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ON MULTIFRACTAL SPECTRUM OF QUASICONFORMAL MAPPINGS

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Abstract. We study the multifractal spectra of quasiconformal mappings, which means that we are interested in the maximum size of the sets in which quasiconformal mapping stretches and rotates according to given parameters. We construct examples of quasiconformal mappings which improve a previous result from [2] in the sense of Hausdorff measure.

1. Introduction

Let $f\colon \mathbf{C} \to \mathbf{C}$ be a quasiconformal mapping and fix parameters $\alpha > 0$ and $\gamma \in \mathbf{R}$. We say that the mapping f stretches and rotates according to the parameters α, γ at a point z if there exists a decreasing sequence of positive radii $(r_n)_{n=1}^{\infty}$, with $\lim_{n\to\infty} r_n = 0$, such that

$$\begin{cases} \alpha = \lim_{n\to\infty} \frac{\log|f(z+r_n)-f(z)|}{\log r_n}, \\ \gamma = \lim_{n\to\infty} \frac{\arg(f(z+r_n)-f(z))}{\log|f(z+r_n)-f(z)|}. \end{cases} \tag{1.1}$$

Note that the rotational limit in (1.1) is independent of the choice of the branch of the argument. As a model case for this kind of stretch and rotation one should keep in mind the mapping

$$f_{\alpha(1+i\gamma)}(z) = \begin{cases} \frac{z}{|z|}|z|^{\alpha(1+i\gamma)} & \text{if } |z| \le 1, \\ z & \text{if } |z| > 1. \end{cases} \tag{1.2}$$

which can be calculated to have stretch α and rotation γ at the origin along any sequence (r_n). We can say, roughly speaking, that mapping f satisfies (1.1) at some point z if f stretches and rotates at this point z, along some scales that decrease to zero, like the mapping $f_{\alpha(1+i\gamma)}$ does at the origin.

Given a mapping f and parameters α, γ we denote by $E_f = E_{f,\alpha,\gamma}$ the set of points that satisfy (1.1). For the Hausdorff dimension of these sets the following sharp result is given in [2].

Theorem 1.1. [2, Theorem 5.1] *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a K-quasiconformal mapping, with $K > 1$, and parameters $\alpha > 0$ and $\gamma \in \mathbf{R}$ be such that $\alpha(1+i\gamma) \in \overline{B_K}$, where*

$$B_K = \left\{ \tau \in \mathbf{C}\colon \left| \tau - \frac{1}{2}\left(K + \frac{1}{K}\right) \right| < \frac{1}{2}\left(K - \frac{1}{K}\right) \right\}. \tag{1.3}$$

Then it holds that

$$\dim_H(E_f) \le 1 + \alpha - \frac{K+1}{K-1}\sqrt{(1-\alpha)^2 + \frac{4K}{(K+1)^2}\alpha^2\gamma^2}. \tag{1.4}$$

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Moreover, if $\alpha(1+i\gamma) \notin \overline{B_K}$ *the sets* E_f *are empty, so in this case there are no points* z *satisfying* (1.1) *for any* K*-quasiconformal mapping* f.

As a function of the variable $\alpha(1+i\gamma)$ the function (1.4) is determined as the unique 'cone'-like function on the disc $\overline{B_K}$ that takes value 2 at the point 1, vanishes on the boundary of B_K, and is linear on every line segment joining 1 to the boundary of B_K, see [2] Remark 5.2.

Theorem 1.1, together with the examples constructed in [2] verifying optimality, completely answers the question concerning the Hausdorff dimension of the sets E_f, and hence gives the optimal dimension for sets where K-quasiconformal mapping can stretch and rotate according to given parameters. This raises a natural question whether the sharpness of the dimension in Theorem 1.1 can be extended to the sharpness on the level of Hausdorff measures. The main result of this paper is the following theorem which proves that the sets E_f can have positive Hausdorff measure with the optimal dimension.

Theorem 1.2. *Let* $\alpha > 0$, $\gamma \in \mathbf{R}$, *and* $K > 1$ *be arbitrary parameters for which* $\alpha(1+i\gamma) \in \overline{B_K}$. *Then there exists a* K*-quasiconformal mapping* $\phi\colon \mathbf{C} \to \mathbf{C}$ *such that*

$$H^d(E_\phi) > 0,$$

where

$$d = 1 + \alpha - \frac{K+1}{K-1}\sqrt{(1-\alpha)^2 + \frac{4K}{(K+1)^2}\alpha^2\gamma^2}$$

is the optimal Hausdorff dimension from (1.4), *and* H^d *is the* d*-dimensional Hausdorff measure.*

For proving Theorem 1.2 we will use a Cantor-type construction inspired by Uriarte-Tuero's construction in [4].

The value of the joint rotational and stretching multifractal spectrum for a given class of mappings and parameters $\alpha > 0$, $\gamma \in \mathbf{R}$ is the supremum of the Hausdorff dimension of the sets E_f over all mappings f from the class chosen. We denote the multifractal spectrum for K-quasiconformal mappings by $F_K(\alpha,\gamma)$ and define it by

$$F_K(\alpha,\gamma) = \sup\{\dim_H(E_f)\colon \text{where } f\colon \mathbf{C}\to\mathbf{C} \text{ is a } K\text{-quasiconformal mapping}\} \tag{1.5}$$

for arbitrary parameters α, γ. Then, note that Theorem 1.1, again with the examples verifying optimality, in fact completely characterizes the multifractal spectrum $F_K(\alpha,\gamma)$ and, as mentioned before, in Theorem 1.2 we push the lower bound further to the level of Hausdorff measures.

One could also study the multifractal spectra for a class of mappings generalizing quasiconformal mappings, but then the definition (1.1) might not give the right way to measure the stretch and rotation. This can be seen, for example, from [3], where we show the correct bounds for the pointwise stretch and rotation for mappings with exponentially integrable distortion. So, when studying the multifractal spectra for a more general class of mappings one must first find out the correct way to define the stretch and rotation and then study the sets E_f with respect to this definition.

This paper is organized as follows. We will first recall definitions and some properties of quasiconformal mappings in the Section 2. Then we use a modified version of the Uriarte-Tuero's construction in the Section 3 to prove Theorem 1.2 in the case

$\gamma = 0$. In the Section 4 we will add rotation to our construction and prove Theorem 1.2 in full generality.

2. Prerequisites

We call an orientation preserving homeomorphism $f\colon \Omega_1 \to \Omega_2$ between complex domains $\Omega_1, \Omega_2 \subset \mathbf{C}$ a K-quasiconformal mapping if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega_1)$ and satisfies the distortion inequality

$$|Df(z)|^2 \le K J_f(z)$$

almost everywhere with some $K \ge 1$. Here $|Df(z)| = \max_{|\psi|=1} |Df(z)\psi|$ and $J_f(z)$ denotes the Jacobian of f at the point z.

We say that a quasiconformal mapping $f : \mathbf{C} \to \mathbf{C}$ is *principal* if it is conformal outside a compact set and is normalized by $f(z) = z + \mathcal{O}(\frac{1}{z})$ as $z \to \infty$. When studying the argument of a principal quasiconformal mapping it is natural to pick the principal branch of the logarithm, as in [2]. Namely, we choose the branch so that the notion $\arg(f(z_0 + e^{i\beta} r_0) - f(z_0))$ can be understood as a rotation of $f(z) = f(z_0 + e^{i\beta} r)$ around the point $f(z_0)$, when $z = z_0 + e^{i\beta} r$ moves from $r = +\infty$ to $r = r_0$ along the line which passes trough points z_0 and $z_0 + e^{i\beta} r_0$. This geometric understanding for $\arg(f(z_0 + e^{i\beta} r_0) - f(z_0))$ will serve best our purposes when dealing with principal quasiconformal mappings.

For basic properties of quasiconformal mappings see [1]. Throughout this paper we will denote the unit disc by D, use the notation $aB(b, r) = B(b, ar)$, and denote the radius of a disc B by $r(B)$.

3. Stretch

In this section we will use a Cantor-type construction, which is a modification of the Uriarte-Tuero's construction in [4], to prove Theorem 1.2 in the case $\alpha > 0$, $\gamma = 0$. We will do this construction in detail as we must be able to present clearly the modifications necessary for proving Theorem 1.2, especially when we are dealing with non-trivial rotation which Uriarte-Tuero does not cover in [4]. But before that let us first cover some trivial cases of Theorem 1.2.

If we choose parameters $\alpha = 1$ and $\gamma = 0$ the conformal mapping $f(z) = z$ satisfies (1.1) at every point $z \in \mathbf{C}$ and hence gives an example of a K-quasiconformal mapping for which $H^2(E_f) > 0$ for any $K \ge 1$.

Other trivial case is the choice of parameters α, γ such that $\alpha(1 + i\gamma) \in \partial B_K$. Then we have from Theorem 1.1 that $\dim_H(E_f) = 0$, and hence to show our claim it is enough to find a K-quasiconformal mapping that satisfies (1.1) at one point. But we have already noticed that the mapping $f_{\alpha(1+i\gamma)}$ in (1.2) satisfies the condition (1.1) at the origin and it is well known that it is a K-quasiconformal mapping, see for example [2] Theorem 3.1. Hence this mapping gives the desired example in this case.

With the above cases verified we are left with constructing examples given arbitrary parameters $\alpha > 0$, $\gamma \in \mathbf{R}$, and $K > 1$ such that $(\alpha, \gamma) \in B_K \setminus \{(1, 0)\}$. Thus in the case $\gamma = 0$ we can assume $\alpha \in \left(\frac{1}{K}, 1\right) \cup (1, K)$. With these preparations let us start our construction for a mapping ϕ that satisfies Theorem 1.2 in the case $\alpha < 1$, and afterwards use the inverse of the mapping ϕ to prove the case $\alpha > 1$.

