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FALCONER DISTANCE PROBLEM, ADDITIVE ENERGY AND CARTESIAN PRODUCTS

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Abstract. A celebrated result due to Wolff says if E is a compact subset of $\mathbf{R}^2$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x-y| \colon x, y \in E\}$ is positive if the Hausdorff dimension of E is greater than $\frac{4}{3}$. In this paper we improve the $\frac{4}{3}$ barrier by a small exponent for Cartesian products. In higher dimensions, also in the context of Cartesian products, we reduce Erdogan's $\frac{d}{2}+\frac{1}{3}$ exponent to $\frac{d^2}{2d-1}$. The proof uses a combination of Fourier analysis and additive comibinatorics.

1. Introduction

The Falconer distance conjecture [3] says that if the Hausdorff dimension of $E \subset \mathbf{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x-y| \colon x, y \in E\}$ is positive.

The best known results are due to Wolff [7] in two dimensions and Erdogan [2] in higher dimensions. They proved that the Lebesgue measure of $\Delta(E)$ is positive if the Hausdorff dimension of E is greater than $\frac{d}{2}+\frac{1}{3}$. This was accomplished by showing that if $s \in \left(\frac{d}{2}, \frac{d+2}{2}\right)$ is the Hausdorff dimension of E and μ is a Frostman measure on E which has finite $(s-\epsilon)$-energy $I_{s-\epsilon}(\mu)$ for all $\epsilon > 0$, then for all $\epsilon > 0$,

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 \, d\omega \leq C t^{-\frac{d+2s-2}{4}+\epsilon}. \tag{1.1}$$

In particular, in the two-dimensional case, which is the focus of this paper, the estimate takes the form

$$\int_{S^1} |\widehat{\mu}(t\omega)|^2 \, d\omega \leq C t^{-\frac{s}{2}+\epsilon}. \tag{1.2}$$

This estimate is then plugged into the Mattila integral,

$$\mathcal{M}(\mu) = \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 \, d\omega \right)^2 t^{d-1} \, dt, \tag{1.3}$$

the most effective tool developed so far for the study of the Falconer distance problem.

Mattila proved in [5] that if E is a compact set of Hausdorff dimension $s > \frac{d}{2}$ and μ is Borel measure supported on E such that $\mathcal{M}(\mu) < \infty$, then the Lebesgue measure of $\Delta(E)$ is positive. For discussion about other versions of Mattila integrals, see [4].

Our results are the following.

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Theorem 1.1. *Let $E = A \times B$, where A and B are compact subsets of $\mathbf{R}$ with positive s_A, s_B-dimensional Hausdorff measure, respectively. If $s_A + s_B + \max(s_A, s_B) > 2$, the Lebesgue measure of $\Delta(E)$ is positive. In particular, if $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(B)$ and $\dim_{\mathcal{H}}(A) \neq \dim_{\mathcal{H}}(B)$, $\dim_{\mathcal{H}}(E) > \frac{4}{3} - \frac{|\dim_{\mathcal{H}}(A) - \dim_{\mathcal{H}}(B)|}{3}$ implies $\Delta(E)$ has positive Lebesgue measure.*

To state the result in the case $\dim_{\mathcal{H}}(A) = \dim_{\mathcal{H}}(B)$, we need the following definition.

Definition 1.2. Let A be a compact subset of $\mathbf{R}^d$ of Hausdorff dimension s_A. We say A is Ahlfors–David regular if there exists a Radon measure ν_A on A and a constant $0 < C_{\nu_A} < \infty$ such that

$$C_{\nu_A}^{-1} r^{s_A} < \nu_A(B(x,r)) < C_{\nu_A} r^{s_A}, \ \forall x \in A, \ 0 < r < 1. \tag{1.4}$$

Theorem 1.3. *Suppose $E = A \times B$, $s_A = s_B = \alpha$ and A is Ahlfors–David regular with ν_A, C_{ν_A} such that (1.4) holds. Then there exists $\delta = \delta(C_{\nu_A}) > 0$ such that whenever $\alpha > \frac{2}{3} - \delta$, the Lebesgue measure of $\Delta(E)$ is positive.*

We also obtain an improvement of Erdogan's $\frac{d}{2} + \frac{1}{3}$ exponent in higher dimension for Cartesian products.

