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TWO-WEIGHT NORM INEQUALITIES ON MORREY SPACES

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Abstract. A description of all the admissible weights similar to the Muckenhoupt class A_p is an open problem for the weighted Morrey spaces. In this paper necessary condition and sufficient condition for two-weight norm inequalities on Morrey spaces to hold are separately given for the Hardy–Littlewood maximal operator. Necessary and sufficient condition is also verified for the power weights.

1. Introduction

The purpose of this paper is to develop a theory of weights for the Hardy–Littlewood maximal operator on the Morrey spaces. The Morrey spaces, which were introduced by Morrey in order to study regularity questions which appear in the Calculus of Variations, describe local regularity more precisely than Lebesgue spaces and widely use not only harmonic analysis but also partial differential equations (cf. [4]).

We shall consider all cubes in $\mathbf{R}^n$ which have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. For a cube $Q \in \mathcal{Q}$ we use $l(Q)$ to denote the sides length of Q and $|Q|$ to denote the volume of Q. Let $0 < p < \infty$ and $0 < \lambda < n$ be two real parameters. For $f \in L^p_{\text{loc}}(\mathbf{R}^n)$, define

$$\|f\|_{L^{p,\lambda}} = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{l(Q)^\lambda} \int_Q |f(x)|^p \, dx \right)^{1/p}.$$

The Morrey space $L^{p,\lambda}(\mathbf{R}^n)$ is defined to be the subset of all L^p locally integrable functions f on $\mathbf{R}^n$ for which $\|f\|_{L^{p,\lambda}}$ is finite. It is easy see that $\|\cdot\|_{L^{p,\lambda}}$ becomes the norm if $p \ge 1$ and becomes the quasi norm if $p \in (0,1)$. The completeness of Morrey spaces follows easily by that of Lebesgue spaces. Let f be a locally integrable function on $\mathbf{R}^n$. The Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \in \mathcal{Q}} \fint_Q |f(y)| \, dy 1_Q(x),$$

where $\fint_Q f(x)\,dx$ stands for the usual integral average of f over Q and 1_Q denotes the characteristic function of the cube Q. By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure.

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Given a measurable set E and a weight w, $w(E) = \int_E w(x)\,dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent number of p. Let $0 < p < \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(\mathbf{R}^n, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty.$$

Let $0 < p < \infty$, $0 < \lambda < n$ and w be a weight. We define the weighted Morrey space $L^{p,\lambda}(\mathbf{R}^n, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^{p,\lambda}(w)} = \sup_{Q \in \mathcal{Q}} \left(\frac{1}{l(Q)^\lambda} \int_Q |f(x)|^p w(x)\,dx \right)^{1/p} < \infty.$$

As is well-known, for the Hardy–Littlewood maximal operator M and $p > 1$, Muckenhoupt [9] showed that the weighted inequality

$$\|Mf\|_{L^p(w)} \le C\|f\|_{L^p(w)}$$

holds if and only if

$$[w]_{A_p} = \sup_{Q \in \mathcal{Q}} \frac{w(Q)}{|Q|} \left(⨍_Q w(x)^{-p'/p}\,dx \right)^{p/p'} < \infty.$$

While, for $1 < p \le q < \infty$, Sawyer [15] showed that the weighted inequality

$$\|Mf\|_{L^q(u)} \le C\|f\|_{L^p(v)}$$

holds if and only if

$$\left(\int_Q M[\sigma 1_Q](x)^q u(x)\,dx \right)^{1/q} \le C\sigma(Q)^{1/p} < \infty, \quad \sigma = v^{-p'/p},$$

holds for every cube $Q \in \mathcal{Q}$.

For $p > 1$ one says that a weight w on $\mathbf{R}^n$ belongs to the Muckenhoupt class A_p when $[w]_{A_p} < \infty$. For $p = 1$ one says that a weight w on $\mathbf{R}^n$ belongs to the Muckenhoupt class A_1 when

$$[w]_{A_1} = \sup_{Q \in \mathcal{Q}} \frac{⨍_Q w(x)\,dx}{\operatorname{ess\,inf}_{x \in Q} w(x)} < \infty.$$

A description of all the admissible weights similar to the Muckenhoupt class A_p is an open problem for the weighted Morrey space $L^{p,\lambda}(\mathbf{R}^n, w)$ (see [12]). In [5], we proved the following partial answer to the problem.

Proposition 1.1. [5, Theorem 2.1] *Let $1 < p < \infty$, $0 < \lambda < n$ and w be a weight. Then, for every cube $Q \in \mathcal{Q}$, the weighted inequality*

$$\left(\frac{1}{l(Q)^\lambda} \int_Q Mf(x)^p w(x)\,dx \right)^{1/p} \le C \sup_{\substack{Q' \in \mathcal{Q} \\ Q' \supset Q}} \left(\frac{1}{l(Q')^\lambda} \int_{Q'} |f(x)|^p w(x)\,dx \right)^{1/p}$$

holds if and only if

$$\sup_{\substack{Q, Q' \in \mathcal{Q} \\ Q \subset Q'}} \frac{w(Q)}{l(Q)^\lambda} \frac{l(Q')^\lambda}{|Q'|} \left(⨍_{Q'} w(x)^{-p'/p}\,dx \right)^{p/p'} < \infty.$$

This proposition says that the weighted inequality

$$\|Mf\|_{L^{p,\lambda}(w)} \le C\|f\|_{L^{p,\lambda}(w)} \tag{1.1}$$

holds if

$$\sup_{Q\in\mathcal{Q}} \|w1_Q\|_{L^{1,\lambda}} \frac{l(Q)^\lambda}{|Q|} \left(\fint_Q w(x)^{-p'/p}\,dx \right)^{p/p'} < \infty. \tag{1.2}$$

One sees that the power weights $w = |\cdot|^\alpha$ belong to the Muckenhoupt class A_p if and only if $-n < \alpha < (p-1)n$. While, the power weights $w = |\cdot|^\alpha$ satisfy (1.2) if and only if $\lambda - n \le \alpha < (p-1)n$. Let H be the Hilbert transform defined by

$$Hf(x) = \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{\mathbf{R}} \frac{1_{(\varepsilon,\infty)}(|x-y|)}{x-y} f(y)\,dy.$$

For $1 < p < \infty$ and $0 < \lambda < 1$, Samko [11] showed that the weighted inequality

$$\|Hf\|_{L^{p,\lambda}(w)} \le C\|f\|_{L^{p,\lambda}(w)}, \quad w = |\cdot|^\alpha,$$

holds if and only if $\lambda-1 \le \alpha < \lambda+(p-1)$. Thus, our sufficient condition (1.2) seems to be quite strong. In this paper we introduce another sufficient condition and necessary condition for which (1.1) to hold (Proposition 4.1). The conditions justify the power weights $w = |\cdot|^\alpha$ fulfill (1.1) if and only if $\lambda - n \le \alpha < \lambda + (p-1)n$ (Proposition 4.2). More precisely, in this paper we introduce sufficient condition and necessary condition for which two-weight Morrey norm inequalities to hold (Theorem 3.1), which is closely related to Sawyer's two-weight theorem. As an appendix, we show two-weight norm inequality in the upper triangle case $0 < q < p < \infty$, $1 < p < \infty$ (Proposition 5.1).

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive constant c independent of appropriate quantities.