Step 1. Let $\alpha < 1$ and $K > \frac{1}{\alpha}$ be arbitrary and let $r < \frac{1}{e}$ be a small constant, which we fix later. Choose first $m_{1,1}$ disjoint discs $B(z^i_{1,1}, r) \subset D$, $i = 1, 2, \ldots, m_{1,1}$, such that one of them is centered at the origin, and then $m_{1,2}$ discs $B(z^i_{1,2}, r^2) \subset D$, $i = 1, 2, \ldots, m_{1,2}$, disjoint among themselves and with the previous discs. We continue in a similar manner until we choose m_{1,l_1} discs $B(z^i_{1,l_1}, r^{l_1}) \subset D$, $i = 1, 2, \ldots, m_{1,l_1}$, disjoint among themselves and with all the previously chosen discs, such that the union of all discs chosen covers a big portion of the unit disc, namely

$$c_1 = m_{1,1}(r^1)^2 + m_{1,2}(r^2)^2 + \cdots + m_{1,l_1}(r^{l_1})^2 = 1 - \epsilon_1,$$

where ϵ_1 is a small constant which we choose later.

Next, we will associate with every disc $B(z^i_{1,j_1}, r^{j_1})$, where $j_1 = 1, 2, \ldots, l_1$, and $i = 1, 2, \ldots, m_{1,j_1}$, a positive parameter σ_{1,j_1}. The parameter σ_{1,j_1} depends only on the radius of the disc $B(z^i_{1,j_1}, r^{j_1})$, and hence it does not depend on the parameters i. We will fix the parameters σ_{1,j_1} later, but they will all be small, say smaller than $\frac{1}{100}$. Then, for every j_1 and every $i = 1, 2, \ldots, m_{j_1}$, define mappings $\varphi^i_{1,j_1}(z) = (\sigma_{1,j_1})^K r^{j_1} z + z^i_{1,j_1}$. Using these mappings we construct the discs

$$D^i_{j_1} = \frac{1}{(\sigma_{1,j_1})^K} \varphi^i_{1,j_1}(D) = B(z^i_{1,j_1}, r^{j_1})$$

and

$$(D^i_{j_1})' = \varphi^i_{1,j_1}(D) = B(z^i_{1,j_1}, (\sigma_{1,j_1})^K r^{j_1}) \subset D^i_{j_1}$$

which form annuli. Finally, as our first approximation for the desired mapping we define

$$g_1(z) = \begin{cases} (\sigma_{1,j_1})^{1-K}(z - z^i_{1,j_1}) + z^i_{1,j_1}, & z \in (D^i_{j_1})', \\ \left| \frac{z - z^i_{1,j_1}}{r(D^i_{j_1})} \right|^{\frac{1}{K}-1} (z - z^i_{1,j_1}) + z^i_{1,j_1}, & z \in D^i_{j_1} \setminus (D^i_{j_1})', \\ z, & \text{otherwise.} \end{cases}$$

Clearly $g_1(z)$ is a principal K-quasiconformal mapping and conformal outside of

$$\bigcup_{j_1=1}^{l_1} \bigcup_{i=1}^{m_{1,j_1}} (D^i_{j_1} \setminus (D^i_{j_1})').$$

The mapping g_1 maps every disc $D^i_{j_1}$ onto itself and every disc $(D^i_{j_1})'$, with radius $(\sigma_{1,j_1})^K r^{j_1}$, onto the disc $(D^i_{j_1})'' = B(z^i_{1,j_1}, \sigma_{1,j_1} r^{j_1})$, with radius $\sigma_{1,j_1} r^{j_1}$, while keeping the rest of the plane fixed. We define $\phi_1(z) = g_1(z)$.

Step 2. The idea is again to fill up a big portion of the unit disc with small discs which we then turn to annuli similarly as in the first step. Then we map these annuli inside the discs $(D^i_{j_1})'$ of the first generation using the mappings φ^i_{1,j_1}, and change our approximating mapping ϕ_1 in the images of the outer discs of the annuli of the second generation to obtain our new approximating function ϕ_2.

To make this precise, choose $m_{2,2}$ disjoint discs $B(z^u_{2,2}, r^2) \subset D$, $u = 1, 2, \ldots, m_{2,2}$, such that one of them is centered at the origin, and then $m_{2,3}$ discs $B(z^u_{2,3}, r^3) \subset D$, $u = 1, 2, \ldots, m_{2,3}$, disjoint among themselves and with the previous discs. We continue like this until we choose m_{2,l_2} discs $B(z^u_{2,l_2}, r^{l_2})$, $u = 1, 2, \ldots, m_{2,l_2}$, disjoint among themselves and with the previous discs, such that discs chosen in this second step cover a big portion of the unit disc, namely

$$c_2 = m_{2,2}(r^2)^2 + m_{2,3}(r^3)^2 + \cdots + m_{2,l_2}(r^{l_2})^2 = 1 - \epsilon_2,$$

where ϵ_2 is a small constant which we choose later.

Here we would remark that at the n:th level of the construction we will always start from smaller discs than we started at the previous levels by choosing discs of radii $r^n, r^{n+1}, \dots, r^{l_n}$, and compatibly to that use the notation $m_{n,n}, m_{n,n+1}, \dots, m_{n,l_n}$ for the number of discs with given radius.

As in the first step we will associate a positive parameter σ_{2,j_2}, which we define later and which are smaller than $\frac{1}{100}$, to every disc $B(z^u_{2,j_2}, r^{j_2})$, with $j_2 = 2, 3, \dots, l_2$, and every possible u. And, as before, the parameters σ_{2,j_n} depend just on the radius of the disc $B(z^u_{2,j_2}, r^{j_2})$, not on the parameter u. Let us again for every $j_2 = 2, 3, \dots, l_2$, and every $u = 1, 2, \dots, m_{2,j_2}$, define mappings $\varphi^u_{2,j_2}(z) = (\sigma_{2,j_2})^K r^{j_2} z + z^u_{2,j_2}$. Then we construct the following discs, which will form annuli,

$$D^{i,u}_{j_1,j_2} = \phi_1\left(\frac{1}{(\sigma_{2,j_2})^K}\varphi^i_{1,j_1} \circ \varphi^u_{2,j_2}(D)\right) = B(z^{i,u}_{j_1,j_2}, r^{j_2}\sigma_{1,j_1} r^{j_1})$$

and

$$(D^{i,u}_{j_1,j_2})' = \phi_1(\varphi^i_{1,j_1} \circ \varphi^u_{2,j_2}(D)) = B(z^{i,u}_{j_1,j_2}, (\sigma_{2,j_2})^K r^{j_2}\sigma_{1,j_1} r^{j_1})$$

for certain $z^{i,u}_{j_1,j_2} \in D$, where $j_2 = 2, 3, \dots, l_2$, $j_1 = 1, 2, \dots, l_1$, $u = 1, 2, \dots, m_{2,j_2}$ and $i = 1, 2, \dots, m_{1,j_1}$. Then let

$$g_2(z) = \begin{cases} (\sigma_{2,j_2})^{1-K}(z - z^{i,u}_{j_1,j_2}) + z^{i,u}_{j_1,j_2}, & z \in (D^{i,u}_{j_1,j_2})', \\ \left|\frac{z - z^{i,u}_{j_1,j_2}}{r(D^{i,u}_{j_1,j_2})}\right|^{\frac{1}{K}-1}(z - z^{i,u}_{j_1,j_2}) + z^{i,u}_{j_1,j_2}, & z \in D^{i,u}_{j_1,j_2} \setminus (D^{i,u}_{j_1,j_2})', \\ z, & \text{otherwise.} \end{cases}$$

Clearly the mapping $g_2(z)$ is K-quasiconformal, and it is conformal outside of the union of annuli

$$\bigcup_{j_1,j_2,i,u} \left(D^{i,u}_{j_1,j_2} \setminus (D^{i,u}_{j_1,j_2})'\right).$$

Additionally it maps every disc $D^{i,u}_{j_1,j_2}$ onto itself and every disc $(D^{i,u}_{j_1,j_2})'$, with radius $(\sigma_{2,j_2})^K r^{j_2}\sigma_{1,j_1} r^{j_1}$, onto the disc $(D^{i,u}_{j_1,j_2})'' = B(z^{i,u}_{j_1,j_2}, \sigma_{2,j_2} r^{j_2}\sigma_{1,j_1} r^{j_1})$, with radius $\sigma_{2,j_2} r^{j_2}\sigma_{1,j_1} r^{j_1}$, while keeping the rest of the plane fixed.

Define $\phi_2 = g_2 \circ \phi_1$ as our second approximation function. Note that ϕ_2 is a K-quasiconformal mapping, since g_2 differs from a conformal mapping only in the discs $D^{i,u}_{j_1,j_2}$ and the mapping ϕ_1 is conformal in the discs $\frac{1}{(\sigma_{2,j_2})^K}\varphi^i_{1,j_1} \circ \varphi^u_{2,j_2}(D)$. Moreover, ϕ_2 maps the discs $\varphi^i_{1,j_1} \circ \varphi^u_{2,j_2}(D)$, with radii $(\sigma_{2,j_2})^K r^{j_2}(\sigma_{1,j_1})^K r^{j_1}$, to the discs $(D^{i,u}_{j_1,j_2})''$, with radii $\sigma_{2,j_2} r^{j_2}\sigma_{1,j_1} r^{j_1}$. This property, which will hold at every step n, will be carried over to our final mapping ϕ and be crucial for obtaining the right stretching condition.

Induction step. Assume that we have constructed $n-1$ previous steps. Choose as before $m_{n,n}$ disjoint discs $B(z^q_{n,n}, r^n) \subset D$, $q = 1, 2, \dots, m_{n,n}$, such that one of them is centered at the origin, and then $m_{n,n+1}$ discs $B(z^q_{n,n+1}, r^{n+1})$, $q = 1, 2, \dots, m_{n,n+1}$, disjoint among themselves and with the previous discs. We again continue this until we choose m_{n,l_n} discs $B(z^q_{n,l_n}, r^{l_n})$, $q = 1, 2, \dots, m_{n,l_n}$, disjoint among themselves and with the previous discs, such that discs chosen in this n:th step cover a large portion of the unit disc, namely

$$c_n = m_{n,n}(r^n)^2 + \cdots + m_{n,l_n}(r^{l_n})^2 = 1 - \epsilon_n,$$

where again ϵ_n is a small constant which we will choose later.