Theorem 1.4. *Suppose that E is a compact subset of $\mathbf{R}^d$ of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_j \subset \mathbf{R}$ has positive s_j-dimensional Hausdorff measure for all $1 \leq j \leq d$. Suppose that $\sum_{j=1}^d s_j > \frac{d^2}{2d-1}$. Then the Lebesgue measure of $\Delta(E)$ is positive.*

1.1. Outline of the proof of Theorems 1.1, 1.3 and 1.4. Our argument consists of three basic steps.

- We first establish Theorem 1.1 which is accomplished using the imbalance inherent in the structure of the Mattila integral.
- The improvement of the $\frac{d}{2} + \frac{1}{3}$ exponent for Cartesian products in higher dimensions (Theorem 1.4) is accomplished in the same way regardless of whether the Hausdorff dimension of the fibers is the same.
- In the case when $\dim_{\mathcal{H}}(A) = \dim_{\mathcal{H}}(B)$, we use a recent result due to Dyatlov and Zahl [1] to show that when A is Ahlfors–David regular, the additive energy of A at scale t^{-1},
$$\nu_A^4\{(a_1, a_2, a_3, a_4)\colon |(a_1 + a_2) - (a_3 + a_4)| \leq t^{-1}\},$$
where ν_A is a Frostman measure on A, satisfies a better than trivial estimate, namely $Ct^{-\dim_{\mathcal{H}}(A) - \delta}$ for some $\delta > 0$, and then show that this leads to a slightly better exponent than $\frac{4}{3}$.

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2. Proof of Theorem 1.1

We shall repeatedly use the following simple estimate.

Lemma 2.1. (Solid Average) *Suppose that ν is a compactly supported Borel measure on $\mathbf{R}^d$ such that $\nu(B(x,r)) \leq Cr^{\alpha}$ for all $x \in \mathbf{R}^d$. Then for any bounded rectangle R,*

$$\int_R |\widehat{\nu}(tu)|^2 \, du \leq C_R t^{-\alpha}.$$

To prove the lemma observe that the left hand side is

$$\leq \int |\widehat{\nu}(tu)|^2 \widehat{\psi}(u)\, du,$$

where ψ is a suitably chosen smooth compactly supported function. This expression equals

$$\iiint e^{2\pi i(x-y)\cdot tu}\widehat{\psi}(u)\, du\, d\nu(x)\, d\nu(y) = \iint \psi(t(x-y))\, d\nu(x)\, d\nu(y) \leq C_R t^{-\alpha}$$

by assumption. This completes the proof of the lemma.

We now parameterize the upper semi-circle S_1^+ in the form

$$\left\{(u, \sqrt{1-u^2})\colon\ -1 \leq u \leq 1\right\}.$$

The argument shall be carried out for this parameterization as the proof for the lower semi-circle is identical.

Let $d\mu(x) = d\nu_A(x_1)\, d\nu_B(x_2)$, where $\nu_A,\ \nu_B$ are Frostman probability measures on A and B, respectively such that

$$\nu_A(B(x,r)) \leq Cr^{s_A}, \quad \nu_B(B(x,r)) \leq Cr^{s_B}.$$

Assume without loss of generality that $s_A \geq s_B$. Also assume E is not a point mass, which implies that either

$$\exists\, a \in \mathbf{R},\ \mu(\{(x_1,x_2)\colon x_1 > a\}), \mu(\{(x_1,x_2)\colon x_1 < a\}) > 0,$$

or

$$\exists\, b \in \mathbf{R},\ \mu(\{(x_1,x_2)\colon x_2 > b\}), \mu(\{(x_1,x_2)\colon x_2 < b\}) > 0.$$

Without loss of generality, we may assume $\mu(\{(x_1,x_2)\colon x_2 > b\}), \mu(\{(x_1,x_2)\colon x_2 < b\}) > 0$ for some $b \in \mathbf{R}$. It follows that

$$\iint \frac{|x_2 - y_2|}{|x-y|}\, d\mu(x)\, d\mu(y) > 0. \tag{2.1}$$

Let $\omega = (\cos(\theta), \sin(\theta))$. Consider the modified Mattila integral

$$\int \left(\int_{S^1} |\widehat{\mu}(t\omega)|^2 |\sin(\theta)|\, d\omega \right)^2 t\, dt. \tag{2.2}$$

Lemma 2.2. *Suppose* (2.1) *holds. Then the finiteness of the integral* (2.2) *implies that the Lebesgue measure of the distance set is positive.*

Proof. To prove this lemma, one simply replaces the distance measure in the derivation of the Mattila integral in [7], given by the relation

$$\int f(t)\, d\nu_0^*(t) = \iint f(|x-y|)\, d\mu(x)\, d\mu(y)$$

by a slightly modified distance measure given by

$$\int f(t)\, d\nu_0(t) = \iint f(|x-y|) \frac{|x_2 - y_2|}{|x-y|}\, d\mu(x)\, d\mu(y).$$

