Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 40, 2015, 851–874

p-TRANSFINITE DIAMETER AND p-CHEBYSHEV CONSTANT IN LOCALLY COMPACT SPACES

Ágota P. Horváth

Budapest University of Technology and Economics, Department of Analysis
Műegyetem rakpart 3, H-1521 Budapest, Hungary; g.horvath.agota@renyi.mta.hu

Abstract. We extend the notion of transfinite diameter and Chebyshev constant to p-potential theory in locally compact spaces and study their relations. As in the classical case, it turns out that provided that the kernel satisfies a certain condition, for any compact sets the energy, the Chebyshev constant and the transfinite diameter are coincide. The investigations follow the linear method developed by e.g. Choquet, Fuglede, Ohtsuka, Farkas and Nagy. Taking into consideration the significance of finite sets of the minimal and almost minimal energy, we examine Fekete and greedy energy sets as well.

1. Introduction

The starting point is a compact set $K \subset \mathbf{R}^3$, and the set of Radon measures supported on K. Taking no notice of its physical meaning, the potential and the energy with respect to μ can be given as

$$U(\mu, x) = \int_{\mathbf{R}^3} \frac{d\mu(y)}{\|x-y\|}, \quad E(\mu) = \int_{\mathbf{R}^3} U(\mu, x) d\mu(x).$$

The problem was finding a measure which minimizes the energy. In 1839 Gauss realized (cf. [1] and the reference therein) that there is a measure μ_0 supported on K, which satisfies

- $\mu_0(K) = \mu(K)$,
- $E(\mu_0) \le E(\mu)$,
- $U(\mu_0, x) =$ constant $=W$ for $x \in K$,
- $U(\mu_0, x) \le W$ for $x \in \mathbf{R}^3 \setminus K$.

On the fundaments of classical theory based the abstract linear potential theory, where $\mathbf{R}^3$ is replaced by some locally compact space X, and the Newtonian kernel by some lower semicontinuos kernel function, $k(x,y)\colon X \times X \to \mathbf{R} \cup \{\infty\}$. This theory is developed by Choquet [5], Fuglede [11], Ohtsuka [19], Yamasaki [24], Carleson [4] and Landkof [16], etc.

Two types of normalization can be found in the literature, normalization with respect to μ (i.e. $\mu(K) = 1$) and with respect to W (i.e. $W = 1$). The second one leads to nonlinear potential theory. While the linear capacity of a set is $C(E) = W(E)^{-1}$, the p-capacity can be defined (cf. [1]) as

$$C_p(E) = \inf \left\{ \int_X f^p \, d\nu \colon f \ge 0,\ f \in L_{\nu,p},\ \int_X k(x,y) f(y)\, d\nu(y) \ge 1\ \forall x \in E \right\},$$

doi:10.5186/aasfm.2015.4047
2010 Mathematics Subject Classification: Primary 31C15, 31C45.
Key words: p-energy, p-transfinite diameter, p-Chebyshev constant, greedy energy points.
Supported by Hungarian National Foundation for Scientific Research, Grant No. K-100461.

where ν is a fixed measure on X. Although there are several further generalizations of nonlinear capacity e.g. for Sobolev and Besov spaces cf. e.g. the works of Hajłasz [13], Costea [7] or the monograph of A. Björn and J. Björn [2]; our investigations are concentrated to the above-mentioned nonlinear capacity.

In the theory of classical potentials there is a nice cluster point where the theory of polynomials and thence the approximation and interpolation theory is connected with potentials and this relationship has a wide importance in practical senses. Namely supposing a certain condition on the kernel, the transfinite diameter, the Chebyshev constant and the capacity of a compact set are coincide. Actually, these notions are the finite-set versions of potential and energy. After the classical investigations of e.g. Fekete [9] and Siciak [22], at the end of the '90-s several applications were inspirated by the monograph of Saff and Totik on logarithmic potentials with external fields [21]. Transfinite diameter and Chebyshev constant in locally compact spaces were examined by Farkas and Nagy [8]. These investigations gave chance of defining some greedy energy points which are asymptotically as good as the minimal energy—or Fekete points, but less difficult to compute them. The power of this discretization method is well illustrated by the applications in metric spaces, for instance computing Hausdorff measures (cf. [12], [3]). It was also pointed out that Fekete sets are optimal in point of view of interpolation (cf. e.g. [21], [15]). The aim of the investigations below is to get something similar in the nonlinear case.

The classical transfinite diameter and Chebyshev constant are linear expressions, that is the first power of a log-polynomial (see below) plays role in them. To avoid using p'^{th} power of log-polynomials, an equivalent form of C_p-capacity will be used. In [1] the following is given: Let X be a measure space equipped with a measure ν, $k(x,y)\colon \mathbf{R}^n \times X \to \mathbf{R} \cup \{\infty\}$ a kernel which is lower semicontinuous on $\mathbf{R}^n$ and measurable on X, μ is a Radon measure on $\mathbf{R}^n$, f is a ν-measurable nonnegative function,

$$\mathcal{E}(\mu, f) = \int_{\mathbf{R}^n} \int_X k(x,y) f(y)\, d\nu(y)\, d\mu(x).$$

By a minimax theorem it can be proved, that for a compact set $K \subset \mathbf{R}^n$

$$C_p(K)^{-\frac{1}{p}} = \sup_{f \in L} \min_{\mu \in \mathcal{M}(K)} \mathcal{E}(\mu, f) = \min_{\mu \in \mathcal{M}(K)} \sup_{f \in L} \mathcal{E}(\mu, f),$$

where $\mathcal{M}(K) = \{\mu \text{ is Radon measure on } K\colon \mu(K) = 1\}$, $L = \{f \geq 0\colon \|f\|_{\nu,p} \leq 1\}$. This expression allows to handle the p-energy like the linear one. By the lower semicontinuity of the kernel, the expression of energy shows that the n-point systems on which the energy is close to the optimal one may consist of only one point, which is very unpleasant in point of interpolation for instance. Indeed taking only the first part of $d(X_n, f)$ (see Definition 1), one can choose always the (almost) minimun point of $F(x) = \int k(x,y) f(y)\, d\nu(y)$. On the other hand, the physical meaning of transfinite diameter also requires n-point systems, where the mutual distance of the points are as large as it is possible.

In the examinations below, we extend the notion of transfinite diameter and Chebyshev constant to the nonlinear case. On behalf of dispersion of minimal energy points we will take into consideration the convex combination of linear and nonlinear energy. If the kernel is infinite at the diagonal, the common infinum will ensure the dispersion of the points. Since in the linear case the two variables of the kernel are in the same space, in the nonlinear part it has to be assumed the same. In the second section the transfinite diameter and (Wiener-type) energy are defined and besides the equivalence, some properties of the energy is studied. These considerations yield non-symmetric kernels which generate some difficulties in connection with the definition of potential which is given in the third section. At first it is given and examined on "one level", with respect to a measurable function f, and then independently of any functions. The properties of the equilibrium potential are also given. The definition of potential gives the possibility of defining Chebysev constant and greedy energy points. These investigations can be found in Sections 3 and 4. In the last section we make some observations with respect to symmetry and examine the behavior of the notions introduced in this note, when the convex combination approaches one of the endpoints.

2. Transfinite diameter and capacity

In this section the frame of our investigations is given. After this the definitions of p-transfinite diameter and p-energy are introduced. Finally we compare these notions. In the classical (linear) case there are several results on comparison of transfinite diameter and capacity. Let us mention here only one example, for instance Szegő verified the equality of the transfinite diameter and the logarithmic capacity in 1924 [23].

Let X be a locally compact Hausdorff space equipped with a regular Borel measure ν, and let the kernel function $k\colon X \times X \longrightarrow \mathbf{R} \cup \{\infty\}$ be lower semicontinuous (l.s.c.) in Fuglede's sense, symmetric, and nonnegative. Let $H \subset X$. At first we define a class of probability measures on H.

$$\mathcal{M}(H) := \{\mu\colon \mu \text{ is a regular Borel measure on } H,\ \mu \text{ has compact support } (\operatorname{supp}\mu \subset H),\ \mu(H) = 1\},$$

and we use the following class of ν-measurable functions:

$$L := \{f \geq 0\colon \int_X f(y)^p \, d\nu((y) \leq 1\}.$$

Following Fuglede's notation (cf. [11, p. 145]) the (upper) integral of a positive l.s.c. function g is defined as

$$\int_X g \, d\mu := \sup_{\substack{0 \leq h \leq g \\ h \in C_c(X)}} \int_X h \, d\mu,$$

where $C_c(X)$ is the set of continuous, compactly supported functions on X. If $g(x,y)$ is defined on $X \times X$, then $h \in C_c(X \times X)$, if $g(x,y)$ is symmetric, then it suffices to take only symmetric functions in the supremum.

We start the n^{th}-diameter of a set, which is the n-point-set version of the energy. At firs all the definitions are given "with respect to f" (where $f \in L$ is an arbitrary function), then they are given independently of f. The last one will be kept in focus.

Definition 1. Let H be a subset of X and $X_n = \{x_1, \ldots, x_n\} \subset H$ a system of nodes in H and $f \in L$ a nonnegative, ν-measurable function on X. Let $\lambda \in (0,1)$.

We define

$$\begin{aligned} d(X_n,f) &:= d_{k,\lambda}(X_n,f) \\ &= (1-\lambda)\int_X f(y)\frac{1}{n}\sum_{i=1}^n k(x_i,y)\,d\nu(y) + \lambda\frac{2}{n(n-1)}\sum_{1\le i<j\le n} k(x_i,x_j), \end{aligned}$$

and the n^{th}-diameter of H

$$d_n(H) := d_{n,k,p,\lambda}(H) = \sup_{f\in L}\inf_{X_n\subset H} d(X_n,f).$$

Notation. If K is a compact subset of X, then

$$\inf_{X_n\subset K} d(X_n,f) = \min_{X_n\subset K} d(X_n,f) = d(X_n^*,f),$$

and $X_n^* = X_{n,k,\lambda}^*(f)$ are the Fekete points of K with respect to f (cf. (3) below). Usually they are not unique.

Below the p-transfinite diameter of a set is introduced. First of all we show that $d_n(H)$ is increasing with n, and so it has a limit in the extended sense. Let $X_{l,n} = X_n \setminus \{x_l\}$. Now

$$\begin{aligned} \frac{1}{n}\sum_{l=1}^n d(X_{l,n},f) =& \frac{1-\lambda}{n}\int_X f(y)\sum_{l=1}^n \frac{1}{n-1}\sum_{\substack{1\le i\le n\\ i\ne l}} k(x_i,y)\,d\nu(y) \\ &+ \frac{\lambda}{n}\sum_{l=1}^n \frac{2}{(n-2)(n-1)}\sum_{\substack{1\le i<j\le n\\ i,j\ne l}} k(x_i,x_j). \end{aligned}$$

Each term $k(x_i,y)$ occurs exactly $n-1$ times, and each term $k(x_i,x_j)$ occurs exactly $n-2$ times. That is choosing arbitrarily an $X_n \subset H$

$$d(X_n,f) = \frac{1}{n}\sum_{l=1}^n d(X_{l,n},f) \ge \inf_{X_{n-1}\subset H} d(X_{n-1},f). \tag{1}$$

Taking infinum in $X_n \subset H$ and supremum in $f \in L$, we get

$$d_n(H) \ge d_{n-1}(H).$$

So we can define the p-transfinite diameter of a set H:

Definition 2.

$$d(H) := d_{k,p,\lambda}(H) = \lim_{n\to\infty} d_n(H).$$

Thereafter the p-energy of a set, that is to say the infinite-set version of the transfinite diameter is defined. To this purpose we introduce some notations.