2. A dual equation

In this section we shall verify a dual equation of Morrey spaces (Lemma 2.4). For any measurable set $E \subset \mathbf{R}^n$ and any $f \in L^p(\mathbf{R}^n)$, we simply have

$$\int_E |f(x)|^p\,dx \le \|f\|_{L^p}^p < \infty.$$

While, if $f \in L^{p,\lambda}(\mathbf{R}^n)$, then for any $Q \in \mathcal{Q}$

$$\int_Q |f(x)|^p\,dx \le \|f\|_{L^{p,\lambda}}^p l(Q)^\lambda.$$

This implies that for any family of counterable cubes $\{Q_j\} \subset \mathcal{Q}$ such that $E \subset \bigcup_j Q_j$, we have

$$\int_E |f(x)|^p\,dx \le \sum_j \int_{Q_j} |f(x)|^p\,dx \le \|f\|_{L^{p,\lambda}}^p \sum_j l(Q_j)^\lambda. \tag{2.1}$$

In general, if $E \subset \mathbf{R}^n$ and $0 < \alpha \le n$, then the α-dimensional Hausdorff content of E is defined by

$$H^\alpha(E) = \inf\left\{ \sum_j l(Q_j)^\alpha \right\},$$

where the infimum is taken over all coverings of E by countable families of cubes $\{Q_j\} \subset \mathcal{Q}$. Thanks to this definition, we get by (2.1)

$$\int_E |f(x)|^p\,dx \le \|f\|_{L^{p,\lambda}}^p H^\lambda(E). \tag{2.2}$$

The Choquet integral of $\phi \ge 0$ with respect to the Hausdorff content H^α is defined by

$$\int_{\mathbf{R}^n} \phi\,dH^\alpha = \int_0^\infty H^\alpha(\{y \in \mathbf{R}^n \colon \phi(y) > t\})\,dt.$$

Thus, by (2.2), for any $\phi \ge 0$ and any $f \in L^{p,\lambda}(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} |f(x)|^p \phi(x)\,dx = \int_0^\infty \int_{\{y\in\mathbf{R}^n\colon \phi(y)>t\}} |f(x)|^p\,dx\,dt \le \|f\|_{L^{p,\lambda}}^p \int_{\mathbf{R}^n} \phi\,dH^\lambda. \tag{2.3}$$

Following the argument in [2], we introduce another characterization of the Morrey space by (2.3).

Definition 2.1. Let $0 < \lambda < n$. Define the basis $\mathcal{B}_\lambda$ to be the set of all weights b such that $b \in A_1$ and $\int_{\mathbf{R}^n} b\,dH^\lambda \le 1$.

We need the following lemma.

Lemma 2.2. [10, Lemma 1] *Let $0 < \alpha < n$ and $p > \alpha/n$. Then, for some constant C depending only on α, n and p,*

$$\int_{\mathbf{R}^n} M[1_Q]^p\,dH^\alpha \le Cl(Q)^\alpha.$$

Let $0 < \lambda < \lambda_0 < n$ and $f \in L^{p,\lambda}(\mathbf{R}^n)$. It follows from (2.3) and Lemma 2.2 that, for every cube $Q \in \mathcal{Q}$,

$$\begin{aligned}\frac{1}{l(Q)^\lambda}\int_Q |f(x)|^p\,dx &= \frac{1}{l(Q)^\lambda}\int_{\mathbf{R}^n} |f(x)|^p 1_Q(x)\,dx \\ &\le \frac{1}{l(Q)^\lambda}\int_{\mathbf{R}^n} |f(x)|^p M[1_Q](x)^{\lambda_0/n}\,dx \\ &\le \|f\|_{L^{p,\lambda}}^p \frac{1}{l(Q)^\lambda}\int_{\mathbf{R}^n} M[1_Q]^{\lambda_0/n}\,dH^\lambda \le C\|f\|_{L^{p,\lambda}}^p,\end{aligned}$$

which yields

$$\|f\|_{L^{p,\lambda}} \approx \sup_{b\in\mathcal{B}_\lambda}\left(\int_{\mathbf{R}^n} |f(x)|^p b(x)\,dx\right)^{1/p}, \tag{2.4}$$

where we have used the fact that $M[1_Q]^{\lambda_0/n} \in A_1$, since $\lambda_0/n < 1$ (cf. [3, Chapter II]).

Definition 2.3. [2] Let $1 < p < \infty$ and $0 < \lambda < n$. The space $H^{p,\lambda}(\mathbf{R}^n)$ is defined by the set of all measurable functions f on $\mathbf{R}^n$ with the quasi norm

$$\|f\|_{H^{p,\lambda}} = \inf_{b\in\mathcal{B}_\lambda}\left(\int_{\mathbf{R}^n} |f(x)|^p b(x)^{-p/p'}\,dx\right)^{1/p} < \infty.$$

For non-negative functions $f \in L^{p,\lambda}(\mathbf{R}^n)$ and $g \in H^{p',\lambda}(\mathbf{R}^n)$, there holds by Hölder's inequality that

$$\begin{aligned}\int_{\mathbf{R}^n} f(x)g(x)\,dx &= \int_{\mathbf{R}^n} f(x)b(x)^{1/p}g(x)b(x)^{-1/p}\,dx \\ &\le \left(\int_{\mathbf{R}^n} f(x)^p b(x)\,dx\right)^{1/p}\left(\int_{\mathbf{R}^n} g(x)^{p'}b(x)^{-p'/p}\,dx\right)^{1/p'}, \quad b \in \mathcal{B}_\lambda.\end{aligned}$$

This implies by (2.4)

$$\int_{\mathbf{R}^n} f(x)g(x)\,dx \le C\|f\|_{L^{p,\lambda}}\|g\|_{H^{p',\lambda}}. \tag{2.5}$$

In this section we shall verify the following lemma.

Lemma 2.4. *Let $1 < p < \infty$ and $0 < \lambda < n$. Then, for any measurable function g on $\mathbf{R}^n$, we have the estimate (allowing to be infinite)*

$$\|g\|_{H^{p',\lambda}} \approx \sup_f \int_{R^n} |f(x)g(x)|\,dx,$$

where the supremum is taken over all functions $f \in L^{p,\lambda}(\mathbf{R}^n)$ with $\|f\|_{L^{p,\lambda}} \le 1$.

This lemma was first introduced in [2] without the proof. In [6], Izumi et al. give the full proof for the block spaces on the unit circle $\mathbf{T}$ with the help of Functional Analysis. In [14], we give the proof for the block spaces on the Euclidean space $\mathbf{R}^n$.

Definition 2.5. Let $1 < p < \infty$ and $0 < \lambda < n$. The block space $B^{p,\lambda}(\mathbf{R}^n)$ is defined by the set of all measurable functions f on $\mathbf{R}^n$ with the norm

$$\|f\|_{B^{p,\lambda}} = \inf\left\{\|\{c_k\}\|_{l^1} : f = \sum_k c_k a_k\right\} < \infty,$$

where a_k is a (p,λ)-block and $\|\{c_k\}\|_{l^1} = \sum_k |c_k| < \infty$, and the infimum is taken over all possible decompositions of f. Additionally, we say that a function a on $\mathbf{R}^n$ is a (p,λ)-block provided that a is supported on a cube $Q \in \mathcal{Q}$ and satisfies

$$\|a\|_{L^p} \le \frac{1}{l(Q)^{\lambda/p'}}.$$

Lemma 2.6. [14] *Let $1 < p < \infty$ and $0 < \lambda < n$. Then, for any measurable function g on $\mathbf{R}^n$, we have the estimate (allowing to be infinite)*

$$\|g\|_{B^{p',\lambda}} = \sup_f \int_{R^n} |f(x)g(x)|\,dx,$$

where the supremum is taken over all functions $f \in L^{p,\lambda}(\mathbf{R}^n)$ with $\|f\|_{L^{p,\lambda}} \le 1$.