As in [7], define

$$d\nu(t) = e^{i\frac{\pi}{4}} t^{-\frac{1}{2}}\, d\nu_0(t) + e^{-i\frac{\pi}{4}} |t|^{-\frac{1}{2}}\, d\nu_0(-t)$$

and it follows that

$$\widehat{\nu}(t) = \iint |x-y|^{-\frac{1}{2}} \cos(2\pi(t|x-y| - \tfrac{1}{8})) \frac{|x_2 - y_2|}{|x-y|} \, d\mu(x) d\mu(y). \tag{2.3}$$

On the other hand,

$$\int |\widehat{\mu}(t\omega)|^2 |\sin\theta| \, d\theta = \iint \left(\int e^{2\pi i (x-y)\cdot(t\omega)} |\sin\theta| \, d\theta \right) d\mu(x) \, d\mu(y). \tag{2.4}$$

Let θ_{x-y} be the angle between the vector $x-y$ and the x-axis. Then $|\sin\theta_{x-y}| = \frac{|x_2-y_2|}{|x-y|}$. We may assume s, the Hausdorff dimension of E, is not greater than $\frac{3}{2}$. By stationary phase(see, e.g. [8] for details),

$$\begin{aligned} &\int e^{2\pi i (x-y)\cdot(t\omega)} |\sin\theta| \, d\theta \\ &= 2(|t||x-y|)^{-\frac{1}{2}} \cos(2\pi(t|x-y| - \tfrac{1}{8})) |\sin\theta_{x-y}| + O((t|x-y|)^{-\frac{3}{2}}) \\ &= 2(|t||x-y|)^{-\frac{1}{2}} \cos(2\pi(t|x-y| - \tfrac{1}{8})) \frac{|x_2 - y_2|}{|x-y|} + O((t|x-y|)^{-s+\epsilon}). \end{aligned} \tag{2.5}$$

Putting (2.3), (2.4), (2.5) together, one can see

$$\begin{aligned} ||\widehat{\nu}||_2^2 &= \int_{|t|\le 1} |\widehat{\nu}(t)|^2 \, dt + \int_{|t|\ge 1} |\widehat{\nu}(t)|^2 \, dt \\ &\le 1 + \int_1^\infty \left(\int |\widehat{\mu}(t\omega)|^2 |\sin\theta| \, d\theta \right)^2 t \, dt + C I_{s-\epsilon}(\mu), \end{aligned}$$

which proves the lemma since $I_{s-\epsilon}(\mu) < \infty$. □

We now proceed with the estimation of (2.2). It follows that

$$\int_{S^1} |\widehat{\mu}(t\omega)|^2 |\sin(\theta)| \, d\omega$$

is bounded by the sum of two terms of the form

$$\int_{-1}^{1} |\widehat{\nu}_A(tu)|^2 \left| \widehat{\nu}_B \left(\pm t\sqrt{1-u^2} \right) \right|^2 du \le \int_{-1}^{1} |\widehat{\nu}_A(tu)|^2 \, du \le C t^{-s_A} \tag{2.6}$$

by Lemma 2.1. Plugging (2.6) into the modified Mattila integral (2.2) we see that

$$\begin{aligned} \mathcal{M}(\mu) &\le C \int \int_{S^1} |\widehat{\mu}(t\omega)|^2 t \cdot t^{-s_A} \, d\omega \, dt = C \int |\widehat{\mu}(\xi)|^2 |\xi|^{-s_A} \, d\xi \\ &= C' \int \int |x-y|^{-2+s_A} \, d\mu(x) \, d\mu(y) \end{aligned}$$

and this energy integral (see e.g. [8] or [6]) is finite if

$$s_A + s_B > 2 - s_A,$$

as desired.

