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CONFORMAL MEASURES AND LOCALLY CONFORMALLY FLAT METRIC TENSORS

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Abstract. In the paper [Na], Nayatani used a Patterson–Sullivan measure μ of a non-elementary Kleinian group Γ of the second kind to define a metric tensor g^μ on the set of discontinuity $\Omega(\Gamma)$ of Γ which is compatible with the natural conformal structure of $\Omega(\Gamma)$. The metric tensor g^μ is Γ-invariant and so it can be projected to a metric tensor g^μ_M of any Kleinian manifold M contained in the quotient $\Omega(\Gamma)/\Gamma$. Nayatani showed that the sign of the scalar curvature of g^μ is determined by the exponent of convergence δ_Γ of Γ. He showed also that in some situations the isometry group of (M, g^μ_M) coincides with the group of conformal automorphisms of M. We point out in this paper that Nayatani's definitions and arguments can be applied if the Patterson–Sullivan measure μ is replaced by any conformal measure of Γ supported by the limit set $L(\Gamma)$ of Γ. Combining this observation with an existence theorem of conformal measures proved in [AFTu] and [Sul3], we deduce that if Γ is not convex cocompact, then Γ has many metric tensors like g^μ and some of them must have scalar curvatures which are negative everywhere. We also obtain a simple new proof for the known fact that if M is compact and $\delta_\Gamma \leq (n-2)/2$, where $n \geq 3$ is the dimension of M, then Γ is convex cocompact. Finally, we point out generalizations of Nayatani's results (and results of others) regarding the isometry group of (M, g^μ_M).

1. Introduction

Let Γ be a non-elementary Kleinian group acting on the unit ball $\mathbf{B}^{n+1}$ of $\mathbf{R}^{n+1}$, $n \geq 3$, with the limit set $L(\Gamma)$, the non-empty set of discontinuity $\Omega(\Gamma)$ and the exponent of convergence δ_Γ. Let μ be an s-conformal measure of Γ supported by $L(\Gamma)$ for some $s \geq \delta_\Gamma$. (See the next section for the definitions of these notions.)

In this paper, we will study the existence and properties of metric tensors g^μ of $\Omega(\Gamma)$ as defined in (5.1). Metric tensors of this form were introduced by Nayatani in [Na] in the case where μ is a Patterson–Sullivan measure of Γ, i.e. a δ_Γ-conformal measure of Γ obtained by using a classical method of construction invented by Patterson and generalized by Sullivan. This paper is based on the observation that Nayatani's definition can be applied even if μ is not a Patterson–Sullivan measure of Γ, and as a consequence of this observation we obtain generalizations of some results of Nayatani and others.

Our first main result, an easy corollary of one of Nayatani's results, shows that the sign of the scalar curvature of g^μ is determined by s and that if Γ is not convex cocompact, then the scalar curvature of some g^μ is negative everywhere (see Theorem 6.1 and Corollary 6.2). More precisely, the scalar curvature of g^μ is positive everywhere, zero everywhere or negative everywhere if $s < N$, $s = N$ or $s > N$, respectively, where $N = (n-2)/2$, and since there are s-conformal measures of Γ for

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arbitrarily large s if Γ is not convex cocompact, there are in this case metric tensors g^μ whose scalar curvatures are negative everywhere. The proof is obtained by combining an argument given by Nayatani in [Na] with an existence result of conformal measures proved in [AFTu] and [Sul3].

The metric tensor g^μ is Γ-invariant, see Proposition 5.2. It follows that if O is a non-empty, open, connected and Γ-invariant subset of $\Omega(\Gamma)$ such that no other element in Γ except the identity mapping has a fixed point in O, we obtain the Kleinian manifold (M, g^μ_M), where $M = O/\Gamma$ and g^μ_M is the projection of g^μ to M. The metric tensor g^μ_M is compatible with the natural conformal structure of M obtained by projection from $\Omega(\Gamma)$, which means that g^μ_M is a locally conformally flat metric tensor of M.

If M is compact and $\delta_\Gamma \leq (n-2)/2$, we obtain as a consequence of Theorem 6.1 that Γ is convex cocompact (Theorem 7.1). In fact, Γ is convex cocompact if $\delta_\Gamma \leq (n-2)/2$ and $\Omega(\Gamma)/\Gamma$ contains a non-empty compact component, see Corollary 7.2. This result was originally proved by Izeki in [I2], but we give a new much simpler proof.

In [Na], Nayatani considered also the isometry group of (M, g^μ_M). He showed that if μ is a Patterson–Sullivan measure of Γ, if any two δ_Γ-conformal measures of Γ supported by $L(\Gamma)$ are the same up to a multiplicative constant, and if the metric induced by g^μ_M is complete, then the isometry group of (M, g^μ_M) coincides with the group of conformal automorphisms of M. Nayatani's result was later generalized by Matsuzaki and Yabuki in [MatYab1] and [Yab]. In the last section of this paper, we will point out straightforward generalizations of the results of Matsuzaki, Nayatani and Yabuki concerning the isometry group of (M, g^μ_M).

We will start our exposition by recalling basic definitions and facts regarding Kleinian groups and conformal measures of Kleinian groups. We will then study some relations between the conformal measures of a given Kleinian group, the normalizer of the group and the bounded parabolic fixed points of the group, and apply the results in the case of conformal measures of geometrically finite Kleinian groups. The purpose of these studies is to obtain auxiliary results and examples that we will use in our discussion on the isometry group of (M, g^μ_M). Finally, we will prove our main results in the order indicated above.

2. Preliminaries

We will start off by recalling basic facts about Kleinian groups and conformal measures of Kleinian groups. We assume that the reader is familiar with Kleinian groups, but we will give detailed references when discussing conformal measures. For a general discussion on Kleinian groups, see [Be], [Mas], [MatTa] or [R2]. Conformal measures of Kleinian groups are discussed, for example, in [Ni], [P2] and [Sul1].

2.1. Kleinian groups. Let Γ be *a Kleinian group* acting on the unit ball $\mathbf{B}^{n+1}$ of the $(n+1)$-dimensional euclidean space $\mathbf{R}^{n+1}$, where $n \geq 1$, i.e., let Γ be a group of Möbius transformations of $\bar{\mathbf{R}}^{n+1} = \mathbf{R}^{n+1} \cup \{\infty\}$ whose elements map $\mathbf{B}^{n+1}$ onto itself and which is discrete in the natural topology of Möbius transformations of $\bar{\mathbf{R}}^{n+1}$. It is well known that if $\gamma \in \Gamma$, then the restriction of γ to $\mathbf{B}^{n+1}$ is an isometry of the hyperbolic metric of $\mathbf{B}^{n+1}$ and the restriction of γ to the unit sphere $\mathbf{S}^n = \partial \mathbf{B}^{n+1}$ is a conformal automorphism of $\mathbf{S}^n$.

The discreteness of Γ implies that if $x \in \mathbf{B}^{n+1}$, then the orbit $\Gamma x = \{\gamma(x) \colon \gamma \in \Gamma\}$ can accumulate only at $\mathbf{S}^n$, and the fact that Γ acts by hyperbolic isometries on $\mathbf{B}^{n+1}$

implies that the set of accumulation points of Γx is independent of x. The set of accumulation points is called *the limit set of* Γ and the complement of the limit set with respect to $\mathbf{S}^n$ is called *the set of discontinuity of* Γ. We write

$$L(\Gamma) = \overline{\Gamma x} \cap \mathbf{S}^n \text{ for any } x \in \mathbf{B}^{n+1} \quad \text{and} \quad \Omega(\Gamma) = \mathbf{S}^n \setminus L(\Gamma) \tag{2.1}$$

to denote these sets. It is well known that either $L(\Gamma)$ contains at most two points or $L(\Gamma)$ is infinite. If $L(\Gamma)$ is finite, Γ is called *elementary*, and Γ is called *non-elementary* otherwise.

It is clear that $L(\Gamma)$ and $\Omega(\Gamma)$ are Γ-invariant, so we can form the quotient spaces $\mathbf{B}^{n+1}/\Gamma$, $\Omega(\Gamma)/\Gamma$ and $(\mathbf{B}^{n+1} \cup \Omega(\Gamma))/\Gamma$. Note that it is possible that $\Omega(\Gamma) = \emptyset$. If $\Omega(\Gamma) = \emptyset$, then Γ is said to be *of the first kind*, and Γ is said to be *of the second kind* otherwise. The hyperbolic structure of $\mathbf{B}^{n+1}$ projects onto $\mathbf{B}^{n+1}/\Gamma$, so $\mathbf{B}^{n+1}/\Gamma$ is a hyperbolic orbifold.

