Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 457–464

HEINZ–SCHWARZ INEQUALITIES FOR HARMONIC MAPPINGS IN THE UNIT BALL

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Abstract. We first prove the following generalization of Schwarz lemma for harmonic mappings. If u is a harmonic mapping of the unit ball onto itself then $\|u(x) - (1 - \|x\|^2)/(1 + \|x\|^2)^{n/2}u(0)\| \leqslant U(|x|N)$. By using this result we obtain certain sharp estimate of the gradient of a harmonic mapping. Those two results extend some known result from harmonic mapping theory [1]. By using the Schwarz lemma for harmonic mappings we derive Heinz inequality on the boundary of the unit ball by providing a sharp constant C_n in the inequality: $\|\partial_r u(r\eta)\|_{r=1} \geqslant C_n$, $\|\eta\| = 1$, for every harmonic mapping of the unit ball into itself satisfying the condition $u(0) = 0$, $\|u(\eta)\| = 1$.

1. Introduction

Heinz in his classical paper [4] obtained the following result: If u is a harmonic diffeomorphism of the unit disk $\mathbf{U}$ onto itself satisfying the condition $u(0) = 0$, then

$$|u_x(z)|^2 + |u_y(z)|^2 \geqslant \frac{2}{\pi^2}, \quad z \in \mathbf{U}.$$

The proof uses the following representation of harmonic mappings in the unit disk

$$u(z) = f(z) + \overline{g(z)}, \tag{1.1}$$

where f and g are holomorphic functions with $|g'(z)| < |f'(z)|$. It uses the maximum principle for holomorphic functions and the following sharp inequality

$$\liminf_{r\to 1^-} \left|\frac{\partial u(re^{it})}{\partial r}\right| \geqslant \frac{2}{\pi} \tag{1.2}$$

proved by using the Schwarz lemma for harmonic functions. The aim of this paper is to generalize inequality (1.2) for several dimensional case.

If u is a harmonic mapping of the unit ball onto itself, then we do not have any representation of u as in (1.1).

It is well known that a harmonic function (and a mapping) $u \in L^\infty(B^n)$, where $B = B^n$ is the unit ball with the boundary $S = S^{n-1}$, has the following integral representation

$$u(x) = \mathcal{P}[f](x) = \int_{S^{n-1}} P(x,\zeta) f(\zeta)\, d\sigma(\zeta), \tag{1.3}$$

where

$$P(x,\zeta) = \frac{1 - \|x\|^2}{\|x - \zeta\|^n}, \quad \zeta \in S^{n-1},$$

doi:10.5186/aasfm.2016.4126
2010 Mathematics Subject Classification: Primary 31A05; Secondary 42B30.
Key words: Harmonic mappings, Heinz inequality, Schwarz inequality.

is Poisson kernel and σ is the unique normalized rotation invariant Borel measure on S^{n-1} and $\|\cdot\|$ is the Euclidean norm.

We have the following Schwarz lemma for harmonic mappings on the unit ball B^n (see e.g. [1]). If u is a harmonic mapping of the unit ball into itself such that $u(0)=0$, then

$$\|u(x)\| \leqslant U(rN), \tag{1.4}$$

where $r=\|x\|$, $N=(0,\dots,0,1)$ and U is a harmonic function of the unit ball into $[-1,1]$ defined by

$$U(x)=\mathcal{P}[\chi_{S^+}-\chi_{S^-}](x), \tag{1.5}$$

where χ is the indicator function and $S^+=\{x\in S\colon x_n\geqslant 0\}$, $S^-=\{x\in S\colon x_n\leqslant 0\}$. Note that, the standard harmonic Schwarz lemma is formulated for real functions only, but we can reduce the previous statement to the standard one by taking $v(x)=\langle u(x),\eta\rangle$, for some $\|\eta\|=1$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. Indeed, we will prove a certain generalization of (1.4) without the a priory condition $u(0)=0$ (Theorem 2.1). For Schwarz lemma for the derivatives of harmonic mappings on the plane and space we refer to the papers [6, 7]. It is worth to mention here a certain extension of (1.2) for the mappings which are solution of certain elliptic partial differential equations in the plane [2]. For certain boundary Schwarz lemma on the unit ball for holomorphic mappings in $\mathbf{C}^n$ we refer to the paper [9].

By using Hopf theorem it can be proved ([5]) that if u is a harmonic mapping of the unit ball onto itself such that $u(0)=0$ and $\|u(\zeta)\|=1$, then

$$\liminf_{r\to 1}\left\|\frac{\partial u}{\partial r}(r\zeta)\right\|\geqslant C_n,$$

where C_n is a certain positive constant. Our goal is to find the largest constant C_n. This is done in Theorem 2.4 and Theorem 2.5.

