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A SUFFICIENT CONDITION FOR A FINITE FAMILY OF CONTINUOUS FUNCTIONS TO BE TRANSFORMED INTO ψ-CONTRACTIONS

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Abstract. Given a metric space (X,d) and a finite set of continuous functions $f_1, f_2, \ldots, f_N \colon X \to X$, we provide a sufficient condition to find a metric δ on X, equivalent with d, and a comparison function ψ such that the functions $f_i \colon (X,\delta) \to (X,\delta)$ are ψ-contractions. If the metric space (X,d) is complete, the same condition assures the existence of a unique fixed point of the function $F \colon \mathcal{K}(X) \to \mathcal{K}(X)$ given by $\mathcal{F}(C) = \bigcup_{i=1}^{N} f_i(C)$ for each $C \in \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the family of non-empty and compact subsets of X.

1. Introduction

Given a bounded complete metric space (X,d) and a contraction $f \colon X \to X$, the Picard–Banach–Caccioppoli principle implies that f has a unique fixed point x_0 and $\bigcap_{n=1}^{\infty} f^n(X) = \{x_0\}$. As this equality has a topological character, the following question is natural: Let X be a compact metrizable topological space and $f \colon X \to X$ a continuous function having the property that there exists $x_0 \in X$ such that $\bigcap_{n=1}^{\infty} f^n(X) = \{x_0\}$. It is possible to find a metric δ on X generating the given topology of X such that f is contraction with respect to δ? Janoš (see [6]) gave an affirmative answer to this question. See also [3] for a similar result.

Along the same lines of research, Leader (see [9]), providing a generalization of Janoš's result, proved that a continuous function f on a metric space (X,d) is a contraction with fixed point $x_0 \in X$ under some metric δ on X equivalent to d if and only if every orbit $(f^n(x))_{n\in\mathbf{N}}$ converges to x_0 and the convergence is uniform on some neighborhood of x_0.

The natural generalization of the above limit condition for an iterated function system was introduced by Kieninger (see [8]) under the name of point-fibred iterated function systems.

Atkins, Barnsley, Vince and Wilson (see [1]) provided a generalization of the results proved by Janoš and Leader (see also [10]) by giving a characterization of hyperbolic affine iterated function systems defined on $\mathbf{R}^m$.

In order to provide a topological generalization of the notion of attractor of an iterated function system consisting of contractions Kameyama introduced the concept of self-similar system and asked the following fundamental question (see [7]): Given a topological self-similar system $(K, \{f_i\}_{i\in\{1,2,\ldots,N\}})$, does there exist a metric on K compatible to the topology such that all the functions f_i are contractions? Such a

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metric is called a self-similar metric. Kameyama provided a topological self-similar set which does not admit a self-similar metric and, on the other hand, he proved that every totally disconnected self-similar set and every non-recurrent finitely ramified self-similar set have a self-similar metric. In [12], we modified Kameyama's question by weakening the requirement that the functions in the topological self-similar system be contractions to requiring that they be φ-contractions. More precisely we gave an affirmative answer to the following question: given a topological self-similar system $(K,(f_i)_{i\in\{1,2,\ldots,N\}})$ does there exist a metric δ on K which is compatible with the original topology and a comparison function φ such that $f_i\colon (K,\delta)\to(K,\delta)$ is φ-contraction for each $i\in\{1,2,\ldots,N\}$? In [13] we obtained a generalization of the above mentioned affirmative answer to modified Kameyama's question studying the case of a possibly infinite family of functions $(f_i)_{i\in I}$. For related results see [2].

Let (X,d) be a metric space, $N\in\mathbf{N}$ and $f_i\colon X\to X$, $i\in\{1,2,\ldots,N\}$, continuous functions. Inspired by the notions of locally uniformly contractive fixed point (see [10]), point-fibred iterated function system (see [1]) and uniformly point-fibred iterated function system (see [11]), in the present paper we provide a sufficient condition (referred to as Condition C) on the set of functions $\{f_1,f_2,\ldots,f_N\}$ in order to find a metric δ on X, equivalent with d, and a comparison function ψ such that the functions $f_i\colon (X,\delta)\to(X,\delta)$ are ψ-contractions. The Condition C is fulfilled if the functions $f_1,f_2,\ldots,f_N$ are ψ-contractions.

This goal is achieved in the following four steps.

Step 1. Condition C allows us to define a compact subset K of X such that $K=\bigcup_{i=1}^{N} f_i(K)$.

Step 2. We construct a metric ρ on X, equivalent with d, such that $\rho(f_i(x),f_i(y))\le\rho(x,y)$ for each $x,y\in X$ and each $i\in\{1,2,\ldots,N\}$.

Step 3. We construct a metric $\widetilde{\rho}$ on X, equivalent with ρ (so with d), a comparison function φ and an open set U such that $K\subseteq U$ and the functions $f_i\colon (U,\widetilde{\rho})\to(X,\widetilde{\rho})$ are φ-contractions.

Step 4. We construct a metric δ on X (actually a family of metrics), equivalent with d, and a comparison function ψ such that the functions $f_i\colon (X,\delta)\to(X,\delta)$ are ψ-contractions.

Condition C proved to be also a sufficient condition for the existence of a unique fixed point of the function $\mathcal{F}\colon \mathcal{K}(X)\to\mathcal{K}(X)$ given by $\mathcal{F}(C)=\bigcup_{i=1}^{N} f_i(C)$ for each $C\in\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the family of non-empty and compact subsets of X. Actually the above mentioned fixed point is K.

2. Preliminaries

Definition 2.1. (Comparison function) A function $\varphi\colon [0,\infty)\to[0,\infty)$ is called a comparison function if it has the following properties:

(i) φ is increasing (i.e. $t_1<t_2\Rightarrow\varphi(t_1)\le\varphi(t_2)$ for each $t_1,t_2\ge 0$);
(ii) $\varphi(t)<t$ for any $t>0$;
(iii) φ is right-continuous.

Definition 2.2. (φ-contraction) Let (X,d) be a metric space and a function $\varphi\colon [0,\infty)\to[0,\infty)$. A function $f\colon X\to X$ is called a φ-contraction if

$$d(f(x),f(y))\le\varphi(d(x,y)),$$

for all $x,y\in X$.

In the following $\mathbf{N}$ denotes the natural numbers, $\mathbf{N}^*= \mathbf{N} \setminus \{0\}$ and $\mathbf{N}_n^* = \{1, 2, \ldots, n\}$, where $n \in \mathbf{N}^*$. Given two sets A and B, by B^A we mean the set of functions from A to B. By $\Lambda(B)$ we mean the set $B^{\mathbf{N}^*}$ and by $\Lambda_n(B)$ we mean the set $B^{\mathbf{N}_n^*}$. The elements of $\Lambda(B) = B^{\mathbf{N}^*}$ are written as words $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ and the elements of $\Lambda_n(B) = B^{\mathbf{N}_n^*}$ are written as words $\omega = \omega_1\omega_2 \ldots \omega_n$ (n—which is the length of ω—is denoted by $|\omega|$). Hence $\Lambda(B)$ is the set of infinite words with letters from the alphabet B and $\Lambda_n(B)$ is the set of words of length n with letters from the alphabet B. By $\Lambda^*(B)$ we denote the set of all finite words, i.e. $\Lambda^*(B) = \bigcup_{n\in\mathbf{N}^*} \Lambda_n(B) \cup \{\lambda\}$, where λ is the empty word. If $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots \in \Lambda(B)$ or if $\omega = \omega_1\omega_2 \ldots \omega_n \in \Lambda_n(B)$, where $m, n \in \mathbf{N}^*$, $n \geq m$, then the word $\omega_1\omega_2 \ldots \omega_m$ is denoted by $[\omega]_m$. For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$, by $\alpha\beta$ we mean the concatenation of the words α and β, i.e. $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m\beta_{m+1} \ldots$. For $f_i\colon X \to X$, $i \in B$, we denote Id_X by f_λ and $f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_m}$ by $f_{\alpha_1\alpha_2\ldots\alpha_m}$ for each $\alpha_1, \alpha_2, \ldots, \alpha_m \in B$.