Proof of Lemma 2.4. Thanks to Lemma 2.6, we need only verify that $H^{p,\lambda}(\mathbf{R}^n) = B^{p,\lambda}(\mathbf{R}^n)$ with $\|\cdot\|_{H^{p,\lambda}} \approx \|\cdot\|_{B^{p,\lambda}}$. This fact was proved in [1]. But, the direct proof is given here for the completeness.

We will denote by $\mathcal{D}$ the family of all dyadic cubes $Q = 2^{-k}(m + [0,1)^n)$, $k \in \mathbf{Z}$, $m \in \mathbf{Z}^n$. Assume that for non-negative function $f \in H^{p,\lambda}(\mathbf{R}^n)$,

$$\left(\int_{\mathbf{R}^n} f(x)^p b(x)^{-p/p'}\,dx\right)^{1/p} \le 2\|f\|_{H^{p,\lambda}} \quad \text{for some } b \in \mathcal{B}_\lambda. \tag{2.6}$$

Consider $E_k = \{x \in \mathbf{R}^n : b(x) > 2^k\}$, $k \in \mathbf{Z}$. Then,

$$\int_{\mathbf{R}^n} b\, dH^\lambda \approx \sum_k 2^k H^\lambda(E_k) \approx 1. \tag{2.7}$$

By the definition of the Hausdorff content H^λ and its dyadic equivalence (cf. [10]), one can select a set of the pairwise disjoint dyadic cubes $\{Q_{k,j}\} \subset \mathcal{D}$ such that $E_k \subset \bigcup_j Q_{k,j}$ and

$$\sum_j l(Q_{k,j})^\lambda \le 2H^\lambda(E_k). \tag{2.8}$$

Upon defining

$$\delta_{k,j} = Q_{k,j} \setminus \bigcup_i Q_{k+1,i},$$

we see that the sets $\delta_{k,j}$ are pairwise disjoint and $\mathbf{R}^n = \bigcup_{k,j} \delta_{k,j}$. With this, we obtain

$$f = \sum_{k,j} c_{k,j} a_{k,j},$$

where

$$c_{k,j} = l(Q_{k,j})^{\lambda/p'} \left(\int_{\delta_{k,j}} f(x)^p\, dx \right)^{1/p}$$

and

$$a_{k,j} = l(Q_{j,k})^{-\lambda/p'} \left(\int_{\delta_{k,j}} f(x)^p\, dx \right)^{-1/p} f 1_{\delta_{k,j}}.$$

It is easy to check that each $a_{k,j}$ is a (p,λ)-block. To prove that $f \in B^{p,\lambda}(\mathbf{R}^n)$, it remains to verify that $\{c_{k,j}\}$ is summable.

Notice that $b(x) \le 2^{k+1}$ if $x \in \delta_{k,j}$. This yields, by using Hölder's inequality,

$$\begin{aligned}
\|\{c_{k,j}\}\|_{l^1} &\le C \sum_{k,j} l(Q_{k,j})^{\lambda/p'} 2^{k/p'} \left(\int_{\delta_{k,j}} f(x)^p b(x)^{-p/p'}\, dx \right)^{1/p} \\
&\le C \left(\sum_{k,j} l(Q_{k,j})^\lambda 2^k \right)^{1/p'} \left(\int_{\mathbf{R}^n} f(x)^p b(x)^{-p/p'}\, dx \right)^{1/p} \\
&\le C \left(\sum_k 2^k H^\lambda(E_k) \right)^{1/p'} \left(\int_{\mathbf{R}^n} f(x)^p b(x)^{-p/p'}\, dx \right)^{1/p} \le C\|f\|_{H^{p,\lambda}},
\end{aligned}$$

where we have used (2.6)–(2.8). This proves $H^{p,\lambda}(\mathbf{R}^n) \subset B^{p,\lambda}(\mathbf{R}^n)$ with $\|\cdot\|_{B^{p,\lambda}} \le C\|\cdot\|_{H^{p,\lambda}}$.

We now prove converse. Suppose that $f \in B^{p,\lambda}(\mathbf{R}^n)$. So, $f = \sum_j c_j a_j$ with $\{c_j\} \in l^1$ and each a_j is a (p,λ)-block. Assume that Q_j is the support cube of a_j. For $0 < \lambda < \lambda_0 < n$, define

$$b(x) = \|\{c_j\}\|_{l^1}^{-1} \sum_j |c_j| b_j(x)$$

with

$$b_j(x) = \frac{1}{l(Q_j)^\lambda} M[1_{Q_j}](x)^{\lambda_0/n}.$$

Then, we see that

$$\int_{\mathbf{R}^n} b_j \, dH^\lambda \le C \quad \text{and} \quad [b_j]_{A_1} \le C.$$

This means that

$$\int_{\mathbf{R}^n} b \, dH^\lambda \le C \quad \text{and} \quad [b]_{A_1} \le C.$$

Thus, we have $Cb \in \mathcal{B}_\lambda$.

It follows from Hölder's inequality that

$$|f(x)|^p \le \left(\sum_j |c_j| b_j(x) \right)^{p/p'} \left(\sum_j |c_j| b_j(x)^{-p/p'} a_j(x)^p \right).$$

This implies

$$\int_{\mathbf{R}^n} |f(x)|^p b(x)^{-p/p'} \, dx \le \|\{c_j\}\|_{l^1}^{p/p'} \sum_j |c_j| \int_{Q_j} b_j(x)^{-p/p'} a_j(x)^p \, dx.$$

Notice that whenever $x \in Q_j$

$$b_j(x)^{-p/p'} \le l(Q_j)^{\lambda p/p'},$$

which implies

$$\int_{\mathbf{R}^n} |f(x)|^p b(x)^{-p/p'} \, dx \le \|\{c_j\}\|_{l^1}^{p/p'} \sum_j |c_j| l(Q_j)^{\lambda p/p'} \int_{Q_j} |a_j(x)|^p \, dx \le \|\{c_j\}\|_{l^1}^{p}.$$

This proves $B^{p,\lambda}(\mathbf{R}^n) \subset H^{p,\lambda}(\mathbf{R}^n)$ with $\|\cdot\|_{H^{p,\lambda}} \le C \|\cdot\|_{B^{p,\lambda}}$. These complete the proof of Lemma 2.4. □

3. Two-weight norm inequalities

In this section we shall prove the following theorem.

