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IDEAL TOPOLOGIES AND CORRESPONDING APPROXIMATION PROPERTIES

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Abstract. We propose a unifying approach to numerous approximation properties in Banach spaces studied from the 1930s up to our days. To do so, we introduce the concept of *ideal topology* and say that a Banach space E has the $(\mathcal{I}, \mathcal{J}, \tau)$-approximation property if E-valued operators belonging to the operator ideal $\mathcal{I}$ can be approximated, with respect to the ideal topology τ, by operators belonging to the operator ideal $\mathcal{J}$. This concept recovers many classical/recent approximation properties as particular instances and several important known results are particular cases of more general results that are valid in this general framework.

1. Introduction and background

In order to put the problem we deal with in this paper in a proper perspective, we start by giving a brief historic account of the subject.

Aware of the fact that norm limits of finite rank bounded operators in Banach spaces are compact, Hildebrandt in 1931 asked if the converse is true. According to Pietsch [56], *this was the most important question ever asked in Banach space theory*. Hildebrandt's question and the mention Banach himself made to the approximation property in his book [4] mark the starting point of one of the most long standing and productive research lines in Functional Analysis, especially in Banach space theory, namely, the study of the approximation property and its variants. From Mazur's problem in the Scottish Book in 1936, passing through Grothendieck's memoir [30] in 1953, the counterexamples due to Enflo in 1973, Szankowski in 1981 and Willis in 1992 and Casazza's survey [12] in 2001, up to recent striking developments, e.g., Figiel, Johnson and Pełczyński [25] in 2011, Johnson and Szankowski [33] in 2012, Godefroy and Ozawa [27] in 2014, the approximation property and its variants have been a permanent source of challenging problems and of inspiration to generations of functional analysts. The subject is so hot that the following important contributions have appeared while we were writing this paper: Dineen and Mujica [24], Oja and Zolk [53], Kürsten and Pietsch [36].

The original problem led to many developments that can be divided into two great groups: (i) quantitative refinements that led, e.g., to the bounded, metric, uniform, bounded projection, commuting bounded, asymptotically commuting bounded approximation properties; (ii) problems concerning approximation, in different topologies, of bounded operators by operators belonging to different special classes (not

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only finite rank operators). We are concerned here with the developments arising from the second trend.

Considering that a Banach space E has the (classical, original) approximation property if (and only if) E-valued operators can be approximated, with respect to the compact-open topology, by finite rank operators, locally convex non-normed topologies have always been part of the game. The big picture can be described as the approximation of operators by simpler ones with respect to different (locally convex, or at least linear) topologies in the spaces of linear operators.

The first variant of the classical approximation property (AP) in the line we are interested here is the compact approximation property (CAP), which goes back to Banach's book [4], that regards the approximation by compact operators with respect to the compact-open topology. It was only in 1992 that Willis proved that AP $\neq$ CAP, and it was a strong motivation for mathematicians to consider the problem of approximation by operators belonging to different classes. By the time of Willis' counterexample, the study of special classes of linear operators had been successfully systematized by Pietsch with his theory of Operator Ideals [55]. The consideration of problems on the approximation by operators belonging to a given operator ideal was a question of time. Indeed, a number of approximation properties (APs) with respect to operator ideals—and other ones that are somehow related to operator ideals—have been studied in the last three decades, see, e.g., [6, 11, 13, 15, 19, 20, 29, 34, 35, 37, 38, 39, 40, 41, 49, 50, 57, 58, 59, 62, 64]. The reader is also referred to the surveys [51, 52] and to the references therein.

We are interested in the following problem: are all these APs determined by operator ideals and their respective theories particular cases of one single general concept? We propose an idea based on the observation that these APs determined by operator ideals are usually defined (or characterized) by the possibility of approximating operators belonging to a certain class by operators belonging to a smaller class with respect to a certain prescribed topology. In our approach operator ideals play the role of the classes of operators and we tried to figure out the conditions for a topology to be suitable in the sense that: (I) it should give rise to APs enjoying the usual expected properties; (II) the resulting APs should recover many important already studied APs as particular instances; (III) results about the already studied APs should be particular cases of more general results in this new environment. Our proposal is the concept of *ideal topology* (cf. Definition 2.1) and the $(\mathcal{I}, \mathcal{J}, \tau)$-*approximation property*, where $\mathcal{I}, \mathcal{J}$ are operator ideals and τ is an ideal topology, as defined in the abstract. We believe the examples we provide and the results we prove throughout the paper show that ideal topologies and the APs they generate fullfill conditions (I)–(III) above, furnishing in this way a suitable framework to study approximation properties in Banach spaces in a rather unified and general way. The referee kindly pointed out that Lissitsin and Oja [46] launched the more general *convex approximation property*, yielding also a unified approach.

The paper is organized as follows: in Section 2 we define and give plenty of examples of ideal topologies, and we introduce the notion of $(\mathcal{I}, \mathcal{J}, \tau)$-approximation property. Several well studied approximation properties are shown to be particular instances of this just defined general concept. In Section 3 we extend/generalize results from [19, 14] on APs to the language of $(\mathcal{I}, \mathcal{J}, \tau)$-APs. To reinforce the unifying feature of our approach, in Section 4 we introduce the notion of *projective ideal topology* to prove that recent results from [16, 6, 10] on APs in (symmetric) projective tensor products of Banach spaces are particular instances of much more general results in the context of $(\mathcal{I},\mathcal{J},\tau)$-APs.

Throughout the paper $E, E_1, \dots, E_n, F, G, G_1, \dots, G_n$ are Banach spaces over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. The closed convex hull of a subset A of a Banach space is denoted by $\overline{\text{co}}(A)$. By $\mathcal{L}(E;F)$ we denote the Banach space of bounded linear operators from E to F endowed with the usual operator norm. Given $u \in \mathcal{L}(E;F)$ and a bounded subset $A \subseteq E$, we use the standard notation

$$\|u\|_A := \sup_{x\in A} \|u(x)\|.$$

The identity operator on a Banach space E is denoted by id_E and the symbol B_E stands for the closed unit ball of E. Operator ideals are always considered in the sense of Pietsch [18, 55]. By $\mathcal{L}$ we denote the ideal of all bounded operators between Banach spaces and by $\mathcal{F}$ and $\mathcal{K}$ the ideals of finite rank and compact operators, respectively. Given a subset A of a topological space (X,τ), $\overline{A}^{\tau}$ denotes the closure of A in X with respect to τ.

The space of continuous n-linear mappings from $E_1 \times \cdots \times E_n$ to F is denoted by $\mathcal{L}(E_1, \dots, E_n; F)$ ($\mathcal{L}(^nE;F)$ if $E_1 = \cdots = E_n = E$), and the space of continuous n-homogeneous polynomials from E to F by $\mathcal{P}(^nE;F)$; both of them endowed with their usual sup (complete) norms. The completed n-fold projective tensor product of $E_1, \dots, E_n$ is denoted by $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$, and the completed n-fold symmetric projective tensor product of E by $\widehat{\otimes}^n_{s,\pi} E$. An elementary symmetric tensor $x \otimes \overset{(n)}{\cdots} \otimes x$ shall be simply denoted by $\otimes^n x$. Given an n-linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ and a polynomial $P \in \mathcal{P}(^nE;F)$, by A_L and P_L we denote their linearizations, that is,

$$A_L \in \mathcal{L}\left(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F\right), \quad A_L(x_1 \otimes \cdots \otimes x_n) = A(x_1, \dots, x_n) \quad \text{and}$$
$$P_L \in \mathcal{L}\left(\widehat{\otimes}^n_{s,\pi} E; F\right), \quad P_L(\otimes^n x) = P(x).$$

For background on multilinear mappings and homogeneous polynomials we refer to [23, 48], and for projective tensor products of Banach spaces we refer to [18, 26, 61].

2. Ideal topologies

In this section we define the notion of ideal topology, provide a method to generate many useful examples and introduce the approximation property with respect to a pair of operator ideals and a given ideal topology. We show that many approximation properties studied in the literature arise as particular instances of this general concept.

Definition 2.1. An *ideal topology* τ is a correspondence that, for all Banach spaces E and F, assigns a linear topology, still denoted by τ, on the space $\mathcal{L}(E;F)$ such that: for every operator ideal $\mathcal{I}$, if

$$\overline{\mathcal{I}}^{\tau}(E;F) := \overline{\mathcal{I}(E;F)}^{\tau}$$

for all Banach spaces E and F, then $\overline{\mathcal{I}}^{\tau}$ is an operator ideal.

Remark 2.2. Let $\mathcal{I}$ be an arbitrary operator ideal. Since $\mathcal{I}(E;F)$ is a linear subspace of $\mathcal{L}(E;F)$ and $(\mathcal{L}(E;F),\tau)$ is a topological vector space, it is always true that $\overline{\mathcal{I}}^{\tau}(E;F)$ is a linear subspace of $\mathcal{L}(E;F)$. Moreover, it is plain that $\mathcal{F}(E;F) \subseteq \overline{\mathcal{I}}^{\tau}(E;F)$. So, once a linear topology is assigned to each of the spaces $\mathcal{L}(E;F)$, the ideal property of $\overline{\mathcal{I}}^{\tau}$ is all that has to be checked to show that τ is an ideal topology.

Example 2.3. (a) It is folklore that the norm topology, which is the topology of uniform convergence on bounded sets, denoted by $\|\cdot\|$, is an ideal topology.

(b) The topology of pointwise convergence τ_s, which is the topology of uniform convergence on finite sets, is an ideal topology. Indeed, the topology τ_s is linear because it is the locally convex topology generated by the seminorms ported by finite sets (or, equivalently, by singletons). It is straightforward to check that $\overline{\mathcal{I}}^{\tau_s}$ is an operator ideal for every operator ideal $\mathcal{I}$.

Now we give a method to generate ideal topologies ranging from τ_s to $\|\cdot\|$. By BAN we denote the class of all Banach spaces over $\mathbf{K}$.

Proposition 2.4. *Suppose that for every Banach space E it has been assigned a collection $\mathcal{A}(E)$ of bounded subsets of E such that $\{x\} \in \mathcal{A}(E)$ for every $x \in E$ and*

$$u(A) \in \mathcal{A}(F) \text{ for all } E, F \in \mathrm{BAN},\ A \in \mathcal{A}(E) \text{ and } u \in \mathcal{L}(E;F). \tag{1}$$

Then the topology $\tau_{\mathcal{A}}$ of uniform convergence on sets belonging to $\mathcal{A}(E)$, $E \in \mathrm{BAN}$, is an ideal topology. Moreover, $\tau_s \subseteq \tau_{\mathcal{A}} \subseteq \|\cdot\|$.