Notation. Let $f \in L$ and $\mu \in \mathcal{M}(H)$. Denote by

$$\mathcal{E}(\mu,f) := \mathcal{E}_k(\mu,f) = \int_X f(y)\int_X k(x,y)\,d\mu(x)\,d\nu(y),$$

$$I(\mu) := I_k(\mu) = \int_X\int_X k(x,y)\,d\mu(x)\,d\mu(y),$$

where by Fubini's theorem the order of integration can be changed (cf. [11, p. 147, (ii)]).

Let $\lambda \in [0,1]$, and let

$$E(\mu, f) = E_{k,\lambda}(\mu, f) = (1-\lambda)\mathcal{E}(\mu, f) + \lambda I(\mu)$$

the mutual energy of f and μ.

$$\epsilon(f, H) := \epsilon_{k,\lambda}(f, H) = \inf_{\mu \in \mathcal{M}(H)} E(\mu, f),$$
$$\epsilon(\mu) := \epsilon_{k,p,\lambda}(\mu) = \sup_{f \in L} E(\mu, f) = (1-\lambda)\|\mathcal{G}\mu(y)\|_{\nu,p'} + \lambda I(\mu),$$

where

$$\mathcal{G}\mu(y) = \int_X k(x,y)\, d\mu(x), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

cf. [1]. The p-energy of a set H can be given as

Definition 3.

$$\tilde{W}(H) := \tilde{W}_{k,p,\lambda}(H) = \sup_{f \in L} \inf_{\mu \in \mathcal{M}(H)} E(\mu, f) = \sup_{f \in L} \epsilon(f, H),$$

$$W(H) := W_{k,p,\lambda}(H) = \inf_{\mu \in \mathcal{M}(H)} \sup_{f \in L} E(\mu, f) = \inf_{\mu \in \mathcal{M}(H)} \epsilon(\mu).$$

It is easy to see that $\tilde{W}(H) \le W(H)$ always. We show that they are equal for compact sets. To this end we need a lemma on lower semicontinuity of the functions above.

Lemma 1. *Let μ be a positive measure, and $f \in L$. Then the following functions are lower semicontinuous on X or in the weak*-topology:*

$$\mu \to I(\mu) \tag{2}$$

$$x \to \int_X k(x,y)\, d\mu(y), \quad x \to \int_X k(x,y) f(y)\, d\nu(y) \tag{3}$$

$$\mu \to \mathcal{E}(\mu, f) \tag{4}$$

$$\mu \to \int_X k(x,y)\, d\mu(y) \tag{5}$$

$$\mu \to \|\mathcal{G}\mu\|_{\nu,p'} \tag{6}$$

Proof. (2) is a result of Fuglede (cf. [11, 2.2.1 (e)]). The first statement of (3) can be find in [11] as well (p. 149), the second one follows from the first one by the replacement $d\nu^f(y) = f(y)\, d\nu(y)$. It has to be mentioned that in case of positive kernels the assumption of compact support can be omitted here. The proof of (4) and (5) is coincide with the proof of Prop. 2.3.2. (c) and (b) in [1], so they are omitted. Finally, since for all i and $f \in L$ $\mathcal{E}(\mu_i, f) \le \sup_{f \in L} \mathcal{E}(\mu_i, f)$, and so $\liminf_{i\to\infty} \mathcal{E}(\mu_i, f) \le \liminf_{i\to\infty} \sup_{f\in L} \mathcal{E}(\mu_i, f)$, $\forall f \in L$, we can take a supremum in f on the left hand-side, that is by the lower semicontinuity of $\mathcal{E}(\mu, f)$ in μ (i.e. (4)), when $\mu_i \xrightarrow{*} \mu$

$$\sup_{f\in L} \mathcal{E}(\mu, f) \le \sup_{f \in L} \liminf_{i\to\infty} \mathcal{E}(\mu_i, f) \le \liminf_{i\to\infty} \sup_{f\in L} \mathcal{E}(\mu_i, f),$$

which is (6) of Lemma 1. □

After these preliminaries we can state the following theorem.

Theorem 1. *If $H \subset X$ then*

$$d(H) \le \tilde{W}(H), \tag{7}$$

and for a compact set $K \subset X$

$$W(K) \le d(K). \tag{8}$$

Corollary. *For a compact set $K \subset X$, $\tilde{W}(K) = W(K)$.*

Proof of Theorem 1. Let $f \in L$. Since $\mu(X) = 1$ for all $\mu \in \mathcal{M}(H)$,

$$\begin{aligned}
\inf_{X_n \subset H} d(X_n, f) &\le \int_X \cdots \int_X d(X_n, f)\, d\mu(x_1) \cdots d\mu(x_n) \\
&= \frac{1-\lambda}{n} \sum_{i=1}^{n} \int_X \int_X f(y) k(x_i, y)\, d\nu(y)\, d\mu(x_i) \\
&\quad + \frac{2\lambda}{n(n-1)} \sum_{1 \le i < j \le n} \int_X \int_X k(x_i, x_j)\, d\mu(x_i)\, d\mu(x_j) \\
&= (1-\lambda)\mathcal{E}(\mu, f) + \lambda I(\mu).
\end{aligned}$$

Taking infinum in μ, and then supremum in $f \in L$, we have $d_n(H) \le \tilde{W}(H)$, hence $d(H) \le \tilde{W}(H)$.

In proving (8) we can assume that $d(K)$ is finite. For an $f \in L$ let us choose a Fekete point system with respect to f from K. Let $\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^*}$, where $\delta_{x_i^*}$ are the Dirac measures at the points of X_n^*. Let h be a continuous function with compact support such that $0 \le h \le k$. Now

$$\begin{aligned}
E_{h,\lambda}(\mu_n, f) &= d_{h,\lambda}(X_n^*, f) + \frac{\lambda}{n^2} \sum_{i=1}^{n} h(x_i^*, x_i^*) - \frac{2\lambda}{n^2(n-1)} \sum_{1 \le i < j \le n} h(x_i^*, x_j^*) \\
&\le \lambda \frac{\|h\|}{n} + d(X_n^*, f) \le \lambda \frac{\|h\|}{n} + d_n(K),
\end{aligned} \tag{9}$$

where $\|h\| = \sup_{x \in K} |h(x)|$. If n is large enough (i.e. for an arbitrary $\varepsilon > 0$, $n > n_0(\|h\|, \varepsilon)$, then

$$E_{h,\lambda}(\mu_n, f) \le d_{n,k,p,\lambda}(K) + \varepsilon \le d_{k,p,\lambda}(K) + \varepsilon.$$

By Banach–Alaoglu theorem, there is a cluster point μ of the set $\mathcal{M}_{n_0} := \{\mu_n \colon n > n_0\} \subset \mathcal{M}(K)$, which is weak*-compact. Taking into consideration the weak* convergence of the product measure we get

$$E_{h,\lambda}(\mu, f) \le d_{k,p,\lambda}(K) + \varepsilon,$$

thus

$$E_{k,\lambda}(\mu, f) \le d_{k,p,\lambda}(K) + \varepsilon.$$

Taking supremum in $f \in L$, and then minimum in $\mu \in \mathcal{M}(K)$, we get that $W(K) \le d(K) + \varepsilon$ for all $\varepsilon > 0$, which proves the statement. □

We are in position to define the (k, p, λ)-capacity of a set H.

Definition 4. For a set $H \subset X$, let

$$C(H) = C_{k,p,\lambda}(H) := W_{k,p,\lambda}^{-p}(H).$$

Below we make some observations with respect to (k, p)-capacity.

Theorem 2. *(k,p)-capacity is a set function with the following properties:*
(a) $C(\emptyset)=0$.
(b) *If $E_1\subset E_2$ are measurable sets, then $C(E_1)\le C(E_2)$.*
(c) *Let $\cdots\supset K_i\supset K_{i+1}\supset\cdots$ a decreasing sequence of compact sets. Then*

$$C\left(\cap_i K_i\right)=\lim_{i\to\infty}C(K_i).$$

(d) *Let $\cdots\subset B_i\subset B_{i+1}\subset\cdots$ an increasing sequence of measurable sets. Then*

$$C\left(\cup_i B_i\right)=\lim_{i\to\infty}C(B_i).$$

(e) *Let $E\subset X$ is measurable. Then*

$$C(E)=\sup\{C(K)\colon K\subset E,\ K\ \text{ is compact}\}.$$

Proof. By Definition 3, (a) and (b) are obvious.

(c) Because $\{K_i\}$ is a decreasing sequence of (compact) sets, $\{W(K_i)\}$ is increasing, so it has a limit: W, and it is clear that $\cap_i K_i=K\subset K_i$; $W(K)\ge W(K_i)$ $\forall i$, that is $W(K)\ge W$.

On the other hand (by Def. 3) for $\varepsilon>0$ arbitrary and for all i, there is a measure μ_i, $\operatorname{supp}\mu_i\subset K_i\subset K_1$, with $\mu_i(K_i)=\mu_i(K_1)=\mu_i(X)=1$ such that $W(K_i)+\varepsilon>\epsilon(\mu_i)$. So there is a weak*-convergent subsequence of $\{\mu_i\}$; $\mu_{i_l}\overset{*}{\to}\mu_0\in\mathcal{M}(K_1)$, and by the w^*-convergence $\operatorname{supp}\mu_0\subset\cap_l K_{i_l}=K$. Since $\epsilon(\mu_i)$ is l.s.c. (see (2) and (6)), we have for all $\varepsilon>0$,

$$\begin{aligned}W(K)-\varepsilon=\inf_{\mu\in\mathcal{M}(K)}\epsilon(\mu)-\varepsilon&\le\epsilon(\mu_0)-\varepsilon\\&\le\liminf_{l\to\infty}\epsilon(\mu_{i_l})-\varepsilon\le\liminf_{i\to\infty}W(K_i)\le W,\end{aligned}$$

which proves (c).