As before, we will associate positive parameters σ_{n,j_n} to the discs $B(z^q_{n,j_n}, r^{j_n})$, $j_n = n, n+1, \dots, l_n$, and $q = 1, 2, \dots, m_{n,j_n}$, where σ_{n,j_n} depends only on radius and is smaller than $\frac{1}{100}$. We then define mappings $\varphi^q_{n,j_n}(z) = (\sigma_{n,j_n})^K r^{j_n} z + z^q_{n,j_n}$ and using them we construct the discs

$$D^I_J = \phi_{n-1}\left(\frac{1}{(\sigma_{n,j_n})^K}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)\right) = B(z^I_J, r^{j_n}\sigma_{n-1,j_{n-1}} r^{j_{n-1}} \cdots \sigma_{1,j_1} r^{j_1}),$$

and

$$(D^I_J)' = \phi_{n-1}\left(\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)\right) = B(z^I_J, (\sigma_{n,j_n})^K r^{j_n}\sigma_{n-1,j_{n-1}} r^{j_{n-1}} \cdots \sigma_{1,j_1} r^{j_1}),$$

for any multi-indexes $I = (i_1, i_2, \dots, i_n)$ and $J = (j_1, j_2, \dots, j_n)$. Then let

$$g_n(z) = \begin{cases} (\sigma_{n,j_n})^{1-K}(z - z^I_J) + z^I_J, & z \in (D^I_J)', \\ \left|\frac{z - z^I_J}{r(D^I_J)}\right|^{\frac{1}{K}-1}(z - z^I_J) + z^I_J, & z \in D^I_J \setminus (D^I_J)', \\ z, & \text{otherwise.} \end{cases}$$

Clearly $g_n(z)$ is a K-quasiconformal mapping and conformal outside of

$$\bigcup_{I,J}(D^I_J \setminus (D^I_J)').$$

Moreover, the mapping g_n maps the discs D^I_J onto itself and maps the discs $(D^I_J)'$ onto the discs $(D^I_J)'' = B(z^I_J, \sigma_{n,j_n} r^{j_n} \cdots \sigma_{1,j_1} r^{j_1})$, while keeping the rest of the plane fixed. Define then $\phi_n = g_n \circ \phi_{n-1}$, and notice similarly as before that ϕ_n is K-quasiconformal. Furthermore, ϕ_n maps the discs $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$, with radii $(\sigma_{n,j_n})^K r^{j_n} \cdots (\sigma_{1,j_1})^K r^{j_1}$, to the discs $(D^I_J)''$, with radii $\sigma_{n,j_n} r^{j_n} \cdots \sigma_{1,j_1} r^{j_1}$, for every multi-indexes $J = (j_1, j_2, \dots, j_n)$, $I = (i_1, i_2, \dots, i_n)$.

Since each mapping ϕ_n is K-quasiconformal and equals the identity mapping outside of the unit disc there exists the K-quasiconformal limit mapping

$$\phi = \lim_{n\to\infty} \phi_n,$$

where the convergence is locally uniform. And since the mapping ϕ equals the identity mapping outside of the unit disc it is a principal K-quasiconformal mapping. It is clear from the construction that ϕ maps every disc $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ to the disc $\phi_n(\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D))$. Hence we see that ϕ maps the compact set

$$\hat{E}_\phi = \bigcap_{n=1}^{\infty}\left(\bigcup_{I,J}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})\right)$$

to the compact set

$$\phi(\hat{E}_\phi) = \bigcap_{n=1}^{\infty}\left(\bigcup_{I,J}\psi^{i_1}_{1,j_1} \circ \cdots \circ \psi^{i_n}_{n,j_n}(\overline{D})\right),$$

where we have written $\psi^{i_k}_{k,j_k}(z) = z^{i_k}_{k,j_k} + \sigma_{k,j_k} r^{j_k} z$, where $1 \le i_k \le m_{k,j_k}$, $k \le j_k \le l_k$ and $k \in \mathbf{N}$.

There are few additional remarks on the construction we would like to make. Firstly, given an arbitrary disc $B = \varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ it holds that $\phi(z) = \phi_n(z)$ for every $z \in \partial B$, since mappings $g_k(z)$, where $k > n$, differ from the identity mapping only inside the discs of level n. So the mapping ϕ maps the boundary of the disc B as the mapping ϕ_n.

Secondly, let B be as above and denote its center by z_B. Then it holds that $\phi(z_B) = \phi_{n-1}(z_B)$. This follows since we chose in the construction at every level a disc centered at the origin and as mappings $g_m(z)$, for arbitrary level m, keep the centerpoints of the discs D_J^I fixed. So the mapping ϕ maps the center of the disc B as the mapping ϕ_{n-1}.

Next, we follow the ideas of Uriarte-Tuero [4] on how to fix our parameters r, σ_{n,j_n}, and ϵ_n in such a manner that we obtain the desired Hausdorff measure for the set $\hat{E}_\phi$. First choose

$$(\sigma_{n,j_n})^{dK} = (r^{j_n})^{2-d}, \tag{3.1}$$

where r is some constant so small that every $\sigma_{n,j_n} < \frac{1}{100}$, and $d \in (0,2)$ will be the optimal Hausdorff dimension for the set $\hat{E}_\phi$, for which Uriarte-Tuero proves in [4] that $0 < H^d(\hat{E}_\phi) < \infty$. We will later choose specific Hausdorff dimensions related to the equation (1.4), which will depend on α, but for now let us work with a general dimension d. Moreover, in [4] it is also proven that if (3.1) holds, then $0 < H^{d'}(\phi(\hat{E}_\phi)) < \infty$, where d and d' are coupled with the equation

$$d' = \frac{2Kd}{2 + (K-1)d}. \tag{3.2}$$

With these choices we can calculate that

$$((\sigma_{n,j_n})^K r^{j_n})^d = (\sigma_{n,j_n} r^{j_n})^{d'} = (r^{j_n})^2,$$

which has a clear geometric interpretation related to the area of a disc with radius r^{j_n}. Furthermore, with these choices we obtain for every $n \in \mathbf{N}$,

$$\begin{aligned} c_n &= m_{n,n}(r^n)^2 + m_{n,n+1}(r^{n+1})^2 + \cdots + m_{n,l_n}(r^{l_n})^2 \\ &= m_{n,n}[(\sigma_{n,n})^K r^n]^d + m_{n,n+1}[(\sigma_{n,n+1})^K r^{n+1}]^d + \cdots + m_{n,l_n}[(\sigma_{n,l_n})^K r^{l_n}]^d \\ &= m_{n,n}(\sigma_{n,n} r^n)^{d'} + m_{n,n+1}(\sigma_{n,n+1} r^{n+1})^{d'} + \cdots + m_{n,l_n}(\sigma_{n,l_n} r^{l_n})^{d'} = 1 - \epsilon_n. \end{aligned}$$

Then it can be calculated that

$$\begin{aligned} &\sum_{j_1,j_2,\dots j_n} m_{1,j_1} m_{2,j_2} \cdots m_{n,j_n}((\sigma_{1,j_1})^K r^{j_1} \cdots (\sigma_{n,j_n})^K r^{j_n})^d \\ &= \sum_{j_1,j_2,\dots j_n} m_{1,j_1} m_{2,j_2} \cdots m_{n,j_n}(\sigma_{1,j_1} r^{j_1} \cdots \sigma_{n,j_n} r^{j_n})^{d'} = \prod_{u=1}^{n}(1-\epsilon_u). \end{aligned}$$

Finally, let us choose constants ϵ_n such that $\epsilon_n \to 0$ so fast that there exists some constant $c > 0$, for which it holds that

$$\prod_{n=1}^{\infty}(1-\epsilon_n) > c.$$

Then using all the discs $\varphi_{1,j_1}^{i_1} \circ \cdots \circ \varphi_{n,j_n}^{i_n}(\overline{D})$, and $\psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n,j_n}^{i_n}(\overline{D})$ from the level n as δ-covers for the sets $\hat{E}_\phi$ and $\phi(\hat{E}_\phi)$ respectively, it is easy to see that $H^d(\hat{E}_\phi) < 1$ and $H^{d'}(\phi(\hat{E}_\phi)) < 1$. For the lower estimates for the Hausdorff measures Uriarte-Tuero uses a Carleson type packing condition, which he proves in [4] Lemma 3.2 and which we will not present here.

First result. In this section we will prove Theorem 1.2 in the case $\alpha < 1$ and $\gamma = 0$ using the above construction. To this end, use the construction with arbitrary parameters $\alpha < 1$ and $K > \frac{1}{\alpha}$, and with the specific Hausdorff dimension

$$d = 1 + \alpha - \frac{K+1}{K-1}(1-\alpha), \tag{3.3}$$

which is the same as (1.4) with these choices for parameters α, γ. Then we know from [4] that $0 < H^d(\hat{E}_\phi) < \infty$. Since we also know that the mapping ϕ is K-quasiconformal, all that remains to be shown is the condition (1.1) for every point $z \in \hat{E}_\phi$ under the mapping ϕ. We first show that the mapping ϕ stretches the discs $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ in a correct manner for arbitrary multi-indexes J, I. To this end, substitute first the above choice (3.3) for d in the equation (3.1) to obtain

$$(\sigma_{n,j_n})^{K\left(1+\alpha-\frac{K+1}{K-1}(1-\alpha)\right)} = r^{j_n\left(2-\left(1+\alpha-\frac{K+1}{K-1}(1-\alpha)\right)\right)}, \tag{3.4}$$

from which we can solve

$$(\sigma_{n,j_n})^K = r^{\frac{Kj_n(1-\alpha)}{\alpha K-1}} \tag{3.5}$$

and

$$\sigma_{n,j_n} = r^{\frac{j_n(1-\alpha)}{\alpha K-1}}. \tag{3.6}$$

Next we recall, that ϕ maps the disc $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$, with radius $(\sigma_{n,j_n})^K r^{j_n} \cdots (\sigma_{1,j_1})^K r^{j_1}$, to the disc $\psi^{i_1}_{1,j_1} \circ \cdots \circ \psi^{i_n}_{n,j_n}(D)$, with radius $\sigma_{n,j_n} r^{j_n} \cdots \sigma_{1,j_1} r^{j_1}$, and maps the center point to the center point, for arbitrary multi-indexes J, I. Denote the center of the disc $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ with z and let $\bar{z}$ be any point from the boundary of this disc. Then with the above observations, and the calculations (3.5) and (3.6) we obtain

$$\begin{aligned}
\frac{\log|\phi(z)-\phi(\bar{z})|}{\log|z-\bar{z}|} &= \frac{\log(\sigma_{n,j_n} r^{j_n} \cdots \sigma_{1,j_1} r^{j_1})}{\log((\sigma_{n,j_n})^K r^{j_n} \cdots (\sigma_{1,j_1})^K r^{j_1})} \\
&= \frac{\log\left(r^{j_1+\cdots+j_n} r^{(j_1+\cdots+j_n)\frac{1-\alpha}{\alpha K-1}}\right)}{\log\left(r^{j_1+\cdots+j_n} r^{(j_1+\cdots+j_n)\frac{K(1-\alpha)}{\alpha K-1}}\right)} \\
&= \frac{(j_1+\cdots+j_n) + (j_1+\cdots+j_n)\frac{1-\alpha}{\alpha K-1}}{(j_1+\cdots+j_n) + (j_1+\cdots+j_n)\frac{K(1-\alpha)}{\alpha K-1}} \\
&= \frac{1+\frac{1-\alpha}{\alpha K-1}}{1+\frac{K(1-\alpha)}{\alpha K-1}} = \frac{\alpha K-\alpha}{K-1} = \alpha.
\end{aligned} \tag{3.7}$$