Proof. First note that $\tau_{\mathcal{A}}$ is not the discrete topology on $\mathcal{L}(E;F)$ as $\mathcal{A}(E) \neq \emptyset$. So $\tau_{\mathcal{A}}$ is a linear topology because, for all Banach spaces E and F, it is the locally convex topology on $\mathcal{L}(E;F)$ generated by the seminorms ported by the sets belonging to $\mathcal{A}(E)$, that is, by the seminorms

$$u \in \mathcal{L}(E;F) \mapsto \|u\|_A := \sup_{x \in A} \|u(x)\|,$$

where $A \in \mathcal{A}(E)$. Let $\mathcal{I}$ be an operator ideal. By Remark 2.2 we just have to check that $\overline{\mathcal{I}}^{\tau_{\mathcal{A}}}$ enjoys the ideal property. Given operators $u \in \mathcal{L}(E;F)$, $v \in \overline{\mathcal{I}}^{\tau_{\mathcal{A}}}(F;G)$, $0 \neq w \in \mathcal{L}(G;H)$, a subset A of E belonging to $\mathcal{A}(E)$ and $\varepsilon > 0$, by (1) we know that $u(A) \in \mathcal{A}(F)$, so we can take an operator $T \in \mathcal{I}(F;G)$ such that $\|v - T\|_{u(A)} < \frac{\varepsilon}{\|w\|}$. Then $w \circ T \circ u \in \mathcal{I}(E;H)$ by the ideal property of $\mathcal{I}$ and

$$\|w \circ v \circ u - w \circ T \circ u\|_A \leq \|w\| \cdot \|v - T\|_{u(A)} < \varepsilon,$$

proving that $w \circ v \circ u \in \overline{\mathcal{I}}^{\tau_{\mathcal{A}}}(E;H)$. The second assertion is obvious because $\mathcal{A}(E)$ contains the singletons and is contained in the set of all bounded subsets of E. $\square$

The containment of the singletons has a twofold purpose: (i) it is a way—among others, of course—to avoid the (nonlinear) discrete topology on $\mathcal{L}(E;F)$; (ii) it implies that $\tau_s \subseteq \tau_{\mathcal{A}}$, which is a desirable property (cf. Proposition 2.9).

Proposition 2.4 allows us to show that several well known and useful topologies are ideal topologies that can be found in our way from τ_s to $\|\cdot\|$.

Example 2.5. Since bounded linear operators send compact sets to compact sets, the compact-open topology τ_c, which is the topology of uniform convergence on compact sets, is an ideal topology. The same happens for the following classes of subsets of Banach spaces: compact and convex sets, weakly compact sets, weakly compact and convex sets (remember that bounded linear operators are weak-weak continuous). So the topologies of uniform convergence on sets belonging to each of these classes are ideal topologies.

Proposition 2.4 can be used to provide many further useful examples of ideal topologies. Given an operator ideal $\mathcal{I}$ and a Banach space E, according to [63, 28, 37] we define

$$C_{\mathcal{I}}(E) = \{A \subseteq E \colon \exists F, \exists u \in \mathcal{I}(F;E) \text{ such that } A \subseteq u(B_F)\},$$

$$K_{\mathcal{I}}(E) = \{\overline{A} \colon A \subseteq E, \exists F, \exists K \subseteq F \text{ compact}, \exists u \in \mathcal{I}(F;E) \text{ such that } A \subseteq u(K)\}.$$

The sets belonging to $C_{\mathcal{I}}(E)$ are called *$\mathcal{I}$-bounded sets* and the sets belonging to $K_{\mathcal{I}}(E)$ are called *$\mathcal{I}$-compact sets*.

Example 2.6. Let $\mathcal{I}$ be an operator ideal. It is clear that $\mathcal{I}$-bounded sets are norm bounded and that singletons are $\mathcal{I}$-bounded (indeed, this is obvious for $x = 0$, and for $x \neq 0$ just pick a funcional $\varphi \in E'$ such that $\varphi(x) = \|x\|$ and note that $\varphi \otimes x \in \mathcal{I}(E;E)$ and $\varphi \otimes x\,(x/\|x\|) = x$). By the ideal property of $\mathcal{I}$ it follows that bounded linear operators send $\mathcal{I}$-bounded sets to $\mathcal{I}$-bounded sets, so the topology $\tau_{C_{\mathcal{I}}}$ of uniform convergence on $\mathcal{I}$-bounded sets (cf., e.g., [2]) is an ideal topology by Proposition 2.4.

Particular instances of this example of special interest are the following. (i) It is clear that $\tau_{K_{\mathcal{I}}} = \tau_{C_{\mathcal{I}\circ\mathcal{K}}}$, so the topology $\tau_{K_{\mathcal{I}}}$ of uniform convergence on $\mathcal{I}$-compact sets (cf. e.g., [37, 20]) is an ideal topology. In particular, the topology τ_{K_p} of uniform convergence on p-compact sets (cf. e.g., [62]) is an ideal topology. Indeed, if $\mathcal{K}_p$ denotes the ideal of p-compact operators, then $\tau_{K_p} = \tau_{K_{\mathcal{K}_p}}$. (ii) For $q > 0$, a subset A of a Banach space E is a *Bourgain–Reinov q-compact set* (see [11, 59, 1]), in symbols $A \in BR_q(E)$, if there is a E-valued absolutely q-summable sequence $(x_n)_n$ such that A is contained in the closure of the absolutely convex hull of $\{x_1, x_2, \ldots, \}$. By [1], $\tau_{BR_q} = \tau_{C_{\mathcal{K}_{(q,1)}}}$, where $\mathcal{K}_{(q,1)}$ is the ideal of $(q,1)$-compact operators, so the topology τ_{BR_q} of uniform convergence on Bourgain–Reinov q-compact sets is an ideal topology.

With plenty of useful ideal topologies in hands we can define the approximation properties determined by a pair of operators ideals and a given ideal topology.

Definition 2.7. Let $\mathcal{I}, \mathcal{J}$ be operator ideals and τ be an ideal topology. We say that a Banach space E has the:

(a) $(\mathcal{I}, \mathcal{J}, \tau)$-approximation property, $(\mathcal{I}, \mathcal{J}, \tau)$-AP for short, if

$$\mathcal{I}(F;E) \subseteq \overline{\mathcal{J}(F;E)}^{\tau} \quad \text{for every Banach space } F;$$

(b) $(\mathcal{I}, \mathcal{J}, \tau)$-weak approximation property, $(\mathcal{I}, \mathcal{J}, \tau)$-WAP for short, if

$$\mathcal{I}(E;E) \subseteq \overline{\mathcal{J}(E;E)}^{\tau}.$$

The examples below unfold that many well studied approximation properties are particular cases of our general concept. It is good to have in mind the following characterizations, which are immediate consequences of the ideal property of $\overline{\mathcal{I}}^{\tau}$:

$$\begin{aligned} E \text{ has the } (\mathcal{L}, \mathcal{I}, \tau)\text{-AP} &\iff \mathcal{L}(E;E) \subseteq \overline{\mathcal{I}(E;E)}^{\tau} \\ &\iff \mathrm{id}_E \in \overline{\mathcal{I}(E;E)}^{\tau} \\ &\iff E \text{ has the } (\mathcal{L}, \mathcal{I}, \tau)\text{-WAP}. \end{aligned} \tag{2}$$

By $\mathcal{I}^{\text{sur}}$ we mean the surjective hull of the operator ideal $\mathcal{I}$.

Example 2.8. (a) The classical approximation property coincides with the $(\mathcal{K}, \mathcal{F}, \|\cdot\|)$-AP, with the $(\mathcal{L}, \mathcal{F}, \tau_c)$-AP (hence with the $(\mathcal{L}, \mathcal{F}, \tau_c)$-WAP).

(b) The compact approximation property coincides with the $(\mathcal{L}, \mathcal{K}, \tau_c)$-AP (hence with the $(\mathcal{L}, \mathcal{K}, \tau_c)$-WAP).

(c) Let $\mathcal{I}$ be an operator ideal. The $\mathcal{I}$-approximation property of [6] coincides with the $(\mathcal{L}, \mathcal{I}, \tau_c)$-AP (hence with the $(\mathcal{L}, \mathcal{I}, \tau_c)$-WAP).

(d) Let $\mathcal{I}$ be an operator ideal. The $\mathcal{I}$-approximation property of Lassalle and Turco [37] and the approximation property with respect to the operator ideal $\mathcal{I}$ of Delgado and Piñeiro [20] both coincide with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{I}}})$-AP (hence with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{I}}})$-WAP) and with the $(\mathcal{I}^{\rm sur}, \mathcal{F}, \tau_c)$-AP (see [20, Theorem 2.3]).

(e) The p-approximation property of Sinha and Karn [62] (see also [19]), $1 \leq p < \infty$, coincides with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{N}^p}})$-AP, where $\mathcal{N}^p$ is the ideal of p-nuclear operators [37] (hence with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{N}^p}})$-WAP), with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{K}_p}})$-AP), where $\mathcal{K}_p$ is the ideal of p-compact operators (hence with the $(\mathcal{L}, \mathcal{F}, \tau_{K_{\mathcal{K}_p}})$-WAP)).

(f) Let $0 < p \leq 1$, $q = p/(1-p)$ and BR_q be the class of Bourgain–Reinov q-compact subsets of Banach spaces (cf. Example 2.6). The aproximation property of order p of Reinov [57] coincides with the $(\mathcal{L}, \mathcal{F}, \tau_{BR_q})$-AP (hence with the $(\mathcal{L}, \mathcal{F}, \tau_{BR_q})$-WAP) (see [11, 59] and [20, p. 70]).

(g) A long standing problem (see [42, Problem 1.e.9]) asks whether the classical approximation property coincides with the $(\mathcal{K}, \mathcal{F}, \|\cdot\|)$-WAP.

The reason why we are interested in ideal topologies containing the topology τ_s of pointwise convergence (cf. Proposition 2.4) is the following.

Proposition 2.9. *Regardless of the operator ideals $\mathcal{I}$ and $\mathcal{J}$, every Banach space has the $(\mathcal{I}, \mathcal{J}, \tau_s)$-AP and the $(\mathcal{I}, \mathcal{J}, \tau_s)$-WAP.*

Proof. It is easy to see that, for every Banach space E, $\mathrm{id}_E \in \overline{\mathcal{F}(E;E)}^{\tau_s}$ (see [43, Proposition 3.14]). Since $\overline{\mathcal{F}}^{\tau_s}$ is an operator ideal, we have $\overline{\mathcal{F}(F;E)}^{\tau_s} = \mathcal{L}(F;E)$ regardless of the Banach spaces E and F. Now the result is immediate. □

Several usual properties of the known approximation properties extend to this more general context. We give just a couple of illustrative examples.

