Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 639–657

THE HILBERT METRIC ON TEICHMÜLLER SPACE AND EARTHQUAKE

Huiping Pan

Sun Yat-Sen University, School of Mathematics and Computational Science
510275, Guangzhou, P. R. China; chnpanhp@foxmail.com

Abstract. Hamenstädt gave a parametrization of the Teichmüller space of punctured surfaces such that the image under this parametrization is the interior of a polytope. In this paper, we study the Hilbert metric on the Teichmüller space of punctured surfaces based on this parametrization. We prove that every earthquake ray is an almost geodesic under the Hilbert metric.

1. Introduction

Let $S_{g,n}$ be an orientable surface of genus g with n punctures. In this paper, we consider the surfaces of negative Euler characteristic with at least one puncture. A marked hyperbolic structure on $S_{g,n}$ is pair (X, f) where X is a complete hyperbolic metric on a surface S and $f\colon S_{g,n} \to S$ is a homeomorphism. Two marked hyperbolic structures (X_1, f_1) and (X_1, f_2) are called equivalent if there is an isometry in the isotopy class of $f_1^{-1} \circ f_2$. For simplicity, we usually denote a marked hyperbolic metric by X instead of the pair (X, f). The Teichmüller space $T_{g,n}$ is defined as the space of equivalent classes of marked hyperbolic structures on $S_{g,n}$. It is well known that $T_{g,n}$, equipped with the natural topology is homeomorphic to a ball in $\mathbf{R}^{6g-6+2n}$.

Given an open convex domain $D \subset \mathbf{R}^m$, Hilbert defined a natural metric on D, now called the Hilbert metric, such that the straight line segments are geodesic segments under this metric. In [15], Papadopoulos raised a problem: "Realize Teichmüller space as a bounded convex set somewhere and study the Hilbert metric on it". In this paper, we will study this problem.

For the case of closed surfaces, Yamada [29] constructed a space which he called the Teichmüller–Coxeter complex within which the original Teichmüller space sits as an open convex but unbounded subset, based on the Weil–Petersson completion of the Teichmüller space. Then in [30], after introducing a new variational characterization of the Hilbert metric, he defined the Weil–Petersson Hilbert metric on the Teichmüller space, where the background geometry is the one induced by the Weil–Petersson geometry instead of the Euclidean geometry.

For the case of punctured surfaces, Hamenstädt [5] provided a parametrization of the Teichmüller space into $\mathbf{R}P^{6g-6+2n}$ by length functions such that the image of $T_{g,n}$ is the interior of a finite-sided polyhedron (see §2.3). Therefore the Hilbert metric is well defined on $T_{g,n}$. Hamenstädt's parametrization depends on the choice of a preferred triangulation Γ of $S_{g,n}$. More precisely, fix a puncture of $S_{g,n}$ and denote it as O, let $\Gamma = \{\eta_1, \eta_2, \dots, \eta_{6g-5+2n}\}$ be a set of bi-infinite simple curves on $S_{g,n}$ such that for any marked hyperbolic metric X the two ends of η_i, $i = 1, 2, \dots, 6g-5+2n$, go into the puncture O and such that $S_{g,n} \backslash \Gamma$ consists of $4g-3+n$ ideal triangles

doi:10.5186/aasfm.2016.4144
2010 Mathematics Subject Classification: Primary 51F99; Secondary 57N16, 52B70.
Key words: Teichmüller space, Hilbert metric, earthquake, mapping class group.

and $n-1$ once punctured discs. Such a set Γ is called a *preferred triangulation* of $S_{g,n}$. There are countably many choices of preferred triangulations.

In this paper, we study the Hilbert metric d_h^Γ on $T_{g,n}$ based on Hamentädt's parametrization. Before stating our main result, we briefly explain a deformation of hyperbolic metric introduced by Thurston in [23], namely, the *earthquake*.

Let α be a simple closed curve on $S_{g,n}$, and $X \in T_{g,n}$ be a marked hyperbolic metric. Denote by α^* the geodesic representative of α on X. Cutting X along α^* and twisting to the left about distance t, we obtain a new marked hyperbolic metric, denoted by $\mathcal{E}_\alpha^t X$. Note that the notion of "left" twist depend only on the orientation of X (no orientation of α^* is necessary). Thurston extended this construction to any measured geodesic lamination. He proved the following result, one of whose proof can be found in [9].

Proposition 1.1. *There is a (unique) continuous map $\mathcal{ML} \times \mathbf{R} \times T_{g,n} \to T_{g,n}$, associating an element $\mathcal{E}_\alpha^t X \in T_{g,n}$ to (α, t, X), such that $\mathcal{E}_{\lambda\alpha}^t X = \mathcal{E}_\alpha^{\lambda t} X$ for all $\lambda > 0$ and all $\alpha \in \mathcal{ML}$, and such that when α is a simple closed geodesic, $\mathcal{E}_\alpha^t X$ is obtained from X by the earthquake defined above.*

The metric $\mathcal{E}_\alpha^t X$ defined in Proposition 1.1 is said to be obtained from X by a *(left) earthquake of amplitude t along the measured geodesic lamination α*, and the orbits $\{\mathcal{E}_\alpha^t X\}_{t=-\infty}^{\infty}$, $\{\mathcal{E}_\alpha^t X\}_{t=-\infty}^{0}$ and $\{\mathcal{E}_\alpha^t X\}_{t=0}^{\infty}$ are called the *earthquake line directed by α and starting at X*, the *anti-earthquake ray directed by α and starting at X*, and the *earthquake ray directed by α and starting at X*, respectively.

Recall that for a metric space (X, d), an unbounded path $\gamma\colon [0, \infty) \to X$ is called an *almost-geodesic* if for any $\epsilon > 0$, there exists $T > 0$, such that

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t)| < \epsilon$$

for any $t \geq s \geq T$.

Now we state our main result.

Main Theorem. *After reparametrization, every (anti-)earthquake ray is an almost-geodesic in $(T_{g,n}, d_h^\Gamma)$.*

In fact, the image of an earthquake ray under Hamenstädt's parametrization eventually looks like a projective line (see §3).

Outline. This paper is organized as the following. In Section 2, we recall some basic properties of the Hilbert metric and express the Hilbert metric d_h^Γ on the Teichmüller space based on Hamenstädt's parametrization. In Section 3, we prove our main theorem. In Section 4, we study the dependence of the Hilbert metric d_h^Γ on the choice of the preferred triangulation Γ. We will show that a sphere $B(X_0, R)$ centered at $X_0 \in T_{g,n}$ of radius R with respect to d_h^Γ for a preferred triangulation Γ is again a sphere up to an additive constant with respect to $d_h^{\Gamma'}$ for another preferred triangulation Γ', provided that Γ' can be obtained from Γ by a *diagonal-flip* (to be defined in §4). But the additive constant depends on the center point X_0. In Section 5, we study the actions of the mapping class group on $(T_{g,n}, d_h^\Gamma)$. It is well known that when the Teichmüller space is endowed with the Teichmüller metric, the Thurston metric or the Weil-Petersson metric, the mapping class group acts by isometries. But here, we will show that the action of a positive Dehn twist is not isometric (see Corollary 5.4). Instead, it is an almost isometry on an unbounded subset of $(T_{g,n}, d_h^\Gamma)$ (see Corollary 5.2).

Acknowledgements. I would like to thank my advisor Lixin Liu for his careful reading of this manuscript and many useful suggestions. I would like to thank Athanase Papadopoulos for useful conversations during his visit at Sun Yat-Sen University (Zhongshan University). I also thank the referee for numerous comments and suggestions. The work is partially supported by NSFC, No:11271378.

2. The Hilbert metric on the Teichmüller space

2.1. The Hilbert metric. There are two versions of the Hilbert's metric. The first version is the original one due to Hilbert which is defined on a bounded convex domain Ω (see Fig. 1(a)) in $\mathbf{R}^m$. Let x, y be two points in the interior of Ω, the line passing through x, y intersects the boundary $\partial\Omega$ at two points a, b, where x lies between a and y. Then the Hilbert metric is defined as:

$$d_H(x,y) = \frac{1}{2}\log\,[a,b,y,x] = \frac{1}{2}\log\frac{|a-y||b-x|}{|a-x||b-y|}, \tag{1}$$

where $[a,b,y,x]$ represents the cross-ratio of $a,\ x,\ y,\ b$.