(d) As previously, monotonicity yields that $\lim_{i\to\infty}W(B_i):=W\ge W(B)$, where $\cup_i B_i=:B$. For an arbitrary $\varepsilon>0$ there is a $\mu\in\mathcal{M}(B)$ such that $W(B)+\varepsilon\ge\epsilon(\mu)$. Let $K\subset B$ is the (compact) support of μ. By regularity of μ, for all i, we can choose a compact set $K_i\subset B_i\cap K$ such that $\mu(K_i)\to1$. Let $\mu_i=\frac{\mu|_{K_i}}{\mu(K_i)}$. So $\mu_i\in\mathcal{M}(B_i\cap K)$ and because $k(x,y)\ge0$ and by monotonicity,

$$\begin{aligned}W(B_i)\le W(B_i\cap K)\le\epsilon(\mu_i)&=\frac{1}{\mu(K_i)}\left(\int_X\left(\int_{K_i}k(x,y)\,d\mu(x)\right)^{p'}d\nu(y)\right)^{\frac{1}{p'}}\\&+\frac{1}{(\mu(K_i))^2}\int_{K_i}\int_{K_i}k(x,y)\,d\mu(y)\,d\mu(x)\le\frac{1}{(\mu(K_i))^2}\epsilon(\mu).\end{aligned}$$

So

$$\lim_{i\to\infty}W(B_i)\le\epsilon(\mu)\le W(B)+\varepsilon,$$

for all $\varepsilon>0$.

(e) As above, we have to show that for any measurable set $E\subset X$, $W(E)=\inf_{\substack{K\subset E\\ K\text{ is compact}}}W(K)=:W$. Again by monotonicity, it is clear that $W\ge W(E)$. To prove the opposite inequality, for $\varepsilon>0$ arbitrary let us choose a $\mu\in\mathcal{M}(E)$, such that $W(E)+\varepsilon\ge\epsilon(\mu)$, and an $f\in L$ arbitrary. According to [11] Lemma 2.2.2, and the Remark after the lemma, $\mathcal{E}(\mu,f)=\sup_{\substack{K\subset E\\ K\text{ is compact}}}\mathcal{E}(\mu|_K,f)$, and $I(\mu)=$

$\sup_{\substack{K\subset E\\ K \text{ is compact}}} I(\mu|_K)$. By regularity of the measure, we can choose a compact set $K_0 \subset E$ with $(\mu(K_0))^2 > 1-\varepsilon$. Thus

$$\begin{aligned} W(E)+\varepsilon &\geq \epsilon(\mu) \geq E(\mu,f) \\ &\geq \sup_{\substack{K\subset E\\ K \text{ is compact}}} \left((1-\lambda)\mu(K)\mathcal{E}\left(\frac{\mu|_K}{\mu(K)},f\right)+\lambda(\mu(K))^2 I\left(\frac{\mu|_K}{\mu(K)}\right)\right) \\ &\geq (1-\varepsilon)E\left(\frac{\mu|_{K_0}}{\mu(K_0)},f\right). \end{aligned}$$

Taking supremum in f, we have

$$\begin{aligned} W(E)+\varepsilon \geq (1-\varepsilon)\epsilon\left(\frac{\mu|_{K_0}}{\mu(K_0)}\right) &\geq (1-\varepsilon)\inf_{\mu\in\mathcal{M}(K_0)}\epsilon(\mu) \\ = (1-\varepsilon)W(K_0) &\geq (1-\varepsilon)\inf_{\substack{K\subset E\\ K \text{ is compact}}} W(K) = (1-\varepsilon)W. \end{aligned}$$

Here ε was arbitrary, so the proof is complete. □

Corollary. *The (e) part of the previous theorem entails that for each measurable set $E\subset X$,*

$$W(E)=\tilde{W}(E).$$

3. Potential function and Chebyshev constant

In this section we define and study the p-potential function, and its discrete version the so-called log-polynomials. As in the previous section, we start an f-version of these notions, then we get rid of f. The mutual energy of μ and f can be expressed as

$$E(\mu,f)=\int_X\int_X\left((1-\lambda)\int_X k(x,y)f(y)\,d\nu(y)+\lambda k(x,y)\right)d\mu(y)\,d\mu(x).$$

Since the kernel is not symmetric, to our purposes we have to symmetrize it. This leads to a discussion similar to the linear case. Originally the n^{th} Chebyshev constant is the infinum of the supremum-norm on a (compact) set of a monic polynomial of degree n (cf. e.g. [21]). The log-polynomials (similarly to [8]) are the generalization of the negative logarithm of the modulus of polynomials, which were studied in connection with logarithmic capacity. Here and in the next section we have to assume the so-called "relative domination principle". Some discussions on relative domination principle, maximum principle and on symmetry can be found in the final section.

Let us begin with the definition of the potential function with respect to a measure, and a function.

Definition 5. Let $H\subset X$, and $\mu\in\mathcal{M}(H)$, $f\in L$. Then

$$\begin{aligned} U(\mu,f,x) &= U_{k,\lambda}(\mu,f,x) \\ &:= \frac{1-\lambda}{2}\left(\int_X\int_X k(x,y)f(y)\,d\nu(y)\,d\mu(x)+\int_X k(x,y)f(y)\,d\nu(y)\right) \\ &\quad +\lambda\int_X k(x,y)\,d\mu(y). \end{aligned}$$

Obviously

$$\int_X U(\mu, f, x) d\mu(x) = E(\mu, f), \tag{10}$$

and with a $\sigma \in \mathcal{M}(H)$ let us denote

$$\begin{aligned} E(\mu, \sigma, f) &:= \int_X U(\mu, f, x)\, d\sigma(x) \\ &= \frac{1-\lambda}{2}\left(\int_X \int_X k(x,y) f(y)\, d\nu(y)\, d\mu(x) + \int_X \int_X k(x,y) f(y)\, d\nu(y)\, d\sigma(x)\right) \\ &\quad + \lambda \int_X \int_X k(x,y)\, d\mu(y)\, d\sigma(x) = \int_X U(\sigma, f, x)\, d\mu(x). \end{aligned} \tag{11}$$

Below the equilibrium measure and potential with respect to an $f \in L$ is constructed. After the suitable definitions, we can follow the chain of ideas of Fuglede.

Notation. (a) A property P is said to fulfil f-nearly everywhere (f-n.e.) on H, if denoting by

$$N := \{x \in H \colon P \text{ does not fulfil in } x\}, \quad \inf_{\mu \in \mathcal{M}(N)} E(\mu, f) = \infty.$$

(b) $H \subset X$,

$$\mu_*(H) = \sup_{\substack{K \subset H \\ K \text{ is compact}}} \mu(K)$$

is the inner measure of H.

Our main tool is the next statement.

Statement 1. *Let $f \in L$ fixed, $\mu \in \mathcal{M}(X)$, $H \subset X$, $0 \le t \le \infty$. Then the following conditions are equivalent:*

$$U(\mu, f, x) \ge t \quad f\text{-n.e. } x \in H. \tag{12}$$

$$E(\mu, \sigma, f) \ge t \quad \forall \sigma \in \mathcal{M}(H),\ E(\sigma, f) < \infty. \tag{13}$$

In order to prove it, we need the following equivalence (cf. [11, Lemma 2.3.1]):

Lemma 2. *Let $H \subset X$, $f \in L$. The following conditions are equivalent:*

(14) $\epsilon(f, H) = \infty$.

(15) $\epsilon(f, K) = \infty,\ \forall K \subset H,\ K$ *is compact.*

(16) $\mu_*(H) = 0,\ \forall \mu$ *positive measure on* $X,\ E(\mu, f) < \infty$.

(17) $\mu = 0$ *is the only positive measure with* $\operatorname{supp} \mu \subset H,\ E(\mu, f) < \infty$.

(18) $\mu = 0$ *is the only positive measure with* $\operatorname{supp} \mu \subset K \subset H$, K *is compact*, $E(\mu, f) < \infty$.

Proof. (14) $\Rightarrow$ (15) is obvious: if $\epsilon(f, K) < \infty$ for a $K \subset H$, then $\epsilon(f, H) < \infty$, because $\mathcal{M}(K) \subset \mathcal{M}(H)$. (15) implies that $\mu(K) = 0$ for all $K \subset H$, K is compact. Since μ is regular, (16) $\Rightarrow$ (17) and (17) $\Rightarrow$ (18) are obvious. For the proof of (18) $\Rightarrow$ (14), we have to show that if (18) is valid then for all $\mu \in \mathcal{M}(H)$, $E(\mu, f) = \infty$. Since μ is regular and k and f are positive, and by (18) for all compact sets $K \subset H$, and for all $0 \ne \mu$ positive measures, $\operatorname{supp} \mu \subset K$, $E(\mu, f) = \infty$, that is (14) is satisfied. □

Proof of Statement 1. Let $\sigma \in \mathcal{M}(H)$, with $E(\sigma, f) < \infty$, and $N_{f,\sigma} := \{x \in \operatorname{supp}\sigma \colon U(\mu, f, x) < t\} \subset N_f = \{x \in H \colon U(\mu, f, x) < t\}$. So $\epsilon(f, N_{f,\sigma}) \geq \epsilon(f, N_f) = \infty$ by (12), and by (14) $\sigma_*(N_{f,\sigma}) = 0$. Since $N_{f,\sigma}$ is measurable, $\sigma(N_{f,\sigma}) = 0$. Hence

$$\begin{aligned} E(\mu, \sigma, f) = &\int_{N_{f,\sigma}} U(\mu, f, x)\, d\sigma(x) \\ &+ \int_{\operatorname{supp}\sigma \setminus N_{f,\sigma}} U(\mu, f, x)\, d\sigma(x) \geq 0 + t\sigma(\operatorname{supp}\sigma \setminus N_{f,\sigma}) = t. \end{aligned}$$

Let us suppose that (12) doesn't fulfil. Then $\epsilon(f, N_f) < \infty$, that is there is a $\sigma \in \mathcal{M}(N_f)$ such that $E(\sigma, f) < \infty$. With this σ, $E(\mu, \sigma, f) < t$ which is a contradiction. □

The next lemma states the existence of an equilibrium measure with respect to f on a compact set K.

Lemma 3. *For all $f \in L$, and $K \subset X$ compact, there is an extremal measure $\mu_f := \mu_{f,K} \in \mathcal{M}(K)$, such that $\epsilon(f, K) = E(\mu_f, f)$.*

Proof. Let us choose a sequence $\{\mu_n\} \subset \mathcal{M}(K)$, such that $E(\mu_n, f) \to \epsilon(f, K)$. As in Theorem 1, one can choose a w^*-convergent subsequence, denoting by $\{\mu_n\}$ again, and so $\mu_n \overset{*}{\to} \mu_f \in \mathcal{M}(K)$. By (2) and (4)

$$E(\mu_f, f) \leq \liminf_{n\to\infty} (1-\lambda)\mathcal{E}(\mu_n, f) + \liminf_{n\to\infty} \lambda I(\mu_n) \leq \liminf_{n\to\infty} E(\mu_n, f) = \epsilon(f, K). \quad \square$$

Now we are in position to draft the theorem on equilibrium potential with respect to f.