Proposition 2.10. *Let $\mathcal{I}, \mathcal{J}$ be operator ideals and τ be an ideal topology.*

(a) *If the Banach space E has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP ($(\mathcal{I}, \mathcal{J}, \tau)$-WAP, respectively) and the Banach space F is isomorphic to a complemented subspace of E, then F has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP ($(\mathcal{I}, \mathcal{J}, \tau)$-WAP, respectively) as well.*
(b) *Given Banach spaces $E_1, \ldots, E_n$, the finite direct sum $\bigoplus_{j=1}^n E_j$ has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP (the $(\mathcal{I}, \mathcal{J}, \tau)$-WAP, respectively) if and only if E_j has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP (the $(\mathcal{I}_j, \mathcal{J}_j, \tau)$-WAP, respectively) for $j = 1, \ldots, n$.*

Proof. The cases of the $(\mathcal{I}, \mathcal{J}, \tau)$-AP in (a) and (b) follow from the fact that a Banach space has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP if and only if its identity operator belongs to the quotient ideal $\overline{\mathcal{J}}^{\tau} \circ \mathcal{I}^{-1}$ (cf. [55, Theorem 3.2.7]). The case of the $(\mathcal{I}, \mathcal{J}, \tau)$-WAP in (a) is easy and we omit the proof. We just give an argument for the implication of the $(\mathcal{I}, \mathcal{J}, \tau)$-WAP case in (b) that does not follow from (a): assume that E_j has the $(\mathcal{I}, \mathcal{J}, \tau)$-AP for $j = 1, \ldots, n$. Call $F := \bigoplus_{j=1}^n E_j$. For each j let $i_j \colon E_j \longrightarrow F$ and $q_j \colon F \longrightarrow E_j$ be the canonical operators. Given an operator $u \in \mathcal{I}(F;F)$, we have that $q_j \circ u \in \mathcal{I}(F;E_j)$, hence $q_j \circ u \in \overline{\mathcal{J}}^{\tau}(F;E_j)$. Then each $i_j \circ q_j \circ u \in \overline{\mathcal{J}}^{\tau}(F;F)$, so $u = \sum_{j=1}^n i_j \circ q_j \circ u \in \overline{\mathcal{J}}^{\tau}(F;F)$. □

3. Ideal topologies in action

An important aspect of the approximation properties in Banach spaces is the fact that, sometimes, the approximation by two different classes of operators with respect to two different topologies actually coincide. The search for this kind of situation in our case can be rephrased as: when does the equality $(\mathcal{I}_1, \mathcal{J}_1, \tau_1)$-AP $= (\mathcal{I}_2, \mathcal{J}_2, \tau_2)$-AP hold? What about the WAP? There are several trivial coincidences, for example the ones in (2) and the following: let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{J}_1, \mathcal{J}_2$ be operator ideals and τ_1, τ_2 be ideal topologies such that $\mathcal{I}_2 \subseteq \mathcal{I}_1$, $\mathcal{J}_1 \subseteq \mathcal{J}_2$ and $\tau_2 \subseteq \tau_1$. If a Banach space E has the $(\mathcal{I}_1, \mathcal{J}_1, \tau_1)$-AP, then E has the $(\mathcal{I}_2, \mathcal{J}_2, \tau_2)$-AP. The same holds for the corresponding WAPs.

In this section we use the notion of ideal topology to prove some non-trivial coincidences that extend and generalize previous results, mainly from [19] and [14]. The argument of the following lemma shall be repeated several times, so we state it separately for further reference.

Lemma 3.1. *Let $\mathcal{I}$ be an operator ideal, E, F_1 and F_2 be Banach spaces, $\mathcal{A}_i$ be a collection of bounded subsets of F_i and τ_i be the locally convex topology on $\mathcal{L}(F_i; E)$ generated by the seminorms ported by the sets belonging to $\mathcal{A}_i$, $i = 1, 2$. If $R \in \overline{\mathcal{I}(F_1; E)}^{\tau_1}$ and $S \in \mathcal{L}(F_2; F_1)$ is such that $S(A) \in \mathcal{A}_1$ for every $A \in \mathcal{A}_2$, then $R \circ S \in \overline{\mathcal{I}(F_2; E)}^{\tau_2}$.*

Proof. It is clear that τ_i is the topology of uniform convergence on the sets belonging to $\mathcal{A}_i$. Let $\varepsilon > 0$ and $A \in \mathcal{A}_2$ be given. By assumption we have $S(A) \in \mathcal{A}_1$ and $R \in \overline{\mathcal{I}(F_1; E)}^{\tau_1}$, so there exists an operator $T \in \mathcal{I}(F_1; E)$ such that

$$\|T \circ S - R \circ S\|_A = \|T - R\|_{S(A)} < \varepsilon.$$

Since $T \circ S \in \mathcal{I}(F_2; E)$ it follows that $R \circ S \in \overline{\mathcal{I}(F_2; E)}^{\tau_2}$. □

Next we show that some of the implications of [19, Theorem 2.1] hold true in a rather general context. We shall henceforth use the following characterization of the surjective hull of an operator ideal $\mathcal{I}$: given $T \in \mathcal{L}(E; F)$, $T \in \mathcal{I}^{\mathrm{sur}}(E; F)$ if and only if $T(B_E) \in C_{\mathcal{I}}(F)$ if and only if T maps bounded subsets of E to $\mathcal{I}$-bounded subsets of F.

Proposition 3.2. *Let E be a Banach space and let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{J}$ be operator ideals such that $\mathcal{I}_1 \subseteq \mathcal{I}_3 \cap (\mathcal{I}_1 \circ \mathcal{J}^{\mathrm{sur}})$. Consider the following conditions:*

(a) $\mathrm{id}_E \in \overline{\mathcal{I}_2(E; E)}^{\tau_{C_{\mathcal{J}}}}$.
(b) *E has the $(\mathcal{I}_1, \mathcal{I}_2, \|\cdot\|)$-AP.*
(c) *E has the $(\mathcal{I}_1, \mathcal{I}_2, \tau_{C_{\mathcal{J}}})$-AP.*
(d) *E has the $(\mathcal{I}_3, \mathcal{I}_2, \tau_{C_{\mathcal{J}}})$-AP.*

Then (a) $\Longrightarrow$ (d) $\Longrightarrow$ (c) $\Longleftrightarrow$ (b).

Proof. The implications (a) $\Longrightarrow$ (d) $\Longrightarrow$ (c) are obvious; and the implication (b) $\Longrightarrow$(c) holds because $\tau_{C_{\mathcal{J}}} \subseteq \|\cdot\|$ (cf. Example 2.6 and Proposition 2.4). Let us prove (c) $\Longrightarrow$(b): Let F be a Banach space and $T \in \mathcal{I}_1(F; E)$. There are a Banach space G and operators $R \in \mathcal{I}_1(G; E)$ and $S \in \mathcal{J}^{\mathrm{sur}}(F; G)$ such that $T = R \circ S$. Then $R \in \overline{\mathcal{I}_2(G; E)}^{\tau_{C_{\mathcal{J}}}}$ and S maps bounded sets to $\mathcal{J}$-bounded sets. By Lemma 3.1 we have $T = R \circ S \in \overline{\mathcal{I}_2(F; E)}^{\|\cdot\|}$, proving that E has the $(\mathcal{I}_1, \mathcal{I}_2, \|\cdot\|)$-AP. □

Note that in Proposition 3.2 no condition has been imposed on the operator ideal $\mathcal{I}_2$.

The aim now is to show that, under some additional assumptions, the conditions (a)–(d) above are all equivalent. To accomplish this task we take advantage of the quantitative change Lima, Nygaard and Oja [39] made in the classical Davis, Figiel, Johnson and Pełczyński classical factorization scheme [17], which we describe next.

Let E be a Banach space, let K be a closed absolutely convex subset of its unit ball B_E and let $a > 1$. For each $n \in \mathbf{N}$ put $B_n = a^{n/2}K + a^{-n/2}B_E$. As B_n is absolutely convex and absorbent, the gauge (Minkowski functional) $\|\cdot\|_n$ of B_n,

$$\|x\|_n = \inf\{\lambda\colon x \in \lambda B_n\},$$

is a seminorm on E that is equivalent to the original norm $\|\cdot\|$ on E. For $x \in E$ define $\|x\|_K = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2}$ and let the subspace $E_K = \{x \in E\colon \|x\|_K < \infty\}$ of E be endowed with the norm $\|\cdot\|_K$. The function

$$f\colon (1,\infty) \longrightarrow \mathbf{R}\ ,\ f(a) = \sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2},$$

is continuous, strictly decreasing, $\lim\limits_{a\to 1^+} f(a) = \infty$ and $\lim\limits_{a\to\infty} f(a) = 0$. So there is exactly one number $\hat{a} \in (1,\infty)$ such that $f(\hat{a}) = 1$. Let $C_K = \{x \in E : \|x\|_K \leq 1\}$ and let J_K be the identity embedding from E_K to E. Replacing a with $\hat{a}$ in [39, Lemma 1.1], we get

Lemma 3.3. [39, Lemma 1.1] *Let E, K, C_K, E_K and J_K be as above. Then:*

(a) *$K \subseteq C_K \subseteq B_E$.*
(b) *E_K is a Banach space with closed unit ball C_K and $J_K \in \mathcal{L}(E_K; E)$ with $\|J_K\| \leq 1$.*
(c) *J_K'' is injective.*
(d) *$J_K(C_K) = C_K$.*

The key result is the following.

Theorem 3.4. (Lima–Nygaard–Oja Factorization Theorem [39, Theorem 2.2]) *Suppose $T \in \mathcal{L}(F;E)$. Let $K = \frac{1}{\|T\|}\overline{T(B_F)}$ and let $T_K \in \mathcal{L}(F;E_K)$ be defined by $T_K(y) = T(y), y \in F$. Then $T = J_K \circ T_K$.*

The expression $T = J_K \circ T_K$ above shall be referred to as *the LNO factorization of T*.

Definition 3.5. An operator ideal $\mathcal{I}$ has the *Grothendieck property* if whenever A is a bounded subset of a Banach space E such that for every $\varepsilon > 0$ there is a set $A_\varepsilon \in C_{\mathcal{I}}(E)$ with $A \subseteq A_\varepsilon + \varepsilon B_E$, it holds that $A \in C_{\mathcal{I}}(E)$.

Example 3.6. A result due to Jarchow [32, Proposition 2.9], as restated by González and Gutiérrez [28, Proposition 3(c)], proves that any closed surjective operator ideal has the Grothendieck property. Lists of closed surjective operator ideals can be found in [28, 21].

