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CONVERGENCE RESULTS FOR FAMILIES OF UNIVALENT MAPPINGS ON THE UNIT BALL IN $\mathbf{C}^n$

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Abstract. Let $\widetilde{S}_A^t(\mathbf{B}^n)$ be the family of normalized univalent mappings on the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, which have generalized parametric representation with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is a family of measurable mappings from $[0,\infty)$ into $L(\mathbf{C}^n)$ with some particular properties. Also, let $\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T_0,T]})$ be the time-T-reachable family of the control system $\mathcal{C}([T_0,T], (\mathcal{N}_{A(t)})_{t\in[T_0,T]})$, where $A \in \widetilde{\mathcal{A}}$ and $T_0 \geq 0$. In this paper we obtain certain convergence results for the families $\widetilde{S}_A^t(\mathbf{B}^n)$ and $\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T_0,T]})$ with respect to the Hausdorff metric ρ on $H(\mathbf{B}^n)$. These results may be seen as dominated convergence type theorems for time-dependent operators $A \in \widetilde{\mathcal{A}}$. In particular, we obtain related convergence results for the family $S^0_{\mathbf{A}}(\mathbf{B}^n)$ (resp. for the family $\widehat{S}_{\mathbf{A}}(\mathbf{B}^n)$) of mappings with $\mathbf{A}$-parametric representation on $\mathbf{B}^n$ (resp. of spirallike mappings on $\mathbf{B}^n$ with respect to $\mathbf{A}$), in the case that $\mathbf{A} \in L(\mathbf{C}^n)$ is a linear operator with $k_+(\mathbf{A}) < 2m(\mathbf{A})$, where $k_+(\mathbf{A})$ is the Lyapunov index of $\mathbf{A}$ and $m(\mathbf{A}) = \min_{\|z\|=1} \Re\langle \mathbf{A}(z), z\rangle$. We also obtain a convergence result for the Carathéodory family $\mathcal{N}_{\mathbf{A}}$, where $m(\mathbf{A}) > 0$. Finally, we obtain some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$, which yield the equality $\widetilde{S}_A^t(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, for all $t \geq 0$, where $S^0(\mathbf{B}^n)$ is the family of normalized univalent mappings with usual parametric representation on $\mathbf{B}^n$. Certain consequences are also provided.

1. Introduction

Since the early works devoted to Loewner chains and the Loewner differential equation in higher dimensions due to Pfaltzgraff [27] and Poreda [28, 29], many results in this field have been obtained (see [1, 5, 6, 9, 11, 13, 14, 15, 20, 21, 35]). We also mention the main contributions of Bracci [5] related to the existence of bounded support points for the family $S^0(\mathbf{B}^n)$, $n \geq 2$, and of Roth [31] concerning the n-dimensional version of the well known Pontryagin maximum principle. Other recent contributions in the Loewner theory in $\mathbf{C}^n$ may be found in [2, 3, 4, 7, 16, 17, 23, 24, 25, 32].

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Let $\widetilde{\mathcal{A}}$ be the family of all measurable mappings $A\colon [0,\infty) \to L(\mathbf{C}^n)$, which satisfy the following conditions:

(i) $m(A(\tau)) \geq 0$, for a.e. $\tau \geq 0$;
(ii) $\operatorname{ess\,sup}_{s\geq 0}\|A(s)\| < \infty$;
(iii) $\sup_{s\geq 0}\int_s^\infty \|V(s,t)^{-1}\| e^{-2\int_s^t m(A(\tau))\,d\tau}\,dt < \infty$, where $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (2.1).

The authors in [22] have investigated various extremal properties of compact families $\widetilde{S}_A^t(\mathbf{B}^n)$ $(t \geq 0)$ consisting of normalized biholomorphic mappings on the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$ which have generalized parametric representation with respect to time-dependent linear operators $A \in \widetilde{\mathcal{A}}$. We have considered examples and applications which yield that the study of the family $\widetilde{S}_A^t(\mathbf{B}^n)$ for time-dependent operators $A \in \widetilde{\mathcal{A}}$ is basically different from that in the case of constant time-dependent linear operators (see [22]). In the case that $A(t) = \mathbf{A}$, for all $t \geq 0$, where $\mathbf{A} \in L(\mathbf{C}^n)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$, then $\widetilde{S}_A^t(\mathbf{B}^n) = S_{\mathbf{A}}^0(\mathbf{B}^n)$, for all $t \geq 0$, where $S_{\mathbf{A}}^0(\mathbf{B}^n)$ is the family of mappings with $\mathbf{A}$-parametric representation (see [13]). Note that $k_+(\mathbf{A})$ is the Lyapunov index of $\mathbf{A}$ and $m(\mathbf{A}) = \min_{\|z\|=1}\Re\langle \mathbf{A}(z), z\rangle$. If $n = 1$ and $a \in \widetilde{A}$, then $\widetilde{S}_a^t(\mathbf{U}) = S$, for all $t \geq 0$ (see [22]), where S is the family of normalized univalent functions on the unit disc $\mathbf{U}$.

In this paper we consider a certain dependence of the family $\widetilde{S}_A^T(\mathbf{B}^n)$ on $A \in \widetilde{\mathcal{A}}$, where $T \geq 0$. The main results of this paper can be summarized as follows. The notations in the following results will be explained in the next sections.

Theorem 1.1. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq T} m(A(t)) > 0$. Also, let $M > 0$, $\alpha \in L^1([T,\infty),\mathbf{R})$ and $(A_k)_{k\in\mathbf{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\| \leq M$ and $\|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))d\tau} \leq \alpha(t)$, for a.e. $t \geq T$ and for all $k \in \mathbf{N}$, where $V_k(T,\cdot)$ is the unique solution on $[T,\infty)$ of the initial value problem* (2.1) *related to A_k. If $\lim_{k\to\infty} A_k(t) = A(t)$ for a.e. $t \geq T$, then $\lim_{k\to\infty}\rho(\widetilde{S}_{A_k}^T(\mathbf{B}^n), \widetilde{S}_A^T(\mathbf{B}^n)) = 0$.*

Theorem 1.2. *Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and let $(\mathbf{A}_l)_{l\in\mathbf{N}}$ be a sequence in $L(\mathbf{C}^n)$ such that $\mathbf{A}_l \to \mathbf{A}$, as $l \to \infty$. Then there is $l_0 \in \mathbf{N}$ such that $S_{\mathbf{A}_l}^0(\mathbf{B}^n)$ is compact for $l \geq l_0$, and $\rho(S_{\mathbf{A}_l}^0(\mathbf{B}^n), S_{\mathbf{A}}^0(\mathbf{B}^n)) \to 0$, as $l \to \infty$.*

In view of the definition of the family $\widetilde{\mathcal{A}}$, it follows that Theorem 1.1 may be seen as a dominated convergence type theorem. In particular, we obtain a related convergence result for the compact family $\widehat{S}_{\mathbf{A}}(\mathbf{B}^n)$ consisting of spirallike mappings on $\mathbf{B}^n$ with respect to $\mathbf{A}$, in the case that $\mathbf{A} \in L(\mathbf{C}^n)$ is a constant time-dependent linear operator with $k_+(\mathbf{A}) < 2m(\mathbf{A})$. We also obtain a convergence result related to the Carathéodory family $\mathcal{N}_{\mathbf{A}}$, where $m(\mathbf{A}) > 0$.

The authors in [22] obtained extremal properties for the family $\widetilde{S}_A^t(\mathbf{B}^n)$ consisting of normalized univalent mappings on $\mathbf{B}^n$ which have generalized parametric representation with respect to time-dependent operators $A \in \widetilde{\mathcal{A}}$, and deduced certain applications by considering examples of time-dependent normalizations that are step functions. In this paper we shall apply Theorem 1.1 to obtain other results which involve time-dependent operators that are step functions. For example, in the last section we shall obtain some sufficient conditions for a time-dependent operator $A \in \widetilde{\mathcal{A}}$ such that $\widetilde{S}_A^t(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, for all $t \geq 0$, where $S^0(\mathbf{B}^n)$ is the family of normalized univalent mappings with usual parametric representation on $\mathbf{B}^n$.

2. Preliminaries

Let $\mathbf{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbf{C}^n\colon \|z\| < r\}$ is denoted by $\mathbf{B}^n_r$ and the unit ball $\mathbf{B}^n_1$ is denoted by $\mathbf{B}^n$. The closed ball $\{z \in \mathbf{C}^n\colon \|z\| \le r\}$ is denoted by $\overline{\mathbf{B}^n_r}$. In the case $n = 1$, the unit disc $\mathbf{B}^1$ is denoted by $\mathbf{U}$.