For a nonvoid set I, on $\Lambda(I) = (I)^{\mathbf{N}^*}$, we consider the metric $d_\Lambda(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1 - \delta_{\alpha_k}^{\beta_k}}{3^k}$, where $\delta_x^y = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$

Remark 2.1. The convergence in the complete metric space $(\Lambda(I), d_\Lambda)$ is the convergence on components.

Definition 2.3. (Iterated function system) Given a metric space (X, d), an iterated function system is a pair $\mathcal{S} = ((X, d), (f_i)_{i\in\{1,2,\ldots,N\}})$, where $f_i\colon X \to X$ is continuous for each $i \in \{1, 2, \ldots, N\}$.

Definition 2.4. (φ-contractive iterated function system) Given a comparison function $\varphi\colon [0, \infty) \to [0, \infty)$, an iterated function system $\mathcal{S} = ((X, d), (f_i)_{i\in\{1,2,\ldots,N\}})$ is called φ-contractive if f_i is φ-contraction for each $i \in \{1, 2, \ldots, N\}$.

Definition 2.5. (φ-hyperbolic iterated function system). Given a comparison function $\varphi\colon [0, \infty) \to [0, \infty)$, an iterated function system $\mathcal{S} = ((X, d), (f_i)_{i\in\{1,2,\ldots,N\}})$ is called φ-hyperbolic if there exists a metric δ on X, equivalent to d, such that the iterated function system $((X, \delta), (f_i)_{i\in\{1,2,\ldots,N\}})$ is φ-contractive.

Theorem 2.1. (see Theorem 3.11 from [14]) *Given a comparison function $\varphi\colon [0, \infty) \to [0, \infty)$ and a complete metric space (X, d), for each φ-contractive iterated function system $\mathcal{S} = ((X, d), (f_i)_{i\in\{1,2,\ldots,N\}})$ there exists a unique non-empty compact subset $A(\mathcal{S})$ of X such that $A(\mathcal{S}) = \bigcup_{i=1}^{N} f_i(A(\mathcal{S}))$.*

3. The result

Definition 3.1. Let us consider a metric space (X, d), the continuous functions $f_1, \ldots, f_N\colon X \to X$ and a function $\pi\colon \Lambda \to X$, where $\Lambda = \Lambda(\{1, 2, \ldots, N\})$. We say that the condition C (for the metric d) is fulfilled if

$$\forall_{x\in X}\ \exists_{\varepsilon_x>0}\ \forall_{\delta>0}\ \exists_{n_{x,\varepsilon_x,\delta}\in\mathbf{N}}\ \forall_{n\in\mathbf{N},\ n\geq n_{x,\varepsilon_x,\delta}}\ \forall_{\omega\in\Lambda}\ \forall_{y\in B(x,\varepsilon_x)}\ d(f_{[\omega]_n}(y), \pi(\omega)) < \delta.$$

In other words, Condition C says that for each $x \in X$ there exists $\varepsilon_x > 0$ such that

$$\lim_{n\to\infty} f_{[\omega]_n}(y) = \pi(\omega)$$

uniformly with respect to $y \in B(x, \varepsilon_x)$ and $\omega \in \Lambda$.

In the sequel, for the sake of simplicity, we denote $\pi(\omega)$ by π_ω.

Remark 3.1. Condition C is fulfilled if there exists a comparison function ψ such that the functions $f_1, f_2, \ldots, f_N \colon (X,d) \to (X,d)$ are ψ -contractions, where (X,d) is a complete metric space.

Indeed, if $\mathcal{F} \colon \mathcal{B}(X) \to \mathcal{B}(X)$ is given by $\mathcal{F}(B) = \overline{\bigcup_{i=1}^{N} f_i(B)}$ for each $B \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the family of all non-empty bounded closed subsets of X, then there exists a unique $A(\mathcal{S}) \in \mathcal{B}(X)$ such that

$$\mathcal{F}(A(\mathcal{S})) = A(\mathcal{S})$$

and moreover

$$\lim_{n\to\infty} h(\mathcal{F}^{[n]}(Y), A(\mathcal{S})) = 0,$$

for each $Y \in \mathcal{B}(X)$, where h is the Hausdorff–Pompeiu metric (see [4], Theorem 2.5). Therefore the set $Z = A(\mathcal{S}) \cup \left(\bigcup_{n\in\mathbf{N}} \mathcal{F}^{[n]}(Y)\right)$ is bounded. For each $x \in Z$, $\omega \in \Lambda$ and $n \in \mathbf{N}$, with the notation $f_{[\omega]_n}(Z) = Z_{[\omega]_n}$, we have

$$d(f_{[\omega]_n}(x), \pi_\omega) \le \operatorname{diam}(Z_{[\omega]_n}) \le \psi^{[n]}(\operatorname{diam}(Z)),$$

where $\{\pi_\omega\} = \bigcap_{n\in\mathbf{N}} f_{[\omega]_n}(A(\mathcal{S}))$ (see [5]). Hence, as $\lim_{n\to\infty} \psi^{[n]}(\operatorname{diam}(Z)) = 0$ (see [11], Remark 3.4), we obtain that

$$\lim_{n\to\infty} f_{[\omega]_n}(y) = \pi_\omega$$

uniformly with respect to $y \in Y$ and $\omega \in \Lambda$. Thus the Condition C is valid.

The following result is a kind of reverse of Remark 3.1.

Theorem 3.1. *Let us consider (X,d) a metric space, the continuous functions $f_1, \ldots, f_N \colon X \to X$ and a function $\pi \colon \Lambda \to X$, where $\Lambda = \Lambda(\{1,2,\ldots,N\})$, such that the condition C (for the metric d) is fulfilled. Then there exist a comparison function $\psi \colon [0,\infty) \to [0,\infty)$ and a metric δ on X, equivalent with d, such that $f_i \colon (X,\delta) \to (X,\delta)$ is ψ-contraction for each $i \in \{1,2,\ldots,N\}$ (i.e.*

$$\delta(f_i(x), f_i(y)) \le \psi(\delta(x,y))$$

for each $x, y \in X$). Moreover, if the metric space (X,d) is complete, then (X,δ) is complete.

Proof. Our rather long proof is divided into 12 facts. The final of the justification of such a fact is marked by $\square$.

Fact 1. (A metric ρ, equivalent with d, making the functions f_i nonexpansive) *There exists a metric ρ on X, equivalent with d, such that*

$$\rho(f_i(x), f_i(y)) \le \rho(x,y)$$

for each $i \in \{1,2,\ldots,N\}$ and each $x, y \in X$. Consequently we have

$$\rho(f_\omega(x), f_\omega(y)) \le \rho(x,y)$$

for each $x, y \in X$ and each $\omega \in \Lambda^*$.