Proposition 3.7. *Let $T = J_K \circ T_K$ be the LNO factorization of the operator $T \in \mathcal{L}(F;E)$. If the operator ideal $\mathcal{I}$ has the Grothendieck property, then $T \in \mathcal{I}^{\rm sur}(F;E)$ if and only if $J_K \in \mathcal{I}^{\rm sur}(E_K;E)$.*

Proof. Assume that $T \in \mathcal{I}^{\rm sur}(F;E)$. In this case we have $T(B_F) \in C_{\mathcal{I}}(E)$. As, for all $\varepsilon > 0$, $\overline{T(B_F)} \subseteq T(B_F) + \varepsilon B_F$ and $\mathcal{I}$ has the Grothendieck property, we have that $\overline{T(B_F)} \in C_{\mathcal{I}}(E)$, hence $K = \frac{1}{\|T\|}\overline{T(B_F)} \in C_{\mathcal{I}}(E)$. Given $\varepsilon > 0$, since $C_K \subseteq a^{n/2}K + a^{-n/2}B_E$ for every n (see [39, p. 331]), choosing n such that $a^{-n/2} < \varepsilon$ and putting $A_\varepsilon = a^{n/2}K \in C_{\mathcal{I}}(E)$, we have $C_K \subseteq A_\varepsilon + \varepsilon B_E$. The Grothendieck property of $\mathcal{I}$ gives $C_K \in C_{\mathcal{I}}(E)$. By items (b) and (d) of Lemma 3.3 it follows that

$$J_K(B_{E_K}) = J_K(C_K) = C_K \in C_{\mathcal{I}}(E),$$

which proves that $J_K \in \mathcal{I}^{\text{sur}}(E_K; E)$. The converse follows from the ideal property. $\square$

According to [44, Definition 4.3], the proposition above states that the surjective hull of an operator ideal with the Grothendieck property is DJFP-surjective. The referee kindly pointed out that the proof of [44, Proposition 4.2] (essentially) yields Proposition 3.7.

Corollary 3.8. *Let $T = J_K \circ T_K$ be the LNO factorization of the operator $T \in \mathcal{L}(F; E)$. If the operator ideal $\mathcal{I}$ is surjective and has the Grothendieck property (in particular, if $\mathcal{I}$ is closed and surjective), then $T \in \mathcal{I}(F; E)$ if and only if $J_K \in \mathcal{I}(E_K; E)$.*

The next result, which is a variant of [19, Theorem 2.1] and a generalization of [14, Theorem 2.4] (see Corollary 3.10), shows that with additional assumptions the conditions (a)–(d) of Proposition 3.2 are equivalent.

Theorem 3.9. *Let $\mathcal{I}, \mathcal{J}_1, \mathcal{J}_2$ be operator ideals such that $\mathcal{J}_1$ has the Grothendieck property, $\mathcal{I} \supseteq \mathcal{J}_1^{\text{sur}} = \mathcal{J}_1^{\text{sur}} \circ \mathcal{J}_2^{\text{sur}}$ and such that operators belonging to $\mathcal{I}$ map $\mathcal{J}_2$-bounded sets to $\mathcal{J}_1$-bounded sets. The following statements are equivalent for a Banach space E:*

(a) $\mathrm{id}_E \in \overline{\mathcal{F}(E; E)}^{\tau_{C_{\mathcal{J}_1}}}$.
(b) *E has the $(\mathcal{J}_1^{\text{sur}}, \mathcal{F}, \|\cdot\|)$-AP.*
(c) *E has the $(\mathcal{J}_1^{\text{sur}}, \mathcal{F}, \tau_{C_{\mathcal{J}_2}})$-AP.*
(d) *E has the $(\mathcal{I}, \mathcal{F}, \tau_{C_{\mathcal{J}_2}})$-AP.*

Proof. (a) $\Longrightarrow$ (b) Let F be a Banach space and $T \in \mathcal{J}_1^{\text{sur}}(F; E)$. Since T maps bounded sets to $\mathcal{J}_1$-bounded sets and, by assumption, $\mathrm{id}_E \in \overline{\mathcal{F}(E; E)}^{\tau_{C_{\mathcal{J}_1}}}$, Lemma 3.1 yields that $T = \mathrm{id}_E \circ T \in \overline{\mathcal{F}(F; E)}^{\|\cdot\|}$.

(b) $\Longrightarrow$ (a) Let $A \in C_{\mathcal{J}_1}(E)$ and $\varepsilon > 0$ be given. There exists a Banach space F and an operator $T \in \mathcal{J}_1(F; E) \subseteq \mathcal{J}_1^{\text{sur}}(F; E)$ such that $A \subseteq T(B_F)$. Lettting $T = J_K \circ T_K$ be the LNO factorization of T, $J_K \in \mathcal{J}_1^{\text{sur}}(E_K; E)$ by Proposition 3.7 as $\mathcal{J}_1$ has the Grothendieck property. By assumption there exists an operator $S \in \mathcal{F}(E_K, E)$ such that $\|S - J_K\| < \frac{\varepsilon}{2\|T\|}$. Noticing that [50, Lemma 4.2 and Corollary 4.3] hold if K is just closed (and not necessarily weakly compact), applying [50, Corollary 4.3] we get an operator $R' \in \mathcal{F}(E; E)$ such that

$$\left\| R' \circ J_K - \frac{S}{\|S\|} \right\| < \frac{\varepsilon}{2\|S\| \cdot \|T\|}.$$

Thus $R := \|S\| R' \in \mathcal{F}(E; E)$ and $\|R \circ J_K - S\| < \frac{\varepsilon}{2\|T\|}$. For every $x \in K$, since $J_K(x) = x$ and $K \subseteq C_K = B_{E_K}$ (Lemma 3.3(a),(b)), we have

$$\begin{aligned}\|R(x) - x\| &= \|R(J_K(x)) - J_K(x)\| \leq \|(R \circ J_K)(x) - S(x)\| + \|S(x) - J_K(x)\| \\ &\leq \|R \circ J_K - S\| \cdot \|x\|_K + \|S - J_K\| \cdot \|x\|_K \\ &\leq \|R \circ J_K - S\| + \|S - J_K\| < \frac{\varepsilon}{\|T\|}.\end{aligned}$$

Since $A \subseteq T(B_F) \subseteq \|T\| K$, we have

$$\|R - \mathrm{id}_E\|_A \leq \|T\| \cdot \|R - \mathrm{id}_E\|_K < \varepsilon,$$

which proves that $\mathrm{id}_E \in \overline{\mathcal{F}(E; E)}^{\tau_{C_{\mathcal{J}_1}}}$.

(b) $\Longleftrightarrow$(c) The same arguments of the proofs of the corresponding implications in Proposition 3.2 work.

(d) $\Longrightarrow$ (c) It follows from the relation $\mathcal{J}_1^{\rm sur} \subseteq \mathcal{I}$.

(a) $\Longrightarrow$ (d) The assumption that operators in $\mathcal{I}$ map $\mathcal{J}_2$-bounded sets to $\mathcal{J}_1$-bounded sets allows us to repeat the argument of the proof of (a) $\Longrightarrow$(b). ☐

Question. To the best of our knowledge, the following question is open: does the ideal $\mathcal{K}_p$ of p-compact operators have the Grothendieck property?

If the answer to the question above turns out to be positive, then Theorem 3.9 can be regarded as a generalization of [19, Theorem 2.1]. For the moment we can only say that Theorem 3.9 is a variant of [19, Theorem 2.1], in the sense that the same conclusion holds in situations not covered by the original result. Let us see that Theorem 3.9 recovers a result due to Choi, Kim and Lee [14] as a particular case.

Corollary 3.10. *Let $\mathcal{I}$ be an operator ideal containing $\mathcal{K}$. The following statements are equivalent for a Banach space E:*

(a) *E has the approximation property.*
(b) *E has the $(\mathcal{K}, \mathcal{F}, \|\cdot\|)$-AP.*
(c) *E has the $(\mathcal{K}, \mathcal{F}, \tau_c)$-AP.*
(d) *E has the $(\mathcal{I}, \mathcal{F}, \tau_c)$-AP.*

Proof. Just apply Theorem 3.9 with $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{K}$ having in mind that $\mathcal{K}$ has the Grothendieck property because it is closed and surjective (cf. Example 3.6), and that $\mathcal{K} = \mathcal{K} \circ \mathcal{K}$ (cf. the proof of [55, Proposition 3.1.3] or [39, Theorem 2.2]). ☐

The equivalence (a) $\Longleftrightarrow$ (c) of Corollary 3.10 recovers the equivalence (a) $\Longleftrightarrow$ (b) of [14, Theorem 2.4].

4. Projective ideal topologies

In this section we reinforce the unifying feature of our approach to approximation properties via ideal topologies by proving that some recent results of [16, 6, 10] on approximation properties in projective tensor products of Banach spaces are particular instances of much more general results in the realm of ideal topologies. Remember that approximation properties and topological tensor products are closely connected since Grothendieck [30]. It is worth noticing that two results of Çaliskan and Rueda [16] are in fact particular instances of one single result. We start with a refinement of the definition of ideal topology.

Definition 4.1. Let $\mathcal{C}$ be class of Banach spaces, that is, a subclass of BAN. A *$\mathcal{C}$-projective ideal topology* τ is a correspondence that, for all positive integers $n \in \mathbf{N}$ and Banach spaces $E, E_1, \ldots, E_n$ and F, assigns a linear topology, still denoted by τ, on each of the spaces $\mathcal{L}(E;F)$, $\mathcal{P}(^nE;F)$ and $\mathcal{L}(E_1, \ldots, E_n; F)$; such that:

(i) When restricted to the spaces $\mathcal{L}(E;F)$, τ is an ideal topology.
(ii) If $E, E_1, \ldots, E_n$ belong to $\mathcal{C}$, then, for every F, the linear bijections

$$P \in (\mathcal{P}(^nE;F), \tau) \mapsto P_L \in \left(\mathcal{L}\left(\widehat{\otimes}^n_{s,\pi} E; F\right), \tau\right) \quad \text{and}$$
$$A \in (\mathcal{L}(E_1, \ldots, E_n; F), \tau) \mapsto A_L \in \left(\mathcal{L}\left(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F\right), \tau\right)$$

are homeomorphisms. For simplicity, a BAN-projective ideal topology shall be referred to as a projective ideal topology.