Let $L(\mathbf{C}^n)$ denote the space of linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$ with the standard operator norm. Also, let I_n be the identity operator in $L(\mathbf{C}^n)$. If $A \in L(\mathbf{C}^n)$, we denote by A^* the adjoint of the operator A. Let $H(\mathbf{B}^n)$ be the family of holomorphic mappings from $\mathbf{B}^n$ into $\mathbf{C}^n$ with the compact-open topology. If $f \in H(\mathbf{B}^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $S(\mathbf{B}^n)$ be the family of normalized biholomorphic mappings on $\mathbf{B}^n$. If $n = 1$, then the family $S(\mathbf{U})$ is denoted by S.

Next, we use the following notations for an operator $A \in L(\mathbf{C}^n)$ (see e.g. [10, 13]):

$$\begin{aligned} m(A) &= \min\{\Re\langle A(z), z\rangle\colon \|z\| = 1\},\\ k(A) &= \max\{\Re\langle A(z), z\rangle\colon \|z\| = 1\},\\ |V(A)| &= \max\{|\langle A(z), z\rangle|\colon \|z\| = 1\},\\ k_+(A) &= \max\{\Re\lambda\colon \lambda \in \sigma(A)\},\end{aligned}$$

where $\sigma(A)$ is the spectrum of A. Note that $|V(A)|$ is the numerical radius of the operator A and $k_+(A)$ is the upper exponential index (Lyapunov index) of A. Then $m(A) \le k_+(A) \le |V(A)| \le \|A\|$ (see e.g. [16]) and it is known that $\|A\| \le 2|V(A)|$ and $k_+(A) = \lim_{t\to\infty} \frac{\log\|e^{tA}\|}{t}$ (see e.g. [10]).

The following families of holomorphic mappings on the unit ball $\mathbf{B}^n$ play the role of the Carathéodory family in $\mathbf{C}^n$ (see [33]):

$$\begin{aligned} \mathcal{N} &= \{h \in H(\mathbf{B}^n)\colon h(0) = 0,\ \Re\langle h(z), z\rangle \ge 0,\ z \in \mathbf{B}^n\},\\ \widetilde{\mathcal{N}} &= \{h \in \mathcal{N}\colon \Re\langle h(z), z\rangle > 0,\ z \in \mathbf{B}^n \setminus \{0\}\}.\end{aligned}$$

If $A \in L(\mathbf{C}^n)$ with $m(A) \ge 0$, let (see e.g. [13])

$$\mathcal{N}_A = \{h \in \mathcal{N}\colon Dh(0) = A\}.$$

Also, let $\mathcal{M} = \mathcal{N}_{I_n}$. In view of the minimum principle for harmonic mappings, it is easily seen that (see [33])

$$\mathcal{M} = \{h \in \widetilde{\mathcal{N}}\colon Dh(0) = I_n\}.$$

The following growth result was obtained by Graham, Hamada, and Kohr [11] for the family $\mathcal{M}$ (see [13, Lemma 1.2] in the case of mappings $h \in \widetilde{\mathcal{N}}$; see also [34, Proposition 1.2.3] for the family $\mathcal{N}$).

Lemma 2.1. *If $h \in \mathcal{N}$, then*

$$\|h(z)\| \le \frac{4\|z\|}{(1 - \|z\|)^2}|V(Dh(0))|, \quad z \in \mathbf{B}^n.$$

Next, let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be a measurable mapping which is locally integrable on $[0,\infty)$. For every $s \ge 0$, let $V(s,\cdot)\colon [s,\infty) \to L(\mathbf{C}^n)$ be the unique locally absolutely continuous solution of the initial value problem (cf. [34])

$$\frac{\partial V}{\partial t}(s,t) = -A(t)V(s,t),\ \text{a.e. } t \in [s,\infty),\ V(s,s) = I_n. \tag{2.1}$$

Also, let $V(t) = V(0,t)$, for all $t \geq 0$. Then $V(s,t) = V(t)V(s)^{-1}$ for $0 \leq s \leq t < \infty$ (see [8]; cf. [34]).

Remark 2.2. Let $s \geq 0$. If $A(t)$ and $\int_s^t A(\tau)\,d\tau$ commute for $t \geq s$, then

$$V(s,t) = e^{-\int_s^t A(\tau)\,d\tau}, \quad \forall\, t \in [s,\infty),$$

by [8, Exercise VII.2.22].

The following estimates related to a measurable and locally integrable mapping $A\colon [0,\infty) \to L(\mathbf{C}^n)$ will be useful in the forthcoming sections (see [34, Proposition 1.2.1, Remark 1.2.2]; cf. [14, Remark 1.6 (v)]).

Lemma 2.3. *Let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be a measurable mapping that is locally integrable, and let $V(s,t)$ be the unique solution on $[s,\infty)$ of the initial value problem* (2.1) *related to A. Then*

$$e^{\int_s^t m(A(\tau))\,d\tau} \leq \left\|V(s,t)^{-1}\right\| \leq e^{\int_s^t k(A(\tau))\,d\tau}$$

and

$$e^{-\int_s^t k(A(\tau))\,d\tau} \leq \left\|V(s,t)\right\| \leq e^{-\int_s^t m(A(\tau))\,d\tau},$$

for all $t \geq s \geq 0$.

Next, we recall the notion of generalized parametric representation with respect to a time-dependent linear operator (see [22]; cf. [14, Definition 1.5], [34, Proposition 1.5.1]).

Definition 2.4. Let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be a measurable mapping, which is locally integrable, such that $m(A(t)) \geq 0$ for a.e. $t \geq 0$, and let $T \geq 0$. Also, let $V(s,t)$ be the unique solution on $[s,\infty)$ of the initial value problem (2.1) related to A. We say that a mapping $f\colon \mathbf{B}^n \to \mathbf{C}^n$ has generalized parametric representation with respect to A on $[T,\infty)$ if there exists a mapping $h = h(z,t)\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ which satisfies the following conditions:

(i) $h(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \mathbf{B}^n$;
(ii) $h(\cdot,t) \in \mathcal{N}$, for all $t \geq 0$;
(iii) $Dh(0,t) = A(t)$, for all $t \geq 0$;

and such that

$$f(z) = \lim_{t\to\infty} V(T,t)^{-1} v(z,T,t)$$

locally uniformly on $\mathbf{B}^n$, where $v(z,T,\cdot)\colon [T,\infty) \to \mathbf{C}^n$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial v}{\partial t}(z,T,t) = -h(v(z,T,t),t), \text{ a.e. } t \in [T,\infty),\ v(z,T,T) = z,$$

for all $z \in \mathbf{B}^n$. Let $\widetilde{S}_A^T(\mathbf{B}^n)$ be the family of mappings with generalized parametric representation with respect to A on $[T,\infty)$.

Obviously, $\widetilde{S}_A^T(\mathbf{B}^n) \neq \emptyset$, since $\mathrm{id}_{\mathbf{B}^n} \in \widetilde{S}_A^T(\mathbf{B}^n)$, for $T \geq 0$ and every measurable and locally integrable mapping $A\colon [0,\infty) \to L(\mathbf{C}^n)$ such that $m(A(t)) \geq 0$, for a.e. $t \geq 0$.

Definition 2.5. Let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be a measurable mapping, which is locally integrable on $[0,\infty)$, such that $m(A(t)) \geq 0$, for a.e. $t \geq 0$. A mapping $h\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ which satisfies the conditions (i)–(iii) of Definition 2.4 will be called a Herglotz vector field (or a generating vector field) with respect to A (cf. [6] and [9]).

Remark 2.6. Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $m(\mathbf{A}) > 0$ and let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be such that $A(t) = \mathbf{A}$, for all $t \geq 0$. In this case, the family $\widetilde{S}^t_A(\mathbf{B}^n)$ reduces to the family $S^0_{\mathbf{A}}(\mathbf{B}^n)$ of mappings with $\mathbf{A}$-parametric representation on $\mathbf{B}^n$, for all $t \geq 0$ (see [13]). If $\mathbf{A} = I_n$, then $S^0_{\mathbf{A}}(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, where $S^0(\mathbf{B}^n)$ is the family of mappings with the usual parametric representation on $\mathbf{B}^n$ (see [11] and [13]).

Various properties of mappings with generalized parametric representation may be found in [12], [14], and [22].

Definition 2.7. (see [33]) Let $A \in L(\mathbf{C}^n)$ be such that $m(A) > 0$. A mapping $f \in S(\mathbf{B}^n)$ is said to be spirallike with respect to A (denoted by $f \in \widehat{S}_A(\mathbf{B}^n)$) if $e^{-tA}f(\mathbf{B}^n) \subseteq f(\mathbf{B}^n)$, for all $t \geq 0$.

Next we recall the notion of a univalent subordination chain whose normalization is given by a time-dependent linear operator in $\mathbf{C}^n$ (see [14]; cf. [18, Chpater 8]).