A point $x \in L(\Gamma)$ is called *a conical limit point of* Γ if it satisfies the following condition. Given $y \in \mathbf{B}^{n+1}$ and a hyperbolic ray R of $\mathbf{B}^{n+1}$ with x as its endpoint, there is $r > 0$ and elements $\gamma_1, \gamma_2, \ldots \in \Gamma$ such that $\gamma_i(y) \to x$ and $d(\gamma_i(y), R) \leq r$ for every $i \geq 1$, where d is the hyperbolic metric of $\mathbf{B}^{n+1}$ obtained from the element of length $2|dz|/(1-|z|^2)$. The set of conical limit points of Γ is denoted by $L_c(\Gamma)$.

A point $x \in L(\Gamma)$ is called *a parabolic fixed point of* Γ if x is the fixed point of some parabolic element in Γ. Given a parabolic fixed point x of Γ, denote by Γ_x *the stabilizer of* x *in* Γ, i.e.,

$$\Gamma_x = \{\gamma \in \Gamma \colon \gamma(x) = x\}. \tag{2.2}$$

It is known that there is $k_x \in \{1, 2, \ldots, n\}$ such that Γ_x contains a finite index subgroup isomorphic to $\mathbf{Z}^{k_x}$, and the number k_x is called *the rank of* x. The parabolic fixed point x of Γ is called *a bounded parabolic fixed point of* Γ if the quotient space $(L(\Gamma) \setminus \{x\})/\Gamma_x$ is compact.

If it is the case that $L(\Gamma)$ can be written as a pairwise disjoint union

$$L(\Gamma) = L_c(\Gamma) \cup \Gamma p_1 \cup \Gamma p_2 \cup \ldots \cup \Gamma p_m, \tag{2.3}$$

where the points $p_1, p_2, \ldots, p_m$ are bounded parabolic fixed points of Γ such that any parabolic fixed point of Γ is contained in one of the orbits $\Gamma p_1, \Gamma p_2, \ldots, \Gamma p_m$, we say that Γ is *geometrically finite*. If Γ is geometrically finite and has no parabolic fixed points (so $L(\Gamma) = L_c(\Gamma)$), then Γ is called *convex cocompact*. It is true that Γ is convex cocompact if and only if the quotient $(\mathbf{B}^{n+1} \cup \Omega(\Gamma))/\Gamma$ is compact. We refer to [Bo] for a useful discussion on geometrically finite Kleinian groups.

2.2. Conformal measures. Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Given $x, y \in \mathbf{B}^{n+1}$ and $s \geq 0$, define *the Poincaré series of* Γ

$$P^s_\Gamma(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(y))}. \tag{2.4}$$

It is easy to see that the divergence or convergence of $P^s_\Gamma(x, y)$ does not depend on the points x and y, which allows us to define *the exponent of convergence* δ_Γ *of* Γ by

$$\delta_\Gamma = \inf\{s \geq 0 \colon P^s_\Gamma(x, y) < \infty \text{ for some } x, y \in \mathbf{B}^{n+1}\}. \tag{2.5}$$

According to basic results, it is the case that $\delta_\Gamma \in]0, n]$, see Theorem 1.6.1 and Corollary 3.4.5 of [Ni]. If $P^{\delta_\Gamma}_\Gamma(x, y) = \infty$ for some $x, y \in \mathbf{B}^{n+1}$, then Γ is said to be *of divergence type*, and Γ is said to be *of convergence type* if $P^{\delta_\Gamma}_\Gamma(x, y) < \infty$.

Let $s > 0$. We say that a measure μ is *an s-conformal measure of* Γ if the σ-algebra of μ-measurable sets is the σ-algebra of Borel sets of $\bar{\mathbf{R}}^{n+1}$, if μ is positive, finite and supported by $L(\Gamma)$, and if

$$\mu(\gamma A) = \int_A |\gamma'|^s \, d\mu \tag{2.6}$$

for every $\gamma \in \Gamma$ and every μ-measurable set A, where $|\gamma'|$ is the operator norm of the derivative of γ with respect to the euclidean metric (i.e., if $x \in \mathbf{R}^{n+1} \setminus \{\gamma^{-1}(\infty)\}$, then $\gamma'(x)/|\gamma'(x)|$ is an orthogonal matrix). Note that if μ is an s-conformal measure of Γ and ϕ is a μ-measurable function, then

$$\int_{\gamma A} \phi \, d\mu = \int_A (\phi \circ \gamma) |\gamma'|^s \, d\mu \tag{2.7}$$

for every $\gamma \in \Gamma$ and every μ-measurable set A.

If Γ has an s-conformal measure, then $s \geq \delta_\Gamma$ (Corollary 4.5.3 of [Ni] or Corollary 4 of [Sul1]), and conversely, if Γ is not convex cocompact, then Γ has s-conformal measures for every $s \geq \delta_\Gamma$: If $s = \delta_\Gamma$, one can use the classical construction method invented by Patterson (see [P1]) and generalized by Sullivan (see [Sul1]), and if $s > \delta_\Gamma$, one can use the construction method described in Theorem 4.1 of [AFTu] (see also Theorem 2.19 of [Sul3] and the paper [FMatSt]). (Measures constructed by the method of Patterson and Sullivan are called *Patterson–Sullivan measures*.) On the other hand, if Γ is convex cocompact, then Γ has s-conformal measures only if $s = \delta_\Gamma$ (and such measures can again be constructed by using the method of Patterson and Sullivan), and any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant (see Theorem 8 of [Sul1]). In fact, if Γ is geometrically finite, then any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant (Theorem 1 of [Sul2]), and more generally, the same is true if Γ is of divergence type (this is obtained by combining [Sul1] with [Tu2]; Proposition 2 of [Sul2] states that non-elementary geometrically finite Kleinian groups are of divergence type).

3. Conformal measures, normalizers and bounded parabolic fixed points

Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Denote by $\text{Möb}(\mathbf{B}^{n+1})$ the group of all Möbius transformations of $\bar{\mathbf{R}}^{n+1}$ mapping $\mathbf{B}^{n+1}$ onto itself. Denote by $N(\Gamma)$ *the normalizer of* Γ *in* $\text{Möb}(\mathbf{B}^{n+1})$, i.e.,

$$N(\Gamma) = \{\beta \in \text{Möb}(\mathbf{B}^{n+1}) \colon \beta\Gamma\beta^{-1} = \Gamma\}. \tag{3.1}$$

The purpose of this section is to study some relations between the conformal measures of Γ, the normalizer $N(\Gamma)$ and the bounded parabolic fixed points of Γ. The rather technical results of this section will be used in the last section of this paper. We will not try to explain the motivation for the results at this point. Instead, we will formulate the results so that they will be easy to apply when needed. We start with the following lemma.

Lemma 3.2. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$, and let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Let $\beta \in \text{Möb}(\mathbf{B}^{n+1})$. Define the measure $\beta_*^s\mu$ by*

$$\beta_*^s\mu(A) = \int_{\beta^{-1}A} |\beta'|^s \, d\mu \tag{3.3}$$

for every Borel set A of $\bar{\mathbf{R}}^{n+1}$. It is the case that $\beta_*^s\mu$ is an s-conformal measure of $\beta\Gamma\beta^{-1}$.

Proof. It is clear that the σ-algebra of $\beta^s_*\mu$-measurable sets is the σ-algebra of Borel sets of $\bar{\mathbf{R}}^{n+1}$, that $\beta^s_*\mu$ is positive and finite (since $|\beta'|$ is positive and finite in $\mathbf{S}^n$), and that $\beta^s_*\mu$ is supported by $\beta L(\Gamma) = L(\beta\Gamma\beta^{-1})$. Let $\gamma \in \Gamma$. Write $\gamma_\beta = \beta \circ \gamma \circ \beta^{-1}$. Using the chain rule and (2.7), we obtain that

$$\begin{aligned}\beta^s_*\mu(\gamma_\beta A) &= \int_{(\gamma\circ\beta^{-1})A} |\beta'|^s \, d\mu = \int_{\beta^{-1}A} (|\beta'|^s \circ \gamma)|\gamma'|^s \, d\mu = \int_{\beta^{-1}A} |(\beta \circ \gamma)'|^s \, d\mu \\ &= \int_{\beta^{-1}A} |(\gamma_\beta \circ \beta)'|^s \, d\mu = \int_{\beta^{-1}A} (|\gamma'_\beta|^s \circ \beta)|\beta'|^s \, d\mu = \int_A |\gamma'_\beta|^s \, d\beta^s_*\mu\end{aligned}$$

for every $\beta^s_*\mu$-measurable set A. We conclude that $\beta^s_*\mu$ is an s-conformal measure of $\beta\Gamma\beta^{-1}$. □

We are particularly interested in the case where $\beta \in N(\Gamma)$, so $\beta^s_*\mu$ is an s-conformal measure of Γ, and where any two s-conformal measures of Γ are the same up to a multiplicative constant. We record this application of Lemma 3.2 in the following proposition.