This shows that every disc $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ has the right stretch with respect to the center point. Next, we show that the condition (1.1) holds for every point $z \in \hat{E}_\phi$ with parameters $\alpha < 1$ and $\gamma = 0$. We will first show the stretch part using (3.7) and the rotation part will follow straight from the construction.

Let $z \in \hat{E}_\phi$ be arbitrary. Then there exist unique sequences $(i_n)_{n=1}^\infty$ and $(j_n)_{n=1}^\infty$ such that $z = \lim_{n\to\infty} \varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})$. We will next show that the decreasing sequence $(r_n)_{n=2}^\infty$, defined by $r_n = r(\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D}))$, of positive radii satisfies the stretching part of (1.1), namely

$$\alpha = \lim_{n\to\infty} \frac{\log|\phi(z+r_n)-\phi(z)|}{\log(r_n)}. \tag{3.8}$$

Let $n \geq 2$ be arbitrary. Let us denote $B = \varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})$ and $A = \varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$. We then denote the center of B by z_B, the center of A with z_A, and remember that $z \in B$. Next, we choose a point $\bar{z}$ from the boundary of the disc A such that $r_n = |z_A - \bar{z}| = |z - \bar{z}|$.

The idea for proving (3.8) is first to show that for every n it holds that

$$|\phi(z) - \phi(\bar{z})| = \bar{c}_n |\phi(z_A) - \phi(\bar{z})| \tag{3.9}$$

for some constants $\bar{c}_n$, which are bounded away from zero and infinity with bounds that do not depend on n. Note, that the equation (3.9) depends on n as the points z_A and $\bar{z}$ depend on it. After establishing the equation (3.9) we will use the equation (3.7) and properties of the logarithm to prove (3.8).

For this we note, that since $z \in B$ and inequality $r(B) < \frac{r(A)}{100}$ holds for the radii of the discs B and A it follows that there exists a constant $\frac{98}{100} < h < \frac{102}{100}$ such that $|z - \bar{z}| = h|z_B - \bar{z}|$, and hence $|z_B - \bar{z}| = \frac{1}{h}|z_A - \bar{z}|$. From the remarks made after the construction of the mapping ϕ, we know that ϕ maps the points z_B, z_A, and $\bar{z}$ as the mapping ϕ_{n-1}. Moreover, ϕ_{n-1} maps these points as a composition of similarities, since they all lie in the disc A. Thus the above equality can be transferred to the image side

$$|\phi(z_B) - \phi(\bar{z})| = \frac{|\phi(z_A) - \phi(\bar{z})|}{h}. \tag{3.10}$$

On the image side we also know that $\phi(z) \in \phi(B)$, where $\phi(B)$ is the disc $\psi^{i_1}_{1,j_1} \circ \cdots \circ \psi^{i_n}_{n,j_n}(\overline{D})$ with radius $\sigma_{1,j_1} r^{j_1} \cdots \sigma_{n,j_n} r^{j_n}$, and that

$$|\phi(z_A) - \phi(\bar{z})| = \sigma_{1,j_1} r^{j_1} \cdots \sigma_{n-1,j_{n-1}} r^{j_{n-1}}.$$

Thus we see that $r(\phi(B)) < \frac{r(\phi(A))}{100}$. Hence from (3.10) and the fact that h is close to one we obtain that there exists a constant $\frac{9}{10} < \bar{h} < \frac{11}{10}$ such that

$$\bar{h}|\phi(z) - \phi(\bar{z})| = |\phi(z_B) - \phi(\bar{z})|. \tag{3.11}$$

Next, we note that since ϕ is a K-quasiconformal mapping there exists some constant $C(K)$, depending only on K, such that given any points z_0, z_1 and z_2, for which $|z_0 - z_1| = |z_0 - z_2|$, it follows that

$$\frac{1}{C(K)}|\phi(z_0) - \phi(z_1)| < |\phi(z_0) - \phi(z_2)| < C(K)|\phi(z_0) - \phi(z_1)|.$$

When choosing $z_0 = z$, $z_1 = \bar{z}$, and $z_2 = z + r_n$ we obtain that there exists a constant $\frac{1}{C(K)} < c_n < C(K)$ such that

$$|\phi(z) - \phi(z + r_n)| = c_n |\phi(z) - \phi(\bar{z})|. \tag{3.12}$$

Then using (3.10), (3.11) and (3.12) we can show for arbitrary $n \in \mathbf{N}$ that

$$\begin{aligned}
\frac{\log|\phi(z) - \phi(z + r_n)|}{\log r_n} &= \frac{\log|c_n(\phi(z) - \phi(\bar{z}))|}{\log|z - \bar{z}|} \\
&= \frac{\log|\frac{c_n}{\bar{h}}(\phi(z_B) - \phi(\bar{z}))|}{\log|z_A - \bar{z}|} \\
&= \frac{\log|\frac{c_n}{h\bar{h}}(\phi(z_A) - \phi(\bar{z}))|}{\log|z_A - \bar{z}|},
\end{aligned} \tag{3.13}$$

where bounds for the constants c_n, h and $\bar{h}$ do not depend on n. Using the above calculation we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\log|\phi(z)-\phi(z+r_n)|}{\log r_n} &= \lim_{n\to\infty}\frac{\log|\frac{c_n}{h\bar{h}}(\phi(z_A)-\phi(\bar{z}))|}{\log|z_A-\bar{z}|}\\ &= \lim_{n\to\infty}\frac{\log|\phi(z_A)-\phi(\bar{z})|}{\log|z_A-\bar{z}|}+\lim_{n\to\infty}\frac{\log|\frac{c_n}{h\bar{h}}|}{\log|z_A-\bar{z}|}=\alpha,\end{aligned}$$

since the first limit is α by (3.7) and the second one is zero since $|\log|\frac{c_n}{h\bar{h}}||$ is bounded, as $\frac{1}{\bar{c}}<\frac{c_n}{h\bar{h}}<\bar{c}$ for some positive constant $\bar{c}$ that does not depend on n, and $|\log|z_A-\bar{z}||\to\infty$ as $n\to\infty$. This shows the condition (3.8).

Moreover, for this sequence $(r_n)_{n=2}^{\infty}$ it trivially holds that $r_n\to 0$ as $n\to\infty$ by the definition of the mappings $\varphi_{n,j_n}^{i_n}$. Hence this shows the stretching part of (1.1) for arbitrary point $z\in\hat{E}_\phi$.

We would like to note that due to (3.12) the direction is not relevant to the stretch, and we could choose arbitrary sequence $(e^{i\beta}r_n)_{n=1}^{\infty}$ and satisfy the stretching condition.

Next, we will note that for an arbitrary point $z\in\hat{E}_\phi$ and the sequence $(r_n)_{n=2}^{\infty}$ related to this point the rotation condition from (1.1)

$$\lim_{n\to\infty}\frac{\arg(\phi(z+r_n)-\phi(z))}{\log|\phi(z+r_n)-\phi(z)|}=0$$

holds. This follows straight from the construction by noticing that given an arbitrary r_n the function $\phi(z+t)$ does not wind around $\phi(z)$ as t moves from $t=+\infty$ to $t=r_n$ along the line which passes trough the points $z, z+r_n$, and remembering how $\arg(\phi(z+r_n)-\phi(z))$ can be understood for principal quasiconformal mappings.

This proves Theorem 1.2 in the case $\alpha<1$, $\gamma=0$. Next, we will prove the case $\alpha>1$, $\gamma=0$.

Let $\alpha>1$ be arbitrary and assume $K>\alpha$, which is equivalent to $(\alpha,0)\in B_K$. Then it holds that $\frac{1}{\alpha}<1$ and $\frac{1}{K}<\frac{1}{\alpha}$, which allows us to obtain from the previous case that there exists a K-quasiconformal mapping ϕ such that every point $z\in\hat{E}_\phi$ satisfies (1.1) for $\frac{1}{\alpha}$ and $\gamma=0$. Moreover, we have from [4] that $0<H^d(\hat{E}_\phi)<\infty$, for $d=1+\frac{1}{\alpha}-\frac{K+1}{K-1}\left(1-\frac{1}{\alpha}\right)$, and that $0<H^{d'}(\phi(\hat{E}_\phi))<\infty$, where

$$d'=\frac{2Kd}{2+(K-1)d}.$$

Substituting $d=1+\frac{1}{\alpha}-\frac{K+1}{K-1}\left(1-\frac{1}{\alpha}\right)$ to the above equation and simplifying yields

$$d'=1+\alpha-\frac{K+1}{K-1}(\alpha-1),$$

and hence $\phi(\hat{E}_\phi)$ has the desired Hausdorff measure. We will next show that the principal K-quasiconformal mapping ϕ^{-1} satisfies the condition (1.1) at every point $z\in\phi(\hat{E}_\phi)$ with respect to the parameters $\alpha>1,\gamma=0$. First we note that the rotation condition follows similarly as in the previous part, and thus it is the stretching condition that we must concentrate on.