2. Preliminaries and main results

First we prove the following extension and generalization of harmonic Schwarz lemma for B^n, $n\geqslant 3$. The case $n=2$ has been treated and proved by Pavlović [10, Theorem 3.6.1].

Theorem 2.1. *If u is a harmonic mapping of the unit ball onto itself, then*

$$\left\|u(x)-\frac{1-\|x\|^2}{(1+\|x\|^2)^{n/2}}u(0)\right\|\leqslant U(\|x\|N). \tag{2.1}$$

Proof. Assume first that $x=rN$. We have that

$$u(rN)=\int_{S^{n-1}}\frac{1-r^2}{\|\zeta-rN\|^n}f(\zeta)\,d\sigma(\zeta),$$

and so

$$u(rN)-\frac{1-r^2}{(1+r^2)^{n/2}}u(0)=\int_{S^{n-1}}\left(\frac{1-r^2}{\|\zeta-rN\|^n}-\frac{1-r^2}{(1+r^2)^{n/2}}\right)f(\zeta)\,d\sigma(\zeta).$$

Further we have

$$\begin{aligned}\|u(rN)-\frac{1-r^2}{(1+r^2)^{n/2}}u(0)\| &\leqslant \int_{S^{n-1}}\left|\frac{1-r^2}{\|\zeta-rN\|^n}-\frac{1-r^2}{(1+r^2)^{n/2}}\right|d\sigma(\zeta)\\ &=\int_{S^+}\left(\frac{1-r^2}{\|\zeta-rN\|^n}-\frac{1-r^2}{(1+r^2)^{n/2}}\right)d\sigma(\zeta)\\ &\quad+\int_{S^-}\left(\frac{1-r^2}{(1+r^2)^{n/2}}-\frac{1-r^2}{\|\zeta-rN\|^n}\right)d\sigma(\zeta).\end{aligned}$$

Thus

$$\left\|u(rN)-\frac{1-r^2}{(1+r^2)^{n/2}}u(0)\right\|\leqslant U(rN).$$

Now if x is not on the ray $[0,N]$, we choose a unitary transformation O such that $O(N)=x/|x|$. Then we make use of harmonic mapping $v(y)=u(O(y))$ for which we have $v(rN)=u(O(rN))=u(x)$. By making use of the previous proof we obtain (2.1). □

In order to continue, recall the Khavinson question [7]. It deals with the sharp function $g(|x|)$ in the inequality $\|\nabla u(x)\|\leqslant g(|x|)\|u\|_\infty$, where x is an arbitrary point of the unit ball. The variational problem of finding the coefficient $g(|x|)$ has been reduced in [8] to a solution of a minimization problem along a scalar parameter inside a double integral. By using Theorem 2.1, we obtain the following new proof of well-known inequality [11, p. 139, eq. (6)]. Observe that it is an extension of [1, Theorem 6.2.6].

Corollary 2.2. *Under conditions of the previous theorem we have the following inequality*

$$\|\nabla u(x)\|\leqslant 2\frac{\omega_{n-1}}{\omega_n}\frac{1}{1-\|x\|},$$

where ω_n is the volume of B^n. The constant $2\frac{\omega_{n-1}}{\omega_n}$ is sharp. However this inequality is not the sharp pointwise estimate, and thus it doesn't answer to the Khavinson question.

Proof. Let $x\in B^n$ and let $v(y)=u(x+(1-\|x\|)y)$. By applying (2.1) to v we obtain

$$\left\|u(x+(1-\|x\|)y)-\frac{1-\|y\|^2}{(1+\|y\|^2)^{n/2}}u(x)\right\|\leqslant U(\|y\|N).$$

It follows that

$$\left\|\frac{u(x+(1-\|x\|)y)-u(x)}{\|y\|}-\frac{\left(\frac{1-\|y\|^2}{(1+\|y\|^2)^{n/2}}-1\right)}{\|y\|}u(x)\right\|\leqslant\frac{U(\|y\|N)}{\|y\|}.$$

Since

$$\lim_{\|y\|\to 0}\frac{\frac{1-\|y\|^2}{(1+\|y\|^2)^{n/2}}-1}{\|y\|}=0,$$

we obtain that

$$(1-\|x\|)\|\nabla u(x)\|\leqslant \partial_r U(rN)|_{r=0}=2\frac{\omega_{n-1}}{\omega_n}.$$ □