Theorem 3. *Let $f \in L$, K be a compact set in X such that $\epsilon(f, K) < \infty$, and μ_f is the equilibrium measure on K with respect to f. Then*

$$U(\mu_f, f, x) \geq \epsilon(f, K) \quad f\text{-n.e. } x \in K. \tag{19}$$

$$U(\mu_f, f, x) \leq \epsilon(f, K) \quad \forall x \in \operatorname{supp}\mu_f. \tag{20}$$

$$U(\mu_f, f, x) = \epsilon(f, K) \quad \mu_f \text{ a.e. } x \in X. \tag{21}$$

Proof. Let $\sigma \in \mathcal{M}(K)$, and let us suppose that $E(\sigma, f) < \infty$. Then for all $\delta \in [0, 1]$ $\mu(\delta) = (1-\delta)\mu_f + \delta\sigma \in \mathcal{M}(K)$. Let us define a function of δ: $F(\delta) = E(\mu(\delta), f)$. $F(\delta)$ has a minimum at $\delta = 0$. It is clear that $F(\delta)$ is differentiable and 0 has a right neighborhood, where $F'(\delta)$ is nonnegative. Computing this we get:

$$U(\mu(\delta), f, x) = (1-\delta)U(\mu_f, f, x) + \delta U(\sigma, f, x),$$

so

$$\begin{aligned} F(\delta) &= \int_X U(\mu(\delta), f, x)\, d\mu(\delta)(x) \\ &= (1-\delta)^2 E(\mu_f, f) + 2\delta(1-\delta)E(\mu_f, \sigma, f) + \delta^2 E(\sigma, f). \end{aligned}$$

So there is a δ_0, such that for all $0 \leq \delta \leq \delta_0$, then

$$F'(\delta) = -2(1-\delta)E(\mu_f, f) + 2(1-2\delta)E(\mu_f, \sigma, f) + 2\delta E(\sigma, f) \geq 0,$$

that is

$$(1-2\delta)E(\mu_f, \sigma, f) \geq (1-\delta)E(\mu_f, f) - \delta E(\sigma, f).$$

Recalling that $E(\sigma, f)$ is finite, and tending to zero with δ, we get that

$$E(\mu_f, \sigma, f) \geq \epsilon(f, K), \quad \forall \sigma \in \mathcal{M}(K),\ E(\sigma, f) < \infty,$$

which proves (19) by Statement 1.

Let us suppose contrary, that there is an $x_0 \in \operatorname{supp} \mu_f$, such that $U(\mu_f, f, x_0) > \epsilon(f, K)$. Since according to (3) $U(\mu_f, f, x)$ is l.s.c., there is a $\delta > 0$, and a neighborhood G of x_0, such that for all $x \in G \cap \operatorname{supp} \mu_f$, $U(\mu_f, f, x) \geq \epsilon(f, K) + \delta$. By (10)

$$\epsilon(f, K) = \int_G U(\mu_f, f, x)\, d\mu_f(x) + \int_{X \setminus G} U(\mu_f, f, x)\, d\mu_f(x).$$

Because $\epsilon(f, K) < \infty$, by (17) $\mu_f(N) = 0$ and so by (19)

$$\epsilon(f, K) > (\epsilon(f, K) + \delta)\mu_f(G) + \epsilon(f, K)\mu_f(X \setminus G),$$

that is $\mu_f(G) = 0$, and so $x_0 \notin \operatorname{supp} \mu_f$.

Again by the lower semicontinuity of $U(\mu_f, f, \cdot)$, (20) can be expressed as $U(\mu_f, f, x) \leq \epsilon(f, K)$ μ_f a.e. in X, which together with the previous chain of ideas gives (21). □

All the previous definitions and calculations of this section concerned expressions with respect to a fixed $f \in L$. In order to get something which is independent of f, let us take some further observations.

Let us recall that $\mathcal{G}\mu(y) := \int_X k(x, y)\, d\mu(x)$, and $\epsilon(\mu) = (1 - \lambda)\|\mathcal{G}\mu\|_{\nu,p'} + \lambda I(\mu)$. In this part $K \subset X$ is a compact set , with $W(K) < \infty$. Let

$$\mathcal{M}_c(K) := \mathcal{M}_c(p, K) = \{\mu \in \mathcal{M}(K) \colon \|\mathcal{G}\mu\|_{\nu,p'} \leq c\}.$$

At first we have to observe that if $W(K) < \infty$, then there is a c, e.g. $c = \frac{2}{1-\lambda} W(K)$, such that

$$\inf_{\mu \in \mathcal{M}_c(K)} \sup_{f \in L} E(\mu, f) = W(K). \tag{22}$$

Denoting by $\tilde{W}_c(K) := \sup_{f \in L} \inf_{\mu \in \mathcal{M}_c(K)} E(\mu, f)$ we have that if c is like in (22) then $\tilde{W}_c(K) = W(K)$. Indeed, taking into account that $\mathcal{M}_c(K) \subset \mathcal{M}(K)$, and "sup inf $\leq$ inf sup", and the finiteness of the energy of K, we have

$$\tilde{W}(K) \leq \tilde{W}_c(K) \leq \inf_{\mu \in \mathcal{M}_c(K)} \sup_{f \in L} E(\mu, f) = \inf_{\mu \in \mathcal{M}K)} \sup_{f \in L} E(\mu, f) = W(K) = \tilde{W}(K).$$

Let us denote by $\epsilon_c(f, K) := \inf_{\mu \in \mathcal{M}_c(K)} E(\mu, f)$. So there is a sequence of functions $\{f_n\} \subset L$ for which $\lim_{n \to \infty} \epsilon_c(f_n, K) := \tilde{W}(K)$. We can assume that $\{f_n\} \subset C_c(X)$. Since L is w^*-compact, $\{f_n\}$ has a a w^*-convergent subsequence. Let us denote this by $\{f_n\}$ again, that is $f_n \overset{*}{\to} f_e = f_e(K) \in L$. According to (6) $\mathcal{M}_c(K)$ is also w^*-compact, so as in Lemma 3. it can be proved that there are $\mu^c_{f_n} \in \mathcal{M}_c(K)$ such that $E(\mu^c_{f_n}, f_n) = \epsilon_c(f_n, K)$. Furthermore there is a w^*-convergent subsequence of $\{\mu^c_{f_n}\}$ (denoting by $\{\mu^c_{f_n}\}$ again) such that $\mu^c_{f_n} \overset{*}{\to} \mu^c_e \in \mathcal{M}_c(K)$. Let us denote its potential function by $U^c_e(K, x) := U(\mu^c_e, f_e, x)$. We will show, that $U^c_e(K, x)$ has the following property:

Lemma 4. *Let K be a compact set in X such that $W(K) < \infty$. If c has property (22), then*

$$\int_X U^c_e(K, x)\, d\mu^c_e(x) = W(K).$$

Proof. Recalling that $\int_X U_e^c(K,x)\,d\mu_e^c(x) = E(\mu_e^c, f_e)$ we have to show that $E(\mu_{f_n}^c, f_n) \to E(\mu_e^c, f_e)$. Let $\varepsilon > 0$ arbitrary. Then

$$E(\mu_e^c, f_e) \geq E(\mu_{f_e}^c, f_e) \geq E(\mu_{f_e}^c, f_n) - \varepsilon \geq E(\mu_{f_n}^c, f_n) - \varepsilon \geq W(K) - 2\varepsilon,$$

where the first and the third inequalities are valid by the definition of $\mu_{f_e}^c$ and $\mu_{f_n}^c$, and the second and fourth ones fulfil if n is large enough. On the other hand by the monotone convergence theorem for all $\varepsilon > 0$ there is a $0 \leq h \leq k, h \in C_c(X \times X)$ such that

$$E_k(\mu_e^c, f_e) \leq E_h(\mu_e^c, f_e) + \varepsilon \leq E_h(\mu_e^c, f) + 2\varepsilon,$$

where $f \in L \cap C_c(x)$ and $\|f - f_e\|_{\nu,p} \leq \varepsilon$ and the last inequality is ensured by $\| \int_X h(x,y) d\mu_e^c(x)\|_{\nu,p'} \leq \| \int_X k(x,y) d\mu_e^c(x)\|_{\nu,p'} \leq c$. Let us estimate

$$\begin{aligned}\left|E_h(\mu_e^c, f) - E_h(\mu_{f_n}^c, f_n)\right| &\leq \left|E_h(\mu_e^c, f) - E_h(\mu_{f_n}^c, f)\right| + \left|E_h(\mu_{f_n}^c, f) - E_h(\mu_{f_n}^c, f_n)\right| \\ &= I + II.\end{aligned}$$

Since h and f are continuous and $\mu_{f_n}^c \xrightarrow{*} \mu_e^c$, $I \leq \varepsilon$ if n is large enough.

$$\begin{aligned} II &= \int_X (f(y) - f_n(y)) \int_X h(x,y)\, d\mu_{f_n}^c(x)\, d\nu(y) \\ &\leq \int_X (f(y) - f_m(y)) \int_X h(x,y)\, d\mu_{f_n}^c(x)\, d\nu(y) \\ &\quad + \left(\int_X (f_m(y) - f_n(y)) \int_X h(x,y)\, d\mu_{f_n}^c(x)\, d\nu(y) \right. \\ &\quad \left. - \int_X (f_m(y) - f_n(y)) \int_X h(x,y)\, d\mu_e^c(x)\, d\nu(y) \right) \\ &\quad + \int_X (f_m(y) - f_n(y)) \int_X h(x,y)\, d\mu_e^c(x)\, d\nu(y) = III + IV + V. \end{aligned}$$

As previously, by the boundedness of the $L_{\nu,p'}$-norm of the integral of h, $V \leq \varepsilon$ if $n < m$ is large enough. By the choice of f and f_e,

$$\begin{aligned} III &\leq \left| \int_X (f(y) - f_e(y)) \int_X h(x,y)\, d\mu_{f_n}^c(x)\, d\nu(y) \right| \\ &\quad + \left| \int_X (f_e(y) - f_m(y)) \int_X h(x,y)\, d\mu_{f_n}^c(x)\, d\nu(y) \right| \leq 2c\varepsilon, \end{aligned}$$

where n is fixed (and large enough) and m is large enough. Also,

$$IV \leq (\|f_m\|_{\nu,p} + \|f_n\|_{\nu,p}) \left\| \left(\int_X h d\mu_{f_n}^c - \int_X h d\mu_e^c \right) \right\|_{\nu,p'},$$

which tends to zero by the dominated convergence theorem, when n tends to infinity. Finally

$$E_k(\mu_e^c, f_e) \leq E_h(\mu_{f_n}^c, f_n) + C\varepsilon \leq E_k(\mu_{f_n}^c, f_n) + C\varepsilon \leq W(K) + C\varepsilon,$$

if n is large enough. Since ε was arbitrary, comparing the two chains of inequalities the lemma is proved. □

Corollary. *K is a compact set in X such that $W(K) < \infty$. If c has property (22), then*

$$E(\mu_e^c, f_e) = E(\mu_{f_e}^c, f_e).$$

The following theorem is similar to Theorem 3. That method showed the importance of "law of reciprocity" (cf. [11, (11) and p. 150]). We give an other type of proof of the parallel theorem here. Usually this chain of ideas can be found in the literature.

Notation. A property P is said to fulfil nearly everywhere (n.e.) on K, if denoting by $N := \{x \in K\colon P \text{ does not fulfil in } x\}$, $\sup_{f\in L} \inf_{\mu\in\mathcal{M}(N)} E(\mu, f) = \tilde{W}(N) = \infty$.