It is well known that the norm topology is a projective ideal topology. Moreover

Proposition 4.2. *The topology of pointwise convergence τ_s is a projective ideal topology.*

Proof. We already know that τ_s is an ideal topology (Example 2.3(b)). Let $(P_\lambda)_\lambda$ be a net in $\mathcal{P}(^nE;F)$ such that $P_\lambda \xrightarrow{\tau_s} P \in \mathcal{P}(^nE;F)$. We have to prove that $(P_\lambda)_L \xrightarrow{\tau_s} P_L$ in $\mathcal{L}\left(\widehat{\otimes}^n_{s,\pi}E;F\right)$, that is, $(P_\lambda)_L(z) \longrightarrow P_L(z)$ in F for every $z \in \widehat{\otimes}^n_{s,\pi}E$. Assume first that $z = \sum_{j=1}^k \lambda_j \otimes^n x_j$ for some $k \in \mathbf{N}$, $x_1, \ldots, x_k \in E$ and nonzero scalars $\lambda_1, \ldots, \lambda_k \in \mathbf{K}$. Given $\varepsilon > 0$, there exists λ_0 such that

$$\|P_\lambda - P\|_{\{x_1,\ldots,x_k\}} < \frac{\varepsilon}{k \cdot \max\limits_{j=1,\ldots,k} |\lambda_j|} \quad \text{for every } \lambda \geq \lambda_0.$$

So, for $\lambda \geq \lambda_0$,

$$\begin{aligned}\|(P_\lambda)_L(z) - P_L(z)\| &= \left\|(P_\lambda - P)_L\left(\sum_{j=1}^k \lambda_j \otimes^n x_j\right)\right\| \leq \sum_{j=1}^k \|(P_\lambda - P)_L(\lambda_j \otimes^n x_j)\| \\ &= \sum_{j=1}^k |\lambda_j| \cdot \|(P_\lambda - P)_L(\otimes^n x_j)\| = \sum_{j=1}^k |\lambda_j| \cdot \|(P_\lambda - P)(x_j)\| < \varepsilon.\end{aligned}$$

This proves that $(P_\lambda)_L(z) \longrightarrow P_L(z)$ in F. Observe that $(P_\lambda - P)_\lambda$ is collection of continuous n-homogeneous polynomials from the Banach space $\widehat{\otimes}^n_{s,\pi}E$ to the Banach space F. The convergence $P_\lambda \xrightarrow{\tau_s} P$ implies, in particular, that the collection $(P_\lambda - P)_\lambda$ is pointwise bounded, so by the polynomial version of the Banach–Steinhaus Theorem [48, Theorem 2.6] there is $K > 0$ such that $\|P_\lambda - P\| \leq K$ for every λ. Let now z be an arbitrary element of $\widehat{\otimes}^n_{s,\pi}E$. There are sequences $(x_j)_{j=1}^\infty$ in E and $(\lambda_j)_{j=1}^\infty$ in $\mathbf{K}$ such that

$$z = \sum_{j=1}^\infty \lambda_j \otimes^n x_j \quad \text{and} \quad \sum_{j=1}^\infty |\lambda_j| \cdot \|x_j\|^n < \infty$$

(see [26, Proposition 2.2(9)]). Given $\varepsilon > 0$, let n_0 be such that $\sum_{j=n_0+1}^\infty |\lambda_j| \cdot \|x_j\|^n < \frac{\varepsilon}{2K}$. Calling $z' = \sum_{j=1}^{n_0} \lambda_j \otimes^n x_j$, by the first part of the proof we know that $(P_\lambda)_L(z') \longrightarrow P_L(z')$ in F. Let λ_0 be such that $\|(P_\lambda)_L(z') - P_L(z')\| < \frac{\varepsilon}{2}$ whenever $\lambda \geq \lambda_0$. Thus,

$$\begin{aligned}\|(P_\lambda)_L(z) - P_L(z)\| &= \left\|(P_\lambda - P)_L\left(\sum_{j=1}^\infty \lambda_j \otimes^n x_j\right)\right\| \\ &= \left\|(P_\lambda - P)_L\left(\sum_{j=1}^{n_0} \lambda_j \otimes^n x_j\right) + (P_\lambda - P)_L\left(\sum_{j=n_0+1}^\infty \lambda_j \otimes^n x_j\right)\right\| \\ &\leq \left\|(P_\lambda - P)_L\left(\sum_{j=1}^{n_0} \lambda_j \otimes^n x_j\right)\right\| + \sum_{j=n_0+1}^\infty \|(P_\lambda - P)_L(\lambda_j \otimes^n x_j)\| \\ &\leq \frac{\varepsilon}{2} + \sum_{j=n_0+1}^\infty |\lambda_j| \cdot \|P_\lambda - P\| \cdot \|x_j\|^n < \varepsilon,\end{aligned}$$

for every $\lambda \geq \lambda_0$, proving that $(P_\lambda)_L(z) \longrightarrow P_L(z)$ in F.

The converse is easy: given a net $(u_\lambda)_\lambda$ in $\mathcal{L}\left(\widehat{\otimes}^n_{s,\pi}E;F\right)$ such that $u_\lambda \xrightarrow{\tau_s} u \in \mathcal{L}\left(\widehat{\otimes}^n_{s,\pi}E;F\right)$, there are (unique) polynomials $(P_\lambda)_\lambda$ and P in $\mathcal{P}(^nE;F)$ such that

$(P_\lambda)_L = u_\lambda$ for every λ and $P_L = u$. For every $x \in E$,

$$P_\lambda(x) = (P_\lambda)_L(\otimes^n x) = u_\lambda(\otimes^n x) \longrightarrow u(\otimes^n x) = P_L(\otimes^n x) = P(x).$$

This proves that $P_\lambda \xrightarrow{\tau_s} P$ and completes the proof of the polynomial case of condition 4.1(ii). The multilinear case is analogous (for a simple proof of the multilinear Banach–Steinhaus Theorem, see Bernardino [5]). □

Now let us give some further examples of projective ideal topologies that are useful in the study of the approximation properties. For $A \subseteq E$ and $A_j \subseteq E_j$, $j = 1, \ldots, n$, define

$$A_1 \otimes \cdots \otimes A_n := \{x_1 \otimes \cdots \otimes x_n \colon x_j \in A_j, j = 1, \ldots, n\} \subseteq E_1 \otimes \cdots \otimes E_n,$$
$$\otimes_s^n A := \{\otimes^n x \colon x \in A\} \subseteq \otimes_s^n E.$$

Proposition 4.3. *Let $\mathcal{C} \subseteq \mathrm{BAN}$ be given. Suppose that for every Banach space E it has been assigned a collection $\mathcal{A}(E)$ of bounded subsets of E containing the singletons, satisfying* (1) *and such that, for all $n \in \mathbf{N}$ and Banach spaces $E_1, \ldots, E_n, E$ belonging to $\mathcal{C}$, the following hold:*

(i) *Every $A \in \mathcal{A}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$ is contained in a finite union of sets of the form $\overline{\mathrm{co}}(A_1 \otimes \cdots \otimes A_n)$, where $A_j \in \mathcal{A}(E_j)$, $j = 1, \ldots, n$.*
(ii) *If $A_j \in \mathcal{A}(E_j)$ for $j = 1, \ldots, n$, then there is $A \in \mathcal{A}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$ such that $A_1 \otimes \cdots \otimes A_n \subseteq A$.*
(iii) *Every $A \in \mathcal{A}\left(\widehat{\otimes}_{s,\pi}^n E\right)$ is contained in a finite union of sets of the form $\overline{\mathrm{co}}(\otimes_s^n A')$, where $A' \in \mathcal{A}(E)$.*
(iv) *If $A \in \mathcal{A}(E)$, then there is $A' \in \mathcal{A}\left(\widehat{\otimes}_{s,\pi}^n E\right)$ such that $\otimes_s^n A \subseteq A'$.*

By $\tau_{\mathcal{A}}$ we mean the topology on the spaces $\mathcal{L}(E;F)$ and $\mathcal{P}(^nE;F)$ of uniform convergence on sets of $\mathcal{A}(E)$, and the topology on the space $\mathcal{L}(E_1, \ldots, E_n; F)$ of uniform convergence on sets of $\mathcal{A}(E_1) \times \cdots \times \mathcal{A}(E_n)$. Then $\tau_{\mathcal{A}}$ is a $\mathcal{C}$-projective ideal topology.

Proof. We already know that $\tau_{\mathcal{A}}$ is an ideal topology (Proposition 2.4). Let E and F be Banach spaces with $E \in \mathcal{C}$ and let $(P_\lambda)_\lambda$ be a net in $\mathcal{P}(^nE;F)$ such that $P_\lambda \xrightarrow{\tau_{\mathcal{A}}} P \in \mathcal{P}(^nE;F)$. Let $A \in \mathcal{A}\left(\widehat{\otimes}_{s,\pi}^n E\right)$ and $\varepsilon > 0$. By condition (iii) there exist $k \in \mathbf{N}$ and sets $A_1', \ldots, A_k' \in \mathcal{A}(E)$ such that $A \subseteq \bigcup_{j=1}^k \left(\overline{\mathrm{co}}(\otimes_n^s A_j')\right)$. Let λ_0 be such that $\|P_\lambda - P\|_{A_j'} < \varepsilon$, $j = 1, \ldots, k$, whenever $\lambda \geq \lambda_0$. Since $(P_\lambda)_L$ and P_L are continuous linear operators,

$$\|(P_\lambda)_L - P_L\|_A \leq \|(P_\lambda)_L - P_L\|_{\bigcup\limits_{j=1}^k \left(\overline{\mathrm{co}}(\otimes_n^s A_j')\right)} = \max_{j=1,\ldots,k} \|(P_\lambda)_L - P_L\|_{\overline{\mathrm{co}}(\otimes_n^s A_j')} =$$
$$= \max_{j=1,\ldots,k} \|(P_\lambda)_L - P_L\|_{\otimes_n^s A_j'} = \max_{j=1,\ldots,k} \|P_\lambda - P\|_{A_j'} < \varepsilon$$

whenever $\lambda \geq \lambda_0$. This proves that $(P_\lambda)_L \xrightarrow{\tau_c} P_L$ in $\mathcal{L}\left(\widehat{\otimes}_{s,\pi}^n E; F\right)$.