2.1. Hypergeometric functions. In order to formulate and to prove our next results recall the basic definition of hypergeometric functions. For two positive integers p and q and vectors $a=(a_1,\dots,a_p)$ and $b=(b_1,\dots,b_q)$ we set

$$ {}_pF_q[a;b,x]=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k\cdot k!}x^k, $$

where $(y)_k:=\frac{\Gamma(y+k)}{\Gamma(y)}=y(y+1)\dots(y+k-1)$ is the Pochhammer symbol. The hypergeometric series converges at least for $|x|<1$. For basic properties and formulas concerning hypergeometric series we refer to the book [3]. The most important step in the proof of our main results, i.e., of Theorem 2.4 and Theorem 2.5 below, is the following lemma.

Lemma 2.3. *The function* $V(r)=\frac{\partial U(rN)}{\partial r}$, $0\leqslant r\leqslant 1$ *is decreasing on the interval* $[0,1]$ *and we have*

$$ V(r)\geqslant V(1)=C_n:=\frac{n!\left(1+n-(n-2)\,{}_2\mathrm{F}_1\left[\frac12,1,\frac{3+n}{2},-1\right]\right)}{2^{3n/2}\Gamma\left[\frac{1+n}{2}\right]\Gamma\left[\frac{3+n}{2}\right]}. $$

Proof. By using spherical coordinates $\eta=(\eta_1,\dots,\eta_n)$ such that $\eta_n=\cos\theta$, where θ is the angle between the vector x and x_n axis, we obtain from (1.5) that

$$ U(rN)=\frac{\Gamma\left[\frac n2\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi}\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}(\chi_{S^+}(x)-\chi_{S^-}(x))\,d\theta $$

and so

$$ U(rN)=\frac{\Gamma\left[\frac n2\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi/2}\left(\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\cos^{n-2}\theta}{(1+r^2+2r\sin\theta)^{n/2}}\right)d\theta $$

or what can be written as

$$ U(rN)=\frac{\Gamma\left[\frac n2\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi/2}\left(\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2+2r\cos\theta)^{n/2}}\right)d\theta. $$

Let $P=2r/(1+r^2)$. Then

$$ \begin{aligned} &\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2+2r\cos\theta)^{n/2}}\\ &=\frac{(1-r^2)}{(1+r^2)^{n/2}}\sum_{k=0}^{\infty}\left(\binom{-n/2}{k}((-1)^k-1)\cos^k\theta\sin^{n-2}\theta\right)P^k. \end{aligned} $$

Since

$$ \int_0^{\pi/2}\cos^k\theta\sin^{n-2}\theta\,d\theta=\frac{\Gamma\left[\frac{1+k}{2}\right]\Gamma\left[\frac12(-1+n)\right]}{2\Gamma\left[\frac{k+n}{2}\right]}, $$

we obtain

$$ U(rN)=\frac{\Gamma\left[\frac n2\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\frac{(1-r^2)}{(1+r^2)^{n/2}}\sum_{k=0}^{\infty}\frac{\Gamma\left[\frac{1+k}{2}\right]\Gamma\left[\frac{n-1}{2}\right]}{2\Gamma\left[\frac{k+n}{2}\right]}\binom{-n/2}{k}((-1)^k-1)P^k. $$

Hence

$$ U(rN)=r\left(1-r^2\right)\left(1+r^2\right)^{-1-\frac n2}\frac{2\Gamma\left[1+\frac n2\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}G(r), $$

where

$$ G(r)={}_3\mathrm{F}_2\left[1,\frac{2+n}{4},\frac{4+n}{4};\frac32,\frac{1+n}{2};\frac{4r^2}{(1+r^2)^2}\right]. $$

By [3, Eq. 3.1.8] for $a = \frac{n}{2},\ b = \frac{1}{2}(-1+n),\ c = \frac{1}{2}$, we have that

$$G(r) = \frac{(1+r^2)^{1+\frac{n}{2}}\, {}_4F_3\left[\left\{\frac{n}{2}, \frac{1}{2}(-1+n), \frac{1}{2}, 1+\frac{n}{4}\right\}, \left\{\frac{n}{4}, \frac{3}{2}, \frac{1}{2}+\frac{n}{2}\right\}, -r^2\right]}{1-r^2}.$$

So

$$U(rN) = r\frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]} {}_4F_3\left[\left\{\frac{n}{2}, \frac{1}{2}(-1+n), \frac{1}{2}, 1+\frac{n}{4}\right\}, \left\{\frac{n}{4}, \frac{3}{2}, \frac{1}{2}+\frac{n}{2}\right\}, -r^2\right],$$

which can be written as

$$U(rN) = \frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}r + \sum_{k=1}^{\infty} \frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(1+2k)(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]} r^{2k+1}.$$