Proposition 3.4. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$, and let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Suppose that any two s-conformal measures of Γ are the same up to a multiplicative constant. Let $\beta \in N(\Gamma)$ and let $\beta^s_*\mu$ be defined as in (3.3). Then there is a constant $b_\beta > 0$ such that $\beta^s_*\mu = b_\beta\mu$.*

Measures of the form $\beta^s_*\mu$ have been used in [AFTu] and [Tu2], for example. Results equivalent to Proposition 3.4 for δ_Γ-conformal measures of Γ can be found in the papers of Matsuzaki and Yabuki, see Lemma 3.2 of [MatYab1], Lemma 4.1 of [MatYab2] and Lemma 3.1 of [Yab]. See also Lemmas 4.2 and 4.3 of [Na]. The setting of these papers is somewhat different from ours, so we give our own proof.

We continue to consider a non-elementary Kleinian group Γ acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. We will show next that the conclusion of Proposition 3.4 can be valid without the assumption that any two s-conformal measures of Γ are the same up to a multiplicative constant. To do this, we will introduce the following explicit construction of conformal measures that features bounded parabolic fixed points. The construction, which we obtained from the papers [AFTu] and [FTu] (see also [FMatSt]), will be used a number of times in this section.

Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ and let $s \geq \delta_\Gamma$ be such that $P^s_\Gamma(x, y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. We can define a measure μ_p by setting that

$$\mu_p(p) = 1 \quad \text{and} \quad \mu_p(\gamma(p)) = \int_{\{p\}} |\gamma'|^s \, d\mu_p = |\gamma'(p)|^s \mu_p(p) = |\gamma'(p)|^s \tag{3.5}$$

for every $\gamma \in \Gamma$. According to the discussion in Section 4 of [AFTu] and Section 6 of [FTu], μ_p is an s-conformal measure of Γ (see also [FMatSt]).

Let us prove the following variant of Proposition 3.4.

Proposition 3.6. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Suppose that $p \in L(\Gamma)$ is a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \ldots, n\}$. Suppose that every bounded parabolic fixed point of Γ of rank k is contained in the orbit Γp. Let $s \geq \delta_\Gamma$ be such that $P^s_\Gamma(x, y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. Define the s-conformal measure μ_p of Γ by using (3.5). Then it is true that if $\beta \in N(\Gamma)$, then $\beta^s_*\mu_p = b_\beta\mu_p$ for some $b_\beta > 0$.*

Proof. Let $\beta \in N(\Gamma)$. The measure $\beta_*^s\mu_p$ is a purely atomic measure whose atoms are the points in the orbit $\Gamma\beta(p)$, since $\beta\Gamma p = (\beta\Gamma\beta^{-1})\beta(p) = \Gamma\beta(p)$. Since $\beta(p)$ is a bounded parabolic fixed point of $\beta\Gamma\beta^{-1} = \Gamma$ of rank k, we obtain that $\beta(p) \in \Gamma p$ and so $\Gamma\beta(p) = \Gamma p$. We conclude that $\beta_*^s\mu_p$ is a purely atomic s-conformal measure of Γ whose atoms are the points of Γp. It is clear that two such measures are the same up to a multiplicative constant, so the existence of $b_\beta > 0$ follows. □

From the point of view of the discussion contained in the last section of this paper, the situation where the conclusion of Proposition 3.4 or Proposition 3.6 is valid with $b_\beta = 1$ for every $\beta \in N(\Gamma)$, i.e., the situation where $\beta_*^s\mu = \mu$ for every $\beta \in N(\Gamma)$ and some fixed s-conformal measure μ of Γ, is particularly relevant. Note that the condition $\beta_*^s\mu = \mu$ for every $\beta \in N(\Gamma)$ is equivalent to the condition that μ satisfies the conformal transformation rule (2.6) with respect to every $\beta \in N(\Gamma)$. In the following, we will consider the existence of such conformal measures.

Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let $q_\Gamma \in \{2, 3, \ldots, n+1\}$ be the minimal number such that there is a q_Γ-dimensional Γ-invariant hyperbolic subspace of $\mathbf{B}^{n+1}$. The minimality of q_Γ implies easily that such a hyperbolic subspace is unique and we denote it by S_Γ. Note that S_Γ is also the unique hyperbolic subspace of $\mathbf{B}^{n+1}$ of minimal dimension whose closure contains $L(\Gamma)$.

Let Γ^{S_Γ} be the group that contains the restrictions to S_Γ of all the elements in Γ. Γ^{S_Γ} can be regarded as a Kleinian group acting on S_Γ, i.e., as a discrete subgroup of $\mathrm{HI}(S_\Gamma)$, the hyperbolic isometry group of S_Γ. Note that each $\alpha \in \mathrm{HI}(S_\Gamma)$ has a natural extension to $\bar{S}_\Gamma$. We will not distinguish between $\alpha \in \mathrm{HI}(S_\Gamma)$ and its extension to $\bar{S}_\Gamma$.

Let us define

$$N(\Gamma^{S_\Gamma}) = \{\alpha \in \mathrm{HI}(S_\Gamma)\colon \alpha\Gamma^{S_\Gamma}\alpha^{-1} = \Gamma^{S_\Gamma}\} \tag{3.7}$$

and

$$A(\Gamma^{S_\Gamma}) = \{\alpha \in \mathrm{HI}(S_\Gamma)\colon \alpha L(\Gamma) = L(\Gamma)\}. \tag{3.8}$$

Theorem 1.1 of [LWX] implies that $N(\Gamma^{S_\Gamma})$ is always a Kleinian group acting on S_Γ, and Corollary 3.1 of [W] implies that $A(\Gamma^{S_\Gamma})$ is a Kleinian group acting on S_Γ if and only if Γ^{S_Γ} is of the second kind, i.e., $L(\Gamma) \neq \bar{S}_\Gamma \cap \mathbf{S}^n$.

We can now prove the following general result.

Proposition 3.9. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Then there is $s \geq \delta_\Gamma$ and an s-conformal measure μ of Γ such that μ satisfies the conformal transformation rule (2.6) with respect to every $\beta \in N(\Gamma)$. Indeed, μ can be chosen so that μ satisfies (2.6) with respect to every $\beta \in \mathrm{M\ddot{o}b}(\mathbf{B}^{n+1})$ such that $\beta L(\Gamma) = L(\Gamma)$.*

Proof. We use the notation introduced above. We obtain that $N(\Gamma^{S_\Gamma})$ is a Kleinian group acting on S_Γ. Note that $\delta_\Gamma = \delta_{\Gamma^{S_\Gamma}} \leq \delta_{N(\Gamma^{S_\Gamma})}$. We can use the construction of Patterson and Sullivan to construct a $\delta_{N(\Gamma^{S_\Gamma})}$-conformal measure μ of $N(\Gamma^{S_\Gamma})$. It is clear that we can extend μ to be a measure with the following properties. The σ-algebra of μ-measurable sets is the σ-algebra of Borel sets of $\bar{\mathbf{R}}^{n+1}$. The measure μ is positive, finite and supported by $L(\Gamma) = L(N(\Gamma^{S_\Gamma}))$. The measure μ satisfies the conformal transformation rule (2.6) with respect to any $\beta \in \mathrm{M\ddot{o}b}(\mathbf{B}^{n+1})$ which maps S_Γ onto itself and whose restriction to S_Γ is contained in $N(\Gamma^{S_\Gamma})$. It is clear that any $\beta \in N(\Gamma)$ satisfies these conditions, which implies that we have proved the first part of the claim.