The second version is due to Birkhoff which is defined on the cone $\mathcal{C}$ over a bounded convex domain Ω (see [11], [10] and [12] for more details about this version). Recall that a cone $\mathcal{C}$ is called *pointed* if $\mathcal{C}\cap-\mathcal{C}=0$. Let $\mathcal{C}$ be a closed, pointed (convex) cone over a convex bounded domain Ω in $\mathbf{R}^m$. Given two nonzero vectors x and y in $\mathcal{C}$ (see Fig. 1(b)), the *Birkhoff's version of the Hilbert metric*, denoted as d_h is defined as:

$$d_h(x,y) = \frac{1}{2}\log M(x,y)/m(x,y), \tag{2}$$

where

$$M(x,y) = \inf\{\lambda \ge 0\colon \lambda y - x \in \mathcal{C}\},$$
$$m(x,y) = \sup\{\lambda \ge 0\colon x - \lambda y \in \mathcal{C}\}.$$

Denote by o the cone point of $\mathcal{C}$, and suppose that the line $\overline{xy}$ passing through x, y intersects the boundary $\partial\mathcal{C}$ at a, b. To calculate $M(x,y)$ explicitly, we distinguish two cases. The first case is that the points o, y, x are collinear. In this case $M(x,y) = m(x,y) = |x|/|y|$, hence $d_h(x,y) = 0$. The second case is that the points o, x, y are not collinear. We draw an auxiliary line $\overline{xp}$ from x which is parallel to the line $\overline{ob}$ and intersects the line $\overline{oy}$ at p. Then

$$M(x,y) = \frac{|p-o|}{|y-o|} = \frac{|x-b|}{|y-b|}.$$

Similarly we get

$$m(x,y) = \frac{|x-a|}{|y-a|}.$$

Hence

$$\frac{M(x,y)}{m(x,y)} = \frac{|a-y||b-x|}{|a-x||b-y|} = [a,b,y,x].$$

By the property of cross-ratio, we have $d_h(\lambda x, \mu y) = d_h(x,y)$ for any $\lambda > 0, \mu > 0$. It is clear that d_h is not a metric on $\mathcal{C}$ since it does not separate x and λx for any $\lambda > 0$. In fact, d_h is a metric on the projective space $\mathcal{C}/\mathbf{R}^+$.

Yamada [30] gave an alternate definition of the Hilbert metric by supporting hyperplanes. Recall that a convex set Ω can be represented as $\cap_{\pi(b)\in\mathcal{P}} H_{\pi(b)}$ where $H_{\pi(b)}$ is the half space bounded by a supporting hyperplane $\pi(b)$ of Ω at the boundary point b, containing the convex set Ω. Let $\mathcal{P}$ be the set of all the supporting hyperplanes of Ω (see Fig. 1(c)). Yamada showed that the Hilbert metric can be represented as:

$$d_H(x,y) = \frac{1}{2}\left(\sup_{\pi\in\mathcal{P}} \log\frac{d(x,\pi)}{d(y,\pi)} + \sup_{\pi\in\mathcal{P}} \log\frac{d(y,\pi)}{d(x,\pi)}\right). \tag{3}$$

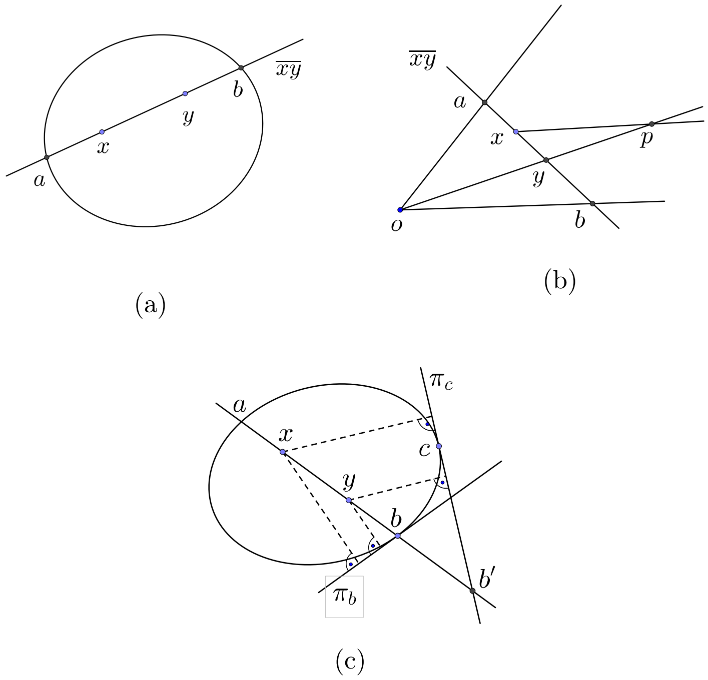


Figure 1. The Hilbert metric.

We briefly explain Yamada's idea here. Let the line $\overline{xy}$ intersects Ω at a, b in the order a, x, y, b. Let $\pi(b)$ be a supporting hyperplane of Ω at b, and let $\pi(c)$ be a hyperplane of Ω at an arbitrary point $c \in \Omega$. Denote by b' the intersection point between $\overline{xy}$ and $\pi(c)$. It is clear that

$$\frac{d(x,\pi(c))}{d(y,\pi(c))} = \frac{|x-b'|}{|y-b'|} \le \frac{|x-b|}{|y-b|} = \frac{d(x,\pi(b))}{d(y,\pi(b))}, \quad \text{for any } c \in \Omega.$$

Similarly, we have

$$\frac{d(y,\pi(c))}{d(x,\pi(c))} \le \frac{d(y,\pi(a))}{d(x,\pi(a))}, \quad \text{for any } c \in \Omega.$$

Since a cone over a bounded convex set is again a convex set, Yamada's idea also applies to the Birkhoff's version of the Hilbert metric. Therefore

$$d_h(x,y) = \frac{1}{2}\left(\sup_{\pi\in\mathcal{P}} \log\frac{d(x,\pi)}{d(y,\pi)} + \sup_{\pi\in\mathcal{P}} \log\frac{d(y,\pi)}{d(x,\pi)}\right), \text{ for any } x,y \in \mathcal{C} \text{ with } [x] \neq [y], \tag{4}$$

where $[x] = \{\lambda x \in \mathcal{C} \colon \lambda > 0\}$, and $\mathcal{P}$ is the set of supporting hyperplanes of the cone $\mathcal{C}$.

Remark. It is easy to see that these two versions of Hilbert metric coincide on the convex bounded domain Ω, i.e. $d_H(x,y) = d_h(x,y)$ for any $x, y \in \Omega$. In this paper, we adopt the Birkhoff's version of the Hilbert metric.

2.2. Measured geodesic lamination. Given a reference metric X_0, a *geodesic lamination* L is a closed subset of $S_{g,n}$ consisting of mutually disjoint simple geodesics which are called *leaves* of this geodesic lamination. A *transverse invariant measure*

μ of a geodesic lamination L is a Radon measure defined on every arc k transverse to the support of L such that μ is invariant with respect to any homotopy of k relative to the leaves of L. A *measured geodesic lamination* is a lamination L endowed with a transverse invariant measure μ. The simplest example of a measured geodesic lamination is a simple closed geodesic, where the transverse invariant measure is the Dirac measure. Each measured geodesic lamination μ induces a functional on the space $\mathcal{S}$ of isotopy classes of nontrival simple closed curves on $S_{g,n}$, which assigns $\inf_{\tilde{\gamma}\in[\gamma]}\int_{\tilde{\gamma}} d\mu$ to any $[\gamma] \in \mathcal{S}$. The amount $\inf_{\tilde{\gamma}\in[\gamma]}\int_{\tilde{\gamma}} d\mu$ is called the *intersection number* of μ with $[\gamma]$ and is denoted by $i(\mu, [\gamma])$. Two measured geodesic laminations μ, μ' are called *equivalent* if $i(\mu, [\gamma]) = i(\mu', [\gamma])$ for any $[\gamma] \in \mathcal{S}$. Denote by $\mathcal{ML}$ the space of equivalent classes of measured geodesic laminations on $S_{g,n}$, and equip $\mathcal{ML}$ with the weak topology of the functional space over $\mathcal{S}$. With this topology, the set of weighted simple closed curves, $\mathbf{R}^+ \times \mathcal{S}$, is dense in $\mathcal{ML}$. The Thurston boundary $\partial T_{g,n}$ of the Teichmüller space $T_{g,n}$ consists of the projective classes of measured geodesic laminations and is homeomorphic to the unit sphere $\mathbf{S}^{6g-6+2n}$ (for more details about measured geodesic laminations we refer to [3] and [17]).