3. Proof of Theorem 1.3

We improve the upper bound of (1.2) to prove the theorem. More precisely, under the assumptions of Theorem 1.3, there exists $\delta = \delta(C_{\nu_A}) > 0$ such that

$$\int_{S^1} |\hat{\mu}(t\omega)|^2 \, d\omega \lesssim t^{-\alpha-\delta},$$

where $\mu = \nu_A \times \nu_B$. First we deal with the case when θ is close to 0. We have

$$\int_0^\delta |\widehat{\nu}_A(t\cos(\theta))|^2 |\widehat{\nu}_B(t\sin(\theta))|^2 \, d\theta \leq \int |\widehat{\nu}_B(tu)|^2 \widehat{\psi}(u/\delta) \, du \tag{3.1}$$

with an appropriately chosen cut-off function ψ. This expression equals

$$\delta \iint \psi(\delta t(u-v)) \, d\nu_B(u) \, d\nu_B(v) \leq C\delta^{1-\alpha} \cdot t^{-\alpha}.$$

Choosing $\delta = t^{-\gamma_0}$, where γ_0 is a small positive number to be determined later, we see that the expression in (3.1) is

$$\leq Ct^{-\gamma_0(1-\alpha)} \cdot t^{-\alpha}. \tag{3.2}$$

We can deal with the neighborhood near $\frac{\pi}{2}$ in the same way, so we omit this part of the calculation.

Now consider

$$\int_I |\widehat{\nu}_A(t\cos(\theta))|^2 |\widehat{\nu}_B(t\sin(\theta))|^2 \, d\theta, \tag{3.3}$$

where I is an interval that excludes both $(0, t^{-\gamma_0})$ and a fixed neighborhood of $\frac{\pi}{2}$. By Cauchy–Schwartz, this expression (3.3) is bounded by

$$C\bigg(\int_I |\widehat{\nu}_A(t\cos(\theta))|^4 \, d\theta\bigg)^{\frac{1}{2}} \cdot \bigg(\int_I |\widehat{\nu}_B(t\sin(\theta))|^4 \, d\theta\bigg)^{\frac{1}{2}}.$$

Making the change of variables $u = \cos(\theta)$ and $u = \sin(\theta)$, respectively, we see that this expression is

$$\leq Ct^{\gamma_0}\bigg(\int |\widehat{\nu}_A(tu)|^4 \widehat{\psi}(u) \, du\bigg)^{\frac{1}{2}} \cdot \bigg(\int |\widehat{\nu}_B(tu)|^4 \widehat{\psi}(u) \, du\bigg)^{\frac{1}{2}} = Ct^{\gamma_0}\sqrt{I} \cdot \sqrt{II},$$

where ψ is a smooth positive function whose Fourier transform has compact support.

Expanding each expression and changing the order of integration, we obtain

$$\begin{aligned} I &= \int\int\int\int \psi(t(u_1 - u_2 + u_3 - u_4)) d\nu_A(u_1) d\nu_A(u_2) \, d\nu_A(u_3) \, d\nu_A(u_4) \\ &\lesssim \nu_A \times \nu_A \times \nu_A \times \nu_A\{(u_1, u_2, u_3, u_4) \in A^4 \colon |(u_1 + u_2) - (u_3 + u_4)| < t^{-1}\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} II &= \int\int\int\int \psi(t(u_1 - u_2 + u_3 - u_4)) d\nu_B(u_1) d\nu_B(u_2) \, d\nu_B(u_3) \, d\nu_B(u_4) \\ &\lesssim \nu_B \times \nu_B \times \nu_B \times \nu_B\{(u_1, u_2, u_3, u_4) \in B^4 \colon |(u_1 + u_2) - (u_3 + u_4)| < t^{-1}\}. \end{aligned} \tag{3.5}$$

Observe that we trivially have

$$I \lesssim t^{-\alpha}; \quad II \lesssim t^{-\alpha}. \tag{3.6}$$

It follows that

$$Ct^{\frac{\gamma_0}{2}}\sqrt{I} \cdot \sqrt{II} \leq Ct^{-\alpha} \leq Ct^{\frac{\gamma_0}{2}} \cdot t^{-\frac{dim_{\mathcal{H}}(A \times B)}{2}},$$

which recovers Wolff's $\frac{4}{3}$ exponent as $\gamma_0 \to 0$. Moreover, the only way this estimate does not beat $\frac{4}{3}$ is if

$$I, II \geq Ct^{-\alpha+\frac{\gamma_0}{2}} \tag{3.7}$$

for a sequence of t's going to infinity. The following theorem due to Dyatlov and Zahl ([1]) shows that this cannot happen for Ahfors–David regular sets.

