an action of $\mathbf{Z}_{4n}$ on X. Then by Proposition 3.6, X is p-hyperelliptic for $p = n\gamma + (a+k)(n-1)/2$. Thus we get the lower bound on p, for the smallest possible values of γ and k. Since $\gamma_{\min} = 0$ or 1 according to whether q is even or odd, and $k_{\min} = 0$ or -1

according to whether a is even or odd, it follows that $p_{\min} = \alpha n + [g - nq - \varepsilon(n-1)]/2$, for $\varepsilon = a \mod 2$ and $\alpha = q \mod 2$. Writing p in the form

$$\begin{aligned} p &= [2n\gamma + (a+k)(n-1)]/2 = [2n\gamma + (g - nq) + k(n-1)]/2 \\ &= [g - (nk_1 + k)]/2, \quad \text{for } k_1 = q - 2\gamma - k, \end{aligned}$$

we get the upper bound on p for the smallest value of the number $nk_1 + k$. Let $\beta = q \mod 4$ for $q \geq 3$. Let $\gamma_{\max}$ be the greatest value of γ having the same parity as q and not exceeding $q/2$ or $(q+1)/2$ according to whether a is even or odd. Then

$$\gamma_{\max} = \begin{cases} (q-\beta)/2 & \text{if } \beta \text{ is even,} \\ (q+1)/2 & \text{if } \beta = \varepsilon = 1, \\ (q - (4-\beta))/2 & \text{in the remaining cases.} \end{cases} \tag{9}$$

Let $a \geq 3$. Then the smallest value of k_1 is ε or $1 - \varepsilon$ according to whether β is even or odd, and it is attained for $k = q - 2\gamma_{\max} - \varepsilon$ or $q - 2\gamma_{\max} - 1 + \varepsilon$ respectively. Thus the upper bound on p is equal to (8). □

Theorem 3.11. *For given integers $q, a \geq 3$, let $\beta = q \mod 4$ and $\varepsilon = a \mod 2$. Then there exists an asymmetric p-hyperelliptic Riemann surface of genus $g = qn + a(n-1)$, for $p = p_{\max} - 2l - s(n-1)$, where $p_{\max}$ is given by (8), and l and s are nonnegative integers satisfying the inequalities listed below.*

$q \equiv 0 \mod 4$	$l \leq \frac{q}{4}$	$2l - \frac{a+\varepsilon}{2} \leq s \leq 2l$
$q \equiv 2 \mod 4$	$l \leq \frac{q-2}{4}$	$2l + 1 - \frac{a+\varepsilon}{2} \leq s \leq 2l + 1$
$q \equiv 3 \mod 4$	$l \leq \frac{q-3}{4}$	$2l - \frac{a-\varepsilon}{2} \leq s \leq 2l$
$q \equiv 1 \mod 4, \varepsilon = 0$	$l \leq \frac{q-5}{4}$	$2l + 1 - \frac{a}{2} \leq s \leq 2l + 1$
$q \equiv 1 \mod 4, \varepsilon = 1$	$l \leq \frac{q-1}{4}$	$2l - 1 - \frac{a+1}{2} \leq s \leq 2l - 1$

Table 3. Conditions on l and s.

Proof. For given $a, q \geq 3$, we shall find all integers $k \geq -1$ having the same parity as a, and $\gamma \geq 0$ having the same parity as q such that $k \leq \min(a, q - 2\gamma)$. By Theorem 3.7, for any sequence (γ, q, a, k), there exists an asymmetric Riemann surface X of genus $g = nq + a(n-1)$ with automorphism group $\mathbf{Z}_{4n}$. By Proposition 3.6, X is p-hyperelliptic, for

$$p = n\gamma + (a+k)(n-1)/2. \tag{10}$$

The greatest value of γ, denoted by $\gamma_{\max}$, is given by (9). The other γ-values can be written in the form $\gamma = \gamma_{\max} - 2l$, for l in the range $0 \leq l \leq \gamma_{\max}/2$. Let $k_{\max}$ denote the greatest value of k corresponding to a given l, and let $\varepsilon = a \mod 2$. Then $k_{\max} = q - 2\gamma - \varepsilon$ or $k_{\max} = q - 2\gamma - 1 + \varepsilon$ according to whether q is even or odd. The other values of k can be written in the form $k = k_{\max} - 2s$ for s in the range $0 \leq s \leq k_{\max}/2$. Since k is in the range $-1 \leq k \leq \min(a, q - 2\gamma)$, it follows that s satisfies the inequalities given in the third column of Table 3. Substituting $\gamma = \gamma_{\max} - 2l$ and $k = k_{\max} - 2s$ into the formula (10), we get $p = p_{\max} - 2l - s(n-1)$. If l and s run the set of all integers satisfying the conditions listed in Table 3, then p takes all possible degrees of hyperellipticity. □

Remark 3.12. If an asymmetric Riemann surface X of genus $g \geq 2$ has automorphism group $\mathbf{Z}_{4n}$, then $g = nq + a(n-1)$, for q as given in Theorem 3.9 and $a \geq -1$. The surface X is (q,n)-gonal and p-hyperelliptic, where the possible values of p are listed in Theorem 3.11 for $a, q \geq 3$, and in Tables 1 or 2 according to whether $q \leq 2$ or $a \leq 2$ respectively.