If $A(\Gamma^{S_\Gamma})$ is a Kleinian group acting on S_Γ, i.e. if $L(\Gamma) \neq \bar{S}_\Gamma \cap \mathbf{S}^n$, we can use the same argument as above, i.e. we first use the construction of Patterson and Sullivan to construct a $\delta_{A(\Gamma^{S_\Gamma})}$-conformal measure μ of $A(\Gamma^{S_\Gamma})$ and then extend μ in a suitable way. On the other hand, if $A(\Gamma^{S_\Gamma})$ is not a Kleinian group acting on S_Γ, which means that $L(\Gamma) = \bar{S}_\Gamma \cap \mathbf{S}^n$, we can choose μ to be the natural $(q_\Gamma - 1)$-dimensional measure of $L(\Gamma)$. □

We will end this section by proving a variant of Proposition 3.9 which uses the measure construction given in (3.5). The benefit of using the measures constructed by (3.5) is that they are relatively simple. It is also the case that in some situations we have a better control over the dimensionality of the constructed measures: If μ is an s-conformal measure of Γ constructed in Proposition 3.9, we know that $s \geq \delta_\Gamma$ but very little else; but we will see that the variant of Proposition 3.9 (i.e., Proposition 3.14) often constructs suitable s-conformal measures of Γ for any $s \geq \delta_\Gamma$ such that $P^s_\Gamma(x, y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. Before formulating and proving Propostition 3.14, we will prove the following three lemmas.

Lemma 3.10. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \dots, n\}$. Suppose that G is a Kleinian group acting on $\mathbf{B}^{n+1}$ such that $\Gamma \subset G$ and $L(G) = L(\Gamma)$. Then p is a bounded parabolic fixed point of G of rank k.*

Proof. It is trivial that p is a parabolic fixed point of G. Let $l \in \{k, k+1, \dots, n\}$ be the rank of p in G. Let α be a Möbius transformation of $\bar{\mathbf{R}}^{n+1}$ that maps $\mathbf{B}^{n+1}$ onto $\mathbf{H}^{n+1}$ and p to ∞, where

$$\mathbf{H}^{n+1} = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \colon x_{n+1} > 0\}$$

is the upper half-space of $\mathbf{R}^{n+1}$. Write $\Gamma^\alpha = \alpha\Gamma\alpha^{-1}$ and $G^\alpha = \alpha G \alpha^{-1}$. In order to show that p is a bounded parabolic fixed point of G of rank k, it is sufficient to show that ∞ is a bounded parabolic fixed point of G^α of rank k.

In the following, we will use a number of well-known results regarding parabolic Kleinian groups acting on $\mathbf{H}^{n+1}$ which have ∞ as the fixed point. We assume that the reader is familiar with these results and so we will not discuss them in detail. For a discussion on these and related results, see [Bo], [R2], [SusSw] or [Tu1], for example.

Recall that Γ^α_∞ and G^α_∞ denote the stabilizers of ∞ in Γ^α and G^α. There is a G^α_∞-invariant l-plane V of $\mathbf{R}^n$ with a compact G^α_∞-quotient. There is also a Γ^α_∞-invariant k-plane $W \subset V$ with a compact Γ^α_∞-quotient. Since ∞ is a bounded parabolic fixed point of Γ^α, it is the case that $L(\Gamma^\alpha) \setminus \{\infty\}$ is contained in a uniform euclidean neighbourhood of W, i.e., the euclidean distance of any $x \in L(\Gamma^\alpha) \setminus \{\infty\}$ from W is uniformly bounded. Since $L(G^\alpha) = L(\Gamma^\alpha)$, we conclude that $L(G^\alpha) \setminus \{\infty\}$ is contained in a uniform euclidean neighbourhood of W. Since G^α_∞ acts on $\mathbf{R}^n$ by euclidean isometries and there is a compact set $C \subset V$ such that $G^\alpha_\infty C = V$, we obtain that if $y \in V$, there is $z \in L(G^\alpha)$ such that $|y - z|$ is uniformly bounded. It follows that $W = V$ and $l = k$, so ∞ is a bounded parabolic fixed point of G^α of rank k. □

Lemma 3.11. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Suppose that $p \in L(\Gamma)$ is a parabolic fixed point of Γ of rank $k \in \{1, 2, \dots, n\}$.*

Suppose that G is a Kleinian group acting on $\mathbf{B}^{n+1}$ that has Γ as a normal subgroup. Then the orbit Gp is a pairwise disjoint union of Γ-orbits of parabolic fixed points of Γ of rank k.

Proof. It is trivial that $\Gamma p \subset Gp$. Suppose that $q \in Gp$, i.e., that $q = g(p)$ for some $g \in G$. Now q is a parabolic fixed point of rank k of $g\Gamma g^{-1} = \Gamma$, and so $\Gamma q \subset Gp$. The claim follows. □

Lemma 3.12. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \dots, n\}$. Let G be a Kleinian group acting on $\mathbf{B}^{n+1}$ such that $\Gamma \subset G$. Suppose that Gp is the pairwise disjoint union of Γp_1, Γp_2, $\dots$, Γp_m, where $p_1, p_2, \dots, p_m$ are bounded parabolic fixed points of Γ of rank k and $p_1 = p$. Let $s \geq \delta_\Gamma$ be such that $P^s_\Gamma(x, y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. Then there is a purely atomic s-conformal measure of G whose atoms are the points of Gp.*

Proof. Note first that $L(G) = L(\Gamma)$, so Lemma 3.10 implies that p is a bounded parabolic fixed point of G of rank k. Define the measure μ by setting that

$$\mu(p) = 1 \quad \text{and} \quad \mu(g(p)) = \int_{\{p\}} |g'|^s \, d\mu = |g'(p)|^s \mu(p) = |g'(p)|^s \tag{3.13}$$

for every $g \in G$. Comparing the definitions (3.5) and (3.13), we see that μ is well-defined and that in order to show that μ is the s-conformal measure of G we are looking for, we need to show only that μ is finite.

Let us define $\mu_1(p_1) = 1$ and $\mu_1(\gamma(p_1)) = |\gamma'(p_1)|^s$ for all $\gamma \in \Gamma$. According to the remark following the definition (3.5), μ_1 is an s-conformal measure of Γ. Observe that μ_1 is the same as the restriction of μ to Γp_1. Next, let $g_2 \in G$ be such that $g_2(p_1) = p_2$. Define $\mu_2(p_2) = \mu(p_2) = |g_2'(p_1)|^s$ and

$$\mu_2(\gamma(p_2)) = \int_{\{p_2\}} |\gamma'|^s \, d\mu_2 = |\gamma'(p_2)|^s \mu_2(p_2) = |\gamma'(p_2)|^s |g_2'(p_1)|^s$$

for every $\gamma \in \Gamma$. Then μ_2 is an s-conformal measure of Γ, and since

$$\mu(\gamma(p_2)) = \mu((\gamma \circ g_2)(p_1)) = |(\gamma \circ g_2)'(p_1)|^s = |\gamma'(p_2)|^s |g_2'(p_1)|^s = \mu_2(\gamma(p_2))$$

for every $\gamma \in \Gamma$, the measure μ_2 is the same as the restriction of μ to Γp_2. We can continue in this manner to construct the s-conformal measures $\mu_3, \mu_4, \dots, \mu_m$ of Γ such that, given $i \in \{1, 2, \dots, m\}$, the restriction of μ to Γp_i is μ_i. Now $\mu = \sum_{i=1}^m \mu_i$, which implies that μ is finite. We obtain that μ is a purely atomic s-conformal measure of G whose atoms are the points in Gp. □

Note that if the situation is as in Lemma 3.12, we obtain that $\delta_G = \delta_\Gamma$, which might be of independent interest. We can now prove the variant of Proposition 3.9 we mentioned earlier. We will use the notation introduced after Proposition 3.6.

Proposition 3.14. *Let Γ be a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \dots, n\}$. Then p is a bounded parabolic fixed point of $N(\Gamma^{S_\Gamma})$ of rank k and the $N(\Gamma^{S_\Gamma})$-orbit of p is a pairwise disjoint union of Γ-orbits of bounded parabolic fixed points of Γ of rank k. Furthermore, there is a purely atomic s-conformal measure μ of $N(\Gamma^{S_\Gamma})$ whose atoms are the points of $N(\Gamma^{S_\Gamma})p$ for every $s \geq \delta_{N(\Gamma^{S_\Gamma})}$ such that $P^s_{N(\Gamma^{S_\Gamma})}(x, y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$, and if the $N(\Gamma^{S_\Gamma})$-orbit of p is the pairwise disjoint union of finitely many Γ-orbits of bounded parabolic fixed points of Γ of rank*

k, we can construct such a measure μ for every $s \geq \delta_\Gamma$ such that $P^s_\Gamma(x,y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. In all cases, the measure μ can be extended to an s-conformal measure of Γ which satisfies the conformal transformation rule (2.6) with respect to any mapping in $N(\Gamma)$.