Definition 2.8. A mapping $f\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ is called a univalent subordination chain if $f(\cdot,t)$ is univalent on $\mathbf{B}^n$, $f(0,t) = 0$ for $t \geq 0$, and $f(\mathbf{B}^n, s) \subseteq f(\mathbf{B}^n, t)$ for $0 \leq s \leq t$. If, in addition, $Df(0,t) = V(t)^{-1}$ for $t \geq 0$, and $\{V(t)f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbf{B}^n$, then we say that f is a normal Loewner chain with respect to A, where $A\colon [0,\infty) \to L(\mathbf{C}^n)$ is a measurable and locally integrable mapping and $V(t) = V(0,t)$ is the unique solution on $[0,\infty)$ of the initial value problem (2.1) related to A.

Note that if $f = f(z,t)$ is a univalent subordination chain, then there exists a unique univalent Schwarz mapping $v = v(z,s,t)$, called the transition mapping associated with f, such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in \mathbf{B}^n,\ 0 \leq s \leq t < \infty.$$

The family $(v_{s,t})$ is also called the evolution family associated with $f(z,t)$, where $v_{s,t}(z) = v(z,s,t)$ (cf. [6]).

Remark 2.9. It is easily seen that if $\mathbf{A} \in L(\mathbf{C}^n)$ and $f \in H(\mathbf{B}^n)$ is a normalized mapping, then $f \in \widehat{S}_{\mathbf{A}}(\mathbf{B}^n)$ if and only if $f(z,t) = e^{t\mathbf{A}}f(z)$ is a normal Loewner chain with respect to $\mathbf{A}$ (see [13]).

In this paper we are concerned with normal Loewner chains whose normalizations depend on operators $A \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is the family of operators $A\colon [0,\infty) \to L(\mathbf{C}^n)$ given in Definition 2.10 below (see [22]):

Definition 2.10. Let $\widetilde{\mathcal{A}}$ be the family of all measurable mappings $A\colon [0,\infty) \to L(\mathbf{C}^n)$, which satisfy the following conditions:

(i) $m(A(\tau)) \geq 0$, for a.e. $\tau \geq 0$;
(ii) $\operatorname{ess\,sup}_{s\geq 0}\|A(s)\| < \infty$;
(iii) $\sup_{s\geq 0}\int_s^\infty \|V(s,t)^{-1}\|e^{-2\int_s^t m(A(\tau))\,d\tau}\,dt < \infty$,

where $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (2.1) related to A.

Remark 2.11. Let $T > 0$, $\mathbf{A} \in L(\mathbf{C}^n)$ and let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be such that $m(A(t)) \geq 0$, for a.e. $t \in [0,T]$, $\operatorname{ess\,sup}_{t\in[0,T]}\|A(t)\| < \infty$ and $A(t) = \mathbf{A}$, for a.e. $t > T$. Then $A \in \widetilde{\mathcal{A}}$ if and only if $k_+(\mathbf{A}) < 2m(\mathbf{A})$, by Lemma 2.3, [9, Remark 2.8] and [14, Remark 2.2]. In particular, $I_n \in \widetilde{\mathcal{A}}$.

Remark 2.12. (i) The authors in [22] proved that if $T \geq 0$, $A \in \widetilde{\mathcal{A}}$, and $g \in H(\mathbf{B}^n)$ is a normalized mapping, then $g \in \widetilde{S}_A^T(\mathbf{B}^n)$ if and only if there exists a normal Loewner chain $f = f(z,t)$ with respect to A such that $g = V(T)f(\cdot,T)$, where $V(t) = V(0,t)$ is the unique locally absolutely continuous solution on $[0,\infty)$ of the initial value problem (2.1) related to A. In particular, if $\mathbf{A} \in L(\mathbf{C}^n)$ is a constant time-dependent operator such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, then $f \in S^0_{\mathbf{A}}(\mathbf{B}^n)$ if and only if there is a normal Loewner chain $f(z,t)$ with respect to $\mathbf{A}$ such that $f = f(\cdot,0)$ (see [13]).

(ii) If $\mathbf{A} \in L(\mathbf{C}^n)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$, then $\widehat{S}_{\mathbf{A}}(\mathbf{B}^n)$ is a compact family in $H(\mathbf{B}^n)$ (see [35]; cf. [13]) and $\widehat{S}_{\mathbf{A}}(\mathbf{B}^n) \subset S^0_{\mathbf{A}}(\mathbf{B}^n)$ (see [13] and [35]).

The results contained in Proposition 2.13 and Lemma 2.14 were obtained in [22].

Proposition 2.13. *Let $a\colon [0,\infty) \to \mathbf{R}$ be a measurable function such that*

$$\operatorname{ess\,inf}_{t\geq 0} a(t) > 0 \quad \text{and} \quad \operatorname{ess\,sup}_{t\geq 0} a(t) < \infty. \tag{2.2}$$

Also, let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and let $A\colon [0,\infty) \to L(\mathbf{C}^n)$ be given by $A(t) = a(t)\mathbf{A}$ for a.e. $t \geq 0$. Then $A \in \widetilde{\mathcal{A}}$ and $\widetilde{S}_A^T(\mathbf{B}^n) = S^0_{\mathbf{A}}(\mathbf{B}^n)$ for $T \geq 0$.

Lemma 2.14. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$. Also, let f be a normal Loewner chain with respect to A, and let v be the transition mapping associated with f. If $h \in \widetilde{S}_A^T(\mathbf{B}^n)$, then $V(t,T)^{-1}h(v(\cdot,t,T)) \in \widetilde{S}_A^t(\mathbf{B}^n)$, for all $t \in [0,T]$. In particular, $V(t,T)^{-1}v(\cdot,t,T) \in \widetilde{S}_A^t(\mathbf{B}^n)$, for all $t \in [0,T]$, where $V(t) = V(0,t)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (2.1) related to A.*

Next, we mention the following growth result for the transition mappings of normal Loewner chains with respect to $A \in \widetilde{\mathcal{A}}$ (see the proof of [22, Proposition 3.10]; cf. [34, Proposition 1.5.2]).

Lemma 2.15. *Let $A \in \widetilde{\mathcal{A}}$ and let f be a normal Loewner chain with respect to A. Also, let v be the transition mapping associated to f. Then for every $r \in (0,1)$, there exists some $C_r > 0$ such that*

$$\begin{aligned}&\big\|V(s,t_2)^{-1}v(z,s,t_2) - V(s,t_1)^{-1}v(z,s,t_1)\big\| \\ &\leq C_r \int_{t_1}^{t_2} \|A(t)\|\|V(s,t)^{-1}\|e^{-2\int_s^t m(A(\tau))\,d\tau}\,dt,\end{aligned}$$

for all $z \in \overline{\mathbf{B}^n_r}$, $s \geq 0$ and $s \leq t_1 < t_2 \leq \infty$, where $V(s) = V(0,s)$ and $V(s,t)$ is the unique solution on $[s,\infty)$ of the initial value problem (2.1) related to A.

We recall the following definitions that have been recently introduced in [22] (cf. [15, 16, 24, 25]; cf. [30], in the case $n = 1$).

Definition 2.16. Let I be an interval and $A \in \widetilde{\mathcal{A}}$. A mapping $h\colon \mathbf{B}^n \times I \to \mathbf{C}^n$ is called a Carathéodory mapping on I with respect to A if the following conditions hold:

(i) $h(\cdot,t) \in \mathcal{N}_{A(t)}$, for all $t \in I$;
(ii) $h(z,\cdot)$ is measurable on I, for all $z \in \mathbf{B}^n$.

Let $\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$ denote the family of Carathéodory mappings on I with respect to A. We say that the Carathéodory mappings on I with respect to A represent the *controls* of the *control system* $\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$, and $(\mathcal{N}_{A(t)})_{t\in I}$ represents the *input family.*

Definition 2.17. Let I be either the interval $[T_0,T_1]$, where $T_1>T_0\geq 0$, or the interval $[T_0,\infty)$, where $T_0\geq 0$, and $A\in\widetilde{\mathcal{A}}$. For every $h\in\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$ we denote by $v(z,T_0,\cdot;h)$ the unique locally absolutely continuous solution on I of the initial value problem

$$\begin{cases}\dfrac{\partial v}{\partial t}(z,T_0,t;h)=-h(v(z,T_0,t;h),t), & \text{for a.e. } t\in I,\\ v(z,T_0,T_0;h)=z, \end{cases}$$

for all $z\in\mathbf{B}^n$.

Note that $v(\cdot,T_0,t;h)$ is a univalent Schwarz mapping with $Dv(0,T_0,t;h)=V(T_0,t)$, for all $t\in I$ (cf. [34] and [22]), where $V(T_0,\cdot)$ is the unique solution on $[T_0,\infty)$ of the initial value problem (2.1) related to A.