Conversely, let $(u_\lambda)_\lambda$ be a net in $\mathcal{L}\left(\widehat{\otimes}_{s,\pi}^n E; F\right)$ such that $u_\lambda \xrightarrow{\tau_{\mathcal{A}}} u \in \mathcal{L}\left(\widehat{\otimes}_{s,\pi}^n E; F\right)$. There are $(P_\lambda)_\lambda$ and P in $\mathcal{P}(^nE;F)$ such that $(P_\lambda)_L = u_\lambda$ for every λ and $P_L = u$. Let $A \in \mathcal{A}(E)$ and $\varepsilon > 0$. By condition (iv) there is a set $A' \in \mathcal{A}\left(\widehat{\otimes}_{s,\pi}^n E\right)$ such that $\otimes_n^s(A) \subseteq A'$. So there is λ_0 such that $\|u_\lambda - u\|_{A'} < \varepsilon$ for $\lambda \geq \lambda_0$. Thus,

$$\|P_\lambda - P\|_A = \|(P_\lambda)_L - P_L\|_{\otimes_n^s A} = \|u_\lambda - u\|_{\otimes_n^s A} \leq \|u_\lambda - u\|_{A'} < \varepsilon,$$

for $\lambda \geq \lambda_0$. This proves that $P_\lambda \xrightarrow{\tau_{\mathcal{A}}} P$ and completes the proof of the polynomial case of condition 4.1(ii). The multilinear case is analogous. □

Example 4.4. Choosing $\mathcal{A}(E)$ as the collection of compact subsets of the Banach space E, let us see that the conditions of Proposition 4.3 are fulfilled. Condition (1) is obvious; every compact subset of $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ is contained in a set of the form $\overline{\mathrm{co}}(A_1 \otimes \cdots \otimes A_n)$, where A_j is compact in E_j for $j = 1, \ldots, n$ (see [22, Proposition 2.1]); and $K_1 \otimes \cdots \otimes K_n$ is compact in $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ whenever K_j is compact in E_j, $j = 1, \ldots, n$ [22, p. 509]. So letting τ_c be the compact-open topology on the spaces $\mathcal{L}(E; F)$ and $\mathcal{P}(^nE; F)$ and the topology on the space $\mathcal{L}(E_1, \ldots, E_n; F)$ of uniform convergent on cartesian products of compact sets, we have by Proposition 4.3 that τ_c is a projective ideal topology.

Example 4.5. Let $\mathcal{A}(E)$ be the collection of convex compact subsets of the Banach space E. Trivially, $\mathcal{A}$ satisfies condition (1). As to condition 4.3(i), given a compact convex set $A \in E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$, as in Example 4.4 there are compact sets $A_j \subseteq E_j$, $j = 1, \ldots n$, such that $A \subseteq \overline{\mathrm{co}}(A_1 \otimes \cdots \otimes A_n)$. Then each $\overline{\mathrm{co}}(A_j)$ is compact and convex in E_j by Mazur's Compactness Theorem [47, Theorem 2.8.15] and $A \subseteq \overline{\mathrm{co}}(\overline{\mathrm{co}}(A_1) \otimes \cdots \otimes \overline{\mathrm{co}}(A_n))$. As to condition 4.3(ii), given convex compacts sets $K_j \subseteq E_j$, $j = 1, \ldots, n$, as in Example 4.4 we know that $K_1 \otimes \cdots \otimes K_n$ is compact in $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$. By Mazur's Theorem we have that $\overline{\mathrm{co}}(K_1 \otimes \cdots \otimes K_n)$ is a compact convex set containing $K_1 \otimes \cdots \otimes K_n$. By Proposition 4.3, the topology $\tau_{\mathcal{A}}$ on spaces of linear operators and polynomials of uniform convergence on compact convex sets and the topology on spaces of multilinear mappings of uniform convergence on cartesian products of compact convex sets is a projective ideal topology.

Example 4.6. Let $\mathcal{D}P$ be the class of all Banach spaces with the Dunford–Pettis property and let WC_π be the class of Banach spaces defined by the following property: for every $n \in \mathbf{N}$ and all $E_1, \ldots, E_n \in WC_\pi$, weakly compact subsets of $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ are contained in a finite union of sets of the form $\overline{\mathrm{co}}(A_1 \otimes \cdots \otimes A_n)$, where A_j is a weakly compact subset of E_j, $j = 1, \ldots, n$ (property WC_π is stronger than property wc_π of Ruess [60, p. 247]). Let $\mathcal{A}(E)$ be the collection of weakly compact subsets of the Banach space E. Since bounded linear operators are weak-weak continuous, $\mathcal{A}$ satisfies condition (1). Condition 4.3(i) is automatically fulfilled for Banach spaces in WC_π. If A_j is weakly compact in E_j, $j = 1, \ldots, n$, and each E_j has the Dunford–Pettis property, then $A_1 \otimes \cdots \otimes A_n$ is weakly compact in $E_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ by [22, Proposition 2.5]. This proves condition 4.3(ii) for Banach spaces in $\mathcal{D}P$. Thus the topology of uniform convergence on weakly compact sets or on products of weakly compact sets is a $(\mathcal{D}P \cap WC_\pi)$-projective ideal topology.

Example 4.7. Let $\mathcal{A}(E)$ be the collection of convex weakly compact subsets of Banach space E. As before, $\mathcal{A}$ satisfies condition (1). The Krein–Smulian Theorem (the closed convex hull of a weakly compact subset of a Banach space is weakly compact as well) yields that condition 4.3(i) is fulfilled for Banach spaces belonging to the class WC_π of Example 4.6. Applying [22, Proposition 2.5] together with the same Krein–Smulian Theorem we have that condition 4.3(ii) is satisfied for the class $\mathcal{D}P$ of Banach spaces with the Dunford–Pettis property. So the topology of uniform convergence on convex weakly compact sets or on products of convex weakly compact sets is a $(\mathcal{D}P \cap WC_\pi)$-projective ideal topology.

Let us put the projective ideal topologies to work. Our first aim is to generalize the results of Çaliskan and Rueda [16, Section 3] and a very recent result from [10].

Definition 4.8. (Composition ideals) For a given operator ideal $\mathcal{I}$, it is said that:

(a) A multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ belongs to the composition multi-ideal $\mathcal{I} \circ L$, in symbols $A \in \mathcal{I} \circ L(E_1, \ldots, E_n; F)$, if there are Banach spaces G, a multilinear mapping $B \in \mathcal{L}(E_1, \ldots, E_n; G)$ and an operator $u \in \mathcal{I}(G; F)$ such that $A = u \circ B$.

(b) A polynomial $P \in \mathcal{P}(^nE; F)$ belongs to the composition polynomial ideal $\mathcal{I} \circ P$, in symbols $P \in \mathcal{I} \circ P(^nE; F)$, if there are a Banach space G, a polynomial $Q \in \mathcal{P}(^nE; G)$ and an operator $u \in \mathcal{I}(G; F)$ such that $P = u \circ Q$.

Further details on these polynomial/multi-ideals can be found in [9].

Proposition 4.9. *Let $\mathcal{I}, \mathcal{J}$ be operator ideals, $\mathcal{C} \subseteq \mathrm{BAN}$, τ be a $\mathcal{C}$-projective ideal topology, $n \in \mathbf{N}$ and E, F be Banach spaces with $E \in \mathcal{C}$. Consider the following conditions:*

(a) $\mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right) \subseteq \overline{\mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)}^{\tau}$.

(b) $\mathcal{I} \circ \mathcal{P}\left(^nE; F\right) \subseteq \overline{\mathcal{J} \circ \mathcal{P}(^nE; F)}^{\tau}$.

(c) $\mathcal{I}\left(E; F\right) \subseteq \overline{\mathcal{J}(E; F)}^{\tau}$.

Then (a) *and* (b) *are equivalent and they imply* (c).

Proof. Let $L\colon (\mathcal{P}(^nE; F), \tau) \longrightarrow \left(\mathcal{L}\left(\widehat{\otimes}^n_{s,\pi} E; F\right), \tau\right)$ be the linearization operator, that is, $L(P) = P_L$.

(a) $\Longrightarrow$ (b) By [9, Proposition 3.2] we know that $L\left(\mathcal{I} \circ \mathcal{P}(^nE; F)\right) = \mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)$ and $L\left(\mathcal{J} \circ \mathcal{P}(^nE; F)\right) = \mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)$. Since L is a homeomorphism, we have

$$\begin{aligned}\mathcal{I} \circ \mathcal{P}\left(^nE; F\right) &= L^{-1}\left(L\left(\mathcal{I} \circ \mathcal{P}\left(^nE; F\right)\right)\right) = L^{-1}\left(\mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)\right) \\ &\subseteq L^{-1}\left(\overline{\mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)}^{\tau}\right) = \overline{L^{-1}\left(\mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)\right)}^{\tau} = \overline{\mathcal{J} \circ \mathcal{P}\left(^nE; F\right)}^{\tau}.\end{aligned}$$

(b) $\Longrightarrow$ (a) In the same fashion,

$$\begin{aligned}\mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right) &= L\left(L^{-1}\left(\mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)\right)\right) = L\left(\mathcal{I} \circ \mathcal{P}\left(^nE; F\right)\right) \subseteq L\left(\overline{\mathcal{J} \circ \mathcal{P}(^nE; F)}^{\tau}\right) \\ &= \overline{L\left(\mathcal{J} \circ \mathcal{P}(^nE; F)\right)}^{\tau} = \overline{\mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)}^{\tau}.\end{aligned}$$

(a) $\Longrightarrow$ (c) Let $u \in \mathcal{I}(E; F)$. As $\widehat{\otimes}^n_{s,\pi} E$ contains a complemented isomorphic copy of E [7, Corollary 4], there are continuous linear operators $j\colon E \longrightarrow \widehat{\otimes}^n_{s,\pi} E$ and $p\colon \widehat{\otimes}^n_{s,\pi} E \longrightarrow E$ such that $p \circ j = \mathrm{id}_E$. Then $u \circ p \in \mathcal{I}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)$ and, by assumption, $u \circ p \in \overline{\mathcal{J}\left(\widehat{\otimes}^n_{s,\pi} E; F\right)}^{\tau}$. The ideal property of $\overline{\mathcal{J}}^{\tau}$ gives $u = u \circ p \circ j \in \overline{\mathcal{J}(E; F)}^{\tau}$. $\square$

Taking $F = \widehat{\otimes}^n_{s,\pi} E$ in Proposition 4.9 we obtain

Theorem 4.10. *Let $\mathcal{I}, \mathcal{J}$ be operator ideals, $\mathcal{C} \subseteq \mathrm{BAN}$, τ be a $\mathcal{C}$-projective ideal topology, $n \in \mathbf{N}$ and $E \in \mathcal{C}$. Consider the following conditions:*

(a) *$\widehat{\otimes}^n_{s,\pi} E$ has the $(\mathcal{I}, \mathcal{J}, \tau)$-WAP.*

(b) $\mathcal{I} \circ \mathcal{P}\left(^nE; \widehat{\otimes}^n_{s,\pi} E\right) \subseteq \overline{\mathcal{J} \circ \mathcal{P}\left(^nE; \widehat{\otimes}^n_{s,\pi} E\right)}^{\tau}$.

(c) $\mathcal{I}\left(E; \widehat{\otimes}^n_{s,\pi} E\right) \subseteq \overline{\mathcal{J}\left(E; \widehat{\otimes}^n_{s,\pi} E\right)}^{\tau}$.

Then (a) *and* (b) *are equivalent and they imply* (c).