Proof. $N(\Gamma^{S_\Gamma})$ is a Kleinian group acting on S_Γ which has Γ^{S_Γ} as a normal subgroup. It follows that $L(N(\Gamma^{S_\Gamma})) = L(\Gamma^{S_\Gamma}) = L(\Gamma)$ and so Lemma 3.10 implies that p is a bounded parabolic fixed point of $N(\Gamma^{S_\Gamma})$ of rank k. Additionally, Lemma 3.11 implies that the $N(\Gamma^{S_\Gamma})$-orbit of p is a pairwise disjoint union of Γ-orbits of bounded parabolic fixed points of Γ of rank k. We can use the definition (3.5) to construct a purely atomic s-conformal measure μ of $N(\Gamma^{S_\Gamma})$ whose atoms are the points in $N(\Gamma^{S_\Gamma})p$, where $s \geq \delta_{N(\Gamma^{S_\Gamma})}$ is such that $P^s_{N(\Gamma^{S_\Gamma})}(x,y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. If $N(\Gamma^{S_\Gamma})p$ is the pairwise disjoint union of finitely many Γ-orbits of bounded parabolic fixed points of Γ of rank k, we can use Lemma 3.12 to show that the measure μ constructed above can be chosen to be an s-conformal measure of $N(\Gamma^{S_\Gamma})$ for any fixed $s \geq \delta_\Gamma$ such that $P^s_\Gamma(x,y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. It is clear that μ can be extended as described in the claim, see the proof of Proposition 3.9. □

We close this section by mentioning a relevant result of Matsuzaki and Yabuki (see [MatYab1] and [MatYab2]) which states that if any two δ_Γ-conformal measures of a non-elementary Kleinian group Γ acting on $\mathbf{B}^{n+1}$, $n \geq 1$, are the same up to a multiplicative constant, then any δ_Γ-conformal measure of Γ satisfies the transformation rule (2.6) with respect to any mapping in $N(\Gamma)$ (the proof is based on special properties of the measure construction of Patterson and Sullivan).

4. Conformal measures of geometrically finite Kleinian groups

In this section, we will take a closer look at conformal measures of geometrically finite Kleinian groups. The results of this section will be used in the last section of this paper.

Let Γ be a non-elementary geometrically finite Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. We mentioned earlier that if Γ is convex cocompact (i.e., Γ does not contain parabolic elements), then $s = \delta_\Gamma$ and any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant. Therefore, we assume that Γ contains parabolic elements. Recall from (2.3) that $L(\Gamma)$ is the pairwise disjoint union of the conical limit set $L_c(\Gamma)$ and finitely many orbits $\Gamma p_1, \Gamma p_2, \ldots, \Gamma p_m$ of bounded parabolic fixed points of Γ.

Suppose first that $s = \delta_\Gamma$. We mentioned earlier that now any two s-conformal measures of Γ are the same up to a multiplicative constant. Moreover, the μ-measure of parabolic fixed points of Γ is zero, see Section 3.5 of [Ni] or Section 2 of [Sul2], so $L_c(\Gamma)$ is of full μ-measure. In fact, Sullivan showed in [Sul2] that μ is sometimes the same as the standard δ_Γ-dimensional Hausdorff covering measure of $L(\Gamma)$ and sometimes the same as the standard δ_Γ-dimensional packing measure of $L(\Gamma)$ up to a multiplicative constant. (If Γ is convex cocompact, μ is the same as either one of these measures up to a multiplicative constant.) More precisely, if $k_{\min}$ and $k_{\max}$ denote the minimum and maximum of the ranks of parabolic fixed points of Γ, then μ is the covering measure up to a multiplicative constant in case $\delta_\Gamma \geq k_{\max}$ and μ is the packing measure up to a multiplicative constant in case $\delta_\Gamma \leq k_{\min}$. Sullivan showed also that if $k_{\min} < \delta_\Gamma < k_{\max}$, then $L(\Gamma)$ has zero measure with respect to the covering measure and infinite measure with respect to the packing measure. We showed in [A-M] that if the covering and packing measure constructions are modified in a suitable way, either one of them can always be used to construct a measure which is the same as μ up to a multiplicative constant. The conclusion is that μ is in every case (including the convex cocompact case) a constant multiple of a measure which is constructed by using the properties of $L(\Gamma)$ as a point set without any reference to the action of Γ and which satisfies the conformal transformation rule (2.6) with respect to any $\beta \in \text{Möb}(\mathbf{B}^{n+1})$ that maps $L(\Gamma)$ onto itself. Let us formulate this result as a proposition.

Proposition 4.1. *Let Γ be a non-elementary geometrically finite Kleinian group acting on $\mathbf{B}^{n+1}$, $n \geq 1$. Suppose that μ is a δ_Γ-conformal measure of Γ. Then μ satisfies the transformation rule (2.6) with respect to any $\beta \in \text{Möb}(\mathbf{B}^{n+1})$ that maps $L(\Gamma)$ onto itself.*

We consider next the case where $s > \delta_\Gamma$. It is not difficult to prove that $\mu(L_c(\Gamma)) = 0$ in this case (see Corollary 20 of [Sul1]), and so μ is the sum of its restrictions to the orbits $\Gamma p_1, \Gamma p_2, \ldots, \Gamma p_m$. In fact, if we use (3.5) to define the measure μ_i, where $i \in \{1, 2, \ldots, m\}$, by $\mu_i(p_i) = 1$ and $\mu_i(\gamma(p_i)) = |\gamma'(p_i)|^s$ for every $\gamma \in \Gamma$, we obtain a purely atomic s-conformal measure of Γ, and it follows that the s-conformal measures of Γ are exactly the measures $\sum_{i=1}^m a_i \mu_i$, where $a_i \geq 0$ for every $i \in \{1, 2, \ldots, m\}$ so that at least one of these numbers is strictly greater than 0.

We end this section by showing that Proposition 3.14 can be applied in the present situation to construct an s-conformal measure of Γ which satisfies the conformal transformation rule (2.6) with respect to any element in $N(\Gamma)$.

Proposition 4.2. *Let Γ be a non-elementary geometrically finite Kleinian group which acts on $\mathbf{B}^{n+1}$, $n \geq 1$, and which contains parabolic elements. Then, given any $s > \delta_\Gamma$, there is an s-conformal measure μ of Γ which satisfies the transformation rule (2.6) for any element in $N(\Gamma)$.*

Proof. Like in the discussion following the proof of Proposition 3.6, let S_Γ be the unique Γ-invariant hyperbolic subspace of $\mathbf{B}^{n+1}$ of minimal dimension. Define Γ^{S_Γ} and $N(\Gamma^{S_\Gamma})$ as in (3.7). Now $N(\Gamma^{S_\Gamma})$ is a Kleinian group acting on S_Γ and Γ^{S_Γ} is a geometrically finite normal subgroup of $N(\Gamma^{S_\Gamma})$. Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \ldots, n\}$. There are only finitely many pairwise disjoint Γ-orbits of bounded parabolic fixed points of Γ, so it follows that the $N(\Gamma^{S_\Gamma})$-orbit of p is the union of finitely many Γ-orbits of bounded parabolic fixed points of Γ of rank k. The existence of μ follows immediately from Proposition 3.14. □

We remark that if the situation is as in the proof of Proposition 4.2, Theorem 1 of [SusSw] implies that Γ^{S_Γ} is a finite index subgroup of $N(\Gamma^{S_\Gamma})$, which means that $N(\Gamma^{S_\Gamma})$ is geometrically finite with $\delta_{N(\Gamma^{S_\Gamma})} = \delta_\Gamma$. (The fact that Γ^{S_Γ} is of finite index in $N(\Gamma^{S_\Gamma})$ follows also from an argument contained in the proof of Theorem 1 in [R1].) It follows that we can apply (3.5) directly to construct μ without having to refer to Proposition 3.14. Furthermore, if $A(\Gamma^{S_\Gamma})$ (see (3.8)) is a Kleinian group acting on S_Γ, then Theorem 1 of [SusSw] can be used again to conclude that Γ^{S_Γ} is of finite index in $A(\Gamma^{S_\Gamma})$ and hence, since now $A(\Gamma^{S_\Gamma})$ is geometrically finite with $\delta_{A(\Gamma^{S_\Gamma})} = \delta_\Gamma$, the s-conformal measure μ constructed in Proposition 4.2 can be taken to satisfy the transformation rule (2.6) with respect to any $\beta \in \text{Möb}(\mathbf{B}^{n+1})$ mapping $L(\Gamma)$ onto itself.

5. Nayatani's metric tensors

In this section, we will define the primary objects of study of this paper, the metric tensors originally introduced by Nayatani in [Na]. Each of these tensors is associated with a Kleinian group, and we will repeat Nayatani's argument which shows that a given tensor is invariant under the action of the associated Kleinian group (Proposition 5.2). We will also point out, building on earlier results of Matsuzaki, Nayatani and Yabuki, that in some situations the elements in the normalizer of the associated Kleinian group induce scalings of the tensor (Proposition 5.5).

Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 1$. Let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Adapting a definition given in [Na], we define a metric tensor g^μ of $\Omega(\Gamma)$ which is compatible with the natural conformal structure of $\Omega(\Gamma)$ by setting that

$$g^\mu_x = \left(\int_{L(\Gamma)} \left(\frac{2}{|x-y|^2} \right)^s d\mu(y) \right)^{2/s} g^e_x \tag{5.1}$$

for every $x \in \Omega(\Gamma)$, where g^e is the standard euclidean metric tensor of $\mathbf{S}^n$. Originally, Nayatani gave this definition assuming that μ is a δ_Γ-conformal Patterson–Sullivan measure of Γ, but the definition works for any conformal measure of Γ.

It is easy to conclude that g^μ is Γ-invariant, i.e., that $\gamma^* g^\mu = g^\mu$ for every $\gamma \in \Gamma$, where $\gamma^* g^\mu$ is the pullback of g^μ with respect to γ (see page 118 of [Na]). In order to facilitate our discussion, we reproduce here the easy argument establishing the Γ-invariance of g^μ.

Proposition 5.2. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, $n \geq 1$, and let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Let g^μ be as in (5.1). Then $\gamma^* g^\mu = g^\mu$ for every $\gamma \in \Gamma$.*

Proof. Let $\gamma \in \Gamma$ and recall that

$$|\gamma(x) - \gamma(y)|^2 = |\gamma'(x)||\gamma'(y)||x-y|^2$$

for every $x, y \in \mathbf{S}^n$ and that

$$\gamma^* g^e = |\gamma'|^2 g^e.$$

Given $x \in \Omega(\Gamma)$, write

$$\lambda_\mu(x) = \left(\int_{L(\Gamma)} \left(\frac{2}{|x-y|^2} \right)^s d\mu(y) \right)^{2/s} \tag{5.3}$$

and note that (in the following, we use (2.7) and the fact that $L(\Gamma)$ is Γ-invariant)

$$\begin{aligned}
(\lambda_\mu \circ \gamma)(x) &= \left(\int_{\gamma L(\Gamma)} \left(\frac{2}{|\gamma(x) - y|^2} \right)^s d\mu(y) \right)^{2/s} \\
&= \left(\int_{L(\Gamma)} \left(\frac{2}{|\gamma(x) - \gamma(y)|^2} \right)^s |\gamma'(y)|^s \, d\mu(y) \right)^{2/s} \\
&= \left(\int_{L(\Gamma)} \left(\frac{2}{|\gamma'(x)||\gamma'(y)||x-y|^2} \right)^s |\gamma'(y)|^s \, d\mu(y) \right)^{2/s} = \frac{\lambda_\mu(x)}{|\gamma'(x)|^2}.
\end{aligned}$$

We conclude that

$$\gamma^* g^\mu = \gamma^*(\lambda_\mu g^e) = (\lambda_\mu \circ \gamma) \gamma^* g^e = \frac{\lambda_\mu}{|\gamma'|^2} |\gamma'|^2 g^e = \lambda_\mu g^e = g^\mu, \tag{5.4}$$

i.e., that g^μ is Γ-invariant. □

The following result related to Proposition 5.2 will be useful in our discussion in the last section of this paper.

Proposition 5.5. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, $n \geq 1$, and let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Let g^μ be as in (5.1). Suppose that for every $\beta \in N(\Gamma)$ there is $b_\beta > 0$ such that $\beta^s_* \mu = b_\beta \mu$. Then it is true that if $\beta \in N(\Gamma)$, there is $c_\beta > 0$ such that $\beta^* g^\mu = c_\beta g^\mu$.*

Proof. Let $\beta \in N(\Gamma)$. According to our assumption, there is $b_\beta > 0$ such that $\beta^s_* \mu = b_\beta \mu$. It follows that

$$\int_{\beta A} \phi \, d\mu = \int_{\beta A} \phi b_\beta^{-1} \, d\beta^s_* \mu = \int_A (\phi \circ \beta) b_\beta^{-1} |\beta'|^s \, d\mu$$

for every μ-measurable set A and every μ-measurable function ϕ. Recall the definition of the function λ_μ from (5.3). Arguing as in the proof of Proposition 5.2, we conclude that

$$\lambda_\mu \circ \beta = \frac{b_\beta^{-2/s} \lambda_\mu}{|\beta'|^2},$$

and using (5.4) we see that

$$\beta^* g^\mu = c_\beta g^\mu,$$

where $c_\beta = b_\beta^{-2/s}$. □

The condition of Proposition 5.5 is satisfied, for example, when any two s-conformal measures of Γ are the same up to a multiplicative constant (e.g., Γ is of divergence type and $s = \delta_\Gamma$), see Proposition 3.4, or when the situation is as described in Proposition 3.6. Results equivalent to Proposition 5.5 with the assumption that any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant have been proved by Nayatani (see Lemmas 4.2 and 4.3 and Proposition 4.4 of [Na]) and Matsuzaki and Yabuki (see Lemmas 3.2 and 3.3 of [MatYab1] and Lemma 3.1 of [Yab]).

6. Scalar curvatures of Nayatani's tensors

We are now in a position to discuss our first main result. Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Let g^μ be the metric tensor of $\Omega(\Gamma)$ defined in (5.1).

One of the main results in [Na] is the following. Suppose for the time being that μ is a δ_Γ-conformal Patterson–Sullivan measure of Γ. Write $N = (n-2)/2$. It is now the case that if $\delta_\Gamma < N$, $\delta_\Gamma = N$ or $\delta_\Gamma > N$, then the scalar curvature of g^μ is positive everywhere, zero everywhere or negative everywhere, respectively. Also, if $\delta_\Gamma > n-2$, then the Ricci curvature of g^μ is negative everywhere. In [Na], this result is contained in Theorem 3.3.

The proof of Theorem 3.3 in [Na] is a straightforward calculation, see pages 120 and 121 of [Na]. From the point of view of this paper, it is essential to note that Nayatani's argument remains valid if we replace the δ_Γ-conformal Patterson–Sullivan measure of Γ with any s-conformal measure of Γ. We obtain the following result.

Theorem 6.1. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Suppose that μ is an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. Let g^μ be the metric tensor defined in (5.1). Write $N = (n-2)/2$. It is now true that if $s < N$, $s = N$ or $s > N$, then the scalar curvature of g^μ is positive* *everywhere, zero everywhere or negative everywhere, respectively. Additionally, if $s > n-2$, then the Ricci curvature of g^μ is negative everywhere.*

Recall that Theorem 4.1 of [AFTu] and Theorem 2.19 of [Sul3] imply that if Γ is not convex cocompact, then there are s-conformal measures of Γ for every $s \geq \delta_\Gamma$. We may deduce the following.

Corollary 6.2. *Let the situation be as in Theorem 6.1. Suppose additionally that Γ is not convex cocompact. Then we can always choose a metric tensor g^μ whose scalar curvature is negative everywhere, and g^μ can be chosen so that its Ricci curvature is negative everywhere as well. Additionally, if $\delta_\Gamma < N$, we can choose a metric tensor g^μ whose scalar curvature is positive everywhere and another tensor whose scalar curvature is zero everywhere.*

7. Compact Kleinian manifolds

Suppose that Γ is a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Let M be a non-empty, open and connected subset of $\Omega(\Gamma)/\Gamma$ and let O be a component of the preimage of M under the natural projection $\Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$. Write $\Gamma_O = \{\gamma \in \Gamma \colon \gamma O = O\}$ and suppose that no non-trivial element in Γ_O has a fixed point in O. Then $M = O/\Gamma_O$ is a manifold with a natural conformal structure, i.e., the projection of the natural conformal structure of O. The manifold M with its natural conformal structure is called *a Kleinian manifold*.

The results of the previous section are particularly interesting in the situation where M is compact. In this situation, it is the case that if g_1 and g_2 are conformally equivalent metric tensors of M, then the scalar curvature of g_1 is positive everywhere, zero everywhere or negative everywhere and the sign of the scalar curvature of g_2 is the same as that of g_1. This result is well-known, see Lemma 1.1 of [ScYau] for example. Also, the result is mentioned on page 123 in [Na].

Since the metric tensors defined in (5.1) are obtained by scaling the standard metric tensor, any two of them are conformally equivalent. Also, since these tensors are Γ-invariant, see Proposition 5.2, they can be projected to M. Therefore, if M is compact, the signs of the scalar curvatures of the projections to M of any two metric tensors defined in (5.1) are the same.