Now, we consider the notion of the reachable family with respect to time-dependent linear operators (see [22]).

Definition 2.18. Let $T_0\geq 0$ and $A\in\widetilde{\mathcal{A}}$. For every $T>T_0$ we denote the *normalized time-T-reachable family* of the *control system* $\mathcal{C}\big([T_0,T],(\mathcal{N}_{A(t)})_{t\in[T_0,T]}\big)$ by

$$\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n},(\mathcal{N}_{A(t)})_{t\in[T_0,T]})=\Big\{V(T_0,T)^{-1}v(\cdot,T_0,T;h)\colon h\in\mathcal{C}\big([T_0,T],(\mathcal{N}_{A(t)})_{t\in[T_0,T]}\big)\Big\}.$$

We also denote the *normalized infinite-time-reachable family* of the *control system* $\mathcal{C}\big([T_0,\infty),(\mathcal{N}_{A(t)})_{t\geq T_0}\big)$ by

$$\widetilde{\mathcal{R}}_\infty(\mathrm{id}_{\mathbf{B}^n},(\mathcal{N}_{A(t)})_{t\geq T_0})=\Big\{\lim_{t\to\infty}V(T_0,t)^{-1}v(\cdot,T_0,t;h)\colon h\in\mathcal{C}\big([T_0,\infty),(\mathcal{N}_{A(t)})_{t\geq T_0}\big)\Big\}.$$

Remark 2.19. In view of Definition 2.4 and Lemma 2.14 (ii), we have that $\widetilde{\mathcal{R}}_\infty(\mathrm{id}_{\mathbf{B}^n},(\mathcal{N}_{A(t)})_{t\geq T_0})=\widetilde{S}_A^{T_0}(\mathbf{B}^n)$ and $\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n},(\mathcal{N}_{A(t)})_{t\in[T_0,T]})\subseteq\widetilde{S}_A^{T_0}(\mathbf{B}^n)$, for all $T\in(T_0,\infty)$ (see [22]).

Using arguments similar to those in the proofs of [16, Lemmas 4.12 and 4.13] (see [30, Theorem I.29, Lemma I.37] and [25, Proposition 2.3, Lemmas 3.1 and 3.2]), we obtain the following lemmas. We omit the proofs of Lemmas 2.20 and 2.21.

Lemma 2.20. *Let I be the interval $[T_0,T]$, where $T>T_0\geq 0$, and let $A\in\widetilde{\mathcal{A}}$. Also, let $(h_k)_{k\in\mathbf{N}}$ be a sequence in $\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$. Then there exist a subsequence $(h_{k_m})_{m\in\mathbf{N}}$ of $(h_k)_{k\in\mathbf{N}}$ and $h\in\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$ such that*

$$\int_{T_0}^t h_{k_m}(v(\cdot,T_0,\tau;h),\tau)\,d\tau\to\int_{T_0}^t h(v(\cdot,T_0,\tau;h),\tau)\,d\tau,\quad \text{as } m\to\infty,$$

locally uniformly on $\mathbf{B}^n$, for all $t\in I$.

Lemma 2.21. *Let I be the interval $[T_0,T]$, where $T>T_0\geq 0$, let $A\in\widetilde{\mathcal{A}}$, $M>0$, and let $(A_k)_{k\in\mathbf{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\|\leq M$, for a.e. $t\in I$ and for all $k\in\mathbf{N}$. Let $h\in\mathcal{C}(I,(\mathcal{N}_{A(t)})_{t\in I})$ and $(h_k)_{k\in\mathbf{N}}$ be a sequence such that $h_k\in\mathcal{C}(I,(\mathcal{N}_{A_k(t)})_{t\in I})$, for $k\in\mathbf{N}$, and*

$$\int_{T_0}^t h_k(v(\cdot,T_0,\tau;h),\tau)\,d\tau\to\int_{T_0}^t h(v(\cdot,T_0,\tau;h),\tau)\,d\tau,\quad \text{as } k\to\infty,$$

locally uniformly on $\mathbf{B}^n$, for all $t\in I$. Then

$$v(\cdot,T_0,t;h_k)\to v(\cdot,T_0,t;h),\quad \text{as } k\to\infty,$$

locally uniformly on $\mathbf{B}^n$, for all $t\in I$.

Remark 2.22. Recently, the authors [22] proved that if $T_0 \geq 0$ and $A \in \widetilde{\mathcal{A}}$, then $\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T_0,T]})$ is a compact family, for all $T > T_0$. Moreover, the family $\widetilde{\mathcal{R}}_\infty(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\geq T_0})$ is also compact.

Now, we give the definition of the Hausdorff metric on $H(\mathbf{B}^n)$ (cf. [30]).

Definition 2.23. Let δ be the well known metric on $H(\mathbf{B}^n)$ such that $(H(\mathbf{B}^n), \delta)$ is a Fréchet space with respect to the compact-open topology. For all nonempty subsets V and W of $H(\mathbf{B}^n)$, let

$$\delta(V, W) = \sup_{f\in V} \inf_{g\in W} \delta(f, g).$$

Also, let ρ be the Hausdorff metric on $H(\mathbf{B}^n)$ given by

$$\rho(V, W) = \max\{\delta(V, W), \delta(W, V)\},$$

for all nonempty compact subsets V and W of $H(\mathbf{B}^n)$.

We close this section with the notions of extreme/support points associated with compact subsets of $H(\mathbf{B}^n)$ (see e.g. [8], [30]).

Definition 2.24. Let $E \subseteq H(\mathbf{B}^n)$ be a nonempty compact set.

(i) A point $f \in E$ is called an *extreme point* of E (denoted by $f \in \operatorname{ex} E$) if $f = \lambda g + (1-\lambda)h$, for some $\lambda \in (0, 1)$, $g, h \in E$, implies that $f \equiv g \equiv h$.

(ii) A point $f \in E$ is called a *support point* of E (denoted by $f \in \operatorname{supp} E$) if there exists a continuous linear functional $L \colon H(\mathbf{B}^n) \to \mathbf{C}$ such that $\Re L$ is nonconstant on E and $\Re L(f) = \max_{g\in E} \Re L(g)$.

Remark 2.25. Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$. In view of [13, Theorem 2.15], the family $S^0_{\mathbf{A}}(\mathbf{B}^n)$ is compact. Thus $\operatorname{ex} S^0_{\mathbf{A}}(\mathbf{B}^n) \neq \emptyset$ and $\operatorname{supp} S^0_{\mathbf{A}}(\mathbf{B}^n) \neq \emptyset$.

3. Convergence results for $\widetilde{S}^T_A(\mathbf{B}^n)$ and for reachable families generated by time-dependent operators

In this section we consider a dependence of $\widetilde{S}^T_A(\mathbf{B}^n)$ on $A \in \widetilde{\mathcal{A}}$, where $T \geq 0$ (cf. [22, Proposition 3.15]; cf. [30] for $n = 1$). Note that the following results may be seen as *dominated convergence* type theorems. In the next section we shall apply Theorem 3.2 to obtain other results which involve time-dependent operators that are step functions (cf. Propositions 4.1 and 4.3).

Theorem 3.1. *Let I be the interval $[T_0, T]$, where $T > T_0 \geq 0$, and $A \in \widetilde{\mathcal{A}}$ be such that* $\operatorname{ess\,inf}_{t\in I} m(A(t)) > 0$. *Also, let $M > 0$ and let $(A_k)_{k\in\mathbf{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\| \leq M$, for a.e. $t \in I$ and for all $k \in \mathbf{N}$. If*

$$A_k(t) \to A(t), \quad \text{as } k \to \infty, \text{ for a.e. } t \in I,$$

then

$$\rho(\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in I}), \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})) \to 0, \quad \text{as } k \to \infty.$$

Proof. First, we prove that $\delta(\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in I}), \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})) \to 0$, as $k \to \infty$. Suppose that there are $\varepsilon > 0$ and a nondecreasing sequence of indices $(k_m)_{m\in\mathbf{N}}$ such that for every $m \in \mathbf{N}$ we have

$$\delta(\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_{k_m}(t)})_{t\in I}), \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})) > \varepsilon$$

i.e. for every $m \in \mathbf{N}$ there exists $f_m \in \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_{k_m}(t)})_{t\in I})$ such that for every $g \in \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})$ we have $\delta(f_m, g) > \varepsilon$.