We need two ingredients to recover Proposition 7 and Proposition 8 of [16] as particular instances of Theorem 4.10. Remember that a vector space-valued map has *finite rank* if its range generates a finite dimensional subspace of the target vector space. It is easy to check that a polynomial $P \in \mathcal{P}(^nE; F)$ has finite rank if and only if there are $k \in \mathbf{N}$, $P_1, \ldots, P_k \in \mathcal{P}(^nE)$ and $b_1, \ldots, b_k \in F$ such that

$P = \sum_{j=1}^{k} P_j \otimes b_j$. The space of all such polynomials is denoted by $\mathcal{P}_{\mathcal{F}}(^nE;F)$. Here is the first ingredient.

Lemma 4.11. $\mathcal{F} \circ \mathcal{P} = \mathcal{P}_{\mathcal{F}}$.

Proof. Let $P \in \mathcal{P}(^nE;F)$. Is is easy to check that $[P(E)] = P_L\left(\widehat{\otimes}^n_{s,\pi}E\right)$. So,

$$\begin{aligned} P \in \mathcal{F} \circ \mathcal{P}(^nE;F) &\Longleftrightarrow P_L \in \mathcal{F}\left(\widehat{\otimes}^n_{s,\pi}E;F\right) \Longleftrightarrow \dim P_L\left(\widehat{\otimes}^n_{s,\pi}E\right) < \infty \\ &\Longleftrightarrow \dim[P(E)] < \infty \Longleftrightarrow P \in \mathcal{P}_{\mathcal{F}}(^nE;F), \end{aligned}$$

where the first equivalence follows from [9, Proposition 3.2]. □

Let $\mathcal{P}_{\mathcal{K}}$ denote the class of compact homogeneous polynomials between Banach spaces (bounded sets are sent to relatively compact sets). The second ingredient is a classical result due to Aron and Schottenloher [3] that asserts that

$$\mathcal{P}_{\mathcal{K}} = \mathcal{K} \circ \mathcal{P}. \tag{3}$$

Taking $\tau = \tau_c$, $\mathcal{I} = \mathcal{K}$, $\mathcal{J} = \mathcal{F}$ and $\mathcal{C} = \mathrm{BAN}$ in Theorem 4.10, with the help of Lemma 4.11 and (3) we get

Corollary 4.12. [16, Proposition 7] *Let $n \in \mathbf{N}$ and E be a Banach space. Consider the following conditions:*

(a) $\widehat{\otimes}^n_{s,\pi}E$ *has the* $(\mathcal{K},\mathcal{F},\tau_c)$*-WAP.*
(b) $\mathcal{P}_{\mathcal{K}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right) \subseteq \overline{\mathcal{P}_{\mathcal{F}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right)}^{\tau_c}$.
(c) $\mathcal{K}\left(E;\widehat{\otimes}^n_{s,\pi}E\right) \subseteq \overline{\mathcal{F}\left(E;\widehat{\otimes}^n_{s,\pi}E\right)}^{\tau_c}$.

Then (a) *and* (b) *are equivalent and they imply* (c).

Remark 4.13. Condition (b) in [16, Proposition 7] reads $\mathcal{P}_{\mathcal{K}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right) = \overline{\mathcal{P}_{\mathcal{F}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right)}^{\tau_c}$, but a glance at its proof reveals that it should read $\mathcal{P}_{\mathcal{K}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right) \subseteq \overline{\mathcal{P}_{\mathcal{F}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right)}^{\tau_c}$.

Taking $\tau = \|\cdot\|$, $\mathcal{I} = \mathcal{K}$, $\mathcal{J} = \mathcal{F}$ and $\mathcal{C} = \mathrm{BAN}$ in Theorem 4.10, with the help of Lemma 4.11 and (3) and remembering that $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{K}$ are norm closed, we get

Corollary 4.14. [16, Proposition 8] *Let $n \in \mathbf{N}$ and E be a Banach space. Consider the following conditions:*

(a) $\widehat{\otimes}^n_{s,\pi}E$ *has the* $(\mathcal{K},\mathcal{F},\|\cdot\|)$*-WAP.*
(b) $\mathcal{P}_{\mathcal{K}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right) = \overline{\mathcal{P}_{\mathcal{F}}\left(^nE;\widehat{\otimes}^n_{s,\pi}E\right)}^{\|\cdot\|}$.
(c) $\mathcal{K}\left(E;\widehat{\otimes}^n_{s,\pi}E\right) = \overline{\mathcal{F}\left(E;\widehat{\otimes}^n_{s,\pi}E\right)}^{\|\cdot\|}$.

Then (a) *and* (b) *are equivalent and they imply* (c).

Moreover, the choices $\tau = \|\cdot\|$, $\mathcal{J} = \mathcal{F}$ and $\mathcal{C} = \mathrm{BAN}$ show that Theorem 4.10 also generalizes the very recent result [10, Proposition 2.12].

The results above can be extended to the full projective tensor product. Replacing the projective symmetric tensor product by the projective tensor product, homogeneous polynomials by multilinear mappings and the polynomial ideal $\mathcal{I} \circ \mathcal{P}$ by the multi-ideal $\mathcal{I} \circ \mathcal{L}$, the proof of Proposition 4.9, *mutatis mutandis*, works. Actually the multilinear case is easier as $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ trivially contains complemented copies of each E_j. So

Proposition 4.15. *Let $\mathcal{I}, \mathcal{J}$ be operator ideals, $\mathcal{C} \subseteq$ BAN, τ be $\mathcal{C}$-a projective ideal topology, $n \in \mathbf{N}$ and $E_1, \ldots, E_n$, F be Banach spaces with $E_1, \ldots, E_n \in \mathcal{C}$. Consider the following conditions:*

(a) $\mathcal{I}\left(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F\right) \subseteq \overline{\mathcal{J}\left(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F\right)}^\tau$.
(b) $\mathcal{I} \circ \mathcal{L}\left(E_1, \ldots, E_n; F\right) \subseteq \overline{\mathcal{J} \circ \mathcal{L}(E_1, \ldots, E_n; F)}^\tau$.
(c) $\mathcal{I}\left(E_j; F\right) \subseteq \overline{\mathcal{J}(E_j; F)}^\tau$ *for* $j = 1, \ldots, n$.

Then (a) *and* (b) *are equivalent and they imply* (c).

Taking $F = E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ in Proposition 4.15 we get

Theorem 4.16. *Let $\mathcal{I}, \mathcal{J}$ be operator ideals, $\mathcal{C} \subseteq$ BAN, τ be a $\mathcal{C}$-projective ideal topology, $n \in \mathbf{N}$ and $E_1, \ldots, E_n$ be Banach spaces with $E_1, \ldots, E_n \in \mathcal{C}$. Consider the following conditions:*

(a) $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ *has the* $(\mathcal{I}, \mathcal{J}, \tau)$*-WAP.*
(b) $\mathcal{I} \circ \mathcal{L}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) \subseteq \overline{\mathcal{J} \circ \mathcal{L}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^\tau$.
(c) $\mathcal{I}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) \subseteq \overline{\mathcal{J}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^\tau$ *for* $j = 1, \ldots, n$.

Then (a) *and* (b) *are equivalent and they imply* (c).

By $\mathcal{L}$ we denote the class of all continuous multilinear mappings of finite rank. The same proof of Lemma 4.11 gives the formula $\mathcal{L} \circ \mathcal{F} = \mathcal{L}_\mathcal{F}$. Denoting by $\mathcal{L}_\mathcal{K}$ the class of compact multilinear mappings, a classical result due to Pełczyński [54, Proposition 3] gives the formula $\mathcal{L} \circ \mathcal{K} = \mathcal{L}_\mathcal{K}$. Thus, a multilinear analogue of [16, Proposition 7] is obtained taking $\mathcal{C} =$ BAN, $\tau = \tau_c$, $\mathcal{I} = \mathcal{K}$ and $\mathcal{J} = \mathcal{F}$ in Theorem 4.16.

Corollary 4.17. *Let $n \in \mathbf{N}$ and $E_1, \ldots, E_n$ be Banach spaces. Consider the following conditions:*

(a) $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ *has the* $(\mathcal{K}, \mathcal{F}, \tau_c)$*-WAP.*
(b) $\mathcal{L}_\mathcal{K}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) \subseteq \overline{\mathcal{L}_\mathcal{F}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^{\tau_c}$.
(c) $\mathcal{K}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) \subseteq \overline{\mathcal{F}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^{\tau_c}$ *for* $j = 1, \ldots, n$.

Then (a) *and* (b) *are equivalent and they imply* (c).

And remembering that $\mathcal{L}_\mathcal{K}$ and $\mathcal{K}$ are norm closed, taking $\mathcal{C} =$ BAN, $\tau = \|\cdot\|$, $\mathcal{I} = \mathcal{K}$ and $\mathcal{J} = \mathcal{F}$ in Theorem 4.16 we obtain a multilinear analogue of [16, Proposition 8].

Corollary 4.18. *Let $n \in \mathbf{N}$ and $E_1, \ldots, E_n$ be Banach spaces. Consider the following conditions:*

(a) $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ *has the* $(\mathcal{K}, \mathcal{F}, \|\cdot\|)$*-WAP.*
(b) $\mathcal{L}_\mathcal{K}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) = \overline{\mathcal{L}_\mathcal{F}\left(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^{\|\cdot\|}$.
(c) $\mathcal{K}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right) = \overline{\mathcal{F}\left(E_j; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n\right)}^{\|\cdot\|}$ *for* $j = 1, \ldots, n$.

Then (a) *and* (b) *are equivalent and they imply* (c).

We finish the paper showing that the concept of projective ideal topology allows us to generalize the results of [6, Section 3]. We shall need the so-called factorization method to generate a multi-ideal from a given operator ideal.

Definition 4.19. For a given operator ideal $\mathcal{I}$, a multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to belong to the multi-ideal $\mathcal{L}[\mathcal{I}]$, in symbols $A \in \mathcal{L}[\mathcal{I}](E_1, \ldots, E_n; F)$, if there are Banach spaces $G_1, \ldots, G_n$, a multilinear mapping $B \in$

$\mathcal{L}(G_1, \ldots, G_n; F)$ and operators $u_j \in \mathcal{I}(E_j; G_j)$, $j = 1, \ldots, n$, such that $A = B \circ (u_1, \ldots, u_n)$.

Further details on these multi-ideals can be found in [8].

The examples of projective ideal topologies we have been working with are topologies of uniform convergence on subsets (or products of subsets) belonging to a certain class $\mathcal{A}(E)$ of subsets of the Banach space E, $E \in$ BAN. The condition

(4) If $A_1, A_2 \in \mathcal{A}(E)$, then there is $A \in \mathcal{A}(E)$ such that $A_1 \cup A_2 \subseteq A$,

is fulfilled by all of them. Indeed, it is obvious that the projective ideal topologies of Proposition 4.2 and Examples 4.4 and 4.6 fulfill condition (4). And using that the closed convex hull of a (weakly) compact set is (weakly) compact we have that the projective ideal topologies of Examples 4.5 and 4.7 fulfill condition (4) too. So, imposing condition (4) we keep all our examples of projective ideal topologies.