Thus

$$\frac{\partial U(rN)}{\partial r} = \frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]} + \sum_{k=1}^{\infty} \frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]} r^{2k}.$$

Since

$$\begin{aligned}&\frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]}\\ &\quad= \frac{(-1)^k 2^n\Gamma\left[1+\frac{n}{2}\right]\Gamma\left[k+\frac{n}{2}\right]}{\pi k!\Gamma[n]} + \frac{2(-1)^k(-2+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[k]\Gamma\left[\frac{1+n}{2}\right]},\end{aligned}$$

we obtain that

$$\frac{\partial U(rN)}{\partial r} = \frac{\Gamma\left[1+\frac{n}{2}\right]\left((1+r^2)^{-n/2}(1+n) - (n-2)r^2\, {}_2F_1\left[\frac{1+n}{2}, \frac{2+n}{2}, \frac{3+n}{2}, -r^2\right]\right)}{\sqrt{\pi}\Gamma\left[\frac{3+n}{2}\right]},$$

which in view of the Kummer quadratic transformation, can be written in the form

$$\frac{\partial U(rN)}{\partial r} = \frac{\Gamma\left[1+\frac{n}{2}\right](1+r^2)^{-n/2}\left(1+n-(n-2)r^2\, {}_2F_1\left[\frac{1}{2}, 1, \frac{3+n}{2}, -r^2\right]\right)}{\sqrt{\pi}\Gamma\left[\frac{3+n}{2}\right]}.$$

The function

$$y\,{}_2F_1[1/2, 1, (3+n)/2, -y]$$

increases in y. Namely, its derivative is

$$\begin{aligned}{}_2F_1[1/2, 2, (3+n)/2, -y] &= \sum_{m=0}^{\infty}(-1)^m a(m) y^m\\ &= \sum_{m=0}^{\infty} \frac{(-1)^m(1+m)\Gamma\left[\frac{1}{2}+m\right]\Gamma\left[\frac{3+n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{3}{2}+m+\frac{n}{2}\right]} y^m.\end{aligned}$$

Then $a(m) > 0$ and

$$\frac{a(m)}{a(m+1)} = \frac{(1+m)(3+2m+n)}{(2+m)(1+2m)} > 1,$$

because $1+n+mn > 0$, and so

$${}_2F_1[1/2, 2, (3+n)/2, -y] \geqslant \sum_{m=0}^{\infty}(a(2m) - a(2m+1))y^{2m} > 0.$$

The conclusion is that $\frac{\partial U(rN)}{\partial r}$ is decreasing. In particular,

$$\frac{\partial U(rN)}{\partial r} \geqslant \left.\frac{\partial U(rN)}{\partial r}\right|_{r=1}.$$

For $r = 1$ we have

$$\frac{\partial U(rN)}{\partial r} = C_n = \frac{n!\left(1 + n - (n-2)\,{}_2\mathrm{F}_1\left[\frac{1}{2}, 1, \frac{3+n}{2}, -1\right]\right)}{2^{3n/2}\Gamma\left[\frac{1+n}{2}\right]\Gamma\left[\frac{3+n}{2}\right]}. \qquad \square$$

Theorem 2.4. *If u is a harmonic mapping of the unit ball into itself such that $u(0) = 0$, then for $x \in B$ the following sharp inequality*

$$\frac{1 - \|u(x)\|}{1 - \|x\|} \geqslant C_n$$

holds.

Proof. From Theorem 2.1 we have that $\|u(x)\| \leqslant U(rN)$ and so

$$\frac{1 - \|u(x)\|}{1 - \|x\|} \geqslant \frac{1 - |U(rN)|}{1 - \|x\|}.$$

Further there is $\rho \in (r, 1)$ such that

$$\frac{1 - U(rN)}{1 - \|x\|} = \frac{\partial U(\rho N)}{\partial r},$$

which in view of Lemma 2.3 is bigger that C_n. The proof is completed. $\square$

Theorem 2.5. (a) *If u is a harmonic mapping of the unit ball into itself such that $u(0) = 0$, and for some $\|\zeta\| = 1$ we have $\lim_{r\to 1} \|u(r\zeta)\| = 1$, then*

$$\liminf_{r\to 1^-} \left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\| \geqslant C_n. \tag{2.2}$$