Let $m \in \mathbf{N}$ be arbitrary. Since $f_m \in \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_{k_m}(t)})_{t\in I})$, there exists $h_m \in \mathcal{C}\big(I, (\mathcal{N}_{A_{k_m}(t)})_{t\in I}\big)$ such that $f_m = V_m(T_0, T)^{-1} v(\cdot, T_0, T; h_m)$, where $V_m(T_0, \cdot)$ is the unique solution on $[T_0, \infty)$ of the initial value problem (2.1) related to A_{k_m}. By [19, Lemma 3], for every $r \in (0,1)$ we have

$$\Re\Big\langle \frac{1}{r} h_m(rz,t) - \big(A_{k_m}(t)z - A(t)z\big), z\Big\rangle \geq \Big(m(A_{k_m}(t))\frac{1-r}{1+r} - \big\|A_{k_m}(t) - A(t)\big\|\Big)\|z\|^2,$$

$z \in \mathbf{B}^n$, $t \in I$. For every $l \in \mathbf{N}$, by Egorov's Theorem, there exists a measurable set $N_l \subset I$ such that $\lambda(N_l) \leq \frac{1}{l}$ and $(A_{k_m})_{m\in\mathbf{N}}$ converges to A uniformly on $I \setminus N_l$. Since $\operatorname{ess\,inf}_{t\in I} m(A(t)) > 0$, we deduce that for every $l \in \mathbf{N}$ there is $m_l \in \mathbf{N}$ such that $q_l \colon \mathbf{B}^n \times I \to \mathbf{C}^n$ given by

$$q_l(z,t) = \begin{cases} \frac{1}{r_l} h_{m_l}(r_l z, t) - A_{k_{m_l}}(t)z + A(t)z, & t \in I \setminus N_l \\ A(t)z, & t \in N_l \end{cases},$$

for all $z \in \mathbf{B}^n$, satisfies $q_l \in \mathcal{C}\big(I, (\mathcal{N}_{A(t)})_{t\in I}\big)$, where $r_l = 1 - \frac{1}{l}$.

For every $l \in \mathbf{N}$ and $t \in I$ the following equality holds:

$$(3.1)\quad \frac{1}{r_l} h_{m_l}(r_l z, t) - h_{m_l}(z,t) = \frac{r_l - 1}{r_l}\int_0^1 (Dh_{m_l}(\tau r_l z + (1-\tau)z, t)(z) - h_{m_l}(z,t))\, d\tau.$$

Since $\|A_{k_{m_l}}(t)\| \leq M$, for a.e. $t \in I$ and for all $l \in \mathbf{N}$, we deduce in view Lemma 2.1 that there is a null set $J \subseteq I$ such that $\{h_{m_l}(\cdot, t)\}_{t\in I\setminus J, l\in\mathbf{N}}$ is a normal family. Hence, using (3.1) and the fact that $r_l \to 1$ as $l \to \infty$, we obtain for a.e. $t \in I$ that

$$(3.2)\qquad q_l(\cdot, t) - h_{m_l}(\cdot, t) \to 0, \quad \text{as } l \to \infty, \text{ locally uniformly on } \mathbf{B}^n.$$

Using Lemma 2.20, we deduce that there is $q \in \mathcal{C}\big(I, (\mathcal{N}_{A(t)})_{t\in I}\big)$ such that up to a subsequence, we have

$$(3.3)\qquad \int_{T_0}^t q_l(v(\cdot, T_0, \tau; q), \tau)\, d\tau \to \int_{T_0}^t q(v(\cdot, T_0, \tau; q), \tau)\, d\tau, \quad \text{as } l \to \infty,$$

locally uniformly on $\mathbf{B}^n$, for all $t \in I$. Since $\operatorname{ess\,sup}_{t\in I}\|A(t)\| < \infty$, we deduce by Lemma 2.1 that there is a null set $J' \subseteq I$ such that $\{q_l(\cdot, t)\}_{t\in I\setminus J', l\in\mathbf{N}}$ is a normal family. Hence, in view of (3.2), (3.3) and the Lebesgue dominated convergence theorem, we obtain that

$$(3.4)\qquad \int_{T_0}^t h_{m_l}(v(\cdot, T_0, \tau; q), \tau)\, d\tau \to \int_{T_0}^t q(v(\cdot, T_0, \tau; q), \tau)\, d\tau, \quad \text{as } l \to \infty,$$

locally uniformly on $\mathbf{B}^n$, for all $t \in I$. In view of (3.3) and (3.4), we apply Lemma 2.21 to deduce that

$$(3.5)\quad v(\cdot, T_0, T; h_{m_l}) \to v(\cdot, T_0, T; q) \text{ and } v(\cdot, T_0, T; q_l) \to v(\cdot, T_0, T; q), \text{ as } l \to \infty,$$

locally uniformly on $\mathbf{B}^n$.

Let $V(s,t)$ be the unique solution on $[s, \infty)$ of the initial value problem (2.1) related to A. In view of (3.5) and Weierstrass' convergence theorem, we deduce that $Dv(0, T_0, T; h_{m_l}) \to Dv(0, T_0, T; q)$, as $l \to \infty$. Since $Dv(0, T_0, T; h_{m_l}) = V_{m_l}(T_0, T)$, for $l \in \mathbf{N}$, and $Dv(0, T_0, T; q) = V(T_0, T)$ (see [22]), it follows that $V_{m_l}(T_0, T) \to V(T_0, T)$, as $l \to \infty$. Since $V(T_0, T)$ and $V_{m_l}(T_0, T)$, for $l \in \mathbf{N}$, are invertible operators, it is easy to prove that (cf. [9, Theorem 2.17])

$$(3.6)\qquad V_{m_l}(T_0, T)^{-1} \to V(T_0, T)^{-1}, \quad \text{as } l \to \infty.$$

Let $g = V(T_0,T)^{-1}v(\cdot,T_0,T;q)$ and $g_l = V(T_0,T)^{-1}v(\cdot,T_0,T;q_l)$, for $l \in \mathbf{N}$. In view of (3.5) and (3.6), we deduce that $g_l \to g$ and $f_{m_l} \to g$ locally uniformly on $\mathbf{B}^n$, as $l \to \infty$. Hence $\delta(f_{m_l}, g_l) \to 0$, as $l \to \infty$. However, this is a contradiction, since $g_l \in \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I})$, for all $l \in \mathbf{N}$.

To prove that $\delta(\widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in I}), \widetilde{\mathcal{R}}_T(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in I})) \to 0$, as $k \to \infty$, it suffices to use similar arguments as before. This completes the proof. □

Theorem 3.2. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq T} m(A(t)) > 0$. Also, let $M > 0$, $\alpha \in L^1([T,\infty),\mathbf{R})$ and $(A_k)_{k\in\mathbf{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\| \leq M$ and $\|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))d\tau} \leq \alpha(t)$, for a.e. $t \geq T$ and for all $k \in \mathbf{N}$, where $V_k(T,\cdot)$ is the unique solution on $[T,\infty)$ of the initial value problem (2.1) related to A_k. If*

$$A_k(t) \to A(t), \quad \text{as } k \to \infty, \text{ for a.e. } t \geq T,$$

then

$$\rho(\widetilde{S}^T_{A_k}(\mathbf{B}^n), \widetilde{S}^T_A(\mathbf{B}^n)) \to 0, \quad \text{as } k \to \infty.$$

Proof. First, we prove that for every sequence $(T_k)_{k\in\mathbf{N}}$ in (T,∞) such that $T_k \to \infty$, as $k \to \infty$, we have

$$\rho(\widetilde{\mathcal{R}}_{T_k}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in[T,T_k]}\big), \widetilde{S}^T_{A_k}(\mathbf{B}^n)) \to 0, \text{ as } k \to \infty. \tag{3.7}$$

Fix a sequence $(T_k)_{k\in\mathbf{N}}$ in (T,∞) such that $T_k \to \infty$, as $k \to \infty$. Let $k \in \mathbf{N}$ and $f_k \in \widetilde{S}^T_{A_k}(\mathbf{B}^n)$ be arbitrary. Then there is $h_k \in \mathcal{C}([T,\infty), (\mathcal{N}_{A_k(t)})_{t\geq T})$ such that $f_k = \lim_{t\to\infty} V_k(T,t)^{-1}v(\cdot,T,t;h_k)$. Let $g_k = V_k(T,T_k)^{-1}v(\cdot,T,T_k;h_k)$. Then $g_k \in \widetilde{\mathcal{R}}_{T_k}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in[T,T_k]}\big)$. Taking into account Lemma 2.15, we deduce that for every $r \in (0,1)$ there is $C_r > 0$ such that

$$\|f_k - g_k\|_{\overline{\mathbf{B}^n_r}} \leq C_r \int_{T_k}^\infty \|A_k(t)\| \|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))\,d\tau}\,dt \leq C_r M \int_{T_k}^\infty \alpha(t)\,dt.$$

Hence

$$\delta(\widetilde{S}^T_{A_k}(\mathbf{B}^n), \widetilde{\mathcal{R}}_{T_k}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in[T,T_k]}\big)) \to 0, \quad \text{as } k \to \infty.$$

By Remark 2.19

$$\delta(\widetilde{\mathcal{R}}_{T_k}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_k(t)})_{t\in[T,T_k]}\big), \widetilde{S}^T_{A_k}(\mathbf{B}^n)) = 0, \quad \text{for all } k \in \mathbf{N},$$

and thus we obtain (3.7).