Clearly ϕ^{-1} maps the discs $\psi_{1,j_1}^{i_1}\circ\cdots\circ\psi_{n,j_n}^{i_n}(\overline{D})$, with radii $\sigma_{1,j_1}r^{j_1}\cdots\sigma_{n,j_n}r^{j_n}$, to the discs $\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(\overline{D})$, with radii $(\sigma_{1,j_1})^K r^{j_1}\cdots(\sigma_{n,j_n})^K r^{j_n}$. Then remember from (3.7) that

$$\frac{\log(\sigma_{n,j_n}r^{j_n}\cdots\sigma_{1,j_1}r^{j_1})}{\log((\sigma_{n,j_n})^K r^{j_n}\cdots(\sigma_{1,j_1})^K r^{j_1})}=\frac{1}{\alpha},$$

and hence we see that

$$\frac{\log((\sigma_{n,j_n})^K r^{j_n}\cdots(\sigma_{1,j_1})^K r^{j_1})}{\log(\sigma_{n,j_n}r^{j_n}\cdots\sigma_{1,j_1}r^{j_1})}=\alpha.$$

This shows that the discs $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_n}_{n,j_n}(\overline{D})$ have the right stretch under the mapping ϕ^{-1} with respect to the center points. Moreover, given arbitrary discs $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_n}_{n,j_n}(\overline{D})$ and $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$, with center points z_B, z_A respectively, we notice from the construction of the mapping ϕ that ϕ^{-1} maps points z_B, z_A and $\bar{z}$ as a composition of similarities, where $\bar{z}$ is any point from the boundary of the disc $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$.

Thus to prove that every $z\in\phi(\hat{E}_\phi)$ satisfies the stretching condition in (1.1) we can use the same proof as in the previous case $\alpha<1$, but now for the sequence $(r_n)_{n=2}^{\infty}$ defined by

$$r_n=r(\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})).$$

This follows by choosing $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_n}_{n,j_n}(\overline{D})$ to be the disc B in the proof and $\psi^{i_1}_{1,j_1}\circ\cdots\circ\psi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$ to be the disc A, and continuing as in the proof using the mapping ϕ^{-1} instead of ϕ, while remembering that $\sigma_{n,j_n}<\frac{1}{100}$ for arbitrary choices of n, j_n. This finishes the proof of Theorem 1.2 in the case $\gamma=0$.

4. Rotation

In this section we will introduce rotation to our construction in order to prove Theorem 1.2 when $\gamma\neq 0$. The idea of the proof is that given arbitrary parameters $\alpha>0$ and $\gamma\neq 0$ we will first establish the right stretch by constructing the $\bar{K}$-quasiconformal mapping $\bar{\phi}$ using the construction from the previous section with parameters $\alpha, 0$. Then we will add the right amount of rotation, without changing the stretch, to every step of this construction to create a K-quasiconformal mapping ϕ that proves Theorem 1.2 in the general case. As in section 3 we will first consider the case $\alpha<1$.

Case $\boldsymbol{\alpha<1,\ \gamma\in\mathbf{R}}$. Let $\alpha<1$ and $\gamma\neq 0$ be arbitrary. We can without loss of generality assume that $\gamma>0$ since this only fixes the direction of rotation. Choose arbitrary $\bar{K}$ such that $\frac{1}{\bar{K}}<\alpha$ and denote by $\bar{\phi}$ the $\bar{K}$-quasiconformal mapping constructed in the Section 3 such that the condition (1.1) holds for every point $z\in\hat{E}_{\bar{\phi}}$ with parameters $\alpha, 0$. Then we know from section 3 that $0<H^{\bar{d}}(\hat{E}_{\bar{\phi}})<\infty$ for

$$\bar{d}=1+\alpha-\frac{\bar{K}+1}{\bar{K}-1}(1-\alpha).$$

We then modify inductively the construction of $\bar{\phi}$ by changing every mapping

$$\bar{g}_n(z)=\begin{cases}(\sigma_{n,j_n})^{1-\bar{K}}(z-z^I_J)+z^I_J, & z\in(\bar{D}^I_J)',\\ \left|\frac{z-z^I_J}{r(\bar{D}^I_J)}\right|^{\frac{1}{\bar{K}}-1}(z-z^I_J)+z^I_J, & z\in\bar{D}^I_J\setminus(\bar{D}^I_J)',\\ z, & \text{otherwise.}\end{cases}$$

to the form

$$
(4.1)\qquad g_n(z) = \begin{cases} (\sigma_{n,j_n})^{1-\bar{K}}(z-z_J^I)e^{i\theta_J^I} + z_J^I, & z \in (D_J^I)', \\ \left|\frac{z-z_J^I}{r(D_J^I)}\right|^{\frac{1}{K}-1+i\alpha\gamma\frac{\bar{K}-1}{K(1-\alpha)}}(z-z_J^I)+z_J^I, & z \in D_J^I \setminus (D_J^I)', \\ z, & \text{otherwise.} \end{cases}
$$

where

$$
\begin{aligned}
D_J^I &= \phi_{n-1}\left(\frac{1}{(\sigma_{n,j_n})^{\bar{K}}}\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(D)\right), \\
(D_J^I)' &= \phi_{n-1}\left(\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(D)\right),
\end{aligned}
$$

and θ_J^I is the change of the argument over the annulus $D_J^I \setminus (D_J^I)'$. Here, as in the construction of the Section 3, ϕ_0 can be understood as the identity mapping and $\phi_n = g_n \circ \cdots \circ g_1(z)$. Since $g_n(z)$ is conformal in the discs $(D_J^I)'$ and every $g_n(z)$ is K-quasiconformal for the same $K > \bar{K}$, we notice similarly as in section 3 that every ϕ_n is K-quasiconformal, and that there exists the principal K-quasiconformal limit map ϕ. As we do not change the mappings $\varphi_{n,j_n}^{i_n}$ it holds that $\hat{E}_\phi = \hat{E}_{\bar{\phi}}$, and hence $\hat{E}_\phi$ has positive Hausdorff measure with respect to $\bar{d}$. Moreover, as $\bar{K}$ goes through all possible values $0 < \frac{1}{\bar{K}} < \alpha$, K goes trough all values such that $\alpha(1+i\gamma) \in B_K$, where B_K was defined in (1.3).

Most importantly, we notice that the stretching condition from (1.1) holds for every point $z \in \hat{E}_\phi$ by the same proof and for the same sequence $(r_n)_{n=2}^\infty$ as before. This follows since every mapping $g_n(z)$ still keeps the center point of the discs D_J^I fixed, equals the identity mapping outside of the discs D_J^I, and is a similarity mapping inside the discs $(D_J^I)'$. So the mapping ϕ maps the boundary of the disc $\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(D)$ as the mapping ϕ_n and it maps the center point of the disc $\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(D)$ as the mapping ϕ_{n-1}. Moreover, as we mentioned before, the mappings $\varphi_{n,j_n}^{i_n}$ are not changed, and the stretching parameter for the mappings g_n inside the discs $(D_J^I)'$ is still $(\sigma_{n,j_n})^{1-\bar{K}}$. Hence the discs $\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(D)$ with radii $(\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n,j_n})^{\bar{K}}r^{j_n}$ are mapped to the discs $(B_J^I)''$ with radii $\sigma_{1,j_1}r^{j_1}\cdots\sigma_{n,j_n}r^{j_n}$. Thus the proof for the stretching condition from section 3 can be applied to this situation as well.

What remains to be done is to show the following two things. First that the rotation condition from (1.1) is satisfied in a subset of $\hat{E}_\phi$ that has positive Hausdorff measure with respect to $\bar{d}$, for a subsequence of the sequence $(r_n)_{n=2}^\infty$, defined by

$$
r_n = r(\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n-1,j_{n-1}}^{i_{n-1}}(\overline{D})),
$$

that satisfies the stretching condition. And second that $\bar{d}$ is really the right dimension by proving

$$
(4.2)\qquad 1+\alpha-\frac{\bar{K}+1}{\bar{K}-1}(1-\alpha) = 1+\alpha-\frac{K+1}{K-1}\sqrt{(1-\alpha)^2+\frac{4K}{(K+1)^2}\alpha^2\gamma^2},
$$

where the right hand side is the optimal Hausdorff dimension from Theorem 1.1. When these two things are shown we see that the mapping ϕ is the desired K-quasiconformal mapping that proves Theorem 1.2 in the case $\alpha < 1$, $\gamma \in \mathbf{R}$.

Let us first consider rotation. We defined it at a point z_0 in (1.1) by

$$\lim_{n\to\infty} \frac{\arg(\phi(z_0+r_n)-\phi(z_0))}{\log|\phi(z_0+r_n)-\phi(z_0)|}, \tag{4.3}$$

for some decreasing sequence (r_n). Since ϕ is a principal K-quasiconformal mapping we remind that $\arg(\phi(z_0+r_n)-\phi(z_0))$ can be understood as rotation of $\phi(z)=\phi(z_0+r)$ around the point $\phi(z_0)$, when $z=z_0+r$ travels from $r=+\infty$ to $r=r_n$ along the line which passes trough points z_0 and z_0+r_n.

We aim to show that the relevant part of rotation of $\phi(z)$ around the point $\phi(z_0)$ comes when z moves through annuli $\frac{1}{(\sigma_{n,j_n})^K}\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_n}_{n,j_n}(D)\setminus\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_n}_{n,j_n}(D)$, so when $\phi(z)$ moves trough annuli $D^I_J\setminus(D^I_J)''$, and the rest of rotation can be viewed as an error part which is insignificant compared to $\log|\phi(z_0+r_n)-\phi(z_0)|$ and will vanish as we take the limit in (4.3).

For showing this we must deal with a few difficulties that arise as the sequence $(r_n)_{n=2}^\infty$ is now fixed, and as we do not want points z_0+r_n to lie in the annuli $\frac{1}{(\sigma_{n,j_n})^K}\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})\setminus\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$, since this would make the calculation of rotation trickier. This follows from the fact that the rotation around the point $\phi(z_0)$ that comes when $\phi(z)$ crosses an arbitrary annulus $D^I_J\setminus(D^I_J)''$ depends on the parameter j_n, for which we have no upper bound and, at least to the author's knowledge, no nice way to produce one.