Recall that if Γ is not convex cocompact, then there are s-conformal measures of Γ for every $s \geq \delta_\Gamma$, and that if Γ is convex cocompact, then every conformal measure of Γ is δ_Γ-conformal. Using the results of the previous section, we obtain the following theorem.

Theorem 7.1. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Let M be a compact Kleinian manifold contained in $\Omega(\Gamma)/\Gamma$ as defined at the beginning of this section. Write $N = (n-2)/2$. It is now true that M has a metric tensor which is compatible with the natural conformal structure of M and whose scalar curvature is positive everywhere, zero everywhere or negative everywhere (and the sign of the scalar curvature is the same for every metric tensor of M which is compatible with the natural conformal structure) if and only if $\delta_\Gamma < N$, $\delta_\Gamma = N$ or $\delta_\Gamma > N$, respectively. Additionally, if $\delta_\Gamma \leq N$, or equivalently, if there is a metric tensor of M which is compatible with the natural conformal structure of M and whose scalar curvature is positive everywhere or zero everywhere, then Γ is convex cocompact.*

The first part of Theorem 7.1 is the same as Corollary 3.4 in [Na]. The second part of Theorem 7.1 is contained in Proposition 7 of Izeki's paper [I2]. In order to discuss Izeki's result in greater detail, we formulate it as the following corollary of Theorem 7.1. For the sake of completeness, we repeat Izeki's argument up to the point where we can apply Theorem 7.1. It is clear that applying Theorem 7.1 is much simpler than using the argument given by Izeki on pages 3736 and 3737 of [I2].

Corollary 7.2. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Suppose that $\Omega(\Gamma)/\Gamma$ has a non-empty compact component K and that $\delta_\Gamma \leq (n-2)/2$. Then $\Omega(\Gamma)$ is connected and Γ is convex cocompact.*

Proof. Let O_K be a component of the preimage of K with respect to the natural projection $\Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$. Let $\Gamma_K = \{\gamma \in \Gamma : \gamma O_K = O_K\}$, so $K = O_K/\Gamma_K$. The compactness of K implies that Γ_K is finitely generated, and so Selberg's lemma implies that Γ_K has a torsion-free subgroup Γ'_K of finite index. The exponent of convergence δ'_K of Γ'_K is smaller than or equal to δ_Γ, so it is strictly smaller than $n-1$, since $\delta_\Gamma \leq (n-2)/2$. Also, O_K/Γ'_K is a compact Kleinian manifold. Proposition 3.2 of [I1] implies that $\Omega(\Gamma'_K) = O_K$ and that the Hausdorff dimension of $\mathbf{S}^n \setminus O_K$ is smaller than or equal to δ'_K. In particular, $\mathbf{S}^n \setminus O_K$ has no interior points in $\mathbf{S}^n$, so $\Omega(\Gamma) = O_K$. This means that $\Omega(\Gamma)$ is connected, that $\Gamma_K = \Gamma$, and that $\Omega(\Gamma)/\Gamma = K$. In order to show that Γ is convex cocompact, it is enough to show that the finite index subgroup Γ'_K of Γ is convex cocompact. But the convex cocompactness of Γ'_K follows from Theorem 7.1. □

Theorem 1 of [I2] is actually somewhat stronger than Proposition 7 discussed above. According to Theorem 1 of [I2], $\Omega(\Gamma)$ is connected and Γ is convex cocompact if Γ is a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, where $n \geq 3$, whose exponent of convergence δ_Γ is strictly smaller than $n/2$ and whose quotient space $\Omega(\Gamma)/\Gamma$ contains a non-empty compact component K. In fact, Theorem 1 of [I2] states more generally that if K exists and $\delta_\Gamma < n-1$, then $\Omega(\Gamma)$ is connected and Γ is geometrically finite. (Recall that a convex cocompact Kleinian group is a geometrically finite Kleinian group containing no parabolic elements.) We remark that in the paper [CQYan] it is proved that if Γ is a non-elementary Kleinian group acting on $\mathbf{B}^{n+1}$, $n \geq 3$, such that $\Omega(\Gamma)/\Gamma$ is non-empty and compact, then Γ is geometrically finite if and only if the Hausdorff dimension of $L(\Gamma)$ is strictly smaller than n (recall from [BiJ] the well-known result stating that in general the exponent of convergence of a non-elementary Kleinian group is equal to the Hausdorff dimension of the conical limit set of the group).

8. Isometry groups of Nayatani's metric tensors

Suppose again that Γ is a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Let $O \subset \Omega(\Gamma)$ be a non-empty, open and connected set which is mapped onto itself by every element in Γ. Suppose that no non-trivial element in Γ has a fixed point in O. It follows that the quotient $M = O/\Gamma$ is a Kleinian manifold.

Suppose that μ is an s-conformal measure of Γ for some $s \geq \delta_\Gamma$. From the definition (5.1), we obtain the metric tensor g^μ on $\Omega(\Gamma)$, and we denote by g^μ_M the projection of this metric tensor to M.

If α is a g^μ_M-isometry of M, then it is trivial that α is a conformal automorphism of M. The purpose of this section is to discuss sufficient conditions guaranteeing that the converse is true, i.e., that every conformal automorphism of M is a g^μ_M-isometry.

Our discussion is based on the following characterization of conformal automorphisms of M given by Lemma 4.1 of [Na]. Recall that $N(\Gamma)$ denotes the normalizer of Γ in $\text{Möb}(\mathbf{B}^{n+1})$. Denote by $N_O(\Gamma)$ the subgroup of $N(\Gamma)$ containing the elements mapping O onto itself. If $\beta \in N_O(\Gamma)$, we obtain the induced mapping $\bar\beta\colon M \to M$ given by $\bar\beta(\Gamma x) = \Gamma\beta(x)$ for every $x \in O$. According to Lemma 4.1 of [Na], it is the case that α is a conformal automorphism of M if and only if $\alpha = \bar\beta$ for some $\beta \in N_O(\Gamma)$, and two mappings $\beta_1, \beta_2 \in N_O(\Gamma)$ induce the same conformal automorphism of M if and only if $\beta_2 = \beta_1 \circ \gamma$ for some $\gamma \in \Gamma$.

Let α be an arbitrary conformal automorphism of M. Our aim is to establish sufficient conditions which guarantee that α is a g^μ_M-isometry, i.e., that $\alpha^* g^\mu_M = g^\mu_M$. In the literature, the conclusion $\alpha^* g^\mu_M = g^\mu_M$ is usually obtained in the following way. One shows first that, under suitable circumstances, there is a constant $c_\alpha > 0$ such that $\alpha^* g^\mu_M = c_\alpha g^\mu_M$, and then one shows that it is in fact the case that $c_\alpha = 1$. Of course, it is true that $\alpha^* g^\mu_M = c_\alpha g^\mu_M$ if $\beta^* g^\mu = c_\alpha g^\mu$, where $\beta \in N_O(\Gamma)$ is such that $\bar\beta = \alpha$, so we are led to consider the group $N_O(\Gamma)$, or more generally, the full normalizer $N(\Gamma)$.

The first isometry result on Nayatani's metric tensors was proved by Nayatani himself. In Proposition 4.4 of [Na], he shows that if $s = \delta_\Gamma$, if any δ_Γ-conformal measure of Γ is the same as μ up to a multiplicative constant, and if the metric induced by g^μ_M is complete, then any conformal automorphism of M is a g^μ_M-isometry. The proof consists of two parts. Nayatani shows first that since any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant, it is the case that if $\beta \in N(\Gamma)$, then there is a constant $c_\beta > 0$ such that $\beta^* g^\mu = c_\beta g^\mu$, see Lemmas 4.2 and 4.3 of [Na]. He shows then that if $\beta \in N_O(\Gamma)$ and $c_\beta \neq 1$, then $\bar\beta$ has a fixed point in M (because of the completeness of the induced metric), which in turn can be used to deduce the contradiction that Γ is elementary.

In Nayatani's argument, the existence of the constant c_β follows from the existence of a constant $b_\beta > 0$ such that $\beta^s_* \mu = b_\beta \mu$, see our Proposition 5.5 (in [Na] this is formulated in a different but equivalent way), and the existence of b_β follows if every two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant, see Proposition 3.4.

It is clear that Nayatani's argument can be applied to s-conformal measures of Γ for any $s \geq \delta_\Gamma$ and not just δ_Γ-conformal measures of Γ. Also, it is possible that the constants b_β considered above exist even if it is not necessarily true that any two s-conformal measures of Γ are the same up to a multiplicative constant, see Proposition 3.6. We obtain the following generalization of Nayatani's result.