(b) *If u is a proper harmonic mapping of the unit ball* **onto** *itself such that $u(0) = 0$, then the following sharp inequality*

$$\liminf_{r\to 1^-} \left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\| \geqslant C_n, \quad \|\zeta\| = 1 \tag{2.3}$$

holds. Here and in the sequel $\mathbf{n}$ *is outward-pointing unit normal.*

Proof. Prove (a). Then (b) follows from (a). Let $0 < r < 1$ and $x \in (r\zeta, \zeta)$. There is a $\rho \in (\|x\|, 1)$ such that

$$\frac{1 - \|u(x)\|}{1 - r} = \left.\frac{\partial \|u(r\zeta)\|}{\partial r}\right|_{r=\rho}. \tag{2.4}$$

On the other hand

$$\left\|\frac{\partial u(r\zeta)}{\partial r}\right\| \geqslant \frac{\partial \|u(r\zeta)\|}{\partial r}.$$

Letting $\|x\| = r \to 1$, in view of Thereom 2.4 and (2.4), we obtain that

$$\liminf_{r\to 1} \left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\| \geqslant C_n.$$

To show that the inequality (2.2) is sharp, let

$$h_m(x) = \begin{cases} 1 - x/m, & \text{if } x \in (1/m, 1]; \\ (m-1)x, & \text{if } -1/m \leqslant x \leqslant 1/m; \\ -1 - x/m, & \text{if } x \in [-1, -1/m), \end{cases}$$

and define

$$f_m(x_1, \ldots, x_{n-1}, x_n) = \frac{\sqrt{1 - h_m(x_n)^2}}{\sqrt{1 - x_n^2}}(x_1, \ldots, x_{n-1}, 0) + (0, \ldots, 0, h_m(x_n)).$$

Then f_m is a homeomorphism of the unit sphere onto itself, such that

$$\lim_{m\to\infty} f_m(x) = (0, \ldots, 0, \chi_{S^+}(x) - \chi_{S^-}(x)).$$

Further, $u_m(x) = \mathcal{P}[f_m](x)$ is a harmonic mapping of the unit ball onto itself such that $\lim_{\|x\|\to 1} \|u_m(x)\| = 1$. Thus u_m is proper. Moreover, $u_m(0) = 0$ and $\lim_{m\to\infty} u_m(x) = (0, \ldots, 0, U(x))$. This implies the fact that the constant C_n is sharp. □

By taking $v(x) = u(x) - \frac{1-\|x\|^2}{(1+\|x\|^2)^{n/2}} u(0)$ and following the proof of Theorem 2.5, in view of Theorem 2.1 we obtain the following theorem.

Theorem 2.6. (a) *If u is a harmonic mapping of the unit ball into itself, and for some $\|\zeta\| = 1$ we have $\lim_{r\to 1} \|u(r\zeta)\| = 1$, then*

$$\liminf_{r\to 1^-} \left\| \frac{\partial u}{\partial \mathbf{n}}(r\zeta) + \frac{u(0)}{2^{n/2-1}} \right\| \geqslant C_n. \tag{2.5}$$

(b) *If u is a proper harmonic mapping of the unit ball onto itself, then the sharp inequality* (2.5) *holds for* $\|\zeta\| = 1$.

In particular, when $n = 2$, the inequality (2.5) *reads as*

$$\liminf_{r\to 1^-} \left\| \frac{\partial u}{\partial \mathbf{n}}(r\zeta) + u(0) \right\| \geqslant \frac{2}{\pi}. \tag{2.6}$$

Remark 2.7. The following table shows first few constants C_n and related functions.

n	$U(rN)$	$\partial_r U(rN)$	C_n
2	$\dfrac{4\arctan(r)}{\pi}$	$\dfrac{4}{\pi(1+r^2)}$,	$\dfrac{2}{\pi}$
3	$\dfrac{-1+r^2+\sqrt{1+r^2}}{r\sqrt{1+r^2}}$	$\dfrac{1-\sqrt{1+r^2}-r^2\left(-3+\sqrt{1+r^2}\right)}{r^2\left(1+r^2\right)^{3/2}}$	$\sqrt{2}-1$
4	$\dfrac{2r\left(-1+r^2\right)+2\left(1+r^2\right)^2\arctan r}{\pi r^2\left(1+r^2\right)}$	$\dfrac{4(r+3r^3-\left(1+r^2\right)^2\arctan r)}{\pi r^3\left(1+r^2\right)^2}$	$\dfrac{4-\pi}{\pi}$

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Received 3 July 2015 • Revised received 18 September 2015 • Accepted 2 October 2015