For this reason, we will first find a subset $\bar{E}_\phi\subset\hat{E}_\phi$ that has positive Hausdorff measure with respect to $\bar{d}$ such that, after a possible rotation of the construction, for every $z\in\bar{E}_\phi$ there exists a subsequence $(r_{\bar{n}})$ of the sequence $(r_n)_{n=2}^\infty$ for which it holds that $z+r_{\bar{n}}\in\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_{\bar{n}-1}}_{\bar{n}-1,j_{\bar{n}-1}}(\overline{D})$.

To this end, note that given any point z in an arbitrary disc $B(z_0,r)$ at least one of the points $z+r, z+e^{\frac{i\pi}{2}}r, z+e^{i\pi}r, z+e^{\frac{i3\pi}{2}}r$ lies in the closed disc $\overline{B}(z_0,r)$. From this we see that for every $z\in\hat{E}_\phi$ and for every $n\geq 2$ at least one of the points $z+r_n, z+e^{\frac{i\pi}{2}}r_n, z+e^{i\pi}r_n, z+e^{\frac{i3\pi}{2}}r_n$ lies in the disc $\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_{n-1}}_{n-1,j_{n-1}}(\overline{D})$. Thus for every point $z\in\hat{E}_\phi$ there exists a subsequence $(r_{\bar{n}})$ of $(r_n)_{n=2}^\infty$ such that points $z+e^{\frac{ip\pi}{2}}r_{\bar{n}}$ lie in the discs $\varphi^{i_1}_{1,j_1}\circ\cdots\circ\varphi^{i_{\bar{n}-1}}_{\bar{n}-1,j_{\bar{n}-1}}(\overline{D})$ for some fixed $p\in\{0,1,2,3\}$. Let us remind ourselves of the fact that the stretching condition of (1.1) holds also for the sequence $(e^{\frac{ip\pi}{2}}r_{\bar{n}})$, since it did not depend on the direction, as mentioned before.

Let us denote by $\hat{E}_{\phi,p}$ the set of points in $\hat{E}_\phi$ which have a subsequence $(r_{\bar{n}})$ for fixed $p\in\{0,1,2,3\}$. From the above observations it follows that every point $z\in\hat{E}_\phi$ is contained in at least one of the sets $\hat{E}_{\phi,p}$. Hence it follows that $\hat{E}_\phi=\bigcup_{p=0}^3\hat{E}_{\phi,p}$, and thus from the subadditivity property of the Hausdorff measures we obtain that

$$H^{\bar{d}}(\hat{E}_\phi)\leq\sum_{p=0}^{3}H^{\bar{d}}(\hat{E}_{\phi,p}).$$

Since we know that $H^{\bar{d}}(\hat{E}_\phi)>0$ it follows that there exists $p\in\{0,1,2,3\}$ such that $H^{\bar{d}}(\hat{E}_{\phi,p})>0$. By the rotation of our construction we can assume that $p=0$, and we can choose the set $\hat{E}_{\phi,0}$ to be the subset $\bar{E}_\phi$. From now on we will denote the set $\hat{E}_{\phi,0}$ with $\bar{E}_\phi$ and the desired subsequence $(r_{\bar{n}})$, for a given $z\in\bar{E}_\phi$, with $(r_n)_{n=2}^\infty$ to keep the notation as simple as possible.

Next, we will show that the rotation condition of (1.1) is satisfied for every point $z\in\bar{E}_\phi$. So let $z_0\in\bar{E}_\phi$ and $n\geq 2$ be arbitrary, and study the values of

$\arg(\phi(z_0+r_n)-\phi(z_0))$ using our conventional understanding for argument of principal quasiconformal mappings. We remember from the construction that the point z_0 is the intersection of the nested closed discs $\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})$, which we will denote by $(B^{I_n}_{J_n})'$, and similarly we will denote the discs $\frac{1}{(\sigma_{n,j_n})^{\bar{K}}}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})$ by $B^{I_n}_{J_n}$.

When z travels from $+\infty$ to the point z_0+r_n along the horizontal line which passes trough the point z_0 its path can be divided in to the following parts. The first part consists of the horizontal line from infinity to the boundary of the disc $B^{I_1}_{J_1}$, of the lines L_k between the boundary of the discs $(B^{I_k}_{j_k})'$ to the boundary of the discs $B^{I_{k+1}}_{j_{k+1}}$, where $k=1,2,\dots,n-2$, and of the line from the boundary of the disc $(B^{I_{n-1}}_{j_{n-1}})'$ to the point $z_0+r_n \notin B^{I_n}_{J_n}$. The second part consist of the lines $\bar{L}_k$ that cross the annuli $B^{I_k}_{J_k} \setminus (B^{I_k}_{J_k})'$, where $k=1,2,\dots,n-1$.

Next, we will approximate rotation of $\phi(z)$ around the point $\phi(z_0)$ in these parts. From the argument's viewpoint, when z travels along L_k the image $\phi(z)$ might as well travel along the line $\phi_k(L_k)$, since the mapping $g_{k+1}(z)$ equals the identity mapping outside the open discs $D^{I_{k+1}}_{J_{k+1}} = \phi_k\{B^{I_{k+1}}_{J_{k+1}}\}$. Hence rotation around the point $\phi(z_0)$ has absolute value of at most π as z travels trough L_k for arbitrary $k=1,2,\dots,n-2$. Similarly rotation of $\phi(z)$ around the point $\phi(z_0)$ is of absolute value at most π when z travels from $+\infty$ to the boundary of the disc $B^{I_1}_{J_1}$, and when z travels from the boundary of the disc $(B^{I_{n-1}}_{J_{n-1}})'$ to the point z_0+r_n. Hence the complete rotation, which we will denote by β_n, from the first part is at most of absolute value $n\pi$.

The rest of the rotation comes from crossing the annuli, and we will approximate that next. We can calculate directly from the construction that the rotation when z travels radially over the annulus $B^{I_k}_{J_k} \setminus (B^{I_k}_{J_k})'$, with respect to the center point of the annulus $\phi(B^{I_k}_{J_k} \setminus (B^{I_k}_{J_k})')$, is

$$\alpha\gamma\frac{\bar{K}-1}{\bar{K}(1-\alpha)}\log(\sigma_{k,j_k})^{\bar{K}}.$$

And using the definition $\sigma_{k,j_k} = r^{\frac{j_k(1-\alpha)}{\alpha\bar{K}-1}}$, this simplifies to

$$\alpha\gamma\log((\sigma_{k,j_k})^{\bar{K}} r^{j_k}). \tag{4.4}$$

Since the point $\phi(z_0)$ need not be the center point of the annulus $\phi(B^{I_k}_{J_k} \setminus (B^{I_k}_{J_k})')$ and the line $\bar{L}_k$ does not need to cross the annulus $B^{I_k}_{J_k} \setminus (B^{I_k}_{J_k})'$ radially, we have some error in our approximation with respect to (4.4), but it is easy to see that it must be smaller than 4π in absolute value. Thus the rotation with respect to the point $\phi(z_0)$ over the annuli is

$$\begin{aligned}&\alpha\gamma(\log((\sigma_{1,j_1})^{\bar{K}} r^{j_1}) + \cdots + \log((\sigma_{n-1,j_{n-1}})^{\bar{K}} r^{j_{n-1}})) + \beta_{1,n}\\ &= \alpha\gamma\log((\sigma_{1,j_1})^{\bar{K}} r^{j_1} \cdots (\sigma_{n-1,j_{n-1}})^{\bar{K}} r^{j_{n-1}}) + \beta_{1,n}\end{aligned} \tag{4.5}$$

where $\beta_{1,n}$ is the sum of error terms, for which we have the estimate $|\beta_{1,n}| \le 4\pi(n-1)$.

Summing up all of the rotation we obtain that

$$\arg(\phi(z_0+r_n)-\phi(z_0)) = \alpha\gamma\log((\sigma_{1,j_1})^{\bar{K}} r^{j_1} \cdots (\sigma_{n-1,j_{n-1}})^{\bar{K}} r^{j_{n-1}}) + \beta_n + \beta_{1,n}, \tag{4.6}$$

where the absolute value of $\beta_n + \beta_{1,n}$ is bounded by $5n\pi$.

We would like to note that we have made very crude, but for our purposes sufficient, approximations for the error terms, and in reality they are much smaller.

Now all that remains is to verify by a straight calculation the rotation part of (1.1) for arbitrary $z_0 \in \bar{E}_\phi$. We will use (4.6) and the fact that the stretching part of (1.1) has been proved. We calculate

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\arg(\phi(z_0+r_n)-\phi(z_0))}{\log|\phi(z_0+r_n)-\phi(z_0)|}\\
&= \lim_{n\to\infty} \frac{\alpha\gamma\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})+\beta_n+\beta_{1,n}}{\alpha\log(r_n)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}\Bigg(\frac{\alpha\gamma\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})}{\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})}\\
&\quad+\frac{\beta_n+\beta_{1,n}}{\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})}\Bigg)\\
&= \lim_{n\to\infty}\left(\gamma+\frac{\beta_n+\beta_{1,n}}{\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})}\right).
\end{aligned} \tag{4.7}$$

Hence we need to show that

$$\lim_{n\to\infty}\frac{\beta_n+\beta_{1,n}}{\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})}=0. \tag{4.8}$$

For this we use the estimates $|\beta_n+\beta_{1,n}|\le 5n\pi$ and

$$\begin{aligned}
\left|\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})\right| &\ge \left|\alpha\log(r^{j_1+\cdots+j_{n-1}})\right| \ge \alpha(j_1+\cdots j_{n-1})\\
&\ge \frac{\alpha n(n-1)}{2},
\end{aligned}$$

where the second to last inequality comes from the assumption $r<\frac{1}{e}$ made in the construction of ϕ in section 3 and the last inequality comes from the choices $j_n\ge n$ in the construction. Thus we see that for every n it holds that

$$\frac{|\beta_n+\beta_{1,n}|}{|\alpha\log((\sigma_{1,j_1})^{\bar{K}}r^{j_1}\cdots(\sigma_{n-1,j_{n-1}})^{\bar{K}}r^{j_{n-1}})|}\le\frac{10n\pi}{\alpha n(n-1)}. \tag{4.9}$$

This shows (4.8) and hence proves the rotation condition of (1.1) for every point $z_0\in\bar{E}_\phi$.