Justification of Fact 1. Let us define the function $\rho \colon X \times X \to [0,\infty]$ by

$$\rho(x,y) = \sup_{\omega\in\Lambda^*} d(f_\omega(x), f_\omega(y)),$$

for each $x, y \in X$. According to the hypothesis, for given $x, y \in X$, there exists $n_1 \in \mathbf{N}$ such that the inequalities $d(f_{[\omega]_n}(x), \pi_\omega) < 1$ and $d(f_{[\omega]_n}(y), \pi_\omega) < 1$ are valid for each $n \in \mathbf{N}$, $n \ge n_1$ and $\omega \in \Lambda$. Therefore $d(f_{[\omega]_n}(x), f_{[\omega]_n}(y)) \le d(f_{[\omega]_n}(x), \pi_\omega) + d(\pi_\omega, f_{[\omega]_n}(y)) \le 1 + 1 = 2$ for every $n \in \mathbf{N}$, $n \ge n_1$ and every $\omega \in \Lambda^*$ such that $|\omega| >$

n_1. As the set $\{\omega \in \Lambda^* \mid |\omega| \le n_1\}$ is finite, we conclude that $\sup_{\omega\in\Lambda^*} d(f_\omega(x), f_\omega(y))$ is finite. Hence $\rho\colon X \times X \to [0,\infty)$.

It is clear that:

i) $\rho(x, y) = 0$ if and only if $x = y$ (since $d(x, y) = d(f_\lambda(x), f_\lambda(y)) \le \rho(x, y)$);
ii) $\rho(x, y) = \rho(y, x)$;
iii) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$,

for each $x, y, z \in X$. Therefore ρ is a metric.

We have

$$\rho(f_i(x), f_i(y)) \le \rho(x, y), \tag{1.1}$$

for each $x, y \in X$ and each $i \in \{1, 2, \ldots, N\}$. Indeed, since

$$d(f_\omega(f_i(x)), f_\omega(f_i(y))) \le \rho(x, y)$$

for each $x, y \in X$, $\omega \in \Lambda^*$ and $i \in \{1, 2, \ldots, N\}$, we obtain that

$$\sup_{\omega\in\Lambda^*} d(f_\omega(f_i(x)), f_\omega(f_i(y))) \le \rho(x, y),$$

i.e.

$$\rho(f_i(x), f_i(y)) \le \rho(x, y),$$

for each $x, y \in X$ and each $i \in \{1, 2, \ldots, N\}$.

As we have seen

$$d(x, y) \le \rho(x, y),$$

for each $x, y \in X$.

($*$) Therefore if $(x_n)_{n\in\mathbf{N}}$ is a sequence of elements from X and $l \in X$ such that $\lim\limits_{n\to\infty} \rho(x_n, l) = 0$, then $\lim\limits_{n\to\infty} d(x_n, l) = 0$.

($**$) Now we prove that if $(x_n)_{n\in\mathbf{N}}$ is a sequence of elements from X and $l \in X$ such that $\lim\limits_{n\to\infty} d(x_n, l) = 0$, then $\lim\limits_{n\to\infty} \rho(x_n, l) = 0$.

Indeed, let us note that according to the hypothesis there exists $\varepsilon_l > 0$ having the property that for each $\varepsilon > 0$ there exists $m_{\varepsilon,\varepsilon_l} \in \mathbf{N}$ such that the inequality

$$d(f_{[\omega]_m}(x), \pi_\omega) < \frac{\varepsilon}{2} \tag{1.2}$$

is valid for each $m \in \mathbf{N}$, $m \ge m_{\varepsilon,\varepsilon_l}$, $\omega \in \Lambda$ and $x \in B(l, \varepsilon_l)$. Let us fix $\varepsilon > 0$. Since the set of continuous functions $\{f_\omega \mid \omega \in \Lambda^* \text{ and } |\omega| < m_{\varepsilon,\varepsilon_l}\}$ is finite, we infer that there exists $n^1_\varepsilon \in \mathbf{N}$ such that the inequality

$$d(f_\omega(x_n), f_\omega(l)) < \varepsilon \tag{1.3}$$

is valid for each $n \in \mathbf{N}$, $n \ge n^1_\varepsilon$ and each $\omega \in \Lambda^*$ such that $|\omega| < m_{\varepsilon,\varepsilon_l}$. Since $\lim_{n\to\infty} d(x_n, l) = 0$, there exists $n^2_\varepsilon \in \mathbf{N}$ such that $x_n \in B(l, \varepsilon_l)$ for each $n \in \mathbf{N}$, $n \ge n^2_\varepsilon$. For $\omega \in \Lambda^*$ having the property that $|\omega| \ge m_{\varepsilon,\varepsilon_l}$, $\omega = \omega_1\omega_2\ldots\omega_m$, where $m \in \mathbf{N}$, $m \ge m_{\varepsilon,\varepsilon_l}$, considering $\omega' = \omega_1\omega_2\ldots\omega_m\omega_m\omega_m\ldots\omega_m\ldots \in \Lambda$, we have $[\omega']_m = \omega$, so, according to (1.2), we have $d(f_\omega(x_n), \pi_{\omega'}) < \frac{\varepsilon}{2}$ for each $n \in \mathbf{N}$, $n \ge n^2_\varepsilon$ and $d(f_\omega(l), \pi_{\omega'}) < \frac{\varepsilon}{2}$. Thus

$$d(f_\omega(x_n), f_\omega(l)) \le d(f_\omega(x_n), \pi_{\omega'}) + d(\pi_{\omega'}, f_\omega(l)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{1.4}$$

is valid for each $n \in \mathbf{N}$, $n \ge n^2_\varepsilon$ and each $\omega \in \Lambda^*$ with $|\omega| \ge m_{\varepsilon,\varepsilon_l}$. From (1.3) and (1.4) we conclude that there exists $n_\varepsilon = \max\{n^1_\varepsilon, n^2_\varepsilon\} \in \mathbf{N}$ such that

$$d(f_\omega(x_n), f_\omega(l)) < \varepsilon$$

for each $n \in \mathbf{N}$, $n \geq n_\varepsilon$ and each $\omega \in \Lambda^*$, i.e. there exists $n_\varepsilon \in \mathbf{N}$ such that

$$\rho(x_n, l) = \sup_{\omega \in \Lambda^*} d(f_\omega(x_n), f_\omega(l)) \leq \varepsilon$$

for each $n \in \mathbf{N}$, $n \geq n_\varepsilon$. Therefore $\lim_{n\to\infty} \rho(x_n, l) = 0$.

From $(*)$ and $(**)$ we conclude that d and ρ are equivalent. □

Now let us consider the set $K = \pi(\Lambda) = \{\pi_\omega \mid \omega \in \Lambda\}$.

Fact 2. (The properties of K)

i) *K is compact.*

ii) $K = \bigcup_{i=1}^N f_i(K)$.