Definition 3.1. [1, Dyatlov and Zahl] Let $X \subset [0,1]^d$ and ν be an outer measure on X with $0 < \nu(X) < \infty$. For $r > 0$, define the (scale r) additive energy by

$$\mathcal{E}(X,\nu,r) = \nu \times \nu \times \nu \times \nu\{(u_1,u_2,u_3,u_4) \in X^4 \colon |(u_1+u_2)-(u_3+u_4)| < r\}.$$

Theorem 3.2. [1, Dyatlov and Zahl] *Let $X \subset [0,1]$ be an Ahlfors–David regular set of Hausdorff dimension α and ν be a measure on X such that for some constant $0 < C_\nu < \infty$,*

$$C_\nu^{-1} r^\alpha < \nu(B(x,r)) < C_\nu r^\alpha,\ \forall x \in X,\ 0 < r < 1.$$

Then

$$\mathcal{E}(X,\nu,r) \leq \widetilde{C}\, r^{\alpha+\beta_\nu}$$

for some $\beta_\nu > 0$ and some $\widetilde{C} > 0$. In particular, we can choose

$$\beta_\nu = \alpha\, e^{-\exp[K(1+\log C_\nu)^{1/2}(1-\alpha)^{-1/2}]}$$

where K is an absolute constant; $\widetilde{C}$ depends only on α and C_ν.

From Theorem 3.2, Definition 1.2 and the trivial estimate (3.6) of II, it follows that

$$I \lesssim t^{-\alpha-\beta_{\nu_A}}, II \lesssim t^{-\alpha},$$

where β_{ν_A} is defined in Theorem 3.2. All implicit constants are finite, independent on t. Together with the estimate near $\theta = 0$ (3.2), we can bound (1.2) by

$$Ct^{-\gamma_0(1-\alpha)-\alpha} + Ct^{\frac{\gamma_0}{2}} t^{-\alpha-\gamma},$$

where $\gamma = \gamma(C_{\nu_A}) > 0$. Let $\gamma_0 > 0$ be a small enough, we get

$$\int_{S^1} |\hat{\mu}(t\xi)|^2\, d\xi \lesssim t^{-\alpha-\delta}$$

for some $\delta = \delta(C_{\nu_A}) > 0$.

4. Proof of Theorem 1.4

Let ν_j denote the restriction of the s_j-dimensional Hausdorff measure to A_j and assume without loss of generality that $s_1 \geq s_2 \geq \cdots \geq s_d$. Parameterize the upper half-sphere in the form

$$\left\{\left(u_1, u_2, \ldots, u_{d-1}, \sqrt{1-u_1^2-\cdots-u_{d-1}^2}\right) :\ -1 \leq u_j \leq 1\right\}.$$

Let μ denote the product measure on E, θ_ω be the angle between the vector $\omega \in S^{d-1}$ and the hyperplane $\{x_d = 0\}$. Without loss of generality, we may assume

$$\mu(\{(x_1,\ldots,x_d)\colon x_d > a\}), \mu(\{(x_1,\ldots,x_d)\colon x_d < a\}) > 0$$

for some $a \in \mathbf{R}$. An argument identical to the one in the proof of Lemma 2.2 shows the finiteness of

$$\int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 |\sin\theta_\omega|\, d\omega\right)^2 t^{d-1}\, dt \tag{4.1}$$

implies that the distance set has positive Lebesgue measure. It follows that

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 |\sin\theta_\omega| \, d\omega$$

is bounded by two terms of the form

$$\int_{-1}^{1} \cdots \int_{-1}^{1} |\widehat{\nu}_1(tu_1)|^2 \cdots |\widehat{\nu}_{d-1}(tu_{d-1})|^2 \cdot \left|\widehat{\nu}_d\left(\pm t\sqrt{1-|u|^2}\right)\right|^2 du_1 \cdots du_{d-1}$$
$$\leq \int_{-1}^{1} \cdots \int_{-1}^{1} |\widehat{\nu}_1(tu_1)|^2 \cdots |\widehat{\nu}_{d-1}(tu_{d-1})|^2 \, du_1 \cdots du_{d-1}.$$

By Lemma 2.1, this quantity is

$$\leq Ct^{-s_1-\ldots-s_{d-1}} \leq Ct^{-s\frac{d-1}{d}},$$

where $s = \sum_{j=1}^{d} s_j$.

Plugging this estimate into the Mattila integral (4.1) we obtain

$$C\int\int |\widehat{\mu}(t\omega)|^2 t^{d-1} \cdot t^{-s\frac{d-1}{d}} \, d\omega \, dt = C\int |\widehat{\mu}(\xi)|^2 |\xi|^{-s\frac{d-1}{d}} \, d\xi$$

and this integral is finite if

$$d - s < s\frac{d-1}{d},$$

which is the case if

$$s > \frac{d^2}{2d-1},$$

as desired.

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