In the same manner, since $A \in \widetilde{\mathcal{A}}$, we can also prove that for every sequence $(T_k)_{k\in\mathbf{N}}$ in (T,∞) such that $T_k \to \infty$, as $k \to \infty$, we have

$$\rho(\widetilde{\mathcal{R}}_{T_k}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T,T_k]}\big), \widetilde{S}^T_A(\mathbf{B}^n)) \to 0, \quad \text{as } k \to \infty. \tag{3.8}$$

Let $(A_{k_m})_{m\in\mathbf{N}}$ be an arbitrary subsequence of $(A_k)_{k\in\mathbf{N}}$. Let $(T_l)_{l\in\mathbf{N}}$ be a sequence in (T,∞) such that $T_l \to \infty$, as $l \to \infty$. By Theorem 3.1, we deduce that for every $l \in \mathbf{N}$ there is $m_l \in \mathbf{N}$ such that

$$\rho(\widetilde{\mathcal{R}}_{T_l}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A_{k_{m_l}}(t)})_{t\in[T,T_l]}\big), \widetilde{\mathcal{R}}_{T_l}\big(\mathrm{id}_{\mathbf{B}^n}, (\mathcal{N}_{A(t)})_{t\in[T,T_l]}\big)) \leq \frac{1}{l}.$$

Taking into account (3.7) and (3.8), we deduce that

$$\rho(\widetilde{S}^T_{A_{k_{m_l}}}(\mathbf{B}^n), \widetilde{S}^T_A(\mathbf{B}^n)) \to 0, \quad \text{as } l \to \infty.$$

Since every subsequence of $(\widetilde{S}^T_{A_k}(\mathbf{B}^n))_{k\in\mathbf{N}}$ contains a subsequence that converges to $\widetilde{S}^T_A(\mathbf{B}^n)$, we deduce that $(\widetilde{S}^T_{A_k}(\mathbf{B}^n))_{k\in\mathbf{N}}$ converges to $\widetilde{S}^T_A(\mathbf{B}^n)$. □

Corollary 3.3. *Let $T \geq 0$ and $A \in \widetilde{\mathcal{A}}$ be such that $\operatorname{ess\,inf}_{t\geq T} m(A(t)) > 0$. Also, let $T' > T$, $M > 0$ and $(A_k)_{k\in\mathbf{N}}$ be a sequence in $\widetilde{\mathcal{A}}$ such that $\|A_k(t)\| \leq M$, for a.e. $t \in [T, T']$, and $A_k(t) = A(t)$, for a.e. $t > T'$, and for all $k \in \mathbf{N}$. If*

$$A_k(t) \to A(t), \quad \text{as } k \to \infty, \text{ for a.e. } t \in [T, T'],$$

then

$$\rho(\widetilde{S}^T_{A_k}(\mathbf{B}^n), \widetilde{S}^T_A(\mathbf{B}^n)) \to 0, \quad \text{as } k \to \infty.$$

Proof. Let $V(T,\cdot)$ be the unique solution on $[T,\infty)$ of the initial value problem (2.1) related to A and for every $k \in \mathbf{N}$ let $V_k(T,\cdot)$ be the unique solution on $[T,\infty)$ of the initial value problem (2.1) related to A_k. For every $k \in \mathbf{N}$ we have (see Lemma 2.3)

$$\begin{aligned}
&\|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))\,d\tau} \\
&\leq \|V_k(T,T')^{-1}\| e^{-2\int_T^{T'} m(A_k(\tau))\,d\tau} \|V(T',t)^{-1}\| e^{-2\int_{T'}^t m(A(\tau))\,d\tau} \\
&\leq e^{(T'-T)M} \|V(T',t)^{-1}\| e^{-2\int_{T'}^t m(A(\tau))\,d\tau},
\end{aligned}$$

for all $t \geq T'$. Let $\alpha\colon [T,\infty) \to \mathbf{R}$ be given by

$$\alpha(t) = \begin{cases} e^{(t-T)M}, & t \in [T,T') \\ e^{(T'-T)M}\|V(T',t)^{-1}\| e^{-2\int_{T'}^t m(A(\tau))\,d\tau}, & t \in [T',\infty) \end{cases}.$$

Since $A \in \widetilde{\mathcal{A}}$, we have $\alpha \in L^1([T,\infty),\mathbf{R})$. By Theorem 3.2, the proof is done. □

For constant time-dependent operators (cf. Remark 2.11), we have the following result.

Theorem 3.4. *Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and let $(\mathbf{A}_l)_{l\in\mathbf{N}}$ be a sequence in $L(\mathbf{C}^n)$ such that $\mathbf{A}_l \to \mathbf{A}$, as $l \to \infty$. Then there is $l_0 \in \mathbf{N}$ such that $S^0_{\mathbf{A}_l}(\mathbf{B}^n)$ is compact for $l \geq l_0$, and $\rho(S^0_{\mathbf{A}_l}(\mathbf{B}^n), S^0_{\mathbf{A}}(\mathbf{B}^n)) \to 0$, as $l \to \infty$.*

Proof. First, we observe that for every $l \in \mathbf{N}$ we have

$$m(\mathbf{A}_l) - \|\mathbf{A} - \mathbf{A}_l\| \leq m(\mathbf{A}) \leq m(\mathbf{A}_l) + \|\mathbf{A} - \mathbf{A}_l\|.$$

Hence

$$\limsup_{l\to\infty} m(\mathbf{A}_l) \leq m(\mathbf{A}) \leq \liminf_{l\to\infty} m(\mathbf{A}_l)$$

and thus $\lim_{l\to\infty} m(\mathbf{A}_l) = m(\mathbf{A})$.

Let $\varepsilon = 2m(\mathbf{A}) - k_+(\mathbf{A})$. In view of [13, Remark 2.2], there exists $\delta > 0$ such that

$$\|e^{t\mathbf{A}}\| \leq \delta e^{(2m(\mathbf{A})-\varepsilon/2)t}, \quad t \geq 0. \tag{3.9}$$

Let $l_0 \in \mathbf{N}$ be such that for every $l \geq l_0$ we have

$$2m(\mathbf{A}) - 2m(\mathbf{A}_l) + \delta\|\mathbf{A} - \mathbf{A}_l\| \leq \varepsilon/4. \tag{3.10}$$

Taking into account the proof of [26, Theorem 2.1, pp. 497–498], and using (3.9) and (3.10), we deduce that for every $l \geq l_0$ we have

$$\|e^{t\mathbf{A}_l}\| \leq \delta e^{(2m(\mathbf{A})-\varepsilon/2+\delta\|\mathbf{A}_l-\mathbf{A}\|)t} \leq \delta e^{-t\varepsilon/4} e^{2m(\mathbf{A}_l)t}, \text{ for all } t \geq 0.$$

Let $\alpha\colon [0,\infty) \to \mathbf{R}$ be given by $\alpha(t) = \delta e^{-t\varepsilon/4}$, for all $t \geq 0$. Then $\alpha \in L^1([0,\infty),\mathbf{R})$ and $\|e^{t\mathbf{A}_l}\| e^{-2m(\mathbf{A}_l)t} \leq \alpha(t)$, for all $t \geq 0$ and $l \geq l_0$. So, by [9, Remark 2.8], we have that $k_+(\mathbf{A}_l) < 2m(\mathbf{A}_l)$, and thus $S^0_{\mathbf{A}_l}(\mathbf{B}^n)$ is compact, for all

$l \geq l_0$. Moreover, by Theorem 3.2, we have $\rho(S^0_{\mathbf{A}_l}(\mathbf{B}^n), S^0_{\mathbf{A}}(\mathbf{B}^n)) \to 0$, as $l \to \infty$. This completes the proof. □

Taking into account Theorem 3.4, it is natural to ask the following question.

Question 3.5. Under the assumptions of Theorem 3.4, is it true that

$$\lim_{l\to\infty} \rho(\overline{\mathrm{ex}\, S^0_{\mathbf{A}_l}(\mathbf{B}^n)}, \overline{\mathrm{ex}\, S^0_{\mathbf{A}}(\mathbf{B}^n)}) = 0 \text{ and } \lim_{l\to\infty} \rho(\overline{\mathrm{supp}\, S^0_{\mathbf{A}_l}(\mathbf{B}^n)}, \overline{\mathrm{supp}\, S^0_{\mathbf{A}}(\mathbf{B}^n)}) = 0?$$

In view of Theorem 3.4, we obtain the following convergence result related to the Carathéodory family $\mathcal{N}_{\mathbf{A}}$, where $\mathbf{A} \in L(\mathbf{C}^n)$ with $m(\mathbf{A}) > 0$. This result is motivated by the fact that every mapping $f \in S^0_{\mathbf{A}}(\mathbf{B}^n)$ is generated by a Herglotz vector field $h\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ with respect to $\mathbf{A}$.