2.3. Geometric parametrization of the Teichmüller space. Hamenstädt [5] gave a geometric parametrization of the Teichmüller space $T_{g,m}$. We briefly recall this parametrization. Let $n \geq 1$ and let $X \in T_{g,n}$. Fix one of the punctures of X and denote it by O. As we explained in the introduction, a preferred triangulation Γ is a set of $6g-5+2n$ mutually disjoint simple geodesics $\eta_1, \dots, \eta_{6g-5+2n}$ on S with the two ends of each simple geodesic going into the puncture O and which decompose X into $4g-3+n$ ideal triangles and $n-1$ once-punctured discs. The space of measured lamination $\mathcal{ML}(S)$ on X can be parameterized by the $6g-5+2n$-tuple $(i(\eta_1, \mu), \dots, i(\eta_{6g-5+2n}, \mu)) \in \mathbf{R}^{6g-5+2n}$, where μ is a measured geodesic lamination with compact support and $i(\eta_i, \mu)$ represents the intersection number of μ with η_i. Let $\mathcal{A}$ be the set of all $6g-5+2n$-tuples$(a_1, \dots, a_{6g-5+2n})$ of nonnegative real numbers with the following properties:

(1) $a_i \leq a_j + a_k$ if the geodesics η_i, η_j, η_k are the sides of an ideal triangle on S.
(2) There is at least one ideal triangle on S with sides η_i, η_j, η_k such that $a_i = a_j + a_k$.

In particular, $\mathcal{A}$ is a cone with vertex at the origin over the boundary of a convex finite-sided polyhedron P in the sphere $\mathbf{S}^{6g-6+2n}$. And $\mathcal{A}$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$.

Theorem 2.1. [5] *The map* $\mu \in \mathcal{ML}(S) \to (i(\eta_1, \mu), \dots, i(\eta_{6g-5+2n}, \mu)) \in \mathbf{R}^{6g-5+2n}$ *is a homeomorphism of* $\mathcal{ML}(S)$ *onto* $\mathcal{A}$.

Since for any marked hyperbolic metric $X \in T_{g,m}$, the length of η_i is infinite for any $i = 1, 2, \dots, 6g-5+2n$, we need a little modification to parameterize the Teichmüller space $T_{g,m}$. Recall that every puncture of S admits a standard cusp neighbourhood which is isometric to a cylinder $[-\log 2, \infty) \times S^1$ equipped with the metric $d\rho^2 + e^{-2\rho}dt^2$ ([2]). The ρ-coordinate in this representation is called the height. Let Δ_∞ be an ideal triangle on the upper half plane (see Fig. 2) with sides η_1, η_2, η_3. Each corner of Δ_∞ can be foliated by horocycles. Extend these foliations until they fill all but in the center bounded by three horocycles M_1M_2, M_2M_3, M_3M_2. We call M_i the *midpoint* of η_i with respect to the ideal triangle Δ_∞ for $i = 1, 2, 3$. Choose the height ρ_0 small enough such that $e^{-\rho_0}$ is much smaller than the length of the horocycle M_1M_2 (note that the three horocycles M_1M_2, M_2M_3, M_3M_2 have the same length).

Every geodesic going into the cusp meets the horocycles ρ=constant orthogonally. Hence each choice of a height ρ_0 cuts from η_i a unique compact arc of finite length since both ends of the geodesic η_i go into the cusp for any $i = 1, 2, \dots, 6g-5+2n$. Denote by $l_{\eta_i}(X)$ the length of this subarc of η_i for a marked hyperbolic metric X for any $i = 1, 2, \dots, 6g-5+2n$.

Theorem 2.2. [5] *Let* $\Pi\colon \mathbf{R}^{6g-5+2n}\backslash\{0\} \to \mathbf{R}P^{6g-6+2n}$ *be the canonical projection. The map*

$$\Lambda\colon S \in T_{g,n} \to (l_{\eta_1}(S), \dots, l_{\eta_{6g-5+2n}}(S)) \in \mathbf{R}^{6g-5+2n}$$

is a diffeomorphism of $T_{g,n}$ *onto a hypersurface in* $\mathbf{R}^{6g-5+2n}$*. And the map* $\Pi \circ \Lambda$ *is a diffeomorphism of* $T_{g,n}$ *onto the interior of a finite-sided closed convex polyhedron* P *in* $\mathbf{R}P^{6g-6+2n}$ *which extends to a homeomorphism of* $T_{g,n} \cup \partial T_{g,n}$ *onto* P*.*

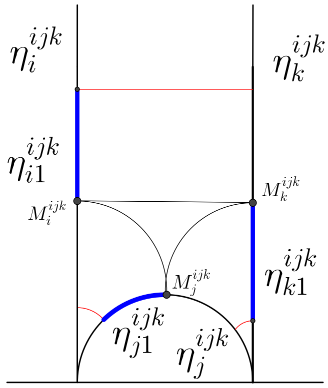


Figure 2. The ideal triangle Δ_∞.

2.4. The Hilbert metric on the Teichmüller space. Now we give the Hilbert metric on the Teichmüller space. Let π_{ijk} be the hyperplane in $\mathbf{R}^{6g-5+2n} = \{(a_1, a_2, \dots, a_{6g-5+2n})\colon a_i \in \mathbf{R}\}$ defined by

$$\pi_{ijk}\colon a_i - a_j + a_k = 0, \tag{5}$$

where the indices $i,\ j,\ k$ satisfy the condition that the corresponding simple geodesics $\eta_i,\ \eta_j,\ \eta_k$ bound an ideal triangle on the surface. Let $\mathcal{P}$ be the set of all such hyperplanes. By Theorem 2.1 and Theorem 2.2, we know that the image of $T_{g,n} \cup \partial T_{g,n}$ under the map $\Pi \circ \Lambda$ is a polyhedron P. Following Yamada's idea, we define the Hilbert metric on $T_{g,n}$ by (4):

$$\begin{aligned} d_h^\Gamma(X_1, X_2) &:= d_h(\Lambda(X_1), \Lambda(X_2)) \\ &= \frac{1}{2}\left(\sup_{\pi\in\mathcal{P}} \frac{d(\Lambda(X_1), \pi_{ijk})}{d(\Lambda(X_2), \pi_{ijk})} + \sup_{\pi\in\mathcal{P}} \frac{d(\Lambda(X_2), \pi_{ijk})}{d(\Lambda(X_1), \pi_{ijk})}\right), \end{aligned} \tag{6}$$

for all $X_1, X_2 \in T_{g,n}$, where $d(\ ,\ )$ represents the Euclidean distance. From (5), we have

$$d(\Lambda(X), \pi_{ijk}) = \frac{l_{\eta_i}(X) - l_{\eta_j}(X) + l_{\eta_i}(X)}{3^{1/2}}.$$

Hence

$$\begin{aligned} d_h^\Gamma(X_1, X_2) = \frac{1}{2}\Bigg(&\sup_{\pi_{ijk}\in\mathcal{P}} \log \frac{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)}{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)} \\ &+ \sup_{\pi_{ijk}\in\mathcal{P}} \log \frac{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)}{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)}\Bigg). \end{aligned} \tag{7}$$

Now we give some explanations for (7). Since the surface $S_{g,n}$ is orientable, we fix an orientation. Denote by Δ_{ijk} the ideal triangle with three sides $\{\eta_i, \eta_j, \eta_k\}$ such that η_i, η_j, η_k appears in the counterclockwise order. Obviously $\Delta_{ijk} = \Delta_{jki} = \Delta_{kij}$. Let $\mathbf{T}_\Gamma$ be the collection of all such triangles corresponding to Γ. The two ends of each of η_i, η_j, η_k go into the cusp O and the intersection of the horocycle $\rho = \rho_0$ with the ideal triangle Δ_{ijk} consists of three components, denoted as $h_i^{ijk}, h_j^{ijk}, h_l^{ijk}$ (the red lines in Fig. 2). It is clear that the sum of the lengths $l_{h_i^{ijk}}(X) + l_{h_j^{ijk}}(X) + l_{h_k^{ijk}}(X)$ over all these $4g-3+n$ ideal triangles is less than the length of the horocycle $\rho = \rho_0$. Denote by M_i^{ijk} the midpoint of η_i with respect to the ideal triangle Δ_{ijk}. M_i together with h_j^{ijk}, h_k^{ijk} cuts η_i into four connected components, two of which are compact, denoted by η_{i1}^{ijk} and η_{i2}^{ijk} with respect to the counterclockwise order (see Fig. 2). It follows that

$$l_{\eta_{i2}^{ijk}}(X) = l_{\eta_{j1}^{ijk}}(X); \quad l_{\eta_{j2}^{ijk}}(X) = l_{\eta_{k1}^{ijk}}(X); \quad l_{\eta_{k2}^{ijk}}(X) = l_{\eta_{i1}^{ijk}}(X).$$

Hence

$$\begin{aligned} l_{\eta_i}(X) - l_{\eta_j}(X) + l_{\eta_k}(X) &= 2l_{\eta_{i1}^{ijk}}(X) = 2l_{\eta_{k2}^{ijk}}(X), \\ l_{\eta_j}(X) - l_{\eta_k}(X) + l_{\eta_i}(X) &= 2l_{\eta_{j1}^{ijk}}(X) = 2l_{\eta_{i2}^{ijk}}(X), \\ l_{\eta_k}(X) - l_{\eta_i}(X) + l_{\eta_j}(X) &= 2l_{\eta_{k1}^{ijk}}(X) = 2l_{\eta_{j2}^{ijk}}(X). \end{aligned} \tag{8}$$