Given operator ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ and Banach spaces $E_1, \ldots, E_n, F$, by

$$\mathcal{I}_1 \otimes \cdots \otimes \mathcal{I}_n(E_1, \ldots, E_n; F)$$

we denote that set of all n-linear mappings $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ for which there are linear operators $T_j \in \mathcal{I}_j(E_j; E_j)$, $j = 1, \ldots, n$, and an n-linear mapping $B \in \mathcal{L}(E_1, \ldots, E_n; F)$ such that $A = B \circ (T_1 \ldots, T_n)$. The next result generalizes [6, Proposition 3.4], which, in its turn, generalizes a classical result due to Heinrich [31, Theorem 3.].

Theorem 4.20. *Let $\mathcal{C} \subseteq$ BAN, $\mathcal{A}$ be as in Proposition 4.3 and satisfying (4), $\tau_{\mathcal{A}}$ be the corresponding $\mathcal{C}$-projective ideal topology, $\mathcal{I}, \mathcal{I}_1, \ldots, \mathcal{I}_n, \mathcal{J}, \mathcal{J}_1, \ldots, \mathcal{J}_n$ be operator ideals with $\mathcal{L}[\mathcal{I}_1, \ldots, \mathcal{I}_n] \subseteq \overline{\mathcal{I}}^{\tau_{\mathcal{A}}} \circ \mathcal{L}$ and $E_1, \ldots, E_n$ be Banach spaces belonging to $\mathcal{C}$ such that*

(5) $\mathcal{J} \circ \mathcal{L}(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n) \subseteq \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$.

If each E_j has the $(\mathcal{J}_j, \mathcal{I}_j, \tau_{\mathcal{A}})$-WAP, then $E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP.

Proof. Let $T \in \mathcal{J}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$. By [9, Proposition 3.2], the n-linear mapping $B \in \mathcal{L}(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$ such that $B_L = T$ belongs to $\mathcal{J} \circ \mathcal{L}$. By (5) there are linear operators $T_j \in \mathcal{J}_j(E_j; E_j)$, $j = 1, \ldots, n$, and an n-linear mapping $D \in \mathcal{L}(E_1, \ldots, E_n; E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n)$ such that $B = D \circ (T_1, \ldots, T_n)$. It follows easily that

$$T = B_L = D_L \circ (T_1 \otimes \cdots \otimes T_n).$$

Given $A \in \mathcal{A}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$, by condition 4.3(i) there are $k \in \mathbf{N}$ and sets $A_j^i \in \mathcal{A}(E_j)$, $j = 1, \ldots, n$, $i = 1, \ldots, k$, such that $A \subseteq \bigcup_{i=1}^k \overline{\text{co}}(A_1^i \otimes \cdots \otimes A_n^i)$. Let $\varepsilon > 0$. By condition (4) there are sets $A_j \in \mathcal{A}(E_j)$ such that $A_j^1 \cup \cdots \cup A_j^k \subseteq A_j$, $j = 1, \ldots, n$. Since sets in $\mathcal{A}$ are bounded there is $M > 0$ such that $\|x\| \le M$ for every $x \in A_j$, $j = 1, \ldots, n$. As E_1 has the $(\mathcal{J}_1, \mathcal{I}_1, \tau_{\mathcal{A}})$-WAP, there is an operator $u_1 \in \mathcal{I}_1(E_1; E_1)$ such that

$$\|u_1 - T_1\|_{A_1} < \frac{\varepsilon}{4nM^{n-1}\|D\| \cdot \|T_2\| \cdots \|T_n\|}.$$

As E_2 has $(\mathcal{J}_2, \mathcal{I}_2, \tau_{\mathcal{A}})$-WAP , there is an operator $u_2 \in \mathcal{I}_2(E_2; E_2)$ such that

$$\|u_2 - T_2\|_{A_2} < \frac{\varepsilon}{4nM^{n-1}\|D\| \cdot \|u_1\| \cdot \|T_3\| \cdots \|T_n\|}.$$

Continuing the process we obtain operators $u_j \in \mathcal{I}_j(E_j; E_j)$ such that

$$\|u_j - T_j\|_{A_j} < \frac{\varepsilon}{4nM^{n-1}\|D\| \cdot \|u_1\| \cdots \|u_{j-1}\| \cdot \|T_{j+1}\| \cdots \|T_n\|}$$

for $j = 1, \dots, n$. Performing a computation identical to the one in the proof of [6, Proposition 3.4] we conclude that

$$\|u_1 \otimes \cdots \otimes u_n(x_1 \otimes \cdots \otimes x_n) - T_1(x_1) \otimes \cdots \otimes T_n(x_n)\| < \frac{\varepsilon}{4\|D\|}, \tag{6}$$

for all $x_1 \in A_1, \dots, x_n \in A_n$. Using that D_L, $u_1 \otimes \cdots \otimes u_n$ and $T_1 \otimes \cdots \otimes T_n$ are all continuous linear operators, from (6) it follows that

$$\begin{aligned}
\|D_L \circ (u_1 \otimes \cdots \otimes u_n) - T\|_A &\le \|D_L \circ (u_1 \otimes \cdots \otimes u_n) - T\|_{\bigcup_{i=1}^k \overline{\text{co}}(A_1^i \otimes \cdots \otimes A_n^i)} \\
&= \max_{i=1,\dots,k} \|D_L \circ (u_1 \otimes \cdots \otimes u_n) - T\|_{\overline{\text{co}}(A_1^i \otimes \cdots \otimes A_n^i)} \\
&\le \|D_L \circ (u_1 \otimes \cdots \otimes u_n) - T\|_{\overline{\text{co}}\left((A_1^1 \cup \cdots \cup A_1^k) \otimes \cdots \otimes (A_n^1 \cup \cdots \cup A_n^k)\right)} \\
&\le \|D_L \circ (u_1 \otimes \cdots \otimes u_n) - T\|_{\overline{\text{co}}(A_1 \otimes \cdots \otimes A_n)} \\
&= \|D_L \circ (u_1 \otimes \cdots \otimes u_n - T_1 \otimes \cdots \otimes T_n)\|_{\overline{\text{co}}(A_1 \otimes \cdots \otimes A_n)} \\
&\le \|D_L\| \cdot \|u_1 \otimes \cdots \otimes u_n - T_1 \otimes \cdots \otimes T_n\|_{\overline{\text{co}}(A_1 \otimes \cdots \otimes A_n)} \\
&= \|D\| \cdot \|u_1 \otimes \cdots \otimes u_n - T\|_{A_1 \otimes \cdots \otimes A_n} \le \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.
\end{aligned}$$

We know that $\overline{\mathcal{I}}^{\tau_{\mathcal{A}}}$ is an operator ideal because $\tau_{\mathcal{A}}$ is an ideal topology, so the assumption $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n] \subseteq \overline{\mathcal{I}}^{\tau_{\mathcal{A}}} \circ \mathcal{L}$ together with [6, Proposition 3.3] yield that $u_1 \otimes \cdots \otimes u_n$ belongs to $\overline{\mathcal{I}}^{\tau_{\mathcal{A}}}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$. Calling on the ideal property of $\overline{\mathcal{I}}^{\tau_{\mathcal{A}}}$ once again we conclude that $D_L \circ (u_1 \otimes \cdots \otimes u_n)$ belongs to $\overline{\mathcal{I}}^{\tau_{\mathcal{A}}}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$ as well. So there is $U \in \mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)$ such that

$$\|U - D_L \circ (u_1 \otimes \cdots \otimes u_n)\|_A < \frac{\varepsilon}{2}.$$

It follows that $\|U - T\|_A < \varepsilon$, which proves that

$$T \in \overline{\mathcal{I}(E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n)}^{\tau_{\mathcal{A}}}$$

and completes the proof. □

When $\mathcal{I}_n = \mathcal{I}$ for every n, we write $\mathcal{L}[\mathcal{I}] := \bigcup_{n=1}^\infty \mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n]$.

Corollary 4.21. *Let $\mathcal{C} \subseteq$ BAN, $\mathcal{A}$ be as in Proposition 4.3 and satisfying (4), $\tau_{\mathcal{A}}$ be the corresponding $\mathcal{C}$-projective ideal topology and $\mathcal{I}$, $\mathcal{J}$ be operator ideals such that $\mathcal{L}[\mathcal{I}] \subseteq \overline{\mathcal{I}}^{\tau_{\mathcal{A}}} \circ \mathcal{L}$. The following are equivalent for a Banach space $E \in \mathcal{C}$ such that $\mathcal{J} \circ \mathcal{L}\left({}^n E; \widehat{\otimes}_\pi^n E\right) \subseteq \otimes^n \mathcal{J}\left({}^n E; \widehat{\otimes}_\pi^n E\right)$ for every n (for some n, respectively):*

(a) *E has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP.*
(b) *$\widehat{\otimes}_\pi^n E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for every n ($\widehat{\otimes}_\pi^k E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for every $k \le n$, respectively).*
(c) *$\widehat{\otimes}_\pi^n E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for some n ($\widehat{\otimes}_\pi^k E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for some $k \le n$, respectively).*
(d) *$\widehat{\otimes}_\pi^{n,s} E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for every n ($\widehat{\otimes}_\pi^{k,s} E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for every $k \le n$, respectively).*
(e) *$\widehat{\otimes}_\pi^{n,s} E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for some n ($\widehat{\otimes}_\pi^{k,s} E$ has the $(\mathcal{J}, \mathcal{I}, \tau_{\mathcal{A}})$-WAP for some $k \le n$, respectively).*

Proof. Repeat the proof of [6, Corollary 3.8] using Theorem 4.20 and Proposition 2.10. □

Since $\text{id}_{E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n} = \text{id}_{E_1} \otimes \cdots \otimes \text{id}_{E_n}$, it is clear that condition (5) holds for $\mathcal{J} = \mathcal{J}_1 = \cdots = \mathcal{J}_n = \mathcal{L}$ and every n. Thus, taking $\mathcal{C} =$ BAN, $\mathcal{J} = \mathcal{J}_1 = \cdots = \mathcal{J}_n = \mathcal{L}$

and letting $\mathcal{A}(E)$ be the collection of compact subsets of the Banach space E, that is, $\tau_{\mathcal{A}} = \tau_c$, Theorem 4.20 recovers [6, Proposition 3.4] and Corollary 4.21 recovers [6, Corollary 3.8] (remember that $(\mathcal{L}, \mathcal{I}, \tau)$-AP = $(\mathcal{L}, \mathcal{I}, \tau)$-WAP).

A number of examples of ideals satisfying $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n] \subseteq \mathcal{J} \circ \mathcal{L}$ and/or $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{J} \circ \mathcal{L}$ can be found in [6, 3.5–3.7].

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