Then all that remains to prove Theorem 1.2 in the case $\alpha<1$, $\gamma\in\mathbf{R}$ is to show the equation (4.2). For this we need to find out the connection between K and $\bar{K}$. From (4.1) we see that every g_n is quasiconformal with the same K as the model map (1.2) with the exponent $\frac{1}{\bar{K}}\left(1+i\frac{\alpha\gamma(\bar{K}-1)}{1-\alpha}\right)$. Therefore by [2] Theorem 3.1 we have

$$\left|\frac{1}{\bar{K}}+\frac{i\alpha\gamma(\bar{K}-1)}{\bar{K}(1-\alpha)}-\frac{1}{2}\left(K+\frac{1}{K}\right)\right|=\frac{1}{2}\left(K-\frac{1}{K}\right). \tag{4.10}$$

Note that this is the equation that characterizes the boundary of the disc B_K. Squaring both sides of (4.10) and reorganizing we obtain

$$\left(1+\frac{\alpha^2\gamma^2}{(1-\alpha)^2}\right)\bar{K}^2-\left(\frac{K^2+1}{K}+\frac{2\alpha^2\gamma^2}{(1-\alpha)^2}\right)\bar{K}+1+\frac{\alpha^2\gamma^2}{(1-\alpha)^2}=0. \tag{4.11}$$

Solving this and choosing the relevant bigger root, as we can calculate from (4.11) that the smaller root is less than one, we obtain

$$\begin{aligned}\bar{K} = &\frac{(K^2+1)(1-\alpha)^2 + 2K\alpha^2\gamma^2}{2K(\alpha^2\gamma^2+(1-\alpha)^2)} \\ &+ \frac{(1-\alpha)^2}{2\alpha^2\gamma^2+2(1-\alpha)^2}\sqrt{\left(\frac{K^2+1}{K}+\frac{2\alpha^2\gamma^2}{(1-\alpha)^2}\right)^2 - 4\left(1+\frac{\alpha^2\gamma^2}{(1-\alpha)^2}\right)^2}.\end{aligned} \tag{4.12}$$

Next, we multiply $(1-\alpha)^2$ inside the square root and simplify the expression obtained this way. We calculate

$$\begin{aligned}&(1-\alpha)^4\left(\left(\frac{K^2+1}{K}+\frac{2\alpha^2\gamma^2}{(1-\alpha)^2}\right)^2 - 4\left(1+\frac{\alpha^2\gamma^2}{(1-\alpha)^2}\right)^2\right) \\ &= \frac{(K^2-1)^2(1-\alpha)^4 + \alpha^2\gamma^2(1-\alpha)^2(4K^3-8K^2+4K)}{K^2} \\ &= \frac{(1-\alpha)^2(K-1)^2}{K^2}((K+1)^2(1-\alpha)^2+4\alpha^2\gamma^2K).\end{aligned}$$

With this we obtain from (4.12) that

$$\bar{K} = \frac{(K^2+1)(1-\alpha)^2+2\alpha^2\gamma^2K+(1-\alpha)(K-1)\sqrt{(K+1)^2(1-\alpha)^2+4\alpha^2\gamma^2K}}{2K(\alpha^2\gamma^2+(1-\alpha)^2)}. \tag{4.13}$$

Then we simplify the equation (4.2) to obtain

$$\begin{aligned}\frac{\bar{K}+1}{\bar{K}-1}(1-\alpha) &= \frac{K+1}{K-1}\sqrt{(1-\alpha)^2+\frac{4K}{(K+1)^2}\alpha^2\gamma^2} \\ \iff \frac{\bar{K}+1}{\bar{K}-1} &= \frac{\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}}{(K-1)(1-\alpha)}.\end{aligned}$$

Next, substitute (4.13) to the left hand side of the above equation to obtain

$$\begin{aligned}&\frac{(K^2+1)(1-\alpha)^2+2\alpha^2\gamma^2K+(1-\alpha)(K-1)\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}+2K(\alpha^2\gamma^2+(1-\alpha)^2)}{(K^2+1)(1-\alpha)^2+2\alpha^2\gamma^2K+(1-\alpha)(K-1)\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}-2K(\alpha^2\gamma^2+(1-\alpha)^2)} \\ &= \frac{(1-\alpha)(K-1)\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}+4K\alpha^2\gamma^2+(1-\alpha)^2(K+1)^2}{(1-\alpha)(K-1)\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}+(1-\alpha)^2(K-1)^2} \\ &= \frac{\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}\left((1-\alpha)(K-1)+\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}\right)}{(1-\alpha)(K-1)\left((1-\alpha)(K-1)+\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}\right)} \\ &= \frac{\sqrt{(K+1)^2(1-\alpha)^2+4K\alpha^2\gamma^2}}{(1-\alpha)(K-1)},\end{aligned}$$

which proves that $\bar{d}$ is the desired dimension. This finishes the proof of Theorem 1.2 in the case $\alpha < 1$, $\gamma \in \mathbf{R}$.

Case $\boldsymbol{\alpha > 1}$, $\boldsymbol{\gamma \in \mathbf{R}}$. Let $\alpha > 1$ and $\gamma \in \mathbf{R}$ be arbitrary. For the same reason as before we can assume without loss of generality that $\gamma > 0$. Choose arbitrary $\bar{K} > \alpha$ and let $\bar{\phi}^{-1}$ be the optimal $\bar{K}$-quasiconformal mapping, for parameters $\alpha, 0$, constructed in the Section 3. From the construction of $\bar{\phi}$ it is clear that $\bar{\phi}^{-1}$ can be constructed with a similar process as the mapping $\bar{\phi}$ using the mappings $\bar{\phi}_n^{-1} =$

$\bar{g}_1^{-1} \circ \cdots \circ \bar{g}_n^{-1}$. From the definition of the mappings $\bar{g}_n(z)$ we can calculate that,

$$\bar{g}_n^{-1}(z) = \begin{cases} (\sigma_{n,j_n})^{\bar{K}-1}(z - z_J^I) + z_J^I, & z \in (D_J^I)'', \\ \left|\frac{z-z_J^I}{r(D_J^I)}\right|^{\bar{K}-1}(z - z_J^I) + z_J^I, & z \in D_J^I \setminus (D_J^I)'', \\ z, & \text{otherwise.} \end{cases}$$

where the discs D_J^I and $(D_J^I)''$ are as in the construction of ϕ. So especially $D_J^I = \frac{1}{\sigma_{n,j_n}}\psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n,j_n}^{i_n}(D)$, with radius $\sigma_{1,j_1}r^{j_1} \cdots \sigma_{n-1,j_{n-1}}r^{j_{n-1}}r^{j_n}$, and $(D_J^I)'' = \psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n,j_n}^{i_n}(D)$, with radius $\sigma_{1,j_1}r^{j_1} \cdots \sigma_{n,j_n}r^{j_n}$.

Then we introduce rotation to the construction of $\bar{\phi}^{-1}$ by replacing every $\bar{g}_n^{-1}(z)$ in the construction with

$$g_n^{-1}(z) = \begin{cases} (\sigma_{n,j_n})^{\bar{K}-1}(z - z_J^I)e^{i\theta_J^I} + z_J^I, & z \in (D_J^I)'', \\ \left|\frac{z-z_J^I}{r(D_J^I)}\right|^{\bar{K}-1+i\alpha\gamma\frac{\bar{K}-1}{\alpha-1}}(z - z_J^I) + z_J^I, & z \in D_J^I \setminus (D_J^I)'', \\ z, & \text{otherwise.} \end{cases}$$

where θ_J^I is the change of argument over the annulus $D_J^I \setminus (D_J^I)''$. In this way we obtain the K-quasiconformal mappings ϕ_n^{-1}, and as their limit we obtain the principal K-quasiconformal mapping ϕ^{-1}, where $K > \bar{K}$. As in the previous case we obtain that every point $z \in \hat{E}_{\phi^{-1}} = \bar{\phi}(\hat{E}_{\bar{\phi}})$ satisfies the stretching condition with the sequence defined by $r_n = r(\psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n-1,j_{n-1}}^{i_{n-1}}(D))$, and that $0 < H^{\bar{d}}(\hat{E}_{\phi^{-1}}) < \infty$ for

$$\bar{d} = 1 + \alpha - \frac{\bar{K}+1}{\bar{K}-1}(\alpha - 1).$$

Moreover, we see that K goes through all possible values such that $\alpha(1 + i\gamma) \in B_K$ as $\bar{K}$ goes trough all values $\bar{K} > \alpha$.

The Proof of the existence of a subset $\bar{E}_{\phi^{-1}} \subset \hat{E}_{\phi^{-1}}$ with positive Hausdorff measure with respect to $\bar{d}$ and whose points satisfy also the rotation condition of (1.1) follows similarly as in the previous case by noticing that the meaningful part of the change of argument over the annulus $\frac{1}{\sigma_{n,j_n}}\psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n,j_n}^{i_n}(\overline{D}) \setminus \psi_{1,j_1}^{i_1} \circ \cdots \circ \psi_{n,j_n}^{i_n}(\overline{D})$ is

$$\alpha\gamma\frac{\bar{K}-1}{\alpha-1}\log(\sigma_{n,j_n}) = \alpha\gamma\log(\sigma_{n,j_n}r^{j_n}),$$

where we have used the fact that $\sigma_{n,j_n} = r^{j_n\frac{\alpha-1}{\bar{K}-\alpha}}$. Thus with a similar proof as before we obtain, for every point $z \in \bar{E}_{\phi^{-1}}$, that

$$\lim_{n\to\infty}\frac{\arg(\phi(z + r_n) - \phi(z))}{\log|\phi(z + r_n) - \phi(z)|} = \lim_{n\to\infty}\frac{\alpha\gamma\log(\sigma_{1,j_1}r^{j_1} \cdots \sigma_{n-1,j_{n-1}}r^{j_{n-1}}) + \beta_n}{\alpha\log(\sigma_{1,j_1}r^{j_1} \cdots \sigma_{n-1,j_{n-1}}r^{j_{n-1}})} = \gamma,$$

where β_n is a similar error term as in the previous case.