Now (7) can be rewritten as

$$\begin{aligned} d_h^\Gamma(X_1, X_2) = \frac{1}{2} \sup_{\Delta_{ijk} \in \mathbf{T}_\Gamma} \max \left\{ \log \frac{l_{\eta_{i1}^{ijk}}(X_1)}{l_{\eta_{i1}^{ijk}}(X_2)}, \log \frac{l_{\eta_{j1}^{ijk}}(X_1)}{l_{\eta_{j1}^{ijk}}(X_2)}, \log \frac{l_{\eta_{k1}^{ijk}}(X_1)}{l_{\eta_{k1}^{ijk}}(X_2)} \right\} \\ + \frac{1}{2} \sup_{\Delta_{ijk} \in \mathbf{T}_\Gamma} \max \left\{ \log \frac{l_{\eta_{i1}^{ijk}}(X_2)}{l_{\eta_{i1}^{ijk}}(X_1)}, \log \frac{l_{\eta_{j1}^{ijk}}(X_2)}{l_{\eta_{j1}^{ijk}}(X_1)}, \log \frac{l_{\eta_{k1}^{ijk}}(X_2)}{l_{\eta_{k1}^{ijk}}(X_1)} \right\}. \end{aligned} \tag{9}$$

And the length of the horocycle h_i^{ijk} can be expressed in terms of $l_{\eta_{j2}^{ijk}}$ and $l_{\eta_{k1}^{ijk}}$ as $l_{h_i^{ijk}}(X) = \exp(-l_{\eta_{k1}^{ijk}}(X)) = \exp(-l_{\eta_{j2}^{ijk}}(X))$. Therefore

$$l_{\eta_{i1}^{ijk}} \geq \rho_0, \; l_{\eta_{j1}^{ijk}} \geq \rho_0, \; l_{\eta_{k1}^{ijk}} \geq \rho_0, \quad \text{for any ideal triangle } \Delta_{ijk} \in \mathbf{T}_\Gamma. \tag{10}$$

We summarize our discussions above as the following proposition.

Proposition 2.3. *With the notations above, the Hilbert metric d_h^Γ on the Teichmüller space $T_{g,n}$ can be expressed as the following two forms.*

- *For any $X_1, X_2 \in T_{g,n}$,*

$$\begin{aligned} d_h^\Gamma(X_1, X_2) = \frac{1}{2} \Bigg(\sup_{\Delta_{ijk} \in \mathbf{T}_\Gamma} \log \frac{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)}{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)} \\ + \sup_{\Delta_{ijk} \in \mathbf{T}_\Gamma} \log \frac{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)}{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)} \Bigg), \end{aligned} \tag{11}$$

  where, $l_{\eta_i}(X) - l_{\eta_j}(X) + l_{\eta_k}(X) \geq 2\rho_0$ for any $X \in T_{g,n}$ and any ideal triangle $\Delta_{ijk} \in \mathbf{T}_\Gamma$.

- *For any* $X_1, X_2 \in T_{g,n}$,

$$d_h^\Gamma(X_1,X_2) = \frac{1}{2}\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\max\left\{\log\frac{l_{\eta_{i1}^{ijk}}(X_1)}{l_{\eta_{i1}^{ijk}}(X_2)}, \log\frac{l_{\eta_{j1}^{ijk}}(X_1)}{l_{\eta_{j1}^{ijk}}(X_2)}, \log\frac{l_{\eta_{k1}^{ijk}}(X_1)}{l_{\eta_{k1}^{ijk}}(X_2)}\right\}$$
$$+\frac{1}{2}\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\max\left\{\log\frac{l_{\eta_{i1}^{ijk}}(X_2)}{l_{\eta_{i1}^{ijk}}(X_1)}, \log\frac{l_{\eta_{j1}^{ijk}}(X_2)}{l_{\eta_{j1}^{ijk}}(X_1)}, \log\frac{l_{\eta_{k1}^{ijk}}(X_2)}{l_{\eta_{k1}^{ijk}}(X_1)}\right\},$$

where $l_{\eta_{i1}^{ijk}} \geq \rho_0$, $l_{\eta_{j1}^{ijk}} \geq \rho_0$, $l_{\eta_{k1}^{ijk}} \geq \rho_0$ *for any ideal triangle* $\Delta_{ijk} \in \mathbf{T}_\Gamma$.

Remark. The reason that we do not consider the Funk metric is that the Funk metric is not projectively invariant while the Hilbert metric is. If we choose a fixed homogenous coordinate for $\mathbf{R}P^{6g-6+2n}$, we can define a Funk metric on $T_{g,n}$.

3. Earthquake

3.1. Earthquake. For any $X \in T_{g,n}$, $\beta \in \mathcal{S}$, denote by $l_\beta(X)$ the length of the geodesic representative of β on X. In [9], Kerckhoff proved that for any simple closed curve β, $l_\beta(\mathcal{E}_\alpha^t X)$ is a convex function of t along the earthquake line $\{\mathcal{E}_\alpha^t X\}_{t\in\mathbf{R}}$. Based on this result, Bonahon proved [1] that each (anti-)earthquake ray converges to a unique point in the Thurston boundary $\partial T_{g,n}$.

Lemma 3.1. [9] *For any* $X \in T_{g,n}$, $\alpha \in \mathcal{ML}$ *and* $\beta \in \mathcal{S}$, *with* $i(\alpha,\beta) > 0$, *then*

$$\frac{d}{dt} l_\beta(\mathcal{E}_\alpha^t X) = \int_\beta \cos\theta_t \, d\alpha,$$

where θ_t *represents the angle at each intersection point between (the geodesic representatives of)* β *and* α *on* $\mathcal{E}_\alpha^t X$ *measured counter-clockwise from* β *to* α. *Moreover, as* t *tends to* $+\infty$ *(resp.* $-\infty$*), the function* $t \mapsto \cos\theta_t$ *is strictly increasing (resp. decreasing).*

Lemma 3.2. [1] *For every* $X \in T_{g,n}$ *and every* $\alpha \in \mathcal{ML}$,

$$\lim_{t\to\pm\infty} \frac{1}{|t|} i(\mathcal{E}_\alpha^t X, \beta) = i(\alpha,\beta), \quad \text{for any } \beta \in \mathcal{S}.$$

By applying similar arguments as in [9] and [1], we get similar results for $l_{\eta_i}(\mathcal{E}_\alpha^t X)$ (note that $l_{\eta_i}(\mathcal{E}_\alpha^t X)$ is the length of a particular compact subarc of η_i).

Lemma 3.3. *For any* $X \in T_{g,n}$, $\alpha \in \mathcal{ML}$ *and* $\eta_i \in \Gamma$,

$$\frac{d}{dt} l_{\eta_i}(\mathcal{E}_\alpha^t X) = \int_{\eta_i} \cos\theta_t d\alpha,$$

where θ_t *represents the angle at each intersection point between (the geodesic representatives of)* η_i *and* α *on* $\mathcal{E}_\alpha^t X$ *measured counter-clockwise from* η_i *to* α. *Moreover, as* t *tends to* $+\infty$ *(resp.* $-\infty$*), the function* $t \mapsto \cos\theta_t$ *is strictly increasing (resp. decreasing).*

Proof. The proof is exactly the same as that of Lemma 3.1 in [9]. □

Lemma 3.4. *For every* $X \in T_{g,n}$ *and every* $\alpha \in \mathcal{ML}$,

$$\lim_{t\to\pm\infty} \frac{1}{|t|} l_{\eta_i}(\mathcal{E}_\alpha^t X) = i(\alpha,\eta_i), \text{ for any } \eta_i \in \Gamma.$$

Proof. The proof is exactly the same as that of Lemma 3.2 in [1]. Also, it follows from Lemma 3.5. □

In fact, we can prove a stronger result.