Corollary 3.13. *For any integer $g \geq 2$ such that $g \equiv 0 \mod n-1$, there exists an asymmetric n-gonal and p-hyperelliptic Riemann surface of genus g, where $p = g/2$ or $p = (g+1-n)/2$ according to whether $g/(n-1)$ is even or odd.*

Proof. If $g \equiv 0 \mod n-1$, then $g = nq + a(n-1)$ for $q = 0$ and $a = g/(n-1)$. Let $\gamma = 0$, and let $k = -1$ or $k = 0$ according to whether a is odd or even. Then by Theorem 3.7, for the sequence (γ, q, a, k), there exists an asymmetric Riemann surface X of genus g, where by (10), $p = g/2$ or $p = (g+1-n)/2$ according to whether $k = 0$ or $k = -1$. $\square$

Corollary 3.14. *If g is a nonnegative integer such that $g \equiv 0 \mod n$, then for any integer γ in the range $0 \leq \gamma \leq g/(2n)$ having the same parity as g/n, there exists an asymmetric $(g/n, n)$-gonal and $n\gamma$-hyperelliptic Riemann surface of genus g.*

Proof. Let $q = g/n$, $a = 0$ and $k = 0$, and let γ be an integer having the same parity as q in the range $0 \leq \gamma \leq q/2$. Then by Theorem 3.7, the sequence (γ, q, a, k) corresponds to an action of the group $\mathbf{Z}_{4n}$ on an asymmetric Riemann surface X of genus g. The surface is (q,n)-gonal and p-hyperelliptic, where $p = n\gamma$ by (10). $\square$

3.2. The action of $\mathbf{Z}_8$ on an asymmetric Riemann surface.

Theorem 3.15. *The cyclic group $G = \mathbf{Z}_8$ acts on an asymmetric Riemann surface X of genus $g \geq 2$ with signature*

$$(11) \qquad (\gamma+1; -; [2, \overset{(g-1-3k)/2-4\gamma}{\ldots}, 2, 4, \overset{k}{\ldots}, 4]; \{-\})$$

for some pair $(k,\gamma) \neq (0,0)$ of nonnegative integers such that $(k,\gamma) \neq (0,1)$ for $g = 11$, and $(k,\gamma) \neq (1,0)$ for $g = 6$, and k has parity different from g, and γ has the same parity as g, and $g - 1 - 3k \geq 8\gamma$. The surface X is p-hyperelliptic for $p = 4\gamma + 1 + k$.

Proof. Suppose that an asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$ has an anticonformal automorphism δ of order 8. Then there exists a NEC group Λ such that the quotient group $G = \Lambda/\Gamma$ is isomorphic to $\langle\delta\rangle$. The group Λ has signature

$$s(\Lambda) = (\gamma+1; -; [2, \overset{l}{\ldots}, 2, 4, \overset{k}{\ldots}, 4]; \{-\}),$$

for some nonnegative integers k, l, γ. By the Riemann–Hurwitz formula,

$$(12) \qquad g = 1 + 8\gamma + 2l + 3k$$

which implies

$$(13) \qquad l = (g-1-3k)/2 - 4\gamma.$$

Since l is a nonnegative integer, it follows that k has a different parity than g, and $g - 1 - 3k \geq 8\gamma$.

An epimorphism $\theta\colon \Lambda \to G$ with kernel Γ is defined by the assignments: $x_i \mapsto \delta^4$ for $1 \leq i \leq l$, $x_{l+i} \mapsto \delta^{2p_i}$ for $1 \leq i \leq k$, and $d_i \mapsto \delta^{t_j}$ for $1 \leq j \leq \gamma+1$, where $p_i \in \{1,3\}$ and the t_j are odd integers. Thus θ maps the product of all elliptic generators to δ^{2t} for $t = [g - 1 - 3k - 8\gamma + \sum_{i=1}^{k} p_i]$. Since k has parity different than g, it follows that t has the same parity as the sum $\sum_{i=1}^{k} p_i$, and consequently, it has parity different than g. Thus the long relation is preserved if and only if γ has the same parity as g. Suppose that the last condition is satisfied. Then taking $p_i = 0$ for $i = 1, \ldots, k$, and $(t_1, \ldots, t_\gamma) = (1, -1, \ldots, 1, -1)$ and $t_{\gamma+1} = -t$, we get a well-defined epimorphism for even g. However, the case when $\gamma = 0$ requires k to be nonzero, because otherwise all periods are equal to 2 in the signature of Λ^+ and so θ is not surjective. For odd g, we can define θ by taking all $p_i = 1$, and $(t_1, \ldots, t_{\gamma-1}) = (1, -1, \ldots, 1, -1)$, and $t_\gamma = -t + 1$, and $t_{\gamma+1} = -1$.