Justification of Fact 2. i) We are going to prove that the function $\pi\colon \Lambda \to X$ is continuous. Indeed, let us consider a fixed $\omega \in \Lambda$ and an arbitrary sequence $(\omega_n)_{n\in\mathbf{N}}$ of elements of Λ such that $\lim_{n\to\infty} \omega_n = \omega$. For a fixed element $x_0 \in X$, according to the hypothesis, for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbf{N}$ such that the inequality $d(f_{[\omega']_n}(x_0), \pi_{\omega'}) < \frac{\varepsilon}{2}$ is valid for each $n \in \mathbf{N}$, $n \geq n_\varepsilon$ and each $\omega' \in \Lambda$. As convergence in (Λ, d_Λ) is convergence on components and $\{1, 2, \ldots, N\}$ is finite, there exists $m_\varepsilon \in \mathbf{N}$ such that $[\omega_n]_{n_\varepsilon} = [\omega]_{n_\varepsilon}$ for each $n \in \mathbf{N}$, $n \geq m_\varepsilon$. Therefore, for $n \in \mathbf{N}$, $n \geq m_\varepsilon$, we have

$$d(\pi_{\omega_n}, \pi_\omega) \leq d(f_{[\omega_n]_{n_\varepsilon}}(x_0), \pi_{\omega_n}) + d(f_{[\omega]_{n_\varepsilon}}(x_0), \pi_\omega) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. $\lim_{n\to\infty} \pi_{\omega_n} = \pi_\omega$. Since (Λ, d_Λ) is a compact metric space (as a product of compact spaces), $\pi(\Lambda) = K$ is compact.

ii) Let us note that

$$f_i(\pi_\omega) = \pi_{i\omega} \tag{2.1}$$

for each $i \in \{1, 2, \ldots, N\}$ and each $\omega \in \Lambda$. Indeed, for a fixed $x \in X$, by taking into account the continuity of f_i, we have

$$f_i(\pi_\omega) = f_i\left(\lim_{n\to\infty} f_{[\omega]_n}(x)\right) = \lim_{n\to\infty} f_i(f_{[\omega]_n}(x)) = \lim_{n\to\infty} f_{[i\omega]_n}(x) = \pi_{i\omega}.$$

Therefore $f_i(K) \subseteq K$ for each $i \in \{1, 2, \ldots, N\}$ and consequently

$$\bigcup_{i=1}^N f_i(K) \subseteq K. \tag{2.2}$$

If $\omega = \omega_1\omega_2\ldots\omega_m\omega_{m+1}\ldots$, with the notation $\omega' = \omega_2\ldots\omega_m\omega_{m+1}\ldots$, we have $\pi_\omega = \pi_{\omega_1\omega'} \overset{(2.1)}{=} f_{\omega_1}(\pi_{\omega'}) \in f_{\omega_1}(K) \subseteq \bigcup_{i=1}^N f_i(K)$, so

$$K \subseteq \bigcup_{i=1}^N f_i(K). \tag{2.3}$$

From (2.2) and (2.3) we obtain that $K = \bigcup_{i=1}^N f_i(K)$. □

Fact 3. (If Condition C is valid for d, then it is also valid for ρ) *The condition C is also valid for ρ.*

Justification of Fact 3. According to the hypothesis, we have

$$\forall_{x\in X}\ \exists_{\varepsilon_x>0}\ \forall_{\delta>0}\ \exists_{n_{x,\varepsilon_x,\delta}\in\mathbf{N}}\ \forall_{n\in\mathbf{N},\ n\geq n_{x,\varepsilon_x,\delta}}\ \forall_{\omega\in\Lambda}\ \forall_{y\in B(x,\varepsilon_x)}\ d(f_{[\omega]_n}(y), \pi_\omega) < \frac{\delta}{2},$$

so

$$\forall_{x\in X}\ \exists_{\varepsilon_x>0}\ \forall_{\delta>0}\ \exists_{n_{x,\varepsilon_x,\delta}\in\mathbf{N}}\ \forall_{n\in\mathbf{N},\ n\geq n_{x,\varepsilon_x,\delta}}\ \forall_{\omega\in\Lambda}\ \forall_{v\in\Lambda^*}\ \forall_{y\in B(x,\varepsilon_x)}$$

$$d(f_{[v\omega]_{|v|+n}}(y), \pi_{v\omega}) < \frac{\delta}{2},$$

since $|v| + n \geq n \geq n_{x,\varepsilon_x,\delta}$.

Taking into account that, based on (2.1), the inequality $d(f_{[v\omega]_{|v|+n}}(y), \pi_{v\omega}) < \frac{\delta}{2}$ can be rewritten as $d(f_v(f_{[\omega]_n}(y)), f_v(\pi_\omega)) < \frac{\delta}{2}$, we get that

$$\forall_{x\in X}\ \exists_{\varepsilon_x>0}\ \forall_{\delta>0}\ \exists_{n_{x,\varepsilon_x,\delta}\in\mathbf{N}}\ \forall_{n\in\mathbf{N},\ n\geq n_{x,\varepsilon_x,\delta}}\ \forall_{\omega\in\Lambda}\ \forall_{y\in B(x,\varepsilon_x)}$$

$$\rho(f_{[\omega]_n}(y), \pi_\omega) = \sup_{v\in\Lambda^*} d(f_v(f_{[\omega]_n}(y)), f_v(\pi_\omega)) \leq \frac{\delta}{2} < \delta,$$

i.e. Condition C is valid for ρ. □

Fact 4. (The construction of the open set U) *There exists an open set U such that $K \subseteq U$ and for each $\delta > 0$ there exists $n_\delta \in \mathbf{N}$ such that the inequality*

$$\rho(f_{[\omega]_n}(y), \pi_\omega) < \delta$$

is valid for each $n \in \mathbf{N}$, $n \geq n_\delta$, $\omega \in \Lambda$ and $y \in U$.

Justification of Fact 4. Since K is compact, there exist $p \in \mathbf{N}$ and $\pi_{\omega_1}, \pi_{\omega_2}, \ldots, \pi_{\omega_p}$ such that

$$K \subseteq B(\pi_{\omega_1}, \varepsilon_{\pi_{\omega_1}}) \cup B(\pi_{\omega_2}, \varepsilon_{\pi_{\omega_2}}) \cup \cdots \cup B(\pi_{\omega_p}, \varepsilon_{\pi_{\omega_p}}),$$

where $\varepsilon_{\pi_{\omega_1}}, \varepsilon_{\pi_{\omega_2}}, \ldots, \varepsilon_{\pi_{\omega_p}}$ are given by the Condition C. Let us denote by U the open set $B(\pi_{\omega_1}, \varepsilon_{\pi_{\omega_1}}) \cup B(\pi_{\omega_2}, \varepsilon_{\pi_{\omega_2}}) \cup \cdots \cup B(\pi_{\omega_p}, \varepsilon_{\pi_{\omega_p}})$. Now we can choose $n_\delta = \max\{n_{\pi_{\omega_1},\varepsilon_{\pi_{\omega_1}},\delta}, n_{\pi_{\omega_2},\varepsilon_{\pi_{\omega_2}},\delta}, \ldots, n_{\pi_{\omega p},\varepsilon_{\pi_{\omega_p}},\delta}\}$ since for each $y \in U$ there exists $j_0 \in \{1, 2, \ldots, p\}$ such that $y \in B(\pi_{\omega_{j_0}}, \varepsilon_{\pi_{\omega j_0}})$, so, according to Fact 3, $\rho(f_{[\omega]_n}(y), \pi_\omega) < \delta$ for each $n \in \mathbf{N}$, $n \geq n_\delta$, $\omega \in \Lambda$. □

Let $(a_n)_{n\in\mathbf{N}}$ be a bounded strictly increasing sequence of positive real numbers such that: α) $a_0 > 1$; β) $\frac{a_1}{a_0} \leq 2$; γ) $(\frac{a_{n+1}}{a_n})_{n\in\mathbf{N}}$ is strictly decreasing and let us denote by l the limit of the sequence $(a_n)_{n\in\mathbf{N}}$ (for example, we can take $a_n = \prod_{k=0}^{n}(1+x^{k+1})$, where $x \in (0, 1)$). Let us also consider $(b_k)_{k\in\mathbf{N}}$ a sequence of positive real numbers such that $\frac{b_k}{4} < b_{k+1} < \frac{b_k}{2}$ for each $k \in \mathbf{N}$. It is clear that $(b_k)_{k\in\mathbf{N}}$ is decreasing and that its limit is 0.