Theorem 3.1. *Let $1 < p < \infty$, $0 < q < \infty$, $0 < \lambda < n$, u, v be weights. Consider the following five statements:*

(a) *There exists a constant $c_1 > 0$ such that*

$$\|Mf\|_{L^{q,\lambda}(u)} \le c_1 \|f\|_{L^{p,\lambda}(v)}$$

holds for every function $f \in L^{p,\lambda}(\mathbf{R}^n, v)$;

(b) *There exists a constant $c_2 > 0$ such that*

$$\frac{1}{|Q|} \|u^{1/q} 1_Q\|_{L^{q,\lambda}} \|v^{-1/p} 1_Q\|_{H^{p',\lambda}} \le c_2$$

holds for every cube $Q \in \mathcal{Q}$;

(c) *There exists a constant $c_3 > 0$ such that*

$$\inf_{b \in \mathcal{B}_\lambda} \left(\sup_{\substack{Q \in \mathcal{Q} \\ Q \subset Q_0}} \frac{1}{\sigma(Q)^{1/p}} \left(\int_Q M[\sigma 1_Q](x)^q u(x) \, dx \right)^{1/q} \right) \le c_3 l(Q_0)^{\lambda/q}, \quad \sigma = (bv)^{-p'/p},$$

holds for every cube $Q_0 \in \mathcal{Q}$;

(d) *There exists a constant $c_4 > 0$ such that, for some $a > 1$,*

$$\inf_{b\in\mathcal{B}_\lambda}\left(\sup_{\substack{Q\in\mathcal{Q}\\ Q\subset Q_0}}\frac{u(Q)^{1/q}}{|Q|^{1/p}}\left(\fint_Q [b(x)v(x)]^{-ap'/p}\,dx\right)^{1/ap'}\right)\le c_4 l(Q_0)^{\lambda/q}$$

holds for every cube $Q_0\in\mathcal{Q}$;

(e) *There exists a constant $c_5 > 0$ such that, for $1/q = 1/r + 1/p$,*

$$\inf_{b\in\mathcal{B}_\lambda}\left(\int_{Q_0} M[\sigma](x)^r u(x)^{r/q}\sigma(x)^{-r/p}\,dx\right)^{1/r}\le c_5 l(Q_0)^{\lambda/q},\quad \sigma=(bv)^{-p'/p},$$

holds for every cube $Q_0\in\mathcal{Q}$.

Then,

(I) (a) *implies* (b) *with $c_2\le Cc_1$;*
(II) *When $1 < p\le q<\infty$,* (b) *and* (c) *imply* (a) *with $c_1\le C(c_2+c_3)$;*
(III) *When $1 < p\le q<\infty$,* (b) *and* (d) *imply* (a) *with $c_1\le C(c_2+c_4)$;*
(IV) *When $0 < q < p < \infty$ and $1 < p < \infty$,* (b) *and* (e) *imply* (a) *with $c_1\le C(c_2+c_5)$.*

We shall prove this theorem in the remainder of this section. Recall that $\mathcal{D}$ denotes the family of all dyadic cubes $Q = 2^{-k}(m+[0,1)^n)$, $k\in\mathbf{Z}$, $m\in\mathbf{Z}^n$. In the following proof, by the argument which uses appropriate averages of the sifted dyadic cubes, we can replace the set of cubes $\mathcal{Q}$ by the set of dyadic cubes $\mathcal{D}$ (cf. [7]). So, the Hardy–Littlewood maximal operator M can be replaced by the dyadic Hardy–Littlewood maximal operator M_d. But, for the sake of simplicity, we will denote M_d by the same M.

3.1. Proof of Theorem 3.1 (I). Assume that the statement (a). Then,

$$\|Mf\|_{L^{q,\lambda}(u)}\le c_1\|f\|_{L^{p,\lambda}(v)}$$

holds for every function $f\in L^{p,\lambda}(\mathbf{R}^n,v)$. For any cube $Q\in\mathcal{D}$ and any function $f\in L^{p,\lambda}(\mathbf{R}^n,v)$,

$$\begin{aligned}\fint_Q |f(x)|\,dx\|u^{1/q}1_Q\|_{L^{q,\lambda}} &= \left\|\fint_Q |f(x)|\,dx u^{1/q}1_Q\right\|_{L^{q,\lambda}}\\ &\le \|M[f1_Q]\|_{L^{q,\lambda}(u)}\le c_1\|f1_Q\|_{L^{p,\lambda}(v)}.\end{aligned}$$

Taking the supremum over all functions f with $\|f1_Q\|_{L^{p,\lambda}(v)}\le 1$, we have by Lemma 2.4

$$\frac{1}{|Q|}\|u^{1/q}1_Q\|_{L^{q,\lambda}}\|v^{-1/p}1_Q\|_{H^{p',\lambda}}\le Cc_1,$$

which is the statement (b).

3.2. Proof of Theorem 3.1 (II). We need more a lemma. Let μ be a positive measure on $\mathbf{R}^n$ and f be a locally μ-integrable function on $\mathbf{R}^n$. The dyadic Hardy–Littlewood maximal operator M_μ is defined by

$$M_\mu f(x)=\sup_{Q\in\mathcal{D}}\fint_Q |f(y)|\,d\mu(y)1_Q(x).$$

Lemma 3.2. [7] *We have the estimate*

$$\|M_\mu f\|_{L^p(\mu)}\le p'\|f\|_{L^p(\mu)},\quad p\in(1,\infty].$$

Assume that $1 < p \le q < \infty$ and the statements (b) and (c). Without loss of generality we may assume that f is non-negative. Recall that M is now the dyadic Hardy–Littlewood maximal operator. Fix a cube Q_0 in $\mathcal{D}$. Then, by a standard argument we have

$$Mf(x) \le C_\infty + M[f1_{Q_0}](x), \quad x \in Q_0,$$

with

$$C_\infty = \sup_{\substack{Q\in\mathcal{D}\\ Q\supsetneq Q_0}} ⨍_Q f(y)\,dy.$$

By the definition of the weighted Morrey norm, we have to evaluate two quantities:

$$C_\infty \left(\frac{1}{l(Q_0)^\lambda}\int_{Q_0} u(x)\,dx\right)^{1/q}; \tag{3.1}$$

$$\left(\frac{1}{l(Q_0)^\lambda}\int_{Q_0} M[f1_{Q_0}](x)^q u(x)\,dx\right)^{1/q}. \tag{3.2}$$

The estimate of (3.1). There holds

$$\left(\frac{1}{l(Q_0)^\lambda}\int_{Q_0} u(x)\,dx\right)^{1/q} \le \|u^{1/q}1_{Q_0}\|_{L^{q,\lambda}},$$

and, for $Q\in\mathcal{D}$ such that $Q\supsetneq Q_0$,

$$⨍_Q f(y)\,dy \le \frac{C}{|Q|}\|v^{-1/p}1_Q\|_{H^{p',\lambda}}\|f1_Q\|_{L^{p,\lambda}(v)},$$

where we have used (2.5). These yield by use of the statement (b)

$$(3.1) \le Cc_2\|f\|_{L^{p,\lambda}(v)}.$$

The estimate of (3.2). Let $\mathcal{D}(Q_0) = \{Q \in \mathcal{D} \colon Q \subset Q_0\}$. Consider, for all $Q\in\mathcal{D}(Q_0)$,

$E(Q) =$

$$\left\{x\in Q\colon M[f1_{Q_0}](x) = ⨍_Q f(y)\,dy\right\} \setminus \bigcup_{\substack{Q'\in\mathcal{D}(Q_0)\\ Q'\supsetneq Q}} \left\{x\in Q'\colon M[f1_{Q_0}](x) = ⨍_{Q'} f(y)\,dy\right\}.$$