Lemma 3.5. *For every $X \in T_{g,n}$ and every $\alpha \in \mathcal{ML}$,*

$$|l_{\eta_i}(\mathcal{E}_\alpha^t X) - |t| i(\alpha, \eta_i)| \leq l_{\eta_i}(X) + e^{-\rho_0}, \text{ for any } \eta_i \in \Gamma \text{ and } t \in \mathbf{R},$$

where $e^{-\rho_0}$ represents the length of the horocycle centered at the puncture O with height ρ_0. Moreover, there is a constant c_i^+, $c_i^- \in [-l_{\eta_i}(X) - e^{-\rho_0}, +l_{\eta_i}(X) + e^{-\rho_0}]$ such that

$$\lim_{t\to+\infty} [l_{\eta_i}(\mathcal{E}_\alpha^t X) - t i(\alpha, \eta_i)] = c_i^+, \quad \lim_{t\to-\infty} [l_{\eta_i}(\mathcal{E}_\alpha^t X) - |t| i(\alpha, \eta_i)] = c_i^-.$$

Proof. By symmetry, it suffices to prove the lemma for $t \geq 0$. We start with the case that α is a simple closed geodesic. Let $f_i(t) \triangleq l_{\eta_i}(\mathcal{E}_\alpha^t X) - t i(\eta_i, \alpha)$. We will show that,

$$|f_i(t)| \leq l_{\eta_i}(X) + e^{-\rho_0}, \text{ for any } \eta_i \in \Gamma \text{ and } t \geq 0. \tag{12}$$

From Lemma 3.3,

$$\frac{df_i(t)}{dt} < 0, \text{ for any } t \geq 0.$$

As a consequence, we get our first inequality immediately,

$$f_i(t) \leq f_i(0) = l_{\eta_i}(X).$$

By the monotonicity of $f_i(t)$, for any positive integer m, we have

$$f_i(t) \geq f_i(ml_\alpha(X)), \text{ for any } t \in [0, ml_\alpha(X)].$$

Therefore, to prove $f_i(t) \geq -l_{\eta_i}(X) - e^{-\rho_0}$ for all $t \geq 0$, it suffices to prove $f_i(ml_\alpha(X)) \geq -l_{\eta_i}(X) - e^{-\rho_0}$ for all $m \in \mathbf{N}^+$.

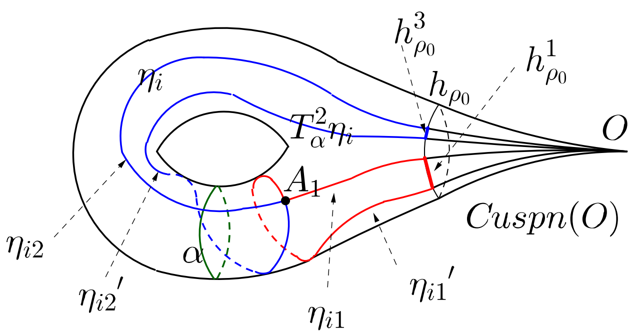


Figure 3. Dehn twist. $T_\alpha^2\beta$ is obtained from β by two times Dehn twist along α.

Recall that $\mathcal{E}_\alpha^{ml_\alpha(X)}X$ is obtained from X by m times Dehn twist along α. Denote by T_α^m the m times Dehn twist along α. Hence $l_{\eta_i}(\mathcal{E}_\alpha^{ml_\alpha(X)}X) = l_{T_\alpha^{-m}\eta_i}(X)$. Denote by h_{ρ_0} the horocycle centered at the puncture O with height ρ_0, and denote by η_i^*, $(T_\alpha^{-m}\eta_i)^*$ the geodesic representatives of η_i, $T_\alpha^{-m}\eta_i$ on X, respectively. It is clear that η_i^* and $(T_\alpha^{-m}\eta_i)^*$ have $mi(\eta_i, \alpha)$ intersection points, one of which is the puncture O. The remaining $mi(\eta_i, \alpha) - 1$ intersection points cut the restrictions of η_i^* and $(T_\alpha^{-m}\eta_i)^*$ on $X \backslash \mathrm{Cuspn}(O)$ into $2mi(\eta_i, \alpha)$ segments $\eta_{i1}, \eta_{i1}', \ldots, \eta_{il}, \eta_{il}'$, where $l = mi(\eta_i, \alpha)$ and $\mathrm{Cuspn}(O)$ represents the cusp neighbourhood of O with boundary horocycle h_{ρ_0}. On the other hand, η_i^* and $(T_\alpha^{-m}\eta_i)^*$ cut the horocycle h_{ρ_0} into four segments $h_{\rho_0}^1, h_{\rho_0}^2, h_{\rho_0}^3, h_{\rho_0}^4$. Two of these four segments, say $h_{\rho_0}^1, h_{\rho_0}^3$, together with above mentioned $2mi(\eta_i,\alpha)$ segments $\eta_{i1},\eta'_{i1},\dots,\eta_{il},\eta'_{il}$ consist $mi(\eta_i,\alpha)$ simple closed curves, and all of them are homotopic to α (see Fig. **??**). More precisely, let $\gamma_1=\eta_{i1}*\eta'_{i1}*h^1_{\rho_0}$, $\gamma_2=\eta_{i2}*\eta'_{i2}$, $\cdots$, $\gamma_{l-1}=\eta_{i,l-1}*\eta'_{i,l-1}$, $\gamma_l=\eta_{il}*\eta'_{il}*h^3_{\rho_0}$. $\gamma_1,\cdots,\gamma_l$ are simple closed curves, and all of them are isotopic to α. It follows that

$$mi(\eta_i,\alpha)l_\alpha(X)\le l_{T_\alpha^{-m}\eta_i}(X)+l_{\eta_i}(X)+e^{-\rho_0},\ \text{for any } m\in\mathbf{N}^+,$$

which means that

$$f_i(ml_\alpha(X))\ge -l_{\eta_i}(X)-e^{-\rho_0},\ \text{for any } m\in\mathbf{N}^+.$$

This completes the proof of (12).

Now we extend these estimates to an arbitrary measured geodesic lamination μ. Recall that $\mathbf{R}^+\times\mathcal{S}$ is dense in $\mathcal{ML}$. There exists $(s_m,\alpha_m)\in\mathbf{R}^+\times\mathcal{S}$ such that $s_m\alpha_m\to\mu$, as $m\to\infty$. It follows from Proposition 1.1 that for any given $t\in\mathbf{R}$, $\mathcal{E}^t_{s_m\alpha_m}X\to\mathcal{E}^t_\mu X$ as $m\to\infty$. Hence for any given $t\in\mathbf{R}$, $l_{\eta_i}(\mathcal{E}^t_{s_m\alpha_m}X)\to l_{\eta_i}(\mathcal{E}^t_\mu X)$ as $m\to\infty$. From (12), we have

$$|ts_mi(\alpha_m,\eta_i)-l_{\eta_i}(\mathcal{E}^t_{s_m\alpha_m}X)|\le l_{\eta_i}(X)+e^{-\rho_0}\ \text{ for any } m\in\mathbf{N}^+,\ \eta_i\in\Gamma \text{ and } t\ge 0.$$

Let m tend to infinity, we have

$$|l_{\eta_i}(\mathcal{E}^t_\mu X)-ti(\mu,\eta_i)|\le l_{\eta_i}(X)+e^{-\rho_0},\ \text{ for any } \eta_i\in\Gamma \text{ and } t\ge 0.$$

The existence of the limit $\lim_{t\to+\infty}[l_{\eta_i}(\mathcal{E}^t_\mu X)-ti(\mu,\eta_i)]$ follows from the boundedness and the monotonicity of $l_{\eta_i}(\mathcal{E}^t_\mu X)-ti(\mu,\eta_i)$. □

Based on the estimates in Lemma 3.5, we can describe the coordinates of an earthquake line $\{\mathcal{E}^t_\mu X\}_{t\in\mathbf{R}}$. Denote by P_1 the hyperplane

$$\{(x_1,x_2,\dots,x_{6g-5+2n}):x_1+x_2+\ldots+x_{6g-5+2n}=1\},$$

and denote by π the projective map

$$\begin{array}{cccc}\pi: & \mathbf{R}^{6g-5+2n}_+ & \longrightarrow & P_1\\ & (x_1,x_2,\dots,x_{6g-5+2n}) & \longmapsto & \dfrac{(x_1,x_2,\dots,x_{6g-5+2n})}{x_1+x_2+\ldots+x_{6g-5+2n}}.\end{array}$$

For a measured geodesic lamination μ, let $I(\mu)=\Sigma_{i=1}^{6g-5+2n}i(\eta_i,\mu)$. It is clear that $\pi\circ\Lambda(\mu)=I(\mu)^{-1}(i(\eta_1,\mu),\dots,i(\eta_{6g-5+2n},\mu))$. Moreover, it follows from Lemma 3.5 that for large enough $|t|$,

$$\pi\circ\Lambda(\mathcal{E}^t_\mu X)=\pi\circ\Lambda(\mu)+\frac{1}{|t|}\frac{c_i}{I(\mu)}+o\left(\frac{1}{|t|}\right).$$

In other words, the images of the earthquake rays $\{\mathcal{E}^t_\mu X\}_{t\ge T}$ and $\{\mathcal{E}^t_\mu X\}_{t\le -T}$ in P_1 look like straight line segments when T is large enough.

Now we prove the main theorem of this paper.

Main Theorem. *After reparametrization, every (anti-)earthquake ray is an almost-geodesic in* $(T_{g,n},d^\Gamma_h)$.