Taking into account Fact 4 and using the method of mathematical induction, we find a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ of natural numbers such that

$$\rho(f_{[\omega]_n}(y), \pi_\omega) < \frac{b_k}{16}$$

for each $n \in \mathbf{N}$, $n \geq n_k$, $\omega \in \Lambda$ and $y \in U$.

Note that

$$\rho(f_{[\omega]_n}(y), f_{[\omega]_n}(x)) < \frac{b_k}{8} \tag{1}$$

for each $n \in \mathbf{N}$, $n \geq n_k$, $\omega \in \Lambda$ and $x, y \in U$ since

$$\rho(f_{[\omega]_n}(y), f_{[\omega]_n}(x)) \leq \rho(f_{[\omega]_n}(y), \pi_\omega) + \rho(\pi_\omega, f_{[\omega]_n}(x)) < \frac{b_k}{16} + \frac{b_k}{16} = \frac{b_k}{8}.$$

We consider the function $\widetilde{\rho}\colon X \times X \to [0, \infty]$ given by

$$\widetilde{\rho}(x, y) = \sup_{\omega\in\Lambda^*} a_{|\omega|}\rho(f_\omega(x), f_\omega(y)),$$

for each $x, y \in X$.

Fact 5. (The properties of $\widetilde{\rho}$)

i)
$$a_0\rho(x,y) \le \widetilde{\rho}(x,y) \le l\rho(x,y)$$
for each $x, y \in X$, *so* $\widetilde{\rho}\colon X \times X \to [0,\infty)$ *and* $\widetilde{\rho}$ *is a metric which is equivalent with* ρ.

ii)
$$\widetilde{\rho}(f_i(x), f_i(y)) \le \widetilde{\rho}(x,y)$$
for each $x, y \in X$ *and each* $i \in \{1, 2, \ldots, N\}$.

iii)
$$\widetilde{\rho}(f_i(x), f_i(y)) \le \max\left\{\sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), l\frac{b_k}{8}\right\}$$
for each $x, y \in U$, $k \in \mathbf{N}$ *and* $i \in \{1, 2, \ldots, N\}$.

iv) *The following implication is valid*
$$l\frac{b_k}{8} < \widetilde{\rho}(f_i(x), f_i(y)) \implies \widetilde{\rho}(f_i(x), f_i(y)) \le \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y),$$
for each $x, y \in U$, $k \in \mathbf{N}$ *and* $i \in \{1, 2, \ldots, N\}$.

Justification of Fact 5. i) For each $x, y \in X$, we have
$$a_0\rho(x,y) = a_{|\lambda|}\rho(f_\lambda(x), f_\lambda(y)) \le \widetilde{\rho}(x,y)$$
and, using Fact 1, we get
$$a_{|\omega|}\rho(f_\omega(x), f_\omega(y)) \le l\rho(x,y)$$
for each $\omega \in \Lambda^*$, hence
$$\widetilde{\rho}(x,y) \le l\rho(x,y).$$

ii) We have
$$\begin{aligned} &a_{|\omega|}\rho(f_\omega(f_i(x)), f_\omega(f_i(y))) = a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \\ &\le a_{|\omega i|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \le \sup_{\omega\in\Lambda^*} a_{|\omega|}\rho(f_\omega(x), f_\omega(y)) = \widetilde{\rho}(x,y) \end{aligned}$$
for each $x, y \in X$, $i \in \{1, 2, \ldots, N\}$ and $\omega \in \Lambda^*$, so we get ii).

iii) We have
$$\begin{aligned} &\widetilde{\rho}(f_i(x), f_i(y)) = \sup_{\omega\in\Lambda^*} a_{|\omega|}\rho(f_\omega(f_i(x)), f_\omega(f_i(y))) \\ &= \max\left\{\sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), \sup_{\omega\in\Lambda^*, |\omega|\ge n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y))\right\} \\ &\overset{(1)}{\le} \max\left\{\sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), l\frac{b_k}{8}\right\}. \end{aligned}$$

iv) If $l\frac{b_k}{8} < \widetilde{\rho}(f_i(x), f_i(y))$, then
$$l\frac{b_k}{8} < \widetilde{\rho}(f_i(x), f_i(y)) \overset{\text{iii)}}{\le} \max\left\{\sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), l\frac{b_k}{8}\right\},$$
so
$$\widetilde{\rho}(f_i(x), f_i(y)) \le \sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)).$$

Since for each $\omega \in \Lambda^*$ such that $|\omega| < n_k$ we have

$$a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)) = a_{|\omega i|}\rho(f_{\omega i}(x), f_{\omega i}(y))\frac{a_{|\omega|}}{a_{|\omega i|}} < \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y),$$

we infer that

$$\sup_{\omega\in\Lambda^*, |\omega|<n_k} a_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y).$$

Consequently, we have

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y). \qquad \square$$

Let us consider $(c_k)_{k\in\mathbf{N}}$, where $c_k = l\frac{a_{n_k+1}}{a_{n_k}}\frac{b_k}{8}$. Note that $c_k \leq l\frac{b_k}{4} < lb_{k+1}$ for each $k \in \mathbf{N}$. Let us define, for each $k \in \mathbf{N}$, the increasing continuous function $\varphi_k\colon [0,\infty) \to [0,\infty)$ given by

$$\varphi_k(t) = \begin{cases} \frac{a_{n_k}}{a_{n_k+1}}t, & \text{if } t \in (c_k, \infty), \\ l\frac{b_k}{8}, & \text{if } t \in [l\frac{b_k}{8}, c_k], \\ t, & \text{if } t \in [0, l\frac{b_k}{8}). \end{cases}$$

Fact 6. (f_i are φ_k-contractions with respect to $\widetilde{\rho}$ on U) *We have*

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \varphi_k(\widetilde{\rho}(x,y))$$

for each $x, y \in U$, each $k \in \mathbf{N}$ and each $i \in \{1, 2, \ldots, N\}$.

Justification of Fact 6. For given $x, y \in U$ and $k \in \mathbf{N}$, we have to consider the following three cases:

c1) $\widetilde{\rho}(x,y) \in [0, l\frac{b_k}{8})$;
c2) $\widetilde{\rho}(x,y) \in [l\frac{b_k}{8}, c_k]$;
c3) $\widetilde{\rho}(x,y) \in (c_k, \infty)$.

In case c1) the inequality to be proved becomes

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \widetilde{\rho}(x,y)$$

which is valid taking into account Fact 5, ii).

In case c2) the inequality to be proved becomes

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq l\frac{b_k}{8}.$$

If this inequality is not true, then

$$l\frac{b_k}{8} < \widetilde{\rho}(f_i(x), f_i(y)),$$

so, using Fact 5, iv), we get

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y),$$

hence we arrive to the contradiction

$$l\frac{b_k}{8} < \widetilde{\rho}(f_i(x), f_i(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}c_k = l\frac{b_k}{8}.$$

In case c3) the inequality to be proved becomes

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y).$$

If this inequality is not true, then

$$\frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y) < \widetilde{\rho}(f_i(x), f_i(y)),$$

so

$$l\frac{b_k}{8} = \frac{a_{n_k}}{a_{n_k+1}}c_k < \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y) < \widetilde{\rho}(f_i(x), f_i(y)),$$

hence, using again Fact 5, iv), we obtain the contradiction

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \frac{a_{n_k}}{a_{n_k+1}}\widetilde{\rho}(x,y). \qquad \square$$

Let us consider the function $\varphi\colon [0,\infty) \to [0,\infty)$ given by

$$\varphi(t) = \inf_{k\in\mathbf{N}} \varphi_k(t),$$

for each $t \in [0,\infty)$.