Proposition 3.6. *Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $m(\mathbf{A}) > 0$, and let $(\mathbf{A}_k)_{k\in\mathbf{N}}$ be a sequence in $L(\mathbf{C}^n)$ such that $\mathbf{A}_k \to \mathbf{A}$, as $k \to \infty$. Then there is $k_0 \in \mathbf{N}$ such that $m(\mathbf{A}_k) > 0$, for all $k \geq k_0$, and $\rho(\mathcal{N}_{\mathbf{A}_k}, \mathcal{N}_{\mathbf{A}}) \to 0$, as $k \to \infty$.*

Proof. Since $\lim_{k\to\infty} m(\mathbf{A}_k) = m(\mathbf{A})$ by the proof of Theorem 3.4, and since $m(\mathbf{A}) > 0$, it follows that there is $k_0 \in \mathbf{N}$ such that $m(\mathbf{A}_k) > 0$, for all $k \geq k_0$. Hence, $\mathcal{N}_{\mathbf{A}_k}$ is well defined, for all $k \geq k_0$.

The fact that $\rho(\mathcal{N}_{\mathbf{A}_k}, \mathcal{N}_{\mathbf{A}}) \to 0$, as $k \to \infty$, follows by arguments similar to those in the proof of Theorem 3.1. Indeed, suppose that there exist $\varepsilon > 0$, a sequence of indices $(k_m)_{m\in\mathbf{N}}$ with $k_m \geq k_0$, $m \in \mathbf{N}$, and a sequence of mappings $(h_m)_{m\in\mathbf{N}}$ with $h_m \in \mathcal{N}_{\mathbf{A}_{k_m}}$, $m \in \mathbf{N}$, such that $\delta(h_m, h) > \varepsilon$, for all $h \in \mathcal{N}_{\mathbf{A}}$. In view of [19, Lemma 3], we deduce that for every $l \in \mathbf{N}$, there exists $m_l \in \mathbf{N}$ such that $q_l \in \mathcal{N}_{\mathbf{A}}$, where $q_l(z) = \frac{1}{r_l} h_{m_l}(r_l z) - \mathbf{A}_{k_{m_l}} z + \mathbf{A}z$, for all $z \in \mathbf{B}^n$, and $r_l = 1 - \frac{1}{l}$. Since $\mathbf{A}_{k_{m_l}} \to \mathbf{A}$, as $l \to \infty$, we deduce that there is $M > 0$ such that $\|\mathbf{A}_{k_{m_l}}\| \leq M$, for all $l \in \mathbf{N}$. Hence, as in the proof of Theorem 3.1, we get that $h_{m_l} - q_l \to 0$, as $l \to \infty$, locally uniformly on $\mathbf{B}^n$, which is a contradiction. Thus, $\delta(\mathcal{N}_{\mathbf{A}_k}, \mathcal{N}_{\mathbf{A}}) \to 0$, as $k \to \infty$. The fact that $\delta(\mathcal{N}_{\mathbf{A}}, \mathcal{N}_{\mathbf{A}_k}) \to 0$, as $k \to \infty$, follows by the same arguments as above. □

We close this section with the following convergence result for the family $\widehat{S}_{\mathbf{A}}(\mathbf{B}^n)$ of spirallike mappings with respect to $\mathbf{A} \in L(\mathbf{C}^n)$, where $k_+(\mathbf{A}) < 2m(\mathbf{A})$.

Proposition 3.7. *Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$, and let $(\mathbf{A}_l)_{l\in\mathbf{N}}$ be a sequence in $L(\mathbf{C}^n)$ such that $\mathbf{A}_l \to \mathbf{A}$, as $l \to \infty$. Then there is $l_0 \in \mathbf{N}$ such that $\widehat{S}_{\mathbf{A}_l}(\mathbf{B}^n)$ is compact for $l \geq l_0$, and $\rho(\widehat{S}_{\mathbf{A}_l}(\mathbf{B}^n), \widehat{S}_{\mathbf{A}}(\mathbf{B}^n)) \to 0$, as $l \to \infty$.*

Proof. By the proof of Theorem 3.4, we have that there exist $l_0 \in \mathbf{N}$ and $\alpha \in L^1([0,\infty), \mathbf{R})$ such that $\|e^{t\mathbf{A}_l}\| e^{-2m(\mathbf{A}_l)t} \leq \alpha(t)$, for all $t \geq 0$ and $l \geq l_0$. In particular, $k_+(\mathbf{A}_l) < 2m(\mathbf{A}_l)$, by [9, Remark 2.8], and thus $\widehat{S}_{\mathbf{A}_l}(\mathbf{B}^n)$ is compact, for all $l \geq l_0$, by [35, Theorem 3.1] (cf. [13]).

Finally, since every spirallike mapping is generated by a Herglotz vector field that is constant in time (see [13]; cf. [10]), we may adapt all arguments in the proof of Theorems 3.1 and 3.2 and deduce that $\lim_{l\to\infty} \rho(\widehat{S}_{\mathbf{A}_l}(\mathbf{B}^n), \widehat{S}_{\mathbf{A}}(\mathbf{B}^n)) = 0$, as desired. □

Question 3.8. In connection with [1] and [35], would be possible to generalize Theorem 3.4 and Proposition 3.7 to the case of non-resonant linear operators?

4. Analytical characterizations of mappings in $\widetilde{S}^t_A(\mathbf{B}^n)$

In this section we obtain some sufficient conditions related to $A \in \widetilde{\mathcal{A}}$, which guarantee the equality $\widetilde{S}^t_A(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, for $t \geq 0$. The first result is a generalization of [13, Theorem 3.12].

Proposition 4.1. *Let* $k \in \mathbf{N}$, $\alpha_1, \dots, \alpha_k > 0$, *and let* $E_1, \dots, E_k \in L(\mathbf{C}^n)$ *be such that* $E_i + E_i^* = 2\alpha_i I_n$, *for all* $i \in \{1, \dots, k\}$. *Also, let* $0 = T_0 < T_1 < \dots < T_{k-1} < T_k = \infty$ *and let* $A\colon [0,\infty) \to L(\mathbf{C}^n)$ *be given by*

$$A(t) = \begin{cases} E_1, & \text{for } t \in [T_0, T_1), \\ \vdots & \\ E_k, & \text{for } t \in [T_{k-1}, T_k). \end{cases}$$

Then $\widetilde{S}^T_A(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, *for all* $T \geq 0$.

Proof. We shall use arguments similar to those in the proof of [13, Theorem 3.12]. Fix $T \geq 0$ and let $i \in \{1, \dots, k\}$ be such that $T \in [T_{i-1}, T_i)$.

First, we prove that $S^0(\mathbf{B}^n) \subseteq \widetilde{S}^T_A(\mathbf{B}^n)$. To this end, let $f \in S^0(\mathbf{B}^n)$. Then there exists a Herglotz vector field $h\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ (with $Dh(0,\cdot) \equiv I_n$) such that $f = \lim_{t\to\infty} e^t v(\cdot, 0, t)$ locally uniformly on $\mathbf{B}^n$, where $v(z,0,\cdot)$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial v}{\partial t}(z,0,t) = -h(v(z,0,t),t), \quad \text{a.e. } t \in [0,\infty), \ v(z,0,0) = z,$$

for all $z \in \mathbf{B}^n$. For each $j \in \{i, \dots, k\}$, let $F_j\colon [0,\infty) \to L(\mathbf{C}^n)$ be recursively given by

$$F_i(t) = e^{(t-T)(\alpha_i I_n - E_i)}, \quad t \geq 0,$$

and

$$F_j(t) = e^{(t-T_{j-1})(\alpha_j I_n - E_j)} F_{j-1}(T_{j-1}), \quad j \neq i, \ t \geq 0.$$

Also, let $\beta_j\colon [0,\infty) \to \mathbf{R}$ be given

$$\beta_i(t) = \alpha_i(t-T) \ \text{ and } \ \beta_j(t) = \alpha_j(t - T_{j-1}) + \beta_{j-1}(T_{j-1}), \quad j \neq i, \ t \geq 0.$$

Let $q\colon \mathbf{B}^n \times [0,\infty) \to \mathbf{C}^n$ be given by

$$q(z,t) = \begin{cases} A(t)z, & t \in [0,T), \\ \alpha_i F_i(t) h(F_i(t)^{-1}z, \beta_i(t)) - (\alpha_i I_n - E_i)z, & t \in [T, T_i), \\ \alpha_{i+1} F_{i+1}(t) h(F_{i+1}(t)^{-1}z, \beta_{i+1}(t)) - (\alpha_{i+1} I_n - E_{i+1})z, & t \in [T_i, T_{i+1}), \\ \vdots & \\ \alpha_k F_k(t) h(F_k(t)^{-1}z, \beta_k(t)) - (\alpha_k I_n - E_k)z, & t \in [T_{k-1}, T_k), \end{cases}$$

for all $z \in \mathbf{B}^n$. Since $E_j + E_j^* = 2\alpha_j I_n$, for all $j \in \{i, \dots, k\}$, we deduce by an inductive argument that $F_j(t)^* = F_j(t)^{-1}$ and $\|F_j(t)^{-1}\| \leq 1$, for all $t \in [T_{j-1}, T_j)$ and $j \in \{i, \dots, k\}$ (cf. [13]). Then it is not difficult to prove that q is well defined and is a Herglotz vector field with respect to A.