Definition 6. Let us call a l.s.c. kernel function normal (with respect to ν), if for all $K \subset X$ compact, $W(K) < \infty$ there is a sequence of functions in L such that $\lim_{n\to\infty} \epsilon(f_n, K) = \tilde{W}(K)$, and $\{\mu_{f_n}\}$ has a subsequence $\{\mu_{f_{n_k}}\}$ such that $\liminf_{n\to\infty} \|\mathcal{G}\mu_{f_{n_k}}\|_{\nu,p'} < \infty$.

Remark. If $\nu(X)$ is finite and k is bounded, then k is obviously normal. If X is compact and $\mathcal{G}\mu$ is bounded for all $\mu \in \mathcal{M}(K)$, then k is normal again. This is the situation in the classical cases of logarithmic and Riesz potentials.

Now with the previous notations we can state the following

Theorem 4. *Let k be normal and K be a compact set in X such that $W(K) < \infty$. Then there is a $c > 0$ such that $U_e^c(K, x)$ is an equilibrium potential and μ_e^c is an equilibrium measure, that is*

$$U_e^c(K, x) \geq W(K) \quad \text{n.e. } x \in K. \tag{23}$$

$$U_e^c(K, x) \leq W(K) \quad \forall x \in \operatorname{supp} \mu_e^c. \tag{24}$$

$$U_e^c(K, x) = W(K) \quad \mu_e^c \text{ a.e. } x \in X. \tag{25}$$

Proof. Let $\{f_n\}$ be as in Definition 6, and $\{\mu_{f_{n_k}}\}$ is the subsequence in question. Let $\{\mu_{f_{n_{k_l}}}\}$ be a w^*-convergent subsequence of it, and let us denote its limit by μ_e. Then, by (6) $\infty > c_0 = \liminf_{n\to\infty} \|\mathcal{G}\mu_{f_{n_k}}\|_{\nu,p'} \geq \|\mathcal{G}\mu_e\|_{\nu,p'}$. We can assume (perhaps omitting some elements of the sequence) that $\|\mathcal{G}\mu_{f_{n_k}}\|_{\nu,p'} \leq 2c_0$. Now let us choose a w^*-convergent subsequence from $\{f_{n_{k_l}}\}$, ant let us denote it's limit by f_e. Let c_1 be a constant with the property (22). Let $c := \max\{2c_0, c_1\}$. Then $\mu_{f_{n_k}} = \mu_{f_{n_k}}^c$ for all k, and $\mu_e = \mu_e^c$, where μ_e is the limit of a w^*-convergent subsequence of $\{\mu_{f_{n_{k_l}}}\}$.

At first we will prove (23) with this c. Following standard arguments let $N := \{x \in K\colon U_e^c(K, x) < W(K)\}$, and $F_n := \left\{x \in K\colon U_e^c(K, x) \leq W(K) - \frac{1}{n}\right\}$. It is clear that F_n-s are compact (U_e^c is l.s.c.) and $\cup_n F_n = N$. Let us suppose indirectly that $\tilde{W}(N) < \infty$. Thence (cf. (c) of Theorem 2) there is an n_0 such that $\forall n > n_0$ $\tilde{W}(F_n) < \infty$. Let us choose an $n > n_0$, and let us define a set $E := \left\{x \in \operatorname{supp} \mu_e^c\colon U_e^c(K, x) \geq W(K) - \frac{1}{2n}\right\}$. Obviously $E \cap F_n = \emptyset$ and $m := \mu_e^c(E) > 0$. By the indirect assumption there is a measure σ, $\operatorname{supp} \sigma \subset F_n$ and $\epsilon(\sigma) < \infty$. Let $\sigma(F_n) = m$, and let us define a signed measure δ, like that $\delta = \sigma$ on F_n, $\delta = -\mu_e^c$ on E, and $\delta = 0$ elsewhere. Let $\mu_\eta = \mu_e^c + \eta\delta \in \mathcal{M}(K)$. We will compute the energy of μ_η with respect to f_e.

$$E(\mu_\eta, f_e) = \int_X U(\mu_\eta, f_e), x)\, d\mu_\eta(x) = (1-\lambda)\int_X\int_X k(x,y) f_e(y)\, d\nu(y)\, d\mu_e^c(x)$$
$$+ \eta(1-\lambda)\int_X\int_X k(x,y) f_e(y)\, d\nu(y)\, d\delta(x) + 2\eta\lambda \int_X\int_X k(x,y)\, d\mu_e^c(y)\, d\delta(x)$$
$$+ \lambda\int_X\int_X k(x,y)\, d\mu_e^c(y)\, d\mu_e^c(x) + \lambda\eta^2 I(\delta) \le W(K) + C\eta^2$$
$$+ \eta\left((1-\lambda)\int_X\int_X k(x,y) f_e(y)\, d\nu(y)\, d\delta(x) + 2\lambda\int_X\int_X k(x,y)\, d\mu_e^c(y)\, d\delta(x)\right),$$

with some positive constant C, where we used that by the finiteness of $I(\mu_e^c)$ and $I(\sigma)$ ($\epsilon(\sigma) < \infty$), $I(\delta) < \infty$. The expression in the bracket is

$$2\left(\frac{1-\lambda}{2}\int_{F_n}\int_X k(x,y) f_e(y)\, d\nu(y) d\sigma(x) + \lambda\int_{F_n}\int_X k(x,y)\, d\mu_e^c(y)\, d\sigma(x)\right.$$
$$\left. - \left(\frac{1-\lambda}{2}\int_E\int_X k(x,y) f_e(y)\, d\nu(y)\, d\mu_e^c(x) + \lambda\int_E\int_X k(x,y)\, d\mu_e^c(y)\, d\mu_e^c(x)\right)\right)$$
$$= 2\int_{F_n} U_e^c(K,x)\, d\sigma(x) - (1-\lambda)\int_X\int_X k(x,y) f_e(y)\, d\nu(y)\, d\mu_e^c(x)$$
$$- \left(2\int_E U_e^c(K,x)\, d\mu_e^c(x) - (1-\lambda)\int_X\int_X k(x,y) f_e(y)\, d\nu(y)\, d\mu_e^c(x)\right)$$
$$\le 2\left(m\left(W(K) - \frac{1}{n}\right) - m\left(W(K) - \frac{1}{2n}\right)\right) < 0.$$

So if η is small enough, then $E(\mu_\eta, f_e) < E(\mu_e^c, f_e) = E(\mu_{f_e}^c, f_e)$. Since $\|\mathcal{G}\mu_e^c\|_{\nu,p'} \le c_0$, recalling that $\epsilon(\sigma), \epsilon(\mu_e^c)$ are finite, if η is small enough, $\mu_\eta \in \mathcal{M}_c$, which leads to a contradiction.

The remainder part of the theorem can be proved as in Theorem 3, the only difference is that we have to show that if $\tilde{W}(N) = \infty$, then $\mu_e^c(N) = 0$. Indeed μ_e^c is a positive measure with $\epsilon(\mu_e^c) < \infty$, because $\|\mathcal{G}\mu_e^c\|_{\nu,p'} < \infty$ and since $W(K) = E(\mu_e^c, f_e) < \infty$, $I(\mu_e^c) < \infty$. Since N is measurable, by the corollary after Theorem 2 $\tilde{W}(N) = \inf_{\mu\in\mathcal{M}(N)} \epsilon(\mu)$ so $\mu_e^c(N)$ has to be equal to zero and we can proceed as in Theorem 3. □

Remark. (1) The assumption that the energy is finite in the theorems on equilibrium potential is essential, i.e. without this assumption the theorem is false, cf. [11, p. 159].

(2) If for f_e given in the proof of the previous theorem $\|\mathcal{G}\mu_{f_e}\|_{\nu,p'} = c_2 < \infty$, then choosing $c > c_2$ in the proof of Lemma 4 instead of $\mu_{f_e}^c$ we can write μ_{f_e}, and then $W(K) = \epsilon(f_e, K)$ and the difference between Theorem 3 and Theorem 4 is in the assumption on the exceptional set.

Now we are in position to define the p-Chebyshev constant.

Notation. Let $f \in L$, $x \in X$, $H \subset X$ and $X_n \subset H$. Then let

$$M(X_n, f, x) := M_{k,\lambda}(X_n, f, x)$$

$$= \frac{1-\lambda}{2n} \sum_{i=1}^{n} \int_X k(x_i, y) f(y)\, d\nu(y) + \frac{1-\lambda}{2} \int_X k(x,y) f(y)\, d\nu(y) + \frac{\lambda}{n} \sum_{i=1}^{n} k(x, x_i),$$

$$M_n(f,H) := M_{n,k,\lambda}(f,H) = \sup_{X_n \subset H} \inf_{x \in H} M(X_n, f, x).$$

We call $p(X_n, f, x)$ as log-polynomial of degree n with respect to f, if

$$p(X_n, f, x) := p_{k,\lambda}(X_n, f, x) = nM(X_n, f, x)$$
$$= \frac{1-\lambda}{2} \int_X \sum_{i=1}^{n} \left(k(x_i, y) + k(x,y)\right) f(y)\, d\nu(y) + \lambda \sum_{i=1}^{n} k(x, x_i),$$
$$\mathcal{P}_n(f,H) := \{p(X_n, f, x) \colon X_n \subset H\}.$$

Obviously, if X_n and Y_m are in H, then $p(X_n, f, x) + p(Y_m, f, x) \in \mathcal{P}_{n+m}(f, H)$.

According to the previous notation

$$(n+m)M_{n+m}(f,H) = \sup_{p(Z_{n+m}, f, \cdot) \in \mathcal{P}_{n+m}(f,H)} \inf_{x \in H} p(Z_{n+m}, f, x)$$
$$\geq \inf_{x \in H} \left(p(X_n, f, x) + p(Y_m, f, x)\right) \geq \inf_{x \in H} p(X_n, f, x) + \inf_{x \in H} p(Y_m, f, x),$$

for all $p(X_n, f, \cdot) \in \mathcal{P}_n(f,H)$ and $p(Y_m, f, x) \in \mathcal{P}_m(f,H)$. That is

$$(n+m)M_{n+m}(f,H) \geq nM_n(f,H) + mM_m(f,H).$$

This inequality entails, that $M_n(f,H)$ has a limit in the extended sense (cf. e.g. [20, Vol. 1, Ch. 3, Ex. 98] and the references therein). So we have the following definition.

Definition 7. By the previous notations, let

$$M(f,H) := \lim_{n \to \infty} M_n(f,H)$$

the p-Chebyshev constant with respect to f, and

$$M(H) := \sup_{f \in L} M(f,H)$$

the p-Chebyshev constant of H.

Definition 8. Let

$$L_c := \{f \in L \colon f \text{ is compactly supported and continuous}\}.$$

We say that a symmetric kernel k satisfies the (L_c-)relative domination principle (cf. [19, p. 141]), if for all $f \in L_c$ and every positive measures μ with compact support and with $I(\mu) < \infty$, if

$$\int_X k(x,y)\, d\mu(y) \leq c_1 - c_2 \int_X k(x,y) f(y)\, d\nu(y), \quad x \in \operatorname{supp} \mu,$$

then

$$\int_X k(x,y)\, d\mu(y) \leq c_1 - c_2 \int_X k(x,y) f(y)\, d\nu(y), \quad x \in X,$$

where c_i are positive constants.