Fact 7. *φ is a comparison function.*

Justification of Fact 7. Since φ_k is increasing for each $k \in \mathbf{N}$, we infer that φ is increasing. For $t_0 > 0$ and $\varepsilon > 0$ such that $t_0 - \varepsilon > 0$ there exists $k_\varepsilon \in \mathbf{N}$ having the property that $l\frac{b_{k_\varepsilon}}{4} < t_0 - \varepsilon$, so $c_k \leq l\frac{b_k}{4} \leq l\frac{b_{k_\varepsilon}}{4} < t_0 - \varepsilon$ for each $k \in \mathbf{N}$, $k > k_\varepsilon$. Then

$$\varphi(t) = \min\left\{\min_{k\in\{0,1,2,\ldots,k_\varepsilon\}} \varphi_k(t), \frac{a_{n_{k_\varepsilon+1}}}{a_{n_{k_\varepsilon+1}+1}}t\right\} \tag{7.1}$$

for each $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

Indeed, for each $k \in \mathbf{N}$, $k > k_\varepsilon$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $\varphi_k(t) = \frac{a_{n_k}}{a_{n_k+1}}t$, so taking into account the fact that $(\frac{a_{n+1}}{a_n})_{n\in\mathbf{N}}$ is decreasing, we infer that $\inf_{k>k_\varepsilon} \varphi_k(t) = \frac{a_{n_{k_\varepsilon+1}}}{a_{n_{k_\varepsilon+1}+1}}t$ and therefore

$$\begin{aligned}\varphi(t) = \inf_{k\in\mathbf{N}} \varphi_k(t) &= \min\left\{\inf_{k\in\{0,1,2,\ldots,k_\varepsilon\}} \varphi_k(t), \inf_{k>k_\varepsilon} \varphi_k(t)\right\}\\ &= \min\left\{\min_{k\in\{0,1,2,\ldots,k_\varepsilon\}} \varphi_k(t), \frac{a_{n_{k_\varepsilon+1}}}{a_{n_{k_\varepsilon+1}+1}}t\right\}.\end{aligned}$$

Hence, from (7.1), we get

$$\varphi(t) < t,$$

for each $t > 0$.

In order to conclude that φ is a comparison function, it remains to prove that φ is right-continuous. We shall prove that φ is continuous. To this end, let us note that the inequality $\varphi(t) < t$ for each $t > 0$ assures us that $\lim_{t>0,t\to 0} \varphi(t) = 0 = \varphi(0)$, so φ is continuous at 0. From (7.1), based on the continuity of the functions φ_k and $t \to \frac{a_{n_{k_\varepsilon+1}}}{a_{n_{k_\varepsilon+1}+1}}t$, we conclude that φ is continuous at each $t_0 > 0$. $\square$

Note that from Fact 6 we get

$$\widetilde{\rho}(f_i(x), f_i(y)) \leq \varphi(\widetilde{\rho}(x,y)) \tag{2}$$

for each $x, y \in U$ and each $i \in \{1, 2, \ldots, N\}$.

We consider the function $n\colon X \to \mathbf{N}$ given by

$$n(x) = \max\{n \in \mathbf{N} \mid \text{there exists } \omega \in \Lambda_n \text{ such that } f_\omega(x) \notin U\} + 1,$$

for each $x \in X$, with the convention that $\max\emptyset = -1$.

Let us remark that n is well defined. Indeed, since K is compact, U is open and $K \subseteq U$, there exist $\eta > 0$ such that $B(K, \eta) \subseteq U$. Taking into account the hypothesis, for every $x \in X$ there exists $n_1 \in \mathbf{N}$ such that $d(f_{[\omega]_n}(x), \pi_\omega) < \eta$ (i.e. $f_{[\omega]_n}(x) \in B(\pi_\omega, \eta) \subseteq U$) for each $n \in \mathbf{N}$, $n \geq n_1$ and each $\omega \in \Lambda$. Hence $\{n \in \mathbf{N} \mid$there exists $\omega \in \Lambda_n$ such that $f_\omega(x) \notin U\} \subseteq \{0, 1, 2, \ldots, n_1 - 1\}$.

Note that if $n(x) = 0$, then $f_\omega(x) \in U$ for every $\omega \in \Lambda^*$ (in particular $x \in U$) and $n(f_j(x)) = 0$ for each $j \in \{1, 2, \ldots, N\}$.

Fact 8. (The properties of n)

i) *For each $x \in X$ there exists $r_x > 0$ such that*

$$n(y) \leq n(x)$$

for each $y \in B(x, r_x)$.

ii) *For each $x \in X$ such that $n(x) \geq 1$ and each $i \in \{1, 2, \ldots, N\}$ we have*

$$n(f_i(x)) \leq n(x) - 1.$$

Justification of Fact 8. i) There exist $r_x^1 > 0$ and $m \in \mathbf{N}$ such that $f_\omega(y) \in U$ for each $y \in B(x, r_x^1)$ and each $\omega \in \Lambda^*$ with $|\omega| > m$. Indeed, since the compact set K is a subset of the open set U, we infer that $\inf_{x \in K} d(x, X - U) \overset{\text{not}}{=} \delta_0 > 0$ and $\{x \in X \mid$ there exists $k_x \in K$ such that $d(x, k_x) < \delta_0\} \subseteq U$. Hence, taking into account condition C, just take $r_x^1 = \varepsilon_x$ and $m = n_{x, \varepsilon_x, \delta_0}$.

Since the set of continuous functions $\{f_\omega \mid \omega \in \Lambda_n,\ n \leq m\}$ is finite, we infer that for each $x \in X$ having the property that $f_\omega(x) \in U$ for each $\omega \in \Lambda_n$, $n \leq m$, there exists $r_x^2 > 0$ such that $f_\omega(y) \in U$ for each $\omega \in \Lambda_n$, $n \leq m$ and each $y \in B(x, r_x^2)$. Therefore, taking $r_x = \min\{r_x^1, r_x^2\}$, we have $\{n \in \mathbf{N} \mid f_\omega(x) \in U$ for each $\omega \in \Lambda_n\} \subseteq \{n \in \mathbf{N} \mid f_\omega(y) \in U$ for each $\omega \in \Lambda_n\}$, for each $y \in B(x, r_x)$. In particular, we get that $n(y) \leq n(x)$ for each $y \in B(x, r_x)$.

ii) With the notation $m = n(f_i(x))$, there exists $\omega_0 \in \Lambda_{m-1}$ such that $f_{\omega_0}(f_i(x)) = f_{\omega_0 i}(x) \notin U$, so, as $\omega_0 i \in \Lambda_m$, we obtain that $m \in \{n \in \mathbf{N} \mid$ there exists $\omega \in \Lambda_n$ such that $f_\omega(x) \notin U\}$. Hence $m + 1 \leq \max\{n \in \mathbf{N} \mid$ there exists $\omega \in \Lambda_n$ such that $f_\omega(x) \notin U\} + 1 = n(x)$. □

For a given $\alpha > 1$, we define the functions $D_\alpha \colon X \times X \to [0, \infty)$ and $\rho_\alpha \colon X \times X \to [0, \infty)$ given by

$$D_\alpha(x, y) = \alpha^{n(x,y)} \widetilde{\rho}(x, y)$$

and

$$\rho_\alpha(x, y) = \inf \left\{ \sum_{i=0}^{n-1} D_\alpha(x_i, x_{i+1}) \mid n \in \mathbf{N}^*,\ \{x_0, x_1, \ldots, x_{n-1}, x_n\} \subseteq X,\ x_0 = x \text{ and } x_n = y \right\},$$

for each $x, y \in X$, where $n(x, y) = \max\{n(x), n(y)\}$.