Proof. By symmetry, it suffices to prove the theorem for every earthquake ray. Let $\{\mathcal{E}^t_\mu(X)\}_{t\ge0}$ be an earthquake ray directed by μ and starting at X. Set $X_t=\mathcal{E}^t_\mu(X)$. Let $f_i(t)=l_{\eta_i}(\mathcal{E}^t_\mu X)-ti(\mu,\eta_i)$. It follows from Lemma 3.5 that

$$|f_i(t)|<l_{\eta_i}(X)+e^{-\rho_0},$$

and that for any $\epsilon>0$ there is a constant $T_1>0$ depending on ϵ, X, Γ and μ such that

$$|f_i(t)-c_i|<\epsilon,\ \text{for any } t>T_1.$$

Moreover, if $i(\mu, \eta_j) = 0$ for some j, $l_{\eta_j}(\mathcal{E}_\mu^t X) \equiv l_{\eta_j}(X)$.

Next we divide the set of ideal triangles corresponding to the preferred triangulation Γ into three types:

- Type A consists of ideal triangles Δ_{ijk} which do not intersect μ, i.e. $i(\eta_i, \mu) = i(\eta_j, \mu) = i(\eta_k, \mu) = 0$;
- Type B consists of ideal triangles Δ_{ijk} whose two sides , say η_i, η_j, intersect α and whose remaining side does not intersect μ, i.e. $i(\eta_i, \mu) = i(\eta_j, \mu) > 0$ and $i(\eta_k, \mu) = 0$;
- Type C consists of ideal triangles Δ_{ijk} with each side intersecting μ, i.e. $i(\eta_i, \mu) > 0, i(\eta_j, \mu) > 0, i(\eta_k, \mu) > 0$.

For each triangle Δ_{ijk}, set

$$(13)\quad d_{ijk} = \frac{c_i - c_j + c_k}{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)} \quad \text{and} \quad \bar{d}_{ijk} = \frac{i(\eta_i, \mu) - i(\eta_j, \mu) + i(\eta_k, \mu)}{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)}.$$

Now we discuss case by case.

Type A. In this case, $i(\eta_i, \mu) = i(\eta_j, \mu) = i(\eta_k, \mu) = 0$.

$$\frac{l_{\eta_i}(X_s) - l_{\eta_j}(X_s) + l_{\eta_k}(X_s)}{l_{\eta_i}(X_t) - l_{\eta_j}(X_t) + l_{\eta_k}(X_t)} \equiv \frac{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)}{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)} \equiv 1, \quad \text{for any } s, t \geq 0.$$

Type B. In this case, $i(\eta_i, \mu) = i(\eta_j, \mu) > 0$ and $i(\eta_k, \mu) = 0$. Recall that for any ideal triangle $\Delta_{i'j'k'}$ (see Proposition 2.3),

$$\frac{l_{\eta_{i'}}(X_s) - l_{\eta_{j'}}(X_s) + l_{\eta_{k'}}(X_s)}{2} \geq \rho_0 > 0.$$

Hence for any $\epsilon > 0$, there is a constant $T_2 > 0$ depending on ϵ, Γ, X and μ such that for any $s, t > T_2$, we have

$$\frac{l_{\eta_i}(X_s) - l_{\eta_j}(X_s) + l_{\eta_k}(X_s)}{l_{\eta_i}(X_t) - l_{\eta_j}(X_t) + l_{\eta_k}(X_t)} = \frac{f_i(s) - f_j(s) + l_{\eta_k}(X)}{f_i(t) - f_j(t) + l_{\eta_k}(X)} \in (1-\epsilon, 1+\epsilon),$$

$$\frac{l_{\eta_k}(X_s) - l_{\eta_i}(X_s) + l_{\eta_j}(X_s)}{l_{\eta_k}(X_t) - l_{\eta_i}(X_t) + l_{\eta_j}(X_t)} \in (1-\epsilon, 1+\epsilon),$$

and

$$\frac{l_{\eta_j}(X_s) - l_{\eta_k}(X_s) + l_{\eta_i}(X_s)}{l_{\eta_j}(X_t) - l_{\eta_k}(X_t) + l_{\eta_i}(X_t)} = \frac{i(\eta_j, \mu)s + i(\eta_i, \mu)s + f_i(s) + f_j(s) - l_{\eta_k}(X)}{i(\eta_j, \mu)t + i(\eta_i, \mu)t + f_i(s) + f_j(s) - l_{\eta_k}(X)}$$
$$\in ((1-\epsilon)\frac{s}{t}, (1+\epsilon)\frac{s}{t}).$$

Moreover, there is $T_2' > 0$ depending on ϵ, Γ, X and μ such that for any $s, t > T_2'$, we have

$$\frac{l_{\eta_i}(X_s) - l_{\eta_j}(X_s) + l_{\eta_k}(X_s)}{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)} = \frac{f_i(s) - f_j(s) + l_{\eta_k}(X)}{l_{\eta_i}(X_0) - l_{\eta_j}(X_0) + l_{\eta_k}(X_0)}$$
$$\in ((1-\epsilon)d_{ijk}, (1+\epsilon)d_{ijk}),$$

$$\frac{l_{\eta_k}(X_s) - l_{\eta_i}(X_s) + l_{\eta_j}(X_s)}{l_{\eta_k}(X_0) - l_{\eta_i}(X_0) + l_{\eta_j}(X_0)} \in ((1-\epsilon)d_{kij}, (1+\epsilon)d_{kij}),$$

and

$$\frac{l_{\eta_j}(X_s) - l_{\eta_k}(X_s) + l_{\eta_i}(X_s)}{l_{\eta_j}(X_0) - l_{\eta_k}(X_0) + l_{\eta_i}(X_0)} \in ((1-\epsilon)s\bar{d}_{jki}, (1+\epsilon)s\bar{d}_{jki}),$$

where d_{ijk} and $\bar{d}_{ijk}$ are defined by (13).

Type C. In this case, $i(\eta_i,\mu)>0, i(\eta_j,\mu)>0, i(\eta_k,\mu)>0$. We distinguish two subcases.

If $i(\eta_i,\mu)-i(\eta_j,\mu)+i(\eta_k,\mu)>0$, it follow from Lemma 3.5 that there is a constant $T_3>0$ depending on ϵ, Γ, X and μ such that for any $s,t>T_3$, we have

$$\begin{aligned}\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_t)-l_{\eta_j}(X_t)+l_{\eta_k}(X_t)} &= \frac{i(\eta_i,\mu)s-i(\eta_j,\mu)s+i(\eta_k,\mu)s+f_i(s)-f_j(s)+f_k(s)}{i(\eta_i,\mu)t-i(\eta_j,\mu)t+i(\eta_k,\mu)t+f_i(s)-f_j(s)+f_k(s)}\\ &\in ((1-\epsilon)\frac{s}{t},(1+\epsilon)\frac{s}{t}),\end{aligned}$$

and

$$\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_0)-l_{\eta_j}(X_0)+l_{\eta_k}(X_0)}\in((1-\epsilon)s\bar{d}_{ijk},(1+\epsilon)s\bar{d}_{ijk}).$$

If $i(\eta_i,\mu)-i(\eta_j,\mu)+i(\eta_k,\mu)=0$, there is a constant $T_3'>0$ depending on ϵ, Γ, X and μ such that for any $s,t>T_3'$, we have

$$\begin{aligned}\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_t)-l_{\eta_j}(X_t)+l_{\eta_k}(X_t)} &= \frac{i(\eta_i,\mu)s-i(\eta_i,\mu)s+i(\eta_k,\mu)s+f_i(s)-f_j(s)+f_k(s)}{i(\eta_i,\mu)t-i(\eta_i,\mu)t+i(\eta_k,\mu)t+f_i(s)-f_j(s)+f_k(s)}\\ &= \frac{f_i(s)-f_j(s)+f_k(s)}{f_i(t)-f_j(t)+f_k(t)}\in(1-\epsilon,1+\epsilon),\end{aligned}$$

and

$$\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_0)-l_{\eta_j}(X_0)+l_{\eta_k}(X_0)}\in((1-\epsilon)d_{ijk},(1+\epsilon)d_{ijk}).$$