A little thought confirms that the sets $E(Q)$ are pairwise disjoint and

$$M[f1_{Q_0}](x) = \sum_{Q\in\mathcal{D}(Q_0)} ⨍_Q f(y)\,dy 1_{E(Q)}(x), \quad x\in Q_0.$$

Take a function g which is non-negative, is supported on Q_0 and satisfies $\|g\|_{L^{q'}(u)} \le 1$. Upon using the duality argument, we shall estimate

$$\sum_{Q\in\mathcal{D}(Q_0)} ⨍_Q f(y)\,dy \int_{E(Q)} g(x)u(x)\,dx. \tag{3.3}$$

Fix $b\in\mathcal{B}_\lambda$ so that

$$\sup_{\substack{Q\in\mathcal{D}\\ Q\subset Q_0}} \frac{1}{\sigma(Q)^{1/p}}\left(\int_Q M[\sigma 1_Q](x)^q u(x)\,dx\right)^{1/q} \le 2c_3 l(Q_0)^{\lambda/q}, \quad \sigma = (bv)^{-p'/p}. \tag{3.4}$$

Then, (3.3) can be rewritten as

$$\sum_{Q\in\mathcal{D}(Q_0)} \frac{\sigma(Q)}{|Q|} \fint_Q f(y)\sigma(y)^{-1}\,d\sigma(y) \int_{E(Q)} g(x)u(x)\,dx,$$

where $d\sigma(y)$ denotes $\sigma(y)\,dy$. We now employ the argument of the principal cubes (cf. [8, 16]).

We define the collection of principal cubes

$$\mathcal{F} = \bigcup_{k=0}^{\infty} \mathcal{F}_k,$$

where $\mathcal{F}_0 = \{Q_0\}$,

$$\mathcal{F}_{k+1} = \bigcup_{F\in\mathcal{F}_k} ch_{\mathcal{F}}(F)$$

and $ch_{\mathcal{F}}(F)$ is defined by the set of all maximal dyadic cubes $Q \subset F$ such that

$$\fint_Q f(y)\sigma(y)^{-1}\,d\sigma(y) > 2 \fint_F f(y)\sigma(y)^{-1}\,d\sigma(y).$$

Observe that

$$\sum_{F'\in ch_{\mathcal{F}}(F)} \sigma(F') \le \left(2\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y)\right)^{-1} \sum_{F'\in ch_{\mathcal{F}}(F)} \int_{F'} f(y)\sigma(y)^{-1}\,d\sigma(y) \le \frac{\sigma(F)}{2},$$

and, hence,

$$\sigma(E_{\mathcal{F}}(F)) = \sigma\left(F \setminus \bigcup_{F'\in ch_{\mathcal{F}}(F)} F'\right) \ge \frac{\sigma(F)}{2}, \tag{3.5}$$

where the sets $E_{\mathcal{F}}(F)$ are pairwise disjoint. We further define the stopping parents

$$\pi_{\mathcal{F}}(Q) = \min\{F \supset Q\colon F \in \mathcal{F}\} \quad \text{for all } Q \in \mathcal{D}(Q_0).$$

It follows that

$$\begin{aligned}(3.3) &= \sum_{F\in\mathcal{F}} \sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \frac{\sigma(Q)}{|Q|} \fint_Q f(y)\sigma(y)^{-1}\,d\sigma(y) \int_{E(Q)} g(x)u(x)\,dx \\ &\le 2\sum_{F\in\mathcal{F}} \fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \frac{\sigma(Q)}{|Q|} \int_{E(Q)} g(x)u(x)\,dx.\end{aligned}$$

From Hölder's inequality,

$$\begin{aligned}&\sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \frac{\sigma(Q)}{|Q|} \int_{E(Q)} g(x)u(x)\,dx \\ &\le \left(\sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \left(\frac{\sigma(Q)}{|Q|}\right)^q \int_{E(Q)} u(x)\,dx\right)^{1/q} \left(\sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \int_{E(Q)} g(x)^{q'}u(x)\,dx\right)^{1/q'}.\end{aligned}$$

From the definition of M, the facts that $E(Q) \subset Q$ and the sets $E(Q)$ are pairwise disjoint,

$$\leq \left(\int_F M[\sigma 1_F](x)^q u(x)\,dx \right)^{1/q} \left(\sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \int_{E(Q)} g(x)^{q'} u(x)\,dx \right)^{1/q'}.$$

From Hölder's inequality again, (3.3) can be majorized by

$$2 \sum_{F\in\mathcal{F}} \fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \left(\int_F M[\sigma 1_F](x)^q u(x)\,dx \right)^{1/q} \left(\sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \int_{E(Q)} g(x)^{q'} u(x)\,dx \right)^{1/q'}$$

$$\leq 2 \left\{ \sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \left(\int_F M[\sigma 1_F](x)^q u(x)\,dx \right)^{1/q} \right)^q \right\}^{1/q}$$

$$\times \left\{ \sum_{F\in\mathcal{F}} \sum_{\substack{Q:\\ \pi_{\mathcal{F}}(Q)=F}} \int_{E(Q)} g(x)^{q'} u(x)\,dx \right\}^{1/q'} =: \text{(i)} \times \text{(ii)}.$$

Since the sets $E(Q)$ are pairwise disjoint,

$$\text{(ii)} = \left(\int_{Q_0} g(x)^{q'} u(x)\,dx \right)^{1/q'} \leq 1.$$

Since $p \leq q$ and $\|\cdot\|_{l^p} \geq \|\cdot\|_{l^q}$,

$$\text{(i)} \leq \left\{ \sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \left(\int_F M[\sigma 1_F](x)^q u(x)\,dx \right)^{1/q} \right)^p \right\}^{1/p}. \tag{3.6}$$

Further,

$$\leq \left\{ \sup_{F\in\mathcal{F}} \frac{1}{\sigma(F)^{1/p}} \left(\int_F M[\sigma 1_F](x)^q u(x)\,dx \right)^{1/q} \right\}$$

$$\times \left\{ \sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \right)^p \sigma(F) \right\}^{1/p}$$

$$\leq 2c_3 l(Q_0)^{\lambda/q} \left\{ \sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y) \right)^p \sigma(F) \right\}^{1/p},$$

where we have used (3.4).

By using the definition of M_σ, (3.5) and the facts that $E_{\mathcal{F}}(F) \subset F$ and the sets $E_{\mathcal{F}}(F)$ are pairwise disjoint,

$$\begin{aligned}
&\left\{\sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y)\right)^p \sigma(F)\right\}^{1/p} \\
&\le C\left\{\sum_{F\in\mathcal{F}} \left(\fint_F f(y)\sigma(y)^{-1}\,d\sigma(y)\right)^p \sigma(E_{\mathcal{F}}(F))\right\}^{1/p} \\
&\le C\left(\int_{\mathbf{R}^n} M_\sigma[f\sigma^{-1}](x)^p\,d\sigma(x)\right)^{1/p}.
\end{aligned}$$

By use of Lemma 3.2,

$$\le C\left(\int_{\mathbf{R}^n}[f(x)\sigma(x)^{-1}]^p\sigma(x)\,dx\right)^{1/p} = C\left(\int_{\mathbf{R}^n} f(x)^p b(x)v(x)\,dx\right)^{1/p} \le C\|f\|_{L^{p,\lambda}(v)},$$

where we have used (2.4).