The signature (11) is not maximal for $(g, k, \gamma) = (11, 0, 1)$ or $(6, 1, 0)$. Using the same arguments as in the proof of Theorem 3.7, it can be proved that these two cases must be rejected, since we get an action on a symmetric surface.

Now let Λ^+ be the canonical Fuchsian subgroup of Λ. Then by (2),

$$s(\Lambda^+) = (\gamma; +; [2, \overset{2l}{\ldots}, 2, 4, \overset{2k}{\ldots}, 4]; \{-\}),$$

and $\Lambda^+/\Gamma = \mathbf{Z}_4$. The element δ^4 is a p-hyperelliptic involution with $s = 2g + 2 - 4p$ fixed points, where p is the genus of $X/\langle\delta^4\rangle$. By Theorem 3.3, we have $4l + 2k = s$ and so $p = (g + 1 - 2l - k)/2$. Substituting (12) into the last equation, we get $p = 4\gamma + 1 + k$. □

Corollary 3.16. *There are five genera less than* 13 *of asymmetric Riemann surfaces with full automorphism group* $\mathbf{Z}_8$*, namely* $4, 8, 9, 10$ *and* 12*, and the degrees of hyperellipticity of such surfaces are listed below in Table 4.*

p	g
2	$4, 8, 10, 12$
4	$10, 12$
5	9
6	12

Table 4. The degrees of hyperellipticity for $g < 13$.

Theorem 3.17. *For given* $g \geq 13$*, let* ε *and* β *be the remainders of* g *on division by* 2 *and* 6*, respectively. Let* $k_{\max}$ *and* s_l *be integers listed in Table* 5 *for* l *in the range* $0 \leq l \leq k_{\max}/2$*. Then for every such* l *and* s *in the range* $0 \leq s \leq s_l$*, there exists an asymmetric* p*-hyperelliptic Riemann surface of genus* g *with full automorphism group* $\mathbf{Z}_8$*, where* $p = 4\varepsilon + 8s + k_{\max} - 2l + 1$.

β	$k_{\max}$	s_l
$\beta = 1$	$(g-1)/3 - 4,$	$(2 + 3l)/8$
$\beta = 4$	$(g-1)/3,$	$3l/8$
$\beta \not\equiv 1 \mod 3$	$(g-\beta)/3 - (1+\varepsilon),$	$(6l + 2 + \beta - 5\varepsilon)/16.$

Table 5. Definitions of $k_{\max}$ and s_l.

Proof. In order to find all degrees of hyperellipticity for a given integer $g \geq 13$, we need to find all pairs $(k, \gamma) \neq (0, 0)$ of nonnegative integers such that k has different parity than g, and γ has the same parity as g, and

$$g - 1 - 3k \geq 8\gamma. \tag{14}$$

By Theorem 3.15, for every such pair, there exists an asymmetric p-hyperelliptic Riemann surface of genus g with automorphism group $\mathbf{Z}_8$, for $p = 4\gamma + k + 1$.

Let $\varepsilon = g \mod 2$ and $\beta = g \mod 6$. We will consider the following three cases: (a) $\beta = 1$, (b) $\beta = 4$ and (c) $\beta \not\equiv 1 \mod 3$. An integer γ having the same parity as g can be written in the form $\gamma = \varepsilon + 2s$ for some nonnegative integer s. By (14), $k \leq (g - 1 - 8\varepsilon)/3$. The greatest integer $k_{\max}$ with parity opposite to that of g and satisfying the last inequality is $(g - 1)/3 - 4$ in case (a), $(g - 1)/3$ in case (b), and $(g - \beta)/3 - (1 + \varepsilon)$ in (c). The other values of k can be written in the form $k_{\max} - 2l$ for some integer l in the range $0 \leq l \leq k_{\max}/2$. For $k = k_{\max} - 2l$ and $\gamma = \varepsilon + 2s$, we have $p = 4\gamma + k + 1 = 4\varepsilon + 8s + k_{\max} - 2l + 1$, and the inequality (14) is satisfied, if $s \leq (2 + 3l)/8$ in case (a), $s \leq (3l - 4\varepsilon)/8$ in case (b), and $s \leq (6l + 2 + \beta - 5\varepsilon)/16$ in case (c). □

Acknowledgements. The authors would like to thank the referee for valuable comments and suggestions, which helped to improve the final version of the paper.

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Received 28 February 2014 • Revised received 10 October 2014 • Accepted 12 January 2015