Therefore, for any $t\geq s\geq\max\{T_1,T_2,T_2',T_3,T_3'\}$,

$$\begin{aligned}d_h^{\Gamma}(X_s,X_t) &= \frac{1}{2}\Big(\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\log\frac{l_{\eta_i}(X_t)-l_{\eta_j}(X_t)+l_{\eta_k}(X_t)}{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}\\ &\quad+\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\log\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_t)-l_{\eta_j}(X_t)+l_{\eta_k}(X_t)}\Big)\\ &\in\Big(\frac{1}{2}\log(1-\epsilon)\frac{t}{s},\frac{1}{2}\log(1+\epsilon)\frac{t}{s}+\frac{1}{2}\log(1+\epsilon)\Big),\end{aligned}\tag{14}$$

and

$$\begin{aligned}d_h^{\Gamma}(X_s,X_0) &= \frac{1}{2}\Big(\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\log\frac{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}{l_{\eta_i}(X_0)-l_{\eta_j}(X_0)+l_{\eta_k}(X_0)}\\ &\quad+\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma}\log\frac{l_{\eta_i}(X_0)-l_{\eta_j}(X_0)+l_{\eta_k}(X_0)}{l_{\eta_i}(X_s)-l_{\eta_j}(X_s)+l_{\eta_k}(X_s)}\Big)\\ &\in\Big(\frac{1}{2}\log\frac{(1-\epsilon)s\bar{d}}{(1+\epsilon)d},\frac{1}{2}\log\frac{(1+\epsilon)s\bar{d}}{(1-\epsilon)d}\Big),\end{aligned}\tag{15}$$

where $d\triangleq\min\{d_{ijk}\colon i(\eta_i,\mu)-i(\eta_j,\mu)+i(\eta_k,\mu)=0,\ i(\eta_i,\mu)+i(\eta_j,\mu)+i(\eta_k,\mu)>0\}$, and $\bar{d}\triangleq\max\{\bar{d}_{ijk}\colon i(\eta_i,\mu)-i(\eta_j,\mu)+i(\eta_k,\mu)>0\}$.

Now, reparametrize the earthquake ray $\{\mathcal{E}_\mu^t X\}_{t\geq0}$ as $\{E_\mu^t X\}_{t\geq0}$ by setting $E_\mu^t X=\mathcal{E}_\mu^{(d/\bar{d})\exp(2t)}X$. It follows from (14) and (15) that for any $\epsilon>0$ there is a $T>0$ depending on ϵ, Γ, X and μ such that

$$|d_h^{\Gamma}(X,E_\mu^s X)-s|<\epsilon,\quad\text{for any } s>T,$$

and

$$|d_h^\Gamma(E_\mu^t X, E_\mu^s X) - (t-s)| < \epsilon, \text{ for any } t \geq s > T.$$

Therefore $\{E_\mu^t X\}_{t\geq 0}$ is an almost geodesic in $(T_{g,n}, d_h^\Gamma)$. □

3.2. The horofunction boundary of $(T_{g,n}, d_h^\Gamma)$. Let (M,d) be a proper geodesic metric space, which is endowed with the topology induced by the metric d. We embed (M,d) into $C(M)$, the space of continuous real-valued functions on X endowed with the topology of uniform convergence on bounded sets, by the map below:

$$\begin{aligned} h\colon M &\longrightarrow C(M) \\ z &\longmapsto [M \ni x \mapsto d(x,z) - d(b,z)], \end{aligned}$$

where $b \in M$ is a base point. The *horofunction boundary* of (M,d) is defined to be

$$\partial\overline{M_b}^{\text{horo}} \triangleq \overline{h(M)}\backslash h(M),$$

where $\overline{h(M)}$ represents the closure of $h(M)$ in $C(M)$. The horofunction boundary is independent of the base point, i.e. $\partial\overline{M_b}^{\text{horo}}$ is homeomorphic to $\partial\overline{M_{b'}}^{\text{horo}}$ for $b, b' \in M$. A function in $\partial\overline{M}^{\text{horo}}$ is called a *horofunction*.

Rieffel ([18]) observed that every almost geodesic converges to a unique point in the horofunction boundary of (X,d). Therefore, we have the following corollary.

Corollary 3.6. *Every (anti-)earthquake ray converges to a unique point in the horofunction boundary of $(T_{g,n}, d_h^\Gamma)$.*

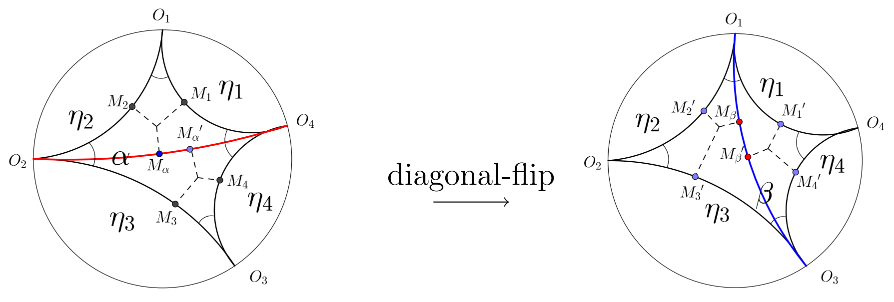
Remark. As we mentioned in the beginning of this section, Bonahon proved [1] that each (anti-)earthquake ray converges to a unique point in the Thurston boundary $\partial T_{g,n}$. It follows from Theorem 2.2 that each (anti-)earthquake ray converges to a point in the Euclidean boundary of the polytope P. In general, let D be a bounded convex domain in the Euclidean space. Foertsch–Karlsson ([4]) proved that every geodesic under the Hilbert metric converges to a boundary point in ∂D in the Euclidean sense. Walsh ([27]) proved that every sequence converging to a point in the horofunction boundary of the Hilbert geometry converges to a point in the Euclidean boundary ∂D. Hence every almost geodesic under the Hilbert metric converges to a boundary point in ∂D.

4. Dependence of d_h^Γ on the preferred triangulation Γ

In this section, we investigate the dependence of the Hilbert metric d_h^Γ on the choice of triangulation Γ. First of all, we define a basic operation for a triangulation, namely, *diagonal-flip*.

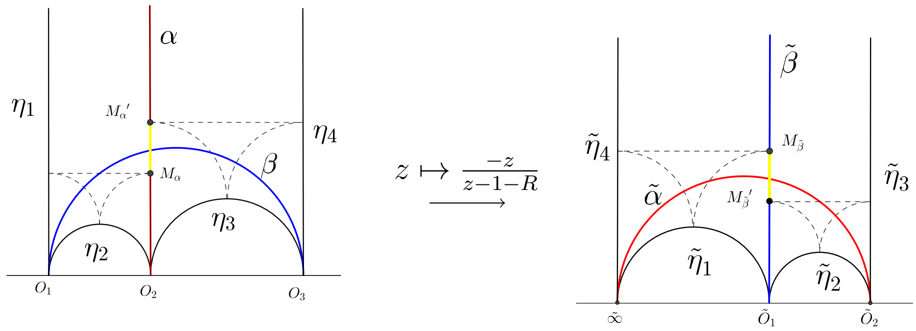
Let Q be an ideal quadrilateral with sides $\eta_1, \eta_2, \eta_3, \eta_4$ and ideal vertices (corners) O_1, O_2, O_3, O_4. Let H_i be a horocycle around the corner O_i whose length is smaller than $e^{-\rho_0}$, $i = 1,2,3,4$, where ρ_0 is chosen in Section 2.3. Let α be a diagonal geodesic connecting O_2, O_4 which triangulates $\mathcal{Q}$ into two triangles Δ_1 and Δ_2, and M_1, M_2, M_3, M_4 the "midpoint" of $\eta_1, \eta_2, \eta_3, \eta_4$ with respect to these two triangles. Further, let M_α and M'_α be the midpoints of α with respect to Δ_1 and Δ_2 respectively. Denote by l_{i1} and l_{i2} the distances from M_i to H_{i-1} and from M_i to H_i respectively. Let β be the other diagonal geodesic, and the notations M'_1, M'_2, M'_3, M'_4, M_β, M'_β, l'_{i1} and l'_{i2} are defined similarly. (see Fig. 4.)

Definition 1. With notations described above, we say that β is obtained from α by a *diagonal-flip* with respect to Q. A triangulation Γ' is said to be obtained from Γ by a diagonal-flip if $\Gamma'\backslash\{\alpha'\} = \Gamma\backslash\{\alpha\}$ and that α' can be obtained from α by a diagonal-flip with respect to some quadrilateral whose four sides are contained in $\Gamma'\backslash\{\alpha'\} = \Gamma\backslash\{\alpha\}$.



(a) (b)

Figure 4. Flip.



(a) (b)

Figure 5. Shearing.

Definition 2. We define the *shearing of Q with respect to α*, denoted by $\mathrm{shr}(\alpha, Q)$, in the following way. The absolute value of $\mathrm{shr}(\alpha, Q)$ is defined to be the distance between M_α and M'_α, and the sign of $\mathrm{shr}(\alpha, Q)$ is defined to "+" if M'_α sits on the left side of M_α observed from Δ_1, otherwise the sign is defined to be "-".

Remark. The "left" notation in the definition of the shearing depends only on the orientation of the ideal quadrilateral.

The lemma below describes some basic properties of a diagonal-flip.

Lemma 4.1. (1) *For the shearing of the ideal quadrilateral Q along β,* $\mathrm{shr}(\beta; Q) = -\mathrm{shr}(\alpha; Q)$.