So altogether we obtain

$$\left(\int_{Q_0} M[f1_{Q_0}](x)^q u(x)\,dx\right)^{1/q} \le Cc_3 l(Q_0)^{\lambda/q}\|f\|_{L^{p,\lambda}(v)}$$

and

$$(3.2) \le Cc_3\|f\|_{L^{p,\lambda}(v)}.$$

These complete the proof of Theorem 3.1 (II).

3.3. Proof of Theorem 3.1 (III). Assume that $1 < p \le q < \infty$ and the statements (b) and (d). Going through the same argument as before, retaining the same notation, we need only evaluate (3.2) especially (3.3). Letting $\sigma \equiv 1$ in (3.6), we see that

$$\text{(i)} \le \left\{\sum_{F\in\mathcal{F}}\left(\fint_F f(y)\,dy u(F)^{1/q}\right)^p\right\}^{1/p}.$$

Fix $b \in \mathcal{B}_\lambda$ so that

$$\sup_{\substack{Q\in\mathcal{D}\\ Q\subset Q_0}} \frac{u(Q)^{1/q}}{|Q|^{1/p}}\left(\fint_Q [b(x)v(x)]^{-ap'/p}\,dx\right)^{1/ap'} \le 2c_4 l(Q_0)^{\lambda/q}. \tag{3.7}$$

Take $c < 1$ is a number that satisfy $(cp)' = ap'$. Hölder's inequality gives

$$\begin{aligned}
\fint_F f(y)\,dy &= \fint_F f(y)[b(y)v(y)]^{1/p}[b(y)v(y)]^{-1/p}\,dy \\
&\le \left(\fint_F f(y)^{cp}[b(y)v(y)]^c\,dy\right)^{1/cp}\left(\fint_F [b(y)v(y)]^{-ap'/p}\,dy\right)^{1/ap'},
\end{aligned}$$

which implies

$$
\begin{aligned}
\text{(i)} &\leq \left\{ \sup_{F\in\mathcal{F}} \frac{u(F)^{1/q}}{|F|^{1/p}} \left(⨍_F [b(y)v(y)]^{-ap'/p}\, dy \right)^{1/ap'} \right\} \\
&\quad \times \left\{ \sum_{F\in\mathcal{F}} \left(⨍_F f(y)^{cp}[b(y)v(y)]^c\, dy \right)^{1/c} |F| \right\}^{1/p} \\
&\leq 2c_4 l(Q_0)^{\lambda/q} \left\{ \sum_{F\in\mathcal{F}} \left(⨍_F f(y)^{cp}[b(y)v(y)]^c\, dy \right)^{1/c} |F| \right\}^{1/p},
\end{aligned}
$$

where we have used (3.7).

The definition of M, the facts that $|F| \leq 2|E_{\mathcal{F}}(F)|$, $E_{\mathcal{F}}(F) \subset F$ and the sets $E_{\mathcal{F}}(F)$ are pairwise disjoint read

$$
\begin{aligned}
&\left\{ \sum_{F\in\mathcal{F}} \left(⨍_F f(y)^{cp}[b(y)v(y)]^c\, dy \right)^{1/c} |F| \right\}^{1/p} \\
&\leq C \left\{ \sum_{F\in\mathcal{F}} \left(⨍_F f(y)^{cp}[b(y)v(y)]^c\, dy \right)^{1/c} |E_{\mathcal{F}}(F)| \right\}^{1/p} \\
&\leq C \left(\int_{\mathbf{R}^n} M[f^{cp}(bv)^c](x)^{1/c}\, dx \right)^{1/p} \\
&\leq C \left(\int_{\mathbf{R}^n} f(x)^p b(x)v(x)\, dx \right)^{1/p} \leq C\|f\|_{L^{p,\lambda}(v)},
\end{aligned}
$$

where we have used the $L^{1/c}$-boundedness of M and (2.4).

So altogether we obtain

$$
\left(\int_{Q_0} M[f1_{Q_0}](x)^q u(x)\, dx \right)^{1/q} \leq Cc_4 l(Q_0)^{\lambda/q} \|f\|_{L^{p,\lambda}(v)}
$$

and

$$
(3.2) \leq Cc_4 \|f\|_{L^{p,\lambda}(v)}.
$$

This completes the proof of Theorem 3.1 (III).

3.4. Proof of Theorem 3.1 (IV). Assume that $0 < q < p < \infty$, $1 < p < \infty$ and the statements (b) and (e). In the same manner as above, retaining the same notation as before, we need only evaluate (3.2). Fix $b \in \mathcal{B}_\lambda$ so that, for $1/q = 1/r + 1/p$,

$$
\left(\int_{Q_0} M[\sigma](x)^r u(x)^{r/q} \sigma(x)^{-r/p}\, dx \right)^{1/r} \leq 2c_5 l(Q_0)^{\lambda/q}, \quad \sigma = (bv)^{-p'/p}. \tag{3.8}
$$

We have for every $Q \in \mathcal{D}(Q_0)$,

$$
\begin{aligned}
⨍_Q f(y)\, dy &= \frac{\sigma(Q)}{|Q|} ⨍_Q f(y)\sigma(y)^{-1}\, d\sigma(y) \\
&\leq M[\sigma](x) M_\sigma[f\sigma^{-1}](x), \quad x \in Q,
\end{aligned}
$$

which implies

$$
Mf(x) \leq M[\sigma](x) M_\sigma[f\sigma^{-1}](x), \quad x \in Q_0.
$$

Thus,

$$\left(\int_{Q_0} M[f1_{Q_0}](x)^q u(x)\,dx\right)^{1/q} \le \left(\int_{Q_0} M[\sigma](x)^q M_\sigma[f\sigma^{-1}](x)^q u(x)\,dx\right)^{1/q}$$
$$= \left(\int_{Q_0} M[\sigma](x)^q u(x)\sigma(x)^{-1} \cdot M_\sigma[f\sigma^{-1}](x)^q\,d\sigma(x)\right)^{1/q}.$$

From Hölder's inequality with the exponent $(p-q)/p + q/p = 1$ and the fact that $1/r = (p-q)/pq$,

$$\le \left(\int_{Q_0} \left(M[\sigma](x)^q u(x)\sigma(x)^{-1}\right)^{p/(p-q)}\,d\sigma(x)\right)^{1/r} \left(\int_{Q_0} M_\sigma[f\sigma^{-1}](x)^p\,d\sigma(x)\right)^{1/p}$$
$$=: \text{(iii)} \times \text{(iv)}.$$

We have by (3.8)

$$\text{(iii)} = \left(\int_{Q_0} M[\sigma](x)^r u(x)^{r/q}\sigma(x)^{-r/p}\,dx\right)^{1/r} \le 2c_5 l(Q_0)^{\lambda/q}$$

and we have by Lemma 3.2

$$\text{(iv)} \le C\left(\int_{\mathbf{R}^n} f(x)^p b(x)v(x)\,dx\right)^{1/p} \le C\|f\|_{L^{p,\lambda}(v)}.$$

These imply

$$\left(\int_{Q_0} M[f1_{Q_0}](x)^q u(x)\,dx\right)^{1/q} \le Cc_5 l(Q_0)^{\lambda/q}\|f\|_{L^{p,\lambda}(v)}$$

and

$$(3.2) \le Cc_5\|f\|_{L^{p,\lambda}(v)}.$$

This completes the proof of Theorem 3.1 (IV).