As the reader can routinely verify ρ_α is a pseudometric on X.

Fact 9. (The properties of ρ_α)

i)

$$\widetilde{\rho}(x, y) \leq \rho_\alpha(x, y),$$

for each $x, y \in X$, so ρ_α is a metric.

ii)

$$\rho_\alpha(x,y) \leq \alpha^{n(x,y)}\widetilde{\rho}(x,y),$$

for each $x, y \in X$.

iii) ρ_α *and* $\widetilde{\rho}$ *are equivalent.*

Justification of Fact 9. i) We have $\widetilde{\rho}(x,y) \leq \sum_{i=0}^{n-1} \widetilde{\rho}(x_i, x_{i+1}) \leq \sum_{i=0}^{n-1} D_\alpha(x_i, x_{i+1})$ for each $n \in \mathbf{N}^*$, $x_i \in X$ for each $i \in \{0, 1, 2, \ldots, n\}$ such that $x_0 = x$ and $x_n = y$, so $\widetilde{\rho}(x,y) \leq \rho_\alpha(x,y)$ for each $x, y \in X$.

ii) We have $\rho_\alpha(x,y) \leq D_\alpha(x,y) = \alpha^{n(x,y)}\widetilde{\rho}(x,y)$, for each $x, y \in X$.

iii) On the one hand, if $(x_n)_{n\in\mathbf{N}}$ is a sequence of elements from X and $l \in X$ is such that $\lim_{n\to\infty} \rho_\alpha(x_n, l) = 0$, then, from i) we get that $\lim_{n\to\infty} \widetilde{\rho}(x_n, l) = 0$. On the other hand, let us consider $(x_n)_{n\in\mathbf{N}}$ a sequence of elements from X and $l \in X$ such that $\lim_{n\to\infty} \widetilde{\rho}(x_n, l) = 0$. Taking into account Fact 8, i), there exists $r_l > 0$ such that $n(y) \leq n(l)$ for each y having the property that $d(y, l) < r_l$. As $\widetilde{\rho}$ and d are equivalent, there exists $n_0 \in \mathbf{N}$ such that $d(x_n, l) < r_l$, so $n(x_n) \leq n(l)$ for each $n \in \mathbf{N}$, $n \geq n_0$. Hence, using ii), we get that $\rho_\alpha(x_n, l) \leq \alpha^{n(x_n,l)}\widetilde{\rho}(x_n, l) \leq \alpha^{n(l)}\widetilde{\rho}(x_n, l)$ for each $n \in \mathbf{N}$, $n \geq n_0$ and consequently $\lim_{n\to\infty} \rho_\alpha(x_n, l) = 0$. Therefore ρ_α and $\widetilde{\rho}$ are equivalent. □

Fact 10. *If* $\varphi_1, \varphi_2\colon [0,\infty) \to [0,\infty)$ *are comparison functions, then the function* $\psi\colon [0,\infty) \to [0,\infty)$ *given by*

$$\psi(t) = \sup\{\varphi_1(t_1) + \varphi_2(t_2) \mid t_1, t_2 \in [0,\infty) \text{ and } t_1 + t_2 \leq t\}$$

for each $t \in [0,\infty)$, *is also a comparison function.*

Justification of Fact 10. First let us prove that ψ is increasing. Indeed, if $t, u \in [0,\infty)$, $t < u$, then for any $t_1, t_2 \in [0,\infty)$ such that $t_1 + t_2 \leq t$, we also have $t_1 + t_2 \leq u$. Hence $\varphi_1(t_1) + \varphi_2(t_2) \leq \sup\{\varphi_1(u_1) + \varphi_2(u_2) \mid u_1, u_2 \in [0,\infty)$ and $u_1 + u_2 \leq u\} = \psi(u)$. Consequently $\psi(t) \leq \psi(u)$.

Now we prove that $\psi(t) < t$ for each $t > 0$. Indeed, for each $t_1, t_2 \in [0,\infty)$ such that $t_1 + t_2 \leq t$ we have $\varphi_1(t_1) + \varphi_2(t_2) \leq t_1 + t_2 \leq t$, so $\psi(t) \leq t$. Hence $\psi(t) \leq t$ for each $t \in [0,\infty)$. For a fixed $t > 0$ and a fixed decreasing sequence $(s_n)_{n\in\mathbf{N}}$ of real numbers converging to 0, for each $n \in \mathbf{N}$ there exist $x_n, y_n \in [0,\infty)$ such that

$$(*) \qquad x_n + y_n \leq t + s_n$$

and $\psi(t + s_n) - s_n < \varphi_1(x_n) + \varphi_2(y_n)$. By passing to subsequences if necessary, we may assume that the bounded sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are monotone. If x is the limit of $(x_n)_{n\in\mathbf{N}}$ and y is limit of $(y_n)_{n\in\mathbf{N}}$, then, by $(*)$, we get $x + y \leq t$. If $(x_n)_{n\in\mathbf{N}}$ is increasing, then the bounded sequence $(\varphi_1(x_n))_{n\in\mathbf{N}}$ is also increasing and $\lim_{n\to\infty} \varphi_1(x_n) \leq \varphi_1(x)$. If $(x_n)_{n\in\mathbf{N}}$ is decreasing, as φ_1 is right continuous, $\lim_{n\to\infty} \varphi_1(x_n) = \varphi_1(x)$. Hence $\lim_{n\to\infty} \varphi_1(x_n) \leq \varphi_1(x)$ and in a similar manner we deduce that $\lim_{n\to\infty} \varphi_2(y_n) \leq \varphi_2(y)$. Then we have

$$(**) \quad \psi(t) \leq \lim_{n\to\infty} \psi(t + s_n) \leq \lim_{n\to\infty} \varphi_1(x_n) + \varphi_2(y_n) + s_n \leq \varphi_1(x) + \varphi_2(y) \leq \psi(t).$$

Thus $\psi(t) = \varphi_1(x) + \varphi_2(y)$. If $x = y = 0$, then $\psi(t) = 0 < t$. If $x \neq 0$ or $y \neq 0$, then $\psi(t) = \varphi_1(x) + \varphi_2(y) < x + y \leq t$.

Finally, we prove that ψ is right continuous. It is clear that ψ is right continuous at 0. In order to prove that ψ is right continuous at $t > 0$ it suffices to prove that for each decreasing sequence $(t_n)_{n\in\mathbf{N}}$ of elements from $[0,\infty)$ such that $\lim_{n\to\infty} t_n = t$, the sequence $(\psi(t_n))_{n\in\mathbf{N}}$ is convergent and $\lim_{n\to\infty}\psi(t_n)=\psi(t)$. This results from $(**)$. □

According to Fact 10, let us consider the comparison function $\psi\colon[0,\infty)\to[0,\infty)$ given by

$$\psi(t)=\sup\{\varphi(t_1)+\frac{t_2}{\alpha}\mid t_1,t_2\in[0,\infty)\text{ and }t_1+t_2\le t\}$$

for each $t\in[0,\infty)$.