(2) *Suppose $l_{11} \geq l_{42}$, then* $\mathrm{shr}(\alpha; Q) > 0$ *and*

$$|l'_{i1} - l_{i1} + \mathrm{shr}(\alpha; Q)| \leq \log 2, \quad |l'_{i2} - l_{i1} - \mathrm{shr}(\alpha; Q)| \leq \log 2, \quad i = 1, 3;$$
$$|l'_{i1} - l_{i1}| \leq \log 2, \qquad |l'_{i2} - l_{i2}| \leq \log 2, \qquad i = 2, 4.$$

Proof. Here we adopt the upper half-plane model for the hyperbolic geometry (see Fig. 5). Since the map $z \mapsto kz$ is an isometry of the hyperbolic metric, we suppose that the Euclidean coordinates of O_1, O_2, O_3 are $(0,0), (1,0), (0, 1+R)$ respectively.

(1) The shearing of Q along α is $\log R$. To calculate the shearing of Q along β, we perform the fractional linear map $f\colon z\mapsto \frac{-z}{z-1-R}$ on the upper half plane which sends O_1, O_2, O_3, ∞ to $(0,0), (1/R, 0), \infty, -1$ respectively . It is clear in this case that the shearing of $f(Q)$ along $\tilde{\beta}=f(\beta)$ is $\log(1/R)$, which means the shearing of Q along β is also $\log(1/R)$ since f is an orientation preserving isometry.

(2) By assumption $l_{11}\geq l_{42}$, hence $R\geq 1$. It is clear that $l_{41}+l_{42}=l'_{41}+l'_{42}$. The coordinates of M_4, M'_4 are $(R+1,R)$ and $(1+R,1+R)$ respectively. In addition, the coordinate of the intersection point between η_4 and H_4 is $(1+R, e^{l_{42}}R)$. Therefore $l'_{42}=\log\frac{e^{l_{42}}R}{1+R}\in[l_{42}-\log 2, l_{42}]$, hence $l'_{41}\in[l_{41}, l_{41}+\log 2]$. Similarly we get $l'_{22}\in[l_{22}-\log 2, l_{22}]$ and $l'_{21}\in[l_{21}, l_{22}+\log 2]$.

For the remaining inequalities, note that $l_{12}=l_{21}$ and $l_{11}+l_{12}=l'_{11}+l'_{42}$, hence $l'_{12}=l'_{21}-\mathrm{shr}(\beta;Q)\in[l_{12}+\mathrm{shr}(\alpha;Q), l_{12}+\mathrm{shr}(\alpha;Q)+\log 2]$, and $l'_{11}\in[l_{11}-\mathrm{shr}(\alpha;Q)-\log 2, l_{11}-\mathrm{shr}(\alpha;Q)]$. Similarly we get $l'_{32}\in[l_{32}+\mathrm{shr}(\alpha;Q), l_{32}+\mathrm{shr}(\alpha;Q)+\log 2]$, and $l'_{31}\in[l_{31}-\mathrm{shr}(\alpha;Q)-\log 2, l_{31}-\mathrm{shr}(\alpha;Q)]$. □

The estimates from the lemma above describe a close relationship between the changes of the lengths of preferred arcs and the shearing along each simple geodesic η_i after a diagonal-flip operation. This relationship provides a clue for investigating the effect of the triangulation Γ on the Hilbert metric d_h^Γ, i.e. investigating the relationship between $d_h^\Gamma(X,Y)$ and $d_h^{\Gamma'}(X,Y)$ for two different triangulations Γ, Γ'. If we fix a point $X_0\in T_{g,n}$, the sphere $B^\Gamma(X_0,R)$ centered at X_0 of radius R with respect to d_h^Γ is an almost-sphere, i.e. a sphere up to an additive constant, with respect to $d_h^{\Gamma'}$, where Γ' can be obtained from Γ by a diagonal-flip.

Proposition 4.2. *Fix $X_0\in T_{g,n}$. Let Γ,Γ' be two preferred triangulations of $S_{g,n}$ such that one can be obtained from the other by a diagonal-flip, then there is a constant $C_{\Gamma,\Gamma',X_0,\rho_0}$ depending on Γ,Γ,X_0,ρ_0 such that*

$$|d_h^\Gamma(X_0,X)-d_h^{\Gamma'}(X_0,X)|\leq C_{\Gamma,\Gamma',X_0,\rho_0},\quad \textit{for any } X\in T_{g,n}.$$

Proof. Set $\Gamma=\{\alpha,\eta_1,\dots,\eta_{6g-6+2n}\}$, $\Gamma'=\{\alpha',\eta_1,\dots,\eta_{6g-6+2n}\}$. First, we prove an inequality which holds for any preferred triangulation Γ,

$$\sup_{\eta\in\Gamma\setminus\alpha} l_\eta(X)\leq\sup_{\eta\in\Gamma} l_\eta(X)\leq 2\sup_{\eta\in\Gamma\setminus\alpha} l_\eta(X),\quad\text{for any } X\in T_{g,n}. \tag{16}$$

Indeed, there are two simple geodesics $\eta_i,\eta_j\in\Gamma$ such that η_i, η_j, α bound an ideal triangle. Recall that

$$l_{\eta_j}(X)-l_\alpha(X)+l_{\eta_i}(X)\geq 2\rho_0>0,$$

then

$$l_\alpha(X)\leq 2\max\{l_{\eta_i}(X), l_{\eta_j}(X)\}.$$

Hence

$$\sup_{\eta\in\Gamma} l_\eta(X)\leq 2\sup_{\eta\in\Gamma\setminus\alpha} l_\eta(X).$$

The first inequality is obvious. Now the proposition follows immediately from Proposition 4.3 and (16). □

Proposition 4.3. *Fix $X_0\in T_{g,n}$, then there is a constant C_{Γ,X_0,ρ_0} depending on Γ, X_0, and ρ_0 such that*

$$\left|d_h^\Gamma(X_0,X)-\frac{1}{2}\sup_{\eta\in\Gamma}\log l_\eta(X)\right|\leq C_{\Gamma,X_0,\rho_0},\quad\textit{for any } X\in T_{g,n}.$$

Proof. Set $\Gamma = \{\eta_1, \dots, \eta_{6g-5+2n}\}$. By (11),

$$d_h^\Gamma(X_1, X_2) = \frac{1}{2}\Big(\sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma} \log \frac{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)}{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)} + \sup_{\Delta_{ijk}\in\mathbf{T}_\Gamma} \log \frac{l_{\eta_i}(X_2) - l_{\eta_j}(X_2) + l_{\eta_k}(X_2)}{l_{\eta_i}(X_1) - l_{\eta_j}(X_1) + l_{\eta_k}(X_1)}\Big).$$

Note that

(17) $$2l_{\eta_i}(X) = [l_{\eta_i}(X) - l_{\eta_j}(X) + l_{\eta_k}(X)] + [l_{\eta_j}(X) - l_{\eta_k}(X) + l_{\eta_i}(X)]$$

for any ideal triangle Δ_{ijk}, and

$$[l_{\eta_i}(X) - l_{\eta_j}(X) + l_{\eta_k}(X)] \geq 2\rho_0, \quad \text{for any } \Delta_{ijk}.$$

Therefore

$$d_h^\Gamma(X_0, X) \leq \frac{1}{2}\Big[\log \frac{\sup_{\eta\in\Gamma} l_\eta(X)}{\rho_0} + \log \frac{\sup_{\eta\in\Gamma} l_\eta(X_0)}{\rho_0}\Big] = \frac{1}{2}\sup_{\eta\in\Gamma} \log l_\eta(X) + \frac{1}{2}\sup_{\eta\in\Gamma}\log l_\eta(X_0) - \log \rho_0.$$

Next we deal with the inverse inequality. Without loss of generality, we assume that $l_{\eta 1}(X) = \sup_{\eta\in\Gamma} l_\eta(X)$ and that η_1, η_2, η_3 bounds an ideal triangle. By (17), at least one of $l_{\eta_1}(X) - l_{\eta_2}(X) + l_{\eta_3}(X)$ and $l_{\eta_2}(X) - l_{\eta_3}(X) + l_{\eta_1}(X)$ is not less than $\sup_{\eta\in\Gamma} l_\eta(X)$. Then

$$d_h^\Gamma(X_0, X) \geq \frac{1}{2}[\log \frac{\sup_{\eta\in\Gamma} l_\eta(X)}{2\sup_{\eta\in\Gamma} l_\eta(X_0)}] = \frac{1}{2}\sup_{\eta\in\Gamma}\log l_\eta(X) - \frac{1}{2}[\log 2 + \sup_{\eta\in\Gamma} l_\eta(X_0)].$$

Set $C_{\Gamma,X_0,\rho_0} \triangleq \max\{\frac{