4. One-weight norm inequalities

We restate Theorem 3.1 in terms of the one-weight setting.

Proposition 4.1. *Let $1 < p < \infty$, $0 < \lambda < n$ and w be a weight. Consider the following four statements:*

(a) *There exists a constant $c_1 > 0$ such that*

$$\|Mf\|_{L^{p,\lambda}(w)} \le c_1\|f\|_{L^{p,\lambda}(w)}$$

holds for every function $f \in L^{p,\lambda}(\mathbf{R}^n, w)$;

(b) *There exists a constant $c_2 > 0$ such that*

$$\frac{1}{|Q|}\|w^{1/p}1_Q\|_{L^{p,\lambda}}\|w^{-1/p}1_Q\|_{H^{p',\lambda}} \le c_2$$

holds for every cube $Q \in \mathcal{Q}$;

(c) *There exists a constant $c_3 > 0$ such that*

$$\inf_{b\in\mathcal{B}_\lambda}\left(\sup_{\substack{Q\in\mathcal{Q}\\ Q\subset Q_0}} \frac{1}{\sigma(Q)}\int_Q M[\sigma 1_Q](x)^p w(x)\,dx\right) \le c_3^p l(Q_0)^\lambda, \quad \sigma = (bw)^{-p'/p},$$

holds for every cube $Q_0 \in \mathcal{Q}$;

(d) *There exists a constant $c_4 > 0$ such that, for some $a > 1$,*

$$\inf_{b \in \mathcal{B}_\lambda} \left(\sup_{\substack{Q \in \mathcal{Q} \\ Q \subset Q_0}} \frac{w(Q)}{|Q|} \left(⨍_Q [b(x) w(x)]^{-ap'/p}\, dx \right)^{p/ap'} \right) \le c_4^p l(Q_0)^\lambda$$

holds for every cube $Q_0 \in \mathcal{Q}$.

Then,

(I) (a) *implies* (b) *with $c_2 \le C c_1$;*
(II) (b) *and* (c) *imply* (a) *with $c_1 \le C(c_2 + c_3)$;*
(III) (b) *and* (d) *imply* (a) *with $c_1 \le C(c_2 + c_4)$.*

From this proposition we have the following.

Proposition 4.2. *Let $1 < p < \infty$, $0 < \lambda < n$ and $w = |\cdot|^\alpha$ be a power weight. Then, the weighted inequality*

$$\|Mf\|_{L^{p,\lambda}(w)} \le C \|f\|_{L^{p,\lambda}(w)}$$

holds if and only if $\lambda - n \le \alpha < \lambda + (p-1)n$.

Proof. Assume that $\lambda - n \le \alpha < \lambda + (p-1)n$.

Proof of (b). We first evaluate

$$\frac{1}{|Q_0|} \|w^{1/p} 1_{Q_0}\|_{L^{p,\lambda}} \|w^{-1/p} 1_{Q_0}\|_{H^{p',\lambda}} \tag{4.1}$$

for

$$Q_0 = \left(c, c + \frac{d}{\sqrt{n}} \right) \times \left(0, \frac{d}{\sqrt{n}} \right)^{n-1} \subset \mathbf{R}^n, \quad c, d > 0.$$

(The restriction to such a Q_0 can be justified by symmetry of the problem.)

Suppose that $d \le c$. Let $0 < \lambda < \lambda_0 < n$ and set

$$b_1(x) = \frac{C}{l(Q_0)^\lambda} M[1_{Q_0}](x)^{\lambda_0/n}.$$

Then, we see that b_1 belongs to $\mathcal{B}_\lambda$ (see Section 2). This implies

$$\|w^{-1/p} 1_{Q_0}\|_{H^{p',\lambda}} \le \left(\int_{Q_0} [|x|^\alpha b_1(x)]^{-p'/p}\, dx \right)^{1/p'} \le C |Q_0|^{1/p'} l(Q_0)^{\lambda/p} \sup_{x \in Q_0} |x|^{-\alpha/p}.$$

While,

$$\|w^{1/p} 1_{Q_0}\|_{L^{p,\lambda}} \le \|1_{Q_0}\|_{L^{p,\lambda}} \sup_{x \in Q_0} |x|^{\alpha/p} = |Q_0|^{1/p} l(Q_0)^{-\lambda/p} \sup_{x \in Q_0} |x|^{\alpha/p}.$$

These yield

$$(4.1) \le C \left(\frac{\sup_{x \in Q_0} |x|}{\inf_{x \in Q_0} |x|} \right)^{|\alpha|/p} \le C \left(\frac{c+d}{c} \right)^{|\alpha|/p} \le C.$$

Suppose that $d > c$. Let $B = \{x \in \mathbf{R}^n \colon |x| < 2d\}$. Take $\lambda_1 > 0$ so that $\alpha < \lambda_1 + (p-1)n < \lambda + (p-1)n$. Set

$$b_2(x) = \frac{\lambda/\lambda_1 - 1}{\lambda/\lambda_1} (4d)^{\lambda_1 - \lambda} |x|^{-\lambda_1} 1_B(x).$$

Then, we see that $|x|^{-\lambda_1} \in A_1$ and that $\int_{\mathbf{R}^n} b_2\, dH^\lambda = 1$. Indeed,

$$\int_{\mathbf{R}^n} |\cdot|^{-\lambda_1} 1_B\, dH^\lambda = (4d)^{\lambda-\lambda_1} + \int_{(4d)^{-\lambda_1}}^{\infty} t^{-\lambda/\lambda_1}\, dt$$
$$= (4d)^{\lambda-\lambda_1} + \frac{1}{\lambda/\lambda_1 - 1}(4d)^{\lambda-\lambda_1} = \frac{\lambda/\lambda_1}{\lambda/\lambda_1 - 1}(4d)^{\lambda-\lambda_1}.$$

Thus, we obtain

$$\begin{aligned}(4.1) &\le \frac{C}{|B|}\|w^{1/p}1_B\|_{L^{p,\lambda}}\|w^{-1/p}1_B\|_{H^{p',\lambda}} \\ &\le \frac{C}{|B|}\left(\int_B (4d)^{-\lambda}|x|^\alpha\, dx\right)^{1/p}\left(\int_B [|x|^\alpha b_2(x)]^{-p'/p}\, dx\right)^{1/p'} \\ &= \frac{C}{|B|}(4d)^{-\lambda_1/p}\left(\int_B |x|^\alpha\, dx\right)^{1/p}\left(\int_B |x|^{\frac{\lambda_1-\alpha}{p-1}}\, dx\right)^{1/p'} \le C,\end{aligned}$$

where we have used $0 < \lambda \le \alpha + n$ and $0 < (\lambda_1 - \alpha)/(p-1) + n$.