Remark. If $\int_X k(x,y) f(y)\, d\nu(y)$ is continuous then the righthand side is l.s.c. If $X = \mathbf{R}$ or $X = \mathbf{C}$, with an integration by parts it can be proved that if $\int_X k(x,y)\, d\nu(y)$ is continuous, then for all $f \in L \cap C_0^1$, $\int_X k(x,y) f(y)\, d\nu(y)$ is continuous as well. If on the righthand side there is a potential, the assumption is the Cartan's maximum principle. If $X = \mathbf{R}^n$ or $X = \mathbf{C}$, and the righthand side is continuous, then it is the principle of domination which is satisfied, e.g., for Riesz kernels of order 2, cf. [16, p. 110].

With this definition we can study the connections among transfinite diameter, energy and Chebyshev constant of a set.

Theorem 5. *Let $H \subset X$. Then*

$$d(H) \le M(H), \tag{26}$$

and if k satisfies the relative domination principle, then

$$M(H) \le \tilde{W}(H). \tag{27}$$

Corollary. *If $K \subset X$ is compact, and k satisfies the relative domination principle, then*

$$d(K) = M(K) = W(K). \tag{28}$$

Proof. Denoting by $d_n(f,H) := \inf_{X_n \subset H} d(X_n, f)$, we show that $d_n(f,H) \le M_n(f,H)$ for all n and f. The inequality is obvious if $M_n(f,H) = \infty$, so we can assume, that it is finite. It ensures the existence of an $X_n \in H$ for which $\int_X k(x_i,y)f(y)d\nu(y)$ and $k(x_i,x_j)$ are finite for all $1 \le i < j \le n$, and so $d_n(f,H)$ is finite too, cf. [8]. Let $\varepsilon > 0$ arbitrary, and $X_n \subset H$ such that $d(X_n,f) \le d_n(f,H)+\varepsilon$. Let $x \in H$ arbitrary.

$$\begin{aligned} &n(n+1)d_n(f,H) \le n(n+1)d_{n+1}(f,H) \le n(n+1)d(\{x\} \cup X_n, f) \\ &= (1-\lambda)n\sum_{i=1}^n \int_X k(x_i,y)f(y)\,d\nu(y) + (1-\lambda)n\int_X k(x,y)f(y)\,d\nu(y) \\ &\quad + 2\lambda\sum_{1\le i<j\le n} k(x_i,x_j) + 2\lambda\sum_{i=1}^n k(x,x_i) = 2p(X_n,f,x) + n(n-1)d(X_n,f). \end{aligned}$$

So

$$p(X_n,f,x) \ge \frac{n(n+1)}{2}d_n(f,H) - \frac{n(n-1)}{2}d(X_n,f) \ge nd_n(f,H) - \frac{n(n-1)}{2}\varepsilon.$$

Since $x \in H$ was arbitrary, taking infinum in x and supremum in $X_n \subset H$, we get that

$$nM_n(f,H) \ge nd_n(f,H) - \frac{n(n-1)}{2}\varepsilon,$$

and since ε was arbitrary, $d_n(f,H) \le M_n(f,H)$ and so $d(f,H) \le M(f,H)$, where $d(f,H) = \lim_{n\to\infty} d_n(f,H)$, which is well-defined by (1). Taking supremum, we have $\sup_{f\in L} d(f,H) \le M(H)$ which is the first statement of the theorem because the supremum and the limit can be changed in the expression of transfinite diameter. Indeed since $d_n(H) \ge d_n(f,H)$ for all n and $f \in L$, $d(H) \ge d(f,H)$ for all $f \in L$, that is $\sup_{f\in L} d(f,H) \le d(H)$. Contrary, for an arbitrary n and ε there is an $f = f(n,\varepsilon) \in L$, like $d_n(H) - \varepsilon \le d_n(f,H) \le d(f,H) \le \sup_{f\in L} d(f,H)$. Thus $d(H) = \sup_{f\in L} d(f,H)$.

At first we will prove the second statement for compact sets, namely we will show that for all n and $f \in L_c$, $M_n(f,K) \le \epsilon(f,K)$. Recalling that μ_f is the equilibrium measure on K with respect to f,

$$\inf_{x\in K} M(X_n,f,x) \le \int_X M(X_n,f,x)\,d\mu_f(x) = \frac{1}{n}\sum_{i=1}^{n} U(\mu_f,f,x_i) \le \epsilon(f,K).$$

We have to explain the last inequality. According to (20) $U(\mu_f,f,x) \le \epsilon(f,K)$ on $\operatorname{supp}\mu_f$. It means that

$$\int_X k(x,y)\,d\mu_f(y) \le \frac{1}{\lambda}\left(c - \frac{1-\lambda}{2}\int_X k(x,y)\,d\nu^f(y)\right), \quad x\in \operatorname{supp}\mu,$$

where $c = \epsilon(f,K) - \frac{1-\lambda}{2}\int_X\int_X k(x,y)f(y)\,d\nu(y)\,d\mu_f(x)$. Since k satisfies the relative domination principle, the inequality fulfils on X as well. As in previously, taking supremum at X_n, limit in n, we get that for all $f\in L_c$ $M(f,K)\le\epsilon(f,K)$. Since L_c is a dense subset of L, taking supremum on both sides in $f\in L_c$ we get the second statement of the theorem for compact sets. For arbitrary sets we will proceed as in [8]: Let $f\in L_c$ again. For a positive ε we choose a measure $\mu\in\mathcal{M}(H)$ such that $E(\mu,f)\le\epsilon(f,H)+\varepsilon$. Let $X_n\subset H$ be arbitrary, and $K := X_n\cup\operatorname{supp}\mu$. Then for all $X_n\subset H$

$$\begin{aligned}\inf_{x\in H} M(X_n,f,x) &\le \inf_{x\in K} M(X_n,f,x) \le M_n(f,K)\\ &\le \epsilon(f,K)\le E(\mu,f)\le\epsilon(f,H)+\varepsilon.\end{aligned}\tag{29}$$

So taking supremum in $X_n\subset H$, we get the result as previously. □

4. Greedy energy or Leja points

Usually a Fekete n-point system of a compact set K ($X_n^*\subset K$) is a result of a minimum problem with n variables. In this section, like in [17] or [12], we define the so-called greedy energy or Leja set ($A_n\subset K$) which can be computed step by step, and in all steps a minimum problem with one variable has to be solved. Furthermore it is pointed out that the behavior of greedy energy sets are asymptotically as good as the behavior of the Fekete sets.

Definition 9. Let $K\subset X$ be a compact set, $f\in L$. A sequence $\{a_n\}_{n=1}^{\infty}\subset K$ is called a greedy energy sequence with respect to k, λ and f, if it is generated in the following way:

- $a_1\in K$ is arbitrary.
- Assuming that $A_n := \{a_1,\dots,a_n\}$ have been selected, a_{n+1} is chosen to satisfy

$$\inf_{x\in K} p(A_n,f,x) = p(A_n,f,a_{n+1}).$$

Remark. In order to get a faster numerical process (e.g. in a metric space), by the lower semicontinuity of the integral, we can choose a_1 as

$$\int_X k(a_1,y)f(y)\,d\nu(y) = \min_{x\in K}\int_X k(x,y)f(y)\,d\nu(y).$$

Theorem 6. *Let us assume that k satisfies the relative domination principle. Then the following statements are satisfied:*

$$\sup_{f\in L}\lim_{n\to\infty} d(A_n,f) = W(K).\tag{30}$$

If $\epsilon(f,K)<\infty$, then the following sequence

$$\mu_n := \frac{1}{n}\sum_{k=1}^{n}\delta_{a_k}$$

has a w^*-convergent subsequence, such that

$$\mu_{n_k} \overset{*}{\to} \mu_f, \tag{31}$$

where μ_f is an equilibrium measure with respect to f, and δ_{a_k} is the Dirac measure concentrated at the k^{th} greedy energy point. Then

$$\sup_{f\in L}\lim_{n\to\infty} M(A_n,f,a_{n+1}) = W(K). \tag{32}$$

Proof. Let $f\in L_c$. First we show that $\lim_{n\to\infty} d(A_n,f)=\epsilon(f,K)$. Then

$$d(A_n,f) = \frac{2}{n(n-1)}\left(\frac{1-\lambda}{2}\sum_{j=2}^{n}\sum_{l=1}^{j-1}\int_X (k(a_l,y)+k(a_j,y))f(y)\,d\nu(y)\right.$$
$$\left.+\lambda\sum_{j=2}^{n}\sum_{l=1}^{j-1}k(a_l,a_j)\right) = \frac{2}{n(n-1)}\sum_{j=2}^{n}p(A_{j-1},f,a_j)$$
$$\le \frac{2}{n(n-1)}\sum_{j=2}^{n}p(A_{j-1},f,x), \quad \forall x\in K.$$

Integrating this inequality against the equilibrium measure of K with respect to f (μ_f) we get

$$d(A_n,f)\le \frac{2}{n(n-1)}\sum_{j=2}^{n}\sum_{l=1}^{j-1}U(\mu_f,f,a_l)\le \epsilon(f,K).$$

In the last inequality, as in the previous section, we used the relative domination principle. Now by the inequality

$$d(X_n^*,f)\le d(A_n,f)\le \epsilon(f,K),$$

which holds for all $f\in L_c$. It turns out from the proof of Theorem 1 (or by the classical result) that for a compact set K $\lim_{n\to\infty} d(X_n^*,f)=\epsilon(f,K)$, and we get (30).

Like in Theorem 1, let $h_m(x)$ and $k_m(x,y)$ be continuous functions, and so

$$I_{m,n} := \int_X\int_X (1-\lambda)h_m(x)+\lambda k_m(x,y)\,d\mu_n(y)\,d\mu_n(x) = \frac{1-\lambda}{n}\sum_{i=1}^{n}h_m(a_i)$$
$$+\frac{\lambda}{n^2}\sum_{i=1}^{n}k_m(a_i,a_i)+\frac{2\lambda}{n^2}\sum_{1\le i<j\le n}k_m(a_i,a_j)\le \frac{1-\lambda}{n}\sum_{i=1}^{n}\int_X k(a_i,y)f(y)\,d\nu(y)$$
$$+\frac{\lambda}{n^2}\sum_{i=1}^{n}k_m(a_i,a_i)+\frac{2\lambda}{n^2}\sum_{1\le i<j\le n}k_m(a_i,a_j) = d(A_n,f)+r(m,n),$$

where

$$r(m,n)=\frac{\lambda}{n^2}\sum_{i=1}^{n}k_m(a_i,a_i)-\frac{2\lambda}{n^2(n-1)}\sum_{1\le i<j\le n}k_m(a_i,a_j)\le \frac{\lambda}{n}M_m+\frac{2\epsilon(f,K)}{n},$$

where M_m is the maximum of $k_m(x,y)$ on K. So if n is large enough, by the previous part $I_{m,n} \le \epsilon(f,K)+\varepsilon$. As earlier, a w^* convergent subsequence can be chosen from $\{\mu_n\}$, and this subsequence tends to a measure $\sigma \in \mathcal{M}(K)$. So

$$\lim_{n\to\infty} I_{m,n} = \int_X\int_X (1-\lambda)h_m(x) + \lambda k_m(x,y)\, d\sigma(y)\, d\sigma(x) \le \epsilon(f,K) \quad \forall m,$$

and then

$$E(\sigma,f) \le \epsilon(f,K),$$

that is σ is an equilibrium measure on K with respect to f.