Theorem 8.1. *Let Γ, M, μ, g^μ and g^μ_M be as defined at the beginning of this section. Suppose that if $\beta \in N_O(\Gamma)$, then there is a constant $b_\beta > 0$ such that $\beta^s_* \mu = b_\beta \mu$. Suppose also that the metric induced by g^μ_M is complete. Then every conformal automorphism of M is a g^μ_M-isometry.*

Nayatani notes in Corollary 4.5 of [Na] that his result is true if Γ is convex cocompact and μ is a Patterson–Sullivan measure of Γ (the convex cocompactness of Γ implies that M is compact (and hence the metric induced by g^μ_M is complete) and that any two δ_Γ-conformal measures of Γ are the same to a multiplicative constant).

We observe that Theorem 8.1 is applicable if M is compact and μ is a purely atomic measure as described in Proposition 3.6.

Regarding Nayatani's isometry result, Maubon showed in [Mau] that if Γ is geometrically finite and μ is δ_Γ-conformal, then the metric induced by g^μ_M is not necessarily complete. It is not possible, therefore, to use Nayatani's argument to conclude that every conformal automorphism of M is a g^μ_M-isometry in this case. However, Yabuki showed in [Yab] that the conclusion holds nevertheless. The first part of Yabuki's argument is the same as Nayatani's, i.e., he concludes that, since any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant, there is a constant $c_\beta > 0$ for any $\beta \in N(\Gamma)$ such that $\beta^* g^\mu = c_\beta g^\mu$. Moreover, the main idea in the second part of Yabuki's argument is the same as in Nayatani's, i.e., Yabuki shows that if $\beta \in N_O(\Gamma)$ and $c_\beta \neq 1$, then $\bar{\beta}$ has a fixed point, which can be used to deduce the contradiction that Γ is elementary. More precisely, Yabuki notes first that g^μ can be extended to a metric tensor of $\mathbf{B}^{n+1} \cup \Omega(\Gamma)$ (it is clear that we can have $x \in \mathbf{B}^{n+1}$ in (5.1); see also Section 2 of [INa]) and that $\bar{\beta}$ can be extended to $(\mathbf{B}^{n+1} \cup \Omega(\Gamma))/\Gamma$. The so-called ε-thick part C^ε_Γ of the so-called convex core of Γ is mapped onto itself by $\bar{\beta}$, and C^ε_Γ is compact because Γ is geometrically finite. Therefore, if $c_\beta \neq 1$, one can show that $\bar{\beta}$ has a fixed point in C^ε_Γ. See Section 3 of [Yab] for the details.

In their paper [MatYab1] (see also [MatYab2]), Matsuzaki and Yabuki generalize the isometry results in [Na] and [Yab] to the case where any two δ_Γ-conformal measures of Γ are the same up to a multiplicative constant (Γ does not have to be geometrically finite or even of divergence type). The first part of the argument is the same as before, i.e. they show that, given $\beta \in N(\Gamma)$, there is a constant $c_\beta > 0$ such that $\beta^* g^\mu = c_\beta g^\mu$, where μ is a δ_Γ-conformal measure of Γ. The argument that shows that in fact $c_\beta = 1$ depends essentially on the properties of the construction used to obtain Patterson–Sullivan measures, and our results in this paper do not provide generalizations in this case.

Next, we wish to point out the following simple condition that guarantees that if the constants c_β, where $\beta \in N(\Gamma)$, considered above exist, then we have in fact that $c_\beta = 1$ in all cases. The condition is that the constants c_β are uniformly bounded away from 0 and ∞. (Observe that $c_{\beta^{-1}} = c_\beta^{-1}$.)

Theorem 8.2. *Let Γ, μ and g^μ be as at the beginning of this section. Suppose that there is a constant $c > 1$ satisfying the following. If $\beta \in N(\Gamma)$, then there is $c_\beta \in [c^{-1}, c]$ such that $\beta^* g^\mu = c_\beta g^\mu$. Then it is in fact true that $c_\beta = 1$ for every $\beta \in N(\Gamma)$.*

Proof. Let $c_{N(\Gamma)} = \sup\{c_\beta \colon \beta \in N(\Gamma)\}$. It is clear that if $\beta_1, \beta_2 \in N(\Gamma)$, then $c_{\beta_1 \circ \beta_2} = c_{\beta_1} c_{\beta_2}$. So if $\beta_0 \in N(\Gamma)$ is fixed, then

$$c_{N(\Gamma)} = \sup\{c_{\beta_0 \circ \beta} \colon \beta \in N(\Gamma)\} = \sup\{c_{\beta_0} c_\beta \colon \beta \in N(\Gamma)\} = c_{\beta_0} c_{N(\Gamma)},$$

and so $c_{\beta_0} = 1$. □

The constants c_β, $\beta \in N(\Gamma)$, are uniformly bounded away from 0 and ∞ for example in the case where the index of Γ in $N(\Gamma)$ is finite, since then there are $\alpha_1, \alpha_2, \ldots, \alpha_k \in N(\Gamma)$ such that any $\beta \in N(\Gamma)$ can be written as $\beta = \alpha_{i_\beta} \circ \gamma_\beta$ for some $i_\beta \in \{1, 2, \ldots, k\}$ and $\gamma_\beta \in \Gamma$, and so $c_\beta = c_{\alpha_{i_\beta}} c_{\gamma_\beta} = c_{\alpha_{i_\beta}}$.

Recall from the proof of Proposition 5.5 that if b_β and c_β are constants as in the above discussion, then they are related by $c_\beta = b_\beta^{-2/s}$ for every $\beta \in N(\Gamma)$. It follows that $c_\beta = 1$ if $b_\beta = 1$. Recall also that the condition $b_\beta = 1$ is equivalent to the condition that μ satisfies the conformal transformation rule (2.6) with respect to β. We conclude that if Γ, μ, M and g^μ_M are as at the beginning of this section, then every conformal automorphism of M is a g^μ_M-isometry in case μ satisfies the conformal transformation rule (2.6) with respect to every $\beta \in N_O(\Gamma)$. According to Proposition 3.9, there always exists such a measure μ. Let us write this down.

Theorem 8.3. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, where $n \geq 3$. Then there exists $s \geq \delta_\Gamma$ and an s-conformal measure μ of Γ such that if g^μ is defined as in (5.1), then $\beta^* g^\mu = g^\mu$ for every $\beta \in N(\Gamma)$. Indeed, μ can be chosen so that $\beta^* g^\mu = g^\mu$ for every $\beta \in \mathrm{M\ddot{o}b}(\mathbf{B}^{n+1})$ that maps $L(\Gamma)$ onto itself. It follows that if M and g^μ_M are as at the beginning of this section, then every conformal automorphism of M is a g^μ_M-isometry.*

Observe that Proposition 3.14 discusses situations where the measure μ of Theorem 8.3 (satisfying $\beta^* g^\mu = g^\mu$ for every $\beta \in N(\Gamma)$) can be chosen so that s is any fixed number that satisfies the condition $P^s_\Gamma(x,y) < \infty$ for some $x, y \in \mathbf{B}^{n+1}$. In particular, Proposition 3.14 is applicable in the case where Γ is a geometrically finite Kleinian group containing parabolic elements, see Proposition 4.2. We mention also that the main result of [MatYab1] discussed above is based on the fact that, in the context of [MatYab1], any Patterson–Sullivan measure of the given Kleinian group Γ satisfies the transformation rule (2.6) with respect to any $\beta \in N(\Gamma)$.

Finally, we point out that we in fact obtain the main result of [Yab] mentioned above as a corollary of the main results of [A-M], see Proposition 4.1. Actually, we obtain the following more general result.

Theorem 8.4. *Let Γ be a non-elementary Kleinian group of the second kind acting on $\mathbf{B}^{n+1}$, $n \geq 3$. Suppose that G is a non-elementary geometrically finite Kleinian group acting on $\mathbf{B}^{n+1}$ such that $L(\Gamma) = L(G)$. Let μ be a δ_G-conformal measure of G. Then μ is a δ_G-conformal measure of Γ and μ satisfies the transformation rule (2.6) with respect to any element in $N(\Gamma)$. It is also true that if g^μ is a metric tensor of $\Omega(\Gamma)$ as defined in (5.1) and if M is a Kleinian manifold contained in $\Omega(\Gamma)/\Gamma$ as described at the beginning of this section, then every conformal automorphism of M is a g^μ_M-isometry.*

We close our exposition by noting that our remarks following Proposition 4.2 discuss situations where the conformal measure μ of the group G in Theorem 8.4 can be chosen to be s-conformal for any fixed $s > \delta_G$.

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