Proof of (d). Next, we evaluate

$$\inf_{b\in\mathcal{B}_\lambda}\left(\frac{1}{l(Q_0)^\lambda}\sup_{\substack{Q\in\mathcal{Q}\\ Q\subset Q_0}}\frac{w(Q)}{|Q|}\left(⨍_Q [b(x)w(x)]^{-ap'/p}\, dx\right)^{p/ap'}\right). \tag{4.2}$$

When $d \le c$, the same estimates of (4.1) are available to those of (4.2). Indeed, for an any cube $Q \subset Q_0$, we have

$$\left(⨍_Q [b_1(x)w(x)]^{-ap'/p}\, dx\right)^{p/ap'} \le Cl(Q_0)^\lambda \sup_{x\in Q_0}|x|^{-\alpha}$$

and

$$\frac{w(Q)}{|Q|} \le \sup_{x\in Q_0}|x|^\alpha.$$

These yield

$$(4.2) \le C\left(\frac{\sup_{x\in Q_0}|x|}{\inf_{x\in Q_0}|x|}\right)^{|\alpha|} \le C\left(\frac{c+d}{c}\right)^{|\alpha|} \le C.$$

When $d > c$ and $\dfrac{d_0}{c_0} := \dfrac{\sup_{x\in Q}|x|}{\inf_{x\in Q}|x|} \le 2$,

$$\begin{aligned}&\frac{1}{l(Q_0)^\lambda}\frac{w(Q)}{|Q|}\left(⨍_Q [b_2(x)w(x)]^{-ap'/p}\, dx\right)^{p/ap'} \\ &\le \frac{C}{(4d)^{\lambda_1}}\sup_{x\in Q}|x|^\alpha \sup_{x\in Q}|x|^{\lambda_1-\alpha} \le C\frac{d_0^{\lambda_1}}{(4d)^{\lambda_1}}\left(\frac{d_0}{c_0}\right)^{|\alpha|} \le C.\end{aligned}$$

When $d > c$ and $\dfrac{d_0}{c_0} := \dfrac{\sup_{x\in Q}|x|}{\inf_{x\in Q}|x|} > 2$, the same estimates of (4.1) are available too. Indeed, it follows by letting $B_0 = \{x \in \mathbf{R}^n\colon |x| < d_0\}$ that

$$\begin{aligned}&\frac{1}{l(Q_0)^\lambda}\frac{w(Q)}{|Q|}\left(\fint_Q [b_2(x)w(x)]^{-ap'/p}\,dx\right)^{p/ap'}\\&\le \frac{C}{(4d)^{\lambda_1}}\fint_{B_0}|x|^\alpha\,dx\left(\fint_{B_0}[|x|^{\alpha-\lambda_1}]^{-ap'/p}\,dx\right)^{p/ap'}\\&= \frac{C}{(4d)^{\lambda_1}}\fint_{B_0}|x|^\alpha\,dx\left(\fint_{B_0}|x|^{a\frac{\lambda_1-\alpha}{p-1}}\,dx\right)^{(p-1)/a}\le C\frac{d_0^{\lambda_1}}{(4d)^{\lambda_1}}\le C,\end{aligned}$$

where we have used $0 < \lambda \le \alpha + n$ and $0 < a(\lambda_1 - \alpha)/(p-1) + n$.

Disproof of (b). Finally, if $\alpha < \lambda - n$, then

$$\lim_{r\to+0}\frac{1}{r^\lambda}\int_{\{|y|<r\}}|x|^\alpha\,dx = \infty,$$

which implies

$$\|w^{1/p}1_{(-1,1)^n}\|_{L^{p,\lambda}} = \infty.$$

Suppose that $\alpha \ge \lambda + (p-1)n$. Notice that $|x|^{(\lambda-n)/p} \in L^{p,\lambda}(\mathbf{R}^n)$ with the norm less than C. This implies by Lemma 2.4

$$\|w^{-1/p}1_{(-1,1)^n}\|_{H^{p',\lambda}} \ge C\int_{(-1,1)^n}|x|^{(\lambda-\alpha-n)/p}\,dx = \infty,$$

where we have used $-n \ge (\lambda - \alpha - n)/p$.

Conclusion. Thus, Proposition 4.2 holds by (I) and (III) of Proposition 4.1. □

5. Appendix

As an appendix, we shall show the following two-weight norm inequality in the upper triangle case $0 < q < p < \infty$, $1 < p < \infty$.

Proposition 5.1. *Let $0 < q < p < \infty$, $1 < p < \infty$ and u, v be weights. Suppose that $v \in A_1$. Then, the weighted inequality*

$$\|Mf\|_{L^q(u)} \le C\|f\|_{L^p(v^{1-p})} \tag{5.1}$$

holds if and only if

$$\|u^{1/q}v^{1/p'}\|_{L^r} < \infty, \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{p}. \tag{5.2}$$

Proof. In the same manner as in the proof of Theorem 3.1, we may assume that f is non-negative and M is the dyadic maximal operator.

Suppose that (5.2) holds. We have for every $Q \in \mathcal{D}$,

$$\begin{aligned}\fint_Q f(y)\,dy &= \frac{v(Q)}{|Q|}\fint_Q f(y)v(y)^{-1}\,dv(y)\\&\le M[v](x)M_v[fv^{-1}](x) \le Cv(x)M_v[fv^{-1}](x), \quad x \in Q,\end{aligned}$$

where we have used $v \in A_1$. This implies

$$Mf(x) \le Cv(x)M_v[fv^{-1}](x), \quad x \in \mathbf{R}^n.$$

Thus,

$$\left(\int_{\mathbf{R}^n} Mf(x)^q u(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbf{R}^n} v(x)^q M_v[fv^{-1}](x)^q u(x)\,dx\right)^{1/q}$$
$$= C\left(\int_{\mathbf{R}^n} v(x)^{q-1}u(x)\cdot M_v[fv^{-1}](x)^q\,dv(x)\right)^{1/q}.$$

From Hölder's inequality with the exponent $(p-q)/p + q/p = 1$ and the fact that $1/r = (p-q)/pq$,

$$\le C\left(\int_{\mathbf{R}^n} \left(v(x)^{q-1}u(x)\right)^{p/(p-q)}\,dv(x)\right)^{1/r}\left(\int_{\mathbf{R}^n} M_v[fv^{-1}](x)^p\,dv(x)\right)^{1/p}$$
$$\le C\left(\int_{\mathbf{R}^n} \left(u(x)^{1/q}v(x)^{1-1/q+1/r}\right)^r\,dx\right)^{1/r}\left(\int_{\mathbf{R}^n} f(x)^p v(x)^{1-p}\,dx\right)^{1/p}$$
$$\le C\|u^{1/q}v^{1/p'}\|_{L^r}\|f\|_{L^p(v^{1-p})},$$

where we have used Lemma 3.2.

Suppose that (5.1) holds. Notice that $q/p + q/r = 1$. Keeping this in mind, we evaluate

$$\int_{\mathbf{R}^n} g(x)v(x)^{q/p'}u(x)\,dx \tag{5.3}$$

with a non-negative function g which satisfies $\|g\|_{L^{p/q}} \le 1$.

It follows from (5.1) that

$$(5.3) = \int_{\mathbf{R}^n} [g(x)^{1/q}v(x)^{1/p'}]^q u(x)\,dx \le \int_{\mathbf{R}^n} M[g^{1/q}v^{1/p'}](x)^q u(x)\,dx$$
$$\le C\left(\int_{\mathbf{R}^n} g(x)^{p/q}v(x)^{p/p'}v(x)^{1-p}\,dx\right)^{q/p} = C\left(\int_{\mathbf{R}^n} g(x)^{p/q}\,dx\right)^{q/p} \le C.$$

This yields (5.2). □

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