Following the chain of ideas of [12], we will show that for every $f \in L_c$,

$$\lim_{n\to\infty} M(A_n,f,a_{n+1}) = \epsilon(f,K). \tag{33}$$

Let us observe that

$$\begin{aligned} p(A_{n+1},f,x) &= p(A_n,f,x) + \frac{1-\lambda}{2}\int_X \left(k(a_{n+1},y)+k(x,y)\right) f(y)\, d\nu(y) + \lambda k(x,a_{n+1}) \\ &\ge p(A_n,f,x) + T, \end{aligned}$$

where $T = \min_{x,y\in K} k(x,y)\left((1-\lambda)\int_X f(y)\,d\nu(y)+\lambda\right) \ge 0$. Here we used the compactness of K and the lower semicontinuity of k. For simplicity, let us denote by $B_n = M(A_n,f,a_{n+1})$. By the definition of greedy energy points, we have that

$$jB_j \le (j+1)B_{j+1} - T, \tag{34}$$

and by the first part

$$\lim_{n\to\infty} \frac{2}{n(n-1)}\sum_{j=1}^{n-1} jB_j = \epsilon(f,K), \tag{35}$$

according to the definition of B_n and recalling (28) we have

$$B_n \le M_n(f,K) \le \epsilon(f,K). \tag{36}$$

Now we will show, that (34)–(36) entails that B_n tends to $\epsilon(f,K)$, which is (33). Let $(\epsilon >)\varepsilon > 0$ and let us assume that n is an index like

$$B_{n-1} < \epsilon(f,K) - \varepsilon. \tag{37}$$

Computing the left hand-side of (35) we get that with $1 > \delta > 0$,

$$\begin{aligned} \frac{2}{n(n-1)}\sum_{j=1}^{n-1} jB_j &= \frac{2}{n(n-1)}\sum_{j=1}^{[(1-\delta)(n-1)]} jB_j + \frac{2}{n(n-1)}\sum_{l=0}^{[\delta(n-1)]}(n-1-l)B_{n-1-l} \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

By (34) and (37)

$$\begin{aligned} \Sigma_2 &\le \frac{2}{n(n-1)}\sum_{l=0}^{[\delta(n-1)]}\left((n-1)(\epsilon(f,K)-\varepsilon) - lT\right) \le \frac{2}{n}(\delta(n-1)+1)(\epsilon(f,K)-\varepsilon) \\ &\quad - T\frac{2}{n(n-1)}\frac{\delta^2(n-1)^2}{2} \le (\epsilon(f,K)-\varepsilon)\left(2\delta\left(1-\frac{1}{n}\right)+\frac{2}{n}\right) - T\delta^2\left(1-\frac{1}{n}\right), \end{aligned}$$

and

$$\Sigma_1 \le \epsilon(f,K)\left((1-\delta)^2\left(1-\frac{1}{n}\right)+\frac{1-\delta}{n}\right),$$

that is

$$\Sigma_1 + \Sigma_2 \le \epsilon(f,K) + \epsilon(f,K)\left(\delta^2 + \frac{2}{n}\right) - 2\delta\varepsilon.$$

Choosing $\delta = \frac{\varepsilon}{\epsilon(f,K)}$,

$$\frac{2}{n(n-1)}\sum_{j=1}^{n-1} jB_j \le \epsilon(f,K) - \frac{\varepsilon^2}{\epsilon(f,K)} + \frac{2\epsilon(f,K)}{n}.$$

That is according to (35), (37) cannot fulfil for infinitely many indices, and together with (36), (33) is proved. Recalling that L_c is dense in L, and taking supremum, (32) is proved. □

Remark. Instead of greedy energy points one can define f-greedy energy sequence $\{a_n^*\}_{n=1}^\infty \subset K_1$, where $K_1 = \operatorname{supp}\mu_f$ or $K_1 := \{x \in K : U(\mu_f, f, x) \le \epsilon(f,K)\}$ and all the infinums are taken on K_1. With this choice Theorem 6 can be proved without any restriction on the kernel (cf. [12]). If the equilibrium measure with respect to f is unique, then in (31) one can write:

$$\mu_{n_k} \overset{*}{\to} \mu_f,$$

where μ_{n_k}-s are the normalized Dirac measures concentrated at the points a_m^*. This is the case e.g. when $k_f^s(x,y)$ is strictly definite (cf. [11] and the definition of $k_f^s(x,y)$ see below).

5. Final remarks

Symmetry, maximum principle. As it is mentioned, our kernel function with respect to an $f \in L$ is

$$k_f(x,y) = (1-\lambda)\int_x k(x,y)f(y)\,d\nu(y) + \lambda k(x,y),$$

which is nonsymmetric. The notion of transfinite diameter and energy are independent of symmetry so it can be introduced and studied for general kernels cf. [8]. If f is in L fixed, following the proof of Theorem 1 it turns out that for a compact set K

$$d(f,K) = \epsilon(f,K),$$

which is known (cf. [8, 6]). From these computations follows the equivalence of $d(K)$ and $\tilde{W}(K)$. In Theorem 1 a minimax theorem is hidden.

In the first part of the third section the potential with respect to a fixed f is examined. These results follow from the results of [11] by the symmetrization of the kernel, namely instead of $k_f(x,y)$ one can use the symmetric kernel

$$\begin{aligned} k_f^s(x,y) &:= \frac{1}{2}\left(k_f(x,y) + k_f(y,x)\right) \\ &= \frac{1-\lambda}{2}\left(\int_X k(x,y)f(y)\,d\nu(y) + \int_X k(x,y)f(x)\,d\nu(x)\right) + \lambda k(x,y). \end{aligned}$$

We gave the details only in order to emphasize the importance of the “law of reciprocity” ($\int U(\sigma)\,d\mu = \int U(\mu)\,d\sigma$), which is ensured by the symmetry of the kernel.

It is proved in [8] that if a symmetric kernel satisfies the Frostman’s maximum principle, then

$$d(K) = M(K) = V(K)$$

for a compact set K, where

$$V(H) = \inf_{\mu \in \mathcal{M}(H)} I(\mu).$$

A positive symmetric kernel satisfies the Frostman's maximum principle (cf. eg. [11, p. 150]), if for every compactly supported positive measures

$$\sup_{x \in \operatorname{supp} \mu} \int_X k(x,y)\, d\mu(y) = \sup_{x \in X} \int_X k(x,y)\, d\mu(y).$$

If $f \in L_c$, then ν^f and so $\frac{1-\lambda}{2}\nu^f + \lambda\mu$ are compactly supported. Recalling that

$$U(\mu, f, x) = c(\mu, f, \lambda) + \int_X k(x,y)\, d\left(\frac{1-\lambda}{2}\nu^f + \lambda\mu\right)(y)$$

it can be seen that the assumption "k satisfies the Frostman's maximum principle" means

$$\sup_{x \in X} U(\mu, f, x) = \sup_{x \in \operatorname{supp} \mu \cup \operatorname{supp} \nu^f} U(\mu, f, x),$$

which is not enough to prove the inequality $M(f,K) \le \epsilon(f,K)$. If we assume that $k_f^s(x,y)$ satisfies the Frostman's maximum principle, that is

$$\sup_{x \in \operatorname{supp} \mu} U(\mu, f, x) = \sup_{x \in X} U(\mu, f, x),$$

then by the result cited above

$$d(f,K) = M(f,K) = \epsilon(f,K).$$

With the notations of Theorem 4, if $\|\mathcal{G}\mu_{f_e}\|_{\nu,p'}$ is finite and $k_{f_e}^s$ satisfies the Frostman's maximum principle, then

$$d(f_e, K) = M(f_e, K) = \epsilon(f_e, K) = W(K).$$

The Frostman's maximum principle for classical kernels was proved by Maria [18] and by Frostman [10], for more general kernels by Carleson [4, p. 14]. For continuous kernels it is proved in [8] and [24], that the equivalence of the Chebyshev constant and the energy entails the maximum principle. Usually the Frostman's maximum principle is weaker then the principle of domination, but to get Theorem 5 it has to be assumed for k_f^s with all $f \in L_c$. Whereas the relative domination principle is assumed on the kernel k, so as our examples showed sometimes it can be checked easier.

Finally by a modification of the usual definition of Chebyshev constant, one can drop the assumption on the kernel. Let $K \subset X$ compact, $f \in L_c$ and $\mu_f := \mu_f(K)$ is an equilibrium measure on K. Let $\operatorname{supp} \mu_f =: K_1 \subset K$, which is compact. Then obviously $\epsilon(f,K) = \epsilon(f,K_1)$, since

$$\epsilon(f,K) = E(\mu_f, f) = \inf_{\mu \in \mathcal{M}(K)} E(\mu, f) \le \inf_{\mu \in \mathcal{M}(K_1)} E(\mu, f) = E(\mu_f, f) = \epsilon(f, K_1).$$

Furthermore $d(f,K) \le d(f,K_1) \le M(f,K_1)$. Since $K_1 = \operatorname{supp} \mu_f(K) = \operatorname{supp} \mu_f(K_1)$, for any $x_i \in K_1$ $U(\mu_f, f, x_i) \le \varepsilon(f, K_1)$ without any further assumptions to the kernel. That is $M(f,K_1) \le \epsilon(f,K_1)$, and finally

$$\epsilon(f,K) = d(f,K) \le d(f,K_1) \le M(f,K_1) \le \epsilon(f,K_1) = \epsilon(f,K).$$

So defining the modified Chebyshev constant for a compact set K with an equilibrium measure μ_f as

$$M^*(f,K) = M(f,K_1),$$

we can state for compact sets without any restriction on the kernel that

$$d(f,K) = M^*(f,K) = \epsilon(f,K),$$

and taking supremum in f

$$d(K) = M^*(K) = \epsilon(K).$$

$\boldsymbol{\lambda}$ tends to 0 or 1. All the previous computations, etc. were valid for all fixed $\lambda \in (0,1)$. Now we will study (and denote) the dependence on λ. Let

$$V_p(H) := \inf_{\mu\in\mathcal{M}(H)} \sup_{f\in L} \mathcal{E}(\mu,f).$$

The first question is that can we get back the original p-energy, or energy when λ tends to zero or one? Secondly the location of the Fekete and greedy energy sets with respect to a fixed n can be examined, when λ tends to one of the endpoints. We will study the $\lambda \to 0$ case, the other case is similar.