Hence all we need to prove is the equation

$$1 + \alpha - \frac{\bar{K}+1}{\bar{K}-1}(\alpha - 1) = 1 + \alpha - \frac{K+1}{K-1}\sqrt{(\alpha-1)^2 + \frac{4K}{(K+1)^2}\alpha^2\gamma^2}.$$

We do this similarly as in the previous case and calculate the connection between $\bar{K}$ and K as follows

$$\begin{aligned}&\left|\bar{K} + i\alpha\gamma\frac{\bar{K}-1}{\alpha-1} - \frac{1}{2}\left(K+\frac{1}{K}\right)\right| = \frac{1}{2}\left(K-\frac{1}{K}\right)\\ \iff &\left(1+\frac{\alpha^2\gamma^2}{(\alpha-1)^2}\right)\bar{K}^2 - \left(\frac{K^2+1}{K}+\frac{2\alpha^2\gamma^2}{(\alpha-1)^2}\right)\bar{K} + 1 + \frac{\alpha^2\gamma^2}{(\alpha-1)^2} = 0.\end{aligned}$$

Solving this and choosing the relevant bigger root we obtain

$$\begin{aligned}\bar{K} = &\frac{(K^2+1)(\alpha-1)^2 + 2K\alpha^2\gamma^2}{2K(\alpha^2\gamma^2+(\alpha-1)^2)}\\ &+ \frac{(\alpha-1)^2}{2\alpha^2\gamma^2+2(\alpha-1)^2}\sqrt{\left(\frac{K^2+1}{K}+\frac{2\alpha^2\gamma^2}{(\alpha-1)^2}\right)^2 - 4\left(1+\frac{\alpha^2\gamma^2}{(\alpha-1)^2}\right)^2}.\end{aligned}$$

Note, that this is the same equation as (4.12) where every term $(1-\alpha)$ has been changed to $(\alpha-1)$, due to the change from $\alpha<1$ to $\alpha>1$. And as the equation (4.2) is the same in both cases, except for the detail that now $\alpha>1$, we can proceed with the same calculation as in the previous case and obtain that $\bar{d}$ is the desired dimension, which concludes the proof when $\alpha>1$, $\gamma\in\mathbf{R}$.

Case $\boldsymbol{\alpha=1,\gamma\in\mathbf{R}}$. As the case $\alpha=1,\gamma=0$ was trivial, due to the mapping $f(z)=z$, we do not have any previous construction to which we could add rotation as in the previous cases. Nevertheless, we will still use the construction from the Section 3 to construct examples verifying optimality. Let $\alpha=1$ and $\gamma\in\mathbf{R}$, and as before we can assume without loss of generality that $\gamma>0$. Then choose arbitrary $d\in(0,2)$ and parameters σ_{n,j_n}, r and $\bar{K}$, as in the construction of the Section 3, such that

$$(\sigma_{n,j_n})^{\bar{K}d} = (r^{j_n})^{2-d}. \tag{4.14}$$

Then, unlike before, we are not interested in the mapping $\bar{\phi}$ but just in the set $\hat{E}_{\bar{\phi}}$ from our construction in the Section 3, defined by

$$\hat{E}_{\bar{\phi}} = \bigcap_{n=1}^{\infty}\left(\bigcup_{I,J}\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(\overline{D})\right),$$

for which we know that $0<H^d(\hat{E}_{\bar{\phi}})<\infty$. We then use the discs $\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(\overline{D})$ and $\frac{1}{(\sigma_{n,j_n})^K}\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n,j_n}^{i_n}(\overline{D})$ as building blocks for our desired mapping ϕ. We construct this mapping as in the Section 3 using the mappings

$$g_n(z) = \begin{cases} e^{i\theta_J^I}(z-z_J^I)+z_J^I, & z\in(D_J^I)',\\ \left|\frac{z-z_J^I}{r(D_J^I)}\right|^{i\gamma\frac{2}{2-d}}(z-z_J^I)+z_J^I, & z\in D_J^I\setminus(D_J^I)',\\ z, & \text{otherwise.}\end{cases} \tag{4.15}$$

where the discs D_J^I, $(D_J^I)'$, and the parameter θ_J^I are chosen as in the case $\alpha<1$, $\gamma\in\mathbf{R}$. Then ϕ will clearly be a principal K-quasiconformal mapping for some K which depends on d and γ. The stretching condition from (1.1) holds for every point $z\in\hat{E}_{\bar{\phi}}$ with a similar proof as before for the sequence defined by

$$r_n = r(\varphi_{1,j_1}^{i_1}\circ\cdots\circ\varphi_{n-1,j_{n-1}}^{i_{n-1}}(\overline{D})).$$

Also the rotation condition from (1.1) holds for every point of a subset of $\hat{E}_{\bar{\phi}}$, that has the right Hausdorff measure, with a similar proof as before. This follows from the observation that the meaningful part of rotation over an arbitrary annulus $\frac{1}{(\sigma_{n,j_n})^K}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D}) \setminus \varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(\overline{D})$ is

$$\gamma\frac{2}{2-d}\log((\sigma_{n,j_n})^{\bar{K}}) = \gamma\log((\sigma_{n,j_n})^{\bar{K}} r^{j_n}),$$

where we have used the equation (4.14).

Hence the only thing left to prove is that d is the right Hausdorff dimension, which we can check from (1.4) to be

$$2 - \frac{2\gamma\sqrt{K}}{K-1}$$

for these choices of parameters α, γ. Next, we must express K using γ and δ, and we do this using the same method as before. We calculate

$$\begin{aligned}\left|1 + i\gamma\frac{2}{2-d} - \frac{1}{2}\left(K + \frac{1}{K}\right)\right| &= \frac{1}{2}\left(K - \frac{1}{K}\right)\\ \iff K^2 - \left(\frac{4\gamma^2}{(2-d)^2} + 2\right)K + 1 &= 0.\end{aligned}$$

Solving this and choosing the relevant bigger root yields

$$K = \frac{2\gamma^2 + (2-d)^2}{(2-d)^2} + \frac{2\gamma}{2-d}\sqrt{\frac{\gamma^2}{(2-d)^2} + 1}. \tag{4.16}$$

We have to show that

$$d = 2 - \frac{2\gamma\sqrt{K}}{K-1},$$

so we substitute the expression from (4.16) in place of K and calculate

$$\begin{aligned} d &= 2 - \frac{2\gamma\sqrt{\frac{2\gamma^2}{(2-d)^2} + 1 + \frac{2\gamma}{2-d}\sqrt{\frac{\gamma^2}{(2-d)^2} + 1}}}{\frac{2\gamma^2}{(2-d)^2} + \frac{2\gamma}{2-d}\sqrt{\frac{\gamma^2}{(2-d)^2} + 1}}\\ \iff 1 &= \frac{2\gamma(2-d)\sqrt{\frac{2\gamma^2}{(2-d)^2} + 1 + \frac{2\gamma}{2-d}\sqrt{\frac{\gamma^2}{(2-d)^2} + 1}}}{2\gamma^2 + 2\gamma(2-d)\sqrt{\frac{\gamma^2}{(2-d)^2} + 1}}\\ \iff \gamma + \sqrt{\gamma^2 + (2-d)^2} &= \sqrt{2\gamma^2 + (2-d)^2 + 2\gamma\sqrt{\gamma^2 + (2-d)^2}}.\end{aligned}$$

This holds since both sides are positive and

$$\left(\gamma + \sqrt{\gamma^2 + (2-d)^2}\right)^2 = 2\gamma^2 + (2-d)^2 + 2\gamma\sqrt{\gamma^2 + (2-d)^2}.$$

This shows that

$$d = 2 - \frac{2\sqrt{K}}{K-1}\gamma,$$

which proves Theorem 1.2 in the case $\alpha = 1$, $\gamma \in \mathbf{R}$. Thus this finishes the proof of Theorem 1.2, since we have covered all possible choices for parameters $\alpha > 0$, $\gamma \in \mathbf{R}$.

Finally, we will show that in the case $\alpha = 1$, $\gamma \neq 0$ and $d \in (0, 2)$ the mapping ϕ is a bilipschitz mapping with the constant L satisfying

$$L - \frac{1}{L} = \frac{2|\gamma|}{2-d}. \tag{4.17}$$

To see this, we first check that ϕ_n is a L-bilipschitz mapping for an arbitrary n.

For the mapping

$$\phi_1(z) = \begin{cases} e^{i\theta^i_{j_1}}(z - z^i_{j_1}) + z^i_{j_1}, & z \in (D^i_{j_1})', \\ \left|\frac{z - z^i_{j_1}}{r(D^i_{j_1})}\right|^{i\gamma\frac{2}{2-d}} (z - z^i_{j_1}) + z^i_{j_1}, & z \in D^i_{j_1} \setminus (D^i_{j_1})', \\ z, & \text{otherwise.} \end{cases}$$

this is straightforward as it is continuous in the complex plane, isometry outside of the annuli $D^i_{j_1} \setminus (D^i_{j_1})'$ and L-bilipschitz in the annuli $D^i_{j_1} \setminus (D^i_{j_1})'$, where the constant L can be verified to satisfy the equality (4.17). Thus the mapping ϕ_1 is bilipschitz with the correct constant. Similarly, for every $n \geq 2$ the mapping g_n, see (4.15), is L-bilipschitz and differs from the identity mapping only inside the discs

$$D^{I_n}_{J_n} = \phi_{n-1}\left(\frac{1}{(\sigma_n, j_n)^{\bar{K}}}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)\right).$$

From the construction of the mapping ϕ we see that the mapping ϕ_{n-1} maps every disc $\frac{1}{(\sigma_n, j_n)^{\bar{K}}}\varphi^{i_1}_{1,j_1} \circ \cdots \circ \varphi^{i_n}_{n,j_n}(D)$ as an isometry. Hence we get by induction, starting from the mapping ϕ_1, that every mapping ϕ_n is L-bilipschitz.

Then, as the mappings ϕ_n and ϕ_n^{-1} converge uniformly to the mappings ϕ and ϕ^{-1}, respectively, it follows that ϕ is also a L-bilipschitz mapping.

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