Let

$$u(z,T,t) = \begin{cases} F_i(t) v(z, 0, \beta_i(t)), & \text{for } z \in \mathbf{B}^n, \ t \in [T, T_i), \\ F_{i+1}(t) v(z, 0, \beta_{i+1}(t)), & \text{for } z \in \mathbf{B}^n, \ t \in [T_i, T_{i+1}), \\ \vdots & \\ F_k(t) v(z, 0, \beta_k(t)), & \text{for } z \in \mathbf{B}^n, \ t \in [T_{k-1}, T_k). \end{cases}$$

We observe that $\frac{d}{dt}F_j(t) = (\alpha_j I_n - E_j)F_j(t)$, for all $j \in \{i, \dots, k\}$ and every $t \in [T_{j-1}, T_j)$, and thus $u(z, T, \cdot)$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial u}{\partial t}(z, T, t) = -q(u(z, T, t), t), \quad \text{a.e. } t \in [T, \infty), \ u(z, T, T) = z,$$

for all $z \in \mathbf{B}^n$.

Let $V(T, \cdot)$ be the unique solution on $[T, \infty)$ of the initial value problem (2.1) related to A. Since $V(T, t) = e^{-\beta_k(t)}F_k(t)$, for all $t \geq T_{k-1}$, and $\lim_{t\to\infty} \beta_k(t) = \infty$, we deduce that $f = \lim_{t\to\infty} e^{\beta_k(t)} v(\cdot, 0, \beta_k(t)) = \lim_{t\to\infty} V(T, t)^{-1} u(\cdot, T, t)$ locally uniformly on $\mathbf{B}^n$. Hence $f \in \widetilde{S}^T_A(\mathbf{B}^n)$. So $S^0(\mathbf{B}^n) \subseteq \widetilde{S}^T_A(\mathbf{B}^n)$.

Using similar arguments as above, we may prove that $\widetilde{S}^T_A(\mathbf{B}^n) \subseteq S^0(\mathbf{B}^n)$ (cf. [13, Remark 3.13]). This completes the proof. □

In view of Propositions 2.13 and 4.1, we obtain the following example.

Example 4.2. Let $E = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $T > 0$. Let $A \in \widetilde{\mathcal{A}}$ be given by

$$A(t) = \begin{cases} E, & \text{for } t \in [0, T), \\ I_2, & \text{for } t \in [T, \infty). \end{cases}$$

Then $\widetilde{S}^s_A(\mathbf{B}^2) = S^0(\mathbf{B}^2)$, for all $s \geq 0$, but there do not exist $\mathbf{A} \in L(\mathbf{C}^2)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and a measurable function $a\colon [0, \infty) \to \mathbf{R}$ such that (2.2) holds and $A(t) = a(t)\mathbf{A}$ for a.e. $t \geq 0$.

Proof. Let $s \geq 0$. Since $E + E^* = 2I_2$, it follows that $\widetilde{S}^s_A(\mathbf{B}^2) = S^0(\mathbf{B}^2)$, by Proposition 4.1. Also, it is elementary to see that there do not exist $\mathbf{A} \in L(\mathbf{C}^2)$ with $k_+(\mathbf{A}) < 2m(\mathbf{A})$ and a measurable function $a\colon [0, \infty) \to \mathbf{R}$ such that (2.2) holds and $A(t) = a(t)\mathbf{A}$ for a.e. $t \geq 0$. □

Using Theorem 3.2 and Proposition 4.1, we may prove the following result. This result is also an improvement of Proposition 4.1 (cf. Proposition 2.13 for $\mathbf{A} = I_n$ and [13, Theorem 3.12]).

Proposition 4.3. *Let $A\colon [0, \infty) \to L(\mathbf{C}^n)$ be a measurable mapping such that $\operatorname{ess\,inf}_{t\geq 0} m(A(t)) > 0$, $\operatorname{ess\,sup}_{t\geq 0} \|A(t)\| < \infty$, and for a.e. $t \geq 0$ there is $\alpha(t) > 0$ such that $A(t) + A(t)^* = 2\alpha(t)I_n$. Then $A \in \widetilde{\mathcal{A}}$ and $\widetilde{S}^T_A(\mathbf{B}^n) = S^0(\mathbf{B}^n)$, for all $T \geq 0$.*

Proof. Let $\sigma, \beta \in (0, \infty)$ be such that $m(A(t)) > \sigma$ and $\|A(t)\| < \beta$, for a.e. $t \geq 0$. In view of the hypothesis, we deduce that there is a measurable function $\alpha\colon [0, \infty) \to \mathbf{R}$ such that $\alpha(t) > 0$ and $A(t) + A(t)^* = 2\alpha(t)I_n$, for a.e. $t \geq 0$. We observe that $k(A(t)) = m(A(t)) = \alpha(t) > \sigma$, for a.e. $t \geq 0$ (cf. [13]).

Fix $T \geq 0$. In view of Lemma 2.3, we have

$$\|V(T, t)^{-1}\| e^{-2\int_T^t m(A(\tau))\, d\tau} \leq e^{-\int_T^t \alpha(\tau)\, d\tau} \leq e^{(T-t)\sigma},$$

for all $t \geq T$, where $V(T, \cdot)$ is the unique solution on $[T, \infty)$ of the initial value problem (2.1) related to A. Thus, we deduce that $A \in \widetilde{\mathcal{A}}$. Since A is measurable, there exists a sequence of step functions $(A_k)_{k\in\mathbf{N}}$ defined on $[0, \infty)$ and with values in the set

$$\big\{E \in L(\mathbf{C}^n) \mid \text{there is } \lambda > \sigma \text{ such that } E + E^* = 2\lambda I_n \text{ and } \|E\| < \beta\big\}$$

such that $A_k(t) \to A(t)$, as $k \to \infty$, for a.e. $t \in [0, \infty)$.

Using similar arguments as above, we may prove that for every $k \in \mathbf{N}$, we have

$$\|V_k(T,t)^{-1}\| e^{-2\int_T^t m(A_k(\tau))\,d\tau} \le e^{(T-t)\sigma},$$

for all $t \ge T$, where $V_k(T,\cdot)$ is the unique solution on $[T,\infty)$ of the initial value problem (2.1) related to A_k. In particular, we deduce that $A_k \in \widetilde{\mathcal{A}}$, for all $k \in \mathbf{N}$. Moreover, if we let $\alpha\colon [T,\infty) \to \mathbf{R}$ be given by $\alpha(t) = e^{(T-t)\sigma}$, for all $t \ge T$, then $\alpha \in L^1([T,\infty),\mathbf{R})$ and thus, by Theorem 3.2, we deduce that

$$\rho(\widetilde{S}^T_{A_k}(\mathbf{B}^n), \widetilde{S}^T_A(\mathbf{B}^n)) \to 0, \quad \text{as } k \to \infty.$$

Taking into account Proposition 4.1, we conclude that $\widetilde{S}^T_A(\mathbf{B}^n) = S^0(\mathbf{B}^n)$. □

In view of Proposition 4.3 and [13, Theorem 3.12], it would be interesting to give an answer to the following question:

Question 4.4. Let $\mathbf{A} \in L(\mathbf{C}^n)$ be such that $k_+(\mathbf{A}) < 2m(\mathbf{A})$. Also, let $A \in \widetilde{\mathcal{A}}$ be such that $k_+(A(t)) < 2m(A(t))$ and $S^0_{A(t)}(\mathbf{B}^n) = S^0_{\mathbf{A}}(\mathbf{B}^n)$, for a.e. $t \ge 0$. Is it true that $\widetilde{S}^T_A(\mathbf{B}^n) = S^0_{\mathbf{A}}(\mathbf{B}^n)$, for all $T \ge 0$?

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