It is clear that if $W_\lambda(H) < \infty$ for a $\lambda \in (0,1)$, then $\exists\, \mu \in \mathcal{M}(H)$ for which $\epsilon_\lambda(\mu) < \infty$ and so $\|\mathcal{G}\mu\|_{\nu,p'}, I(\mu) < \infty$, that is $\forall\lambda \in (0,1)$ $W_\lambda(H) < \infty$ and so $V_p(H), V(H) < \infty$. For a H like this $W_\lambda(H) \geq (1-\lambda)\inf_{\mu\in\mathcal{M}(H)} \|\mathcal{G}\mu\|_{\nu,p'} + \lambda \inf_{\mu\in\mathcal{M}(H)} I(\mu)$, that is

$$\lim_{\lambda\to 0} W_\lambda(H) \geq V_p(H).$$

On the other hand for all $\mu \in \mathcal{M}(H)$ $W_\lambda(H) \leq \epsilon_\lambda(\mu)$, that is $\lim_{\lambda\to 0} W_\lambda(H) \leq \|\mathcal{G}\mu\|_{\nu,p'}$ for all $\mu \in \mathcal{M}(H), I(\mu) < \infty$. Taking infinum we have if $W_\lambda(H) < \infty$ for a $\lambda \in (0,1)$, then

$$\lim_{\lambda\to 0} W_\lambda(H) = V_p(H), \text{ if } \exists M \text{ such that } \inf_{\mu\in\mathcal{M}(H)} \|\mathcal{G}\mu\|_{\nu,p'} = \inf_{\substack{\mu\in\mathcal{M}(H)\\ I(\mu)<M}} \|\mathcal{G}\mu\|_{\nu,p'}.$$

(Obviously $\|\mathcal{G}\mu\|_{\nu,p'} < \infty \not\Rightarrow I(\mu) < \infty$, namely if $k(x,x) = \infty$ and $\|k(x_i,y)\|_{\nu,p'} < \infty$, $i = 1,\ldots,n$, then with $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ $\|\mathcal{G}\mu\|_{\nu,p'} < \infty$ and $I(\mu) = \frac{1}{n^2}\sum_{i,j=1}^n k(x_i, x_j) = \infty$.)

Fekete and greedy energy points are defined for fixed $f \in L$. Actually, this is an unconditional Gauss variational problem, cf. [19, p. 213]. Let K be a compact set. We will investigate the behavior of the n^{th} Fekete set when λ tends to zero. In [1] it is proved that

$$V_p(K) = \sup_{f\in L} \inf_{\mu\in\mathcal{M}(K)} \mathcal{E}(\mu,f) = \sup_{f\in L} \int_X k(x_0,y) f(y) d\nu(y),$$

where by the l.s.c. of the integral and by the compactness of the set $x_0 \in K$ and it depends on f and K. (That is δ_{x_0} is an extremal measure.) The assumption is the same here: If $\epsilon_\lambda(f,K) < \infty$ for a $\lambda \in (0,1)$, then denoting by $V_p(f,K) := \inf_{\mu\in\mathcal{M}(K)} \mathcal{E}(\mu,f) = \int_X k(x_0,y) f(y)\, d\nu(y)$

$$\lim_{\lambda\to 0} \epsilon_\lambda(f,K) = V_p(f,K), \text{ if } \exists M \text{ such that } \inf_{\mu\in\mathcal{M}(K)} \mathcal{E}(\mu,f) = \inf_{\substack{\mu\in\mathcal{M}(H)\\ I(\mu)<M}} \mathcal{E}(\mu,f).$$

Assuming the condition above, let us choose a sequence $\lambda_m \to 0$, let n be fixed, and let us denote by $X^*_{n,m} \subset K$ a Fekete set with respect to λ_m. Let $\mu_m := \frac{1}{n}\sum_{i=1}^n \delta_{x^*_{i,m}}$ the normalized counting measure at the points of $X^*_{n,m}$. It has a w^*-convergent subsequence denoting by $\{\mu_m\}$ again. Its limit is σ_n. As previously, let $0 \le h_f(x) \le \int_X k(x,y)f(y)\,d\nu(y)$, and $0 \le h \le k$ are continuous functions. Let $\varepsilon > 0$ arbitrary. Repeating the computations in (9), if m is large enough

$$\begin{aligned}\int_X h_f(x)\,d\sigma_n(x) - \varepsilon \le \int_X h_f(x)\,d\mu_m(x) \le (1-\lambda_m)\int_X h_f(x)\,d\mu_m(x) + \varepsilon \\ + \lambda_m \int_X\int_X h(x,y)\,d\mu_m(y)\,d\mu_m(x) \le d_{n,\lambda_m}(f,K) + 2\varepsilon \\ \le d_{\lambda_m}(f,K) + 2\varepsilon = \epsilon_{\lambda_m}(f,K) + 2\varepsilon,\end{aligned}$$

where in the last step the previous remark is used. Finally, taking into consideration the assumption and tending to infinity with m, we have that $\int_X h_f(x)d\sigma_n(x) \le V_p(f,K)$, that is

$$\int_X\int_X k(x,y)f(y)\,d\nu(y)\,d\sigma_n(x) \le V_p(f,K),$$

so if the extremal measure is unique, then for all n, $\mu_m \overset{*}{\to} \delta_{x_0}$. This is the situation e.g. when $k(x,y)$ is continuous (then $I(\mu)$ is finite for all $\mu \in \mathcal{M}(K)$), and if $k^s_{1,f}(x,y) := \frac{1}{2}\left(\int_X k(x,y)f(y)\,d\nu(y) + \int_X k(x,y)f(x)\,d\nu(x)\right)$ is definite.

On the other hand let $k(x,y)$ is infinite at the diagonal, $W_\lambda(K)$ is finite for a $\lambda \in (0,1)$, and let us denote by $g(x) := \int_X k(x,y)f(y)\,d\nu(y)$. Let us assume that there is an $x_0 \in K$ such that

$$g(x_0) = \min_{x\in K} g(x) = m < \liminf_{\substack{x\to x_0\\ x\in K\setminus\{x_0\}}} g(x) = M.$$

We can assume that $g(x) > g(x_0)$ on K. Now $V_p(f,K) = g(x_0) = m$. If $X_n \subset K$, then

$$d(X_n,f) = \frac{1-\lambda}{n}\sum_{i-1}^{n} g(x_i) + \lambda\frac{2}{n(n-1)}\sum_{1\le i<j\le n} k(x_i,x_j).$$

For all $X_n \subset K$

$$d(X_n,f) \ge (1-\lambda)\left(M\left(1-\frac{1}{n}\right) + \frac{m}{n}\right) + \min_{X_n\subset K} \lambda\frac{2}{n(n-1)}\sum_{1\le i<j\le n} k(x_i,x_j),$$

and so $d_n(f,K)$ is also greater than or equal to the righthand side. Taking a limit in n and taking account the results of [8], we have $d(f,k) \ge (1-\lambda)M + \lambda V(K)$ that is by the first part of this section, for all $\lambda \in (0,1)$

$$\epsilon_\lambda(f,K) \ge (1-\lambda)M + \lambda V(K),$$

and so

$$\lim_{\lambda\to 0} \epsilon_\lambda(f,K) \ge M > m = V_p(f,K).$$

This is the situation e.g. when $X = \mathbf{C}/\mathbf{R}^n$, $k(x,y)$ is the logarithmic/Newtonian kernel and K is thin at x_0, ν^f is the measure of the potential in question cf. e.g. [21, Lemma 5.2] or [14, 10.3].

Acknowledgement. The author thanks the referee for the careful reading of the manuscript and the suggestions for further work.

References

[1] Adams, D., and L. I. Hedberg: Function spaces and potential theory. - Springer-Verlag, 1996.

[2] Björn, A., and J. Björn: Nonlinear potential theory on metric spaces. - Eur. Math. Soc., 2011.

[3] Borodachov, S. V., D. P. Hardin, and E. B. Saff: Asymptotics for discrete weighted minimal Riesz energy problems on rectificable sets. - Trans. Amer. Math. Soc. 360:3, 2008, 1559–1580.

[4] Carleson, L.: Selected problems on exceptional sets. - D. Van Nostrand Co. Inc., 1967.

[5] Choquet, G.: Theory of capacities. - Ann. Inst. Fourier (Grenoble) 5, 1955, 131–295.

[6] Choquet, G.: Diamétre transfini et comparaison de diverses capacités. - In: Seminaire de Théorie du Potentiel, F. des Sci. de Paris 4-01-4-07, 1958/1959.

[7] Costea, S.: Sobolev capacity and Hausdorff measures in metric measure spaces. - Ann. Acad. Sci. Fenn. Math. 34, 2009, 179–194.

[8] Farkas, B., and B. Nagy: Transfinite diameter, Chebyshev constant and energy on locally compact spaces. - Potential Anal. 28, 2008, 241–260.

[9] Fekete, M.:Über den transfiniten Durchmesser ebener Punktmengen. - Math. Z. 32, 1930, 108–114.

[10] Frostman, O.: Potentiel d'équilibre et capacité des ensembles. - Comm. Sém. Math. Univ. Lund 3, 1935.

[11] Fuglede, B.: On the theory of potentials in locally compact spaces. - Acta Math. 103, 1960, 140–215.

[12] Garcia, A. L., and E. B. Saff: Asymptotics of Greedy energy points. - Math. Comp. 79:272, 2010, 2287–2316.

[13] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. - Potential Anal. 5, 1995, 403–415.

[14] Helms, L. L.: Introduction to potential theory. - John Wiley & Sons, Inc., 1969.

[15] Horváth, Á. P.: Weighted Fejér constants and Fekete sets. - Acta Math. Hungar. 141:4, 2013, 366–382.

[16] Landkof, N. S.: Foundation of modern potential theory. - Springer-Verlag, New York, 1972.

[17] Leja, F.: Propriétes des points extrémaux des ensembles plans et leur application á la représentation conforme. - Ann. Polon. Math. 3, 1957, 319–342.

[18] Maria, M. A.: The potential of a positive mass and the weight function of Wiener. - Proc. Nat. Acad. Sci. USA 20, 1934, 485–489.

[19] Ohtsuka, M.: On potentials in locally compact spaces. - J. Sci. Hiroshima Univ. ser. A-1, 1961, 135–352.

[20] Pólya, Gy., and G. Szegő: Problems and theorems in analysis. - Springer-Verlag, Berlin–Heidelberg, 1998.

[21] Saff, E. B., and V. Totik: Logarithmic potentials with external fields. - Springer-Verlag, 1997.

[22] Siciak, J.: Sur la distribution des points extrémaux dans les ensembles plans. - Ann. Math. Polon. 4, 1957-1958, 214–219.

[23] Szegő, G.: Bemerkungen zu einer Arbeit von Herrn M. Fekete: Über die Verteilung der Wulzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. - Math. Zeitscher 21, 1924, 203–208.

[24] Yamasaki, M.: Relations between capacities and maximum principle. - J. Sci. Hiroshima Univ. Ser. A-1, 1969, 59–69.

Received 20 January 2015 • Accepted 13 March 2015