Fact 11. (f_i are ψ-contractions with respect to ρ_α) *We have*

$$\rho_\alpha(f_j(x),f_j(y))\le\psi(\rho_\alpha(x,y)),$$

for each $j\in\{1,2,\dots,N\}$ *and each* $x,y\in X$.

Justification of Fact 11. Let us consider $x,y\in X$ and $\varepsilon>0$. From the definition of ρ_α, there exist $n\in\mathbf{N}^*$ and $\{x_0,x_1,\dots,x_{n-1},x_n\}\subseteq X$ such that $x_0=x$, $x_n=y$ and

$$\rho_\alpha(x,y)\le\sum_{i=0}^{n-1}D_\alpha(x_i,x_{i+1})<\rho_\alpha(x,y)+\varepsilon. \tag{11.1}$$

Let us note that if there exist $l,k\in\{0,1,2,\dots,n\}$, $l<k$ such that $n(x_l)=n(x_k)=0$, then

$$\begin{aligned}D_\alpha(x_l,x_k)&=\alpha^{n(x_l,x_k)}\widetilde{\rho}(x_l,x_k)=\widetilde{\rho}(x_l,x_k)\\&\le\widetilde{\rho}(x_l,x_{l+1})+\widetilde{\rho}(x_{l+1},x_{l+2})+\ldots+\widetilde{\rho}(x_{k-1},x_k)\\&\le D_\alpha(x_l,x_{l+1})+D_\alpha(x_{l+1},x_{l+2})+\ldots+D_\alpha(x_{k-1},x_k),\end{aligned}$$

so

$$\begin{aligned}&\rho_\alpha(x,y)\\&\le D_\alpha(x_0,x_1)+\ldots+D_\alpha(x_{l-1},x_l)+D_\alpha(x_l,x_k)+D_\alpha(x_k,x_{k+1})+\ldots+D_\alpha(x_{n-1},x_n)\\&\le\sum_{i=0}^{n-1}D_\alpha(x_i,x_{i-1})<\rho_\alpha(x,y)+\varepsilon.\end{aligned}$$

Thus, we can assume that the set $\{x_0,x_1,\dots,x_{n-1},x_n\}$ contains at most two elements x_i and x_j such that $n(x_i)=n(x_j)=0$ and if $i\ne j$, then $|i-j|=1$.

We claim that

$$\rho_\alpha(f_j(x),f_j(y))\le\psi(\rho_\alpha(x,y)+\varepsilon),$$

for each $j\in\{1,2,\dots,N\}$. In order to prove our claim we have to consider two cases:

c1) The set $\{s\mid s\in\{0,1,2,\dots,n\}\text{ and }n(x_s)=0\}$ has at most one element.

c2) The set $\{s\mid s\in\{0,1,2,\dots,n\}\text{ and }n(x_s)=0\}$ has two elements, denoted by x_l and x_{l+1}, where $l\in\{0,1,\dots,n-1\}$.

In case c1) we have

$$\begin{aligned}\rho_\alpha(f_j(x),f_j(y))&\le\sum_{i=0}^{n-1}D_\alpha(f_j(x_i),f_j(x_{i+1}))=\sum_{i=0}^{n-1}\alpha^{n(f_j(x_i),f_j(x_{i+1}))}\widetilde{\rho}(f_j(x_i),f_j(x_{i+1}))\\&\overset{\text{Fact 5, ii)}}{\le}\sum_{i=0}^{n-1}\alpha^{n(f_j(x_i),f_j(x_{i+1}))}\widetilde{\rho}(x_i,x_{i+1})\overset{\text{Fact 8, ii)}}{\le}\sum_{i=0}^{n-1}\alpha^{n(x_i,x_{i+1})-1}\widetilde{\rho}(x_i,x_{i+1})\end{aligned}$$

$$= \frac{1}{\alpha}\sum_{i=0}^{n-1}\alpha^{n(x_i,x_{i+1})}\widetilde{\rho}(x_i,x_{i+1}) = \frac{1}{\alpha}\sum_{i=0}^{n-1}D_\alpha(x_i,x_{i+1})$$

$$\overset{(11.1)}{<} \frac{1}{\alpha}(\rho_\alpha(x,y)+\varepsilon) \le \psi(\rho_\alpha(x,y)+\varepsilon).$$

In case c2) we have

$$\rho_\alpha(f_j(x),f_j(y)) \le \sum_{i=0}^{n-1}D_\alpha(f_j(x_i),f_j(x_{i+1}))$$

$$= D_\alpha(f_j(x_l),f_j(x_{l+1})) + \sum_{i=0,i\neq l}^{n-1} D_\alpha(f_j(x_i),f_j(x_{i+1}))$$

$$\overset{n(f_j(x_l))=n(f_j(x_{l+1}))=0}{=} \widetilde{\rho}(f_j(x_l),f_j(x_{l+1})) + \sum_{i=0,i\neq l}^{n-1} D_\alpha(f_j(x_i),f_j(x_{i+1}))$$

$$\overset{\substack{x_l,x_{l+1}\in U,(2),\\ \text{Fact 5, ii) and Fact 8, ii)}}}{\le} \varphi(\widetilde{\rho}(x_l,x_{l+1})) + \frac{1}{\alpha}\sum_{i=0,i\neq l}^{n-1} D_\alpha(x_i,x_{i+1})$$

$$= \varphi(D_\alpha(x_l,x_{l+1})) + \frac{1}{\alpha}\sum_{i=0,i\neq l}^{n-1} D_\alpha(x_i,x_{i+1})$$

$$\le \psi(\sum_{i=0}^{n-1}D_\alpha(x_i,x_{i+1})) \overset{(11.1)}{\le} \psi(\rho_\alpha(x,y)+\varepsilon).$$

From our claim, taking into account the fact that ψ is right continuous, it follows that

$$\rho_\alpha(f_j(x),f_j(y)) \le \psi(\rho_\alpha(x,y)),$$

for each $j\in\{1,2,\ldots,N\}$. □

Now just take $\rho_\alpha = \delta$.

Fact 12. *If the metric space (X,d) is complete, then (X,δ) is complete.*

Justification of Fact 12. If $(x_n)_{n\in\mathbf{N}}$ is a ρ-Cauchy sequence of elements of X, then, since $d\le\rho$ (see Fact 1), $(x_n)_{n\in\mathbf{N}}$ is d-Cauchy, so there exists $l\in X$ such that $\lim_{n\to\infty} d(x_n,l)=0$. As ρ is equivalent with d, we infer that $\lim_{n\to\infty}\rho(x_n,l)=0$, hence (X,ρ) is complete. Using a similar way of reasoning, based on Fact 5, i), we infer that $(X,\widetilde{\rho})$ is complete and, based on Fact 9, i), that (X,δ) is complete. □

Remark 3.2. The above theorem states the existence of a comparison function ψ having the property that $\mathcal{S}=((X,d),(f_i)_{i\in\{1,2,\ldots,N\}})$ is ψ-hyperbolic (since $\mathcal{S}=((X,\delta),(f_i)_{i\in\{1,2,\ldots,N\}})$ is ψ-contractive). Then, according to Theorem 2.1, taking into account Fact 2 and Fact 12, we infer that $A(\mathcal{S})=K$.

Consequently, Condition C is a sufficient one for the existence of a unique fixed point of the function $\mathcal{F}\colon\mathcal{K}(X)\to\mathcal{K}(X)$ given by

$$\mathcal{F}(C)=\bigcup_{i=1}^{N} f_i(C)$$

for each $C\in\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the family of non-empty and compact subsets of a complete metric space (X,d).

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