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EXAMPLES OF METRIC MEASURE SPACES RELATED TO MODIFIED HARDY–LITTLEWOOD MAXIMAL OPERATORS

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Abstract. We furnish elementary examples of (nondoubling) metric measure spaces, for which the modified Hardy–Littlewood maximal operators M_k and M_k^c, uncentered and centered, fail to be weak type $(1,1)$ for $1 \le k < 3$ and $1 \le k < 2$, respectively.

1. Introduction

The aim of this short note is to complement our previous article [2], where we proved that the modified Hardy–Littlewood maximal operators M_3 and M_2^c, uncentered and centered, are weak type $(1,1)$; see [2, Theorem 3.1]. This was done in the setting of a general metric measure space (X,d,μ) with the sole assumption on the Borel measure μ to be finite on bounded sets (balls of measure zero are admitted). Recall that for a parameter $k \ge 1$ the *modified Hardy–Littlewood maximal operator* $M_k = M_{k,d,\mu}$ is defined by

$$M_k f(x) = \sup_{x\in B} \frac{1}{\mu(kB)} \int_B |f|\, d\mu, \qquad x \in X,$$

where the supremum is taken over all open balls B containing x, and the balls of measure zero are omitted. Here kB denotes the ball concentric with B and of radius k times that of B. The centered version $M_k^c = M_{k,d,\mu}^c$ is defined analogously but the balls included in the definition of M_k^c and related to $x \in X$ are centered at x (and of positive measure).

The results of [2, Theorem 3.1], as acknowledged in [2, p. 447], are sharp in the sense that, in general, any $k < 3$ or any $k < 2$ is not enough in the uncentered or in the centered case, respectively, for M_k or M_k^c to be of weak type $(1,1)$. This was shown by Sawano [1] by a direct construction, see the proof of [1, Proposition 1.1] for the centered case; for the uncentered case it was mentioned in the beginning of [1, Section 3] that a similar construction can be done.

Looking at [1, Section 2.3] one can admit that the construction is tangled. In this note we present extremely elementary examples which prove the abovementioned sharpness.

2. Examples

In both examples X is a countable planar graph, connected and acyclic, equipped with the geodesic distance d, that is the length of the (unique) minimal path joining

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two points; the distance between two points joined by an edge is, by definition, equal to 1. The induced topology is discrete and (X, d) is separable. The Borel (nondoubling) measure μ on X is either the countable measure or a slight modification of it; the measure of any ball is positive and finite.

Example 2.1. (centered case) Consider the graph X with vertices x_n and x_{nj}, and edges joining x_n with x_{n+1}, and x_n with x_{nj}, for any $n \in \mathbf{N}$ and $j \in \{1, 2, \dots, n\}$. See Figure 1. Let μ be the counting measure on X, $\mu(\{x\}) = 1$ for $x \in X$. Consider (X, d, μ) as a metric measure space and fix $1 \le k < 2$. We shall estimate $M_k^c \delta_{x_n}(x_{nj})$, where δ_{x_n} denotes the Dirac delta at x_n. Take $r > 1$ such that $kr < 2$. Let $B = B(x_{nj}, r)$. Then $B = \{x_{nj}, x_n\}$ and $kB = B(x_{nj}, kr) = \{x_{nj}, x_n\}$, hence $\mu(kB)^{-1} \int_B \delta_{x_n}\, d\mu = \frac{1}{2}$. Consequently, $M_k^c \delta_{x_n}(x_{nj}) \ge \frac{1}{2}$ and $\mu(\{M_k^c \delta_{x_n} > \frac{1}{3}\}) \ge n$. Since $\|\delta_{x_n}\|_{\ell^1(X,\mu)} = 1$, M_k^c fails to be weak type $(1, 1)$.

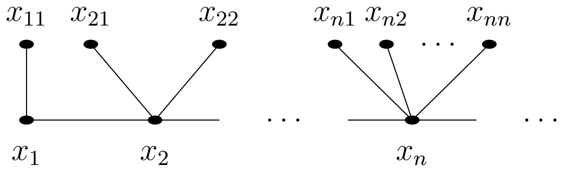

Figure 1.

Example 2.2. (uncentered case) Consider the graph X with vertices x_n, x_{nj} and y_{nj}, and edges joining x_n with x_{n+1}, x_n with x_{nj}, and x_{nj} with y_{nj}, for any $n \in \mathbf{N}$ and $j \in \{1, 2, \dots, n\}$. See Figure 2 (where only the branch starting from x_n is shown). Let μ be the measure on X, such that $\mu(\{x_n\}) = \mu(\{y_{nj}\}) = 1$, and $\mu(\{x_{nj}\}) = \frac{1}{n}$. Consider (X, d, μ) as a metric measure space, fix $1 \le k < 3$ and choose $1 < r < 2$ such that $kr < 3$. Let $B = B(x_{nj}, r)$. Then $B = \{x_n, x_{nj}, y_{nj}\}$ and $kB \subset \{x_{n-1}, x_n, x_{n+1}, x_{n1}, \dots, x_{nn}, y_{nj}\}$, for $n \ge 2$, say. Therefore $\mu(kB)^{-1} \int_B \delta_{x_n}\, d\mu \ge \frac{1}{5}$, and hence $M_k \delta_{x_n}(y_{nj}) \ge \frac{1}{5}$. Consequently, $\mu(\{M_k \delta_{x_n} > \frac{1}{6}\}) \ge n$. Since $\|\delta_{x_n}\|_{\ell^1(X,\mu)} = 1$, M_k fails to be weak type $(1, 1)$.

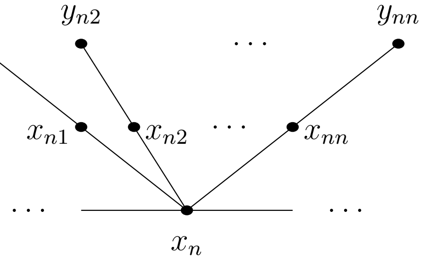

Figure 2.

Finally, notice that since $\|\delta_{x_n}\|_{\ell^p(X,\mu)} = 1$ the above argument also shows that M_k^c for $1 \le k < 2$, and M_k for $1 \le k < 3$, fail to be weak type (p, p) for any $1 \le p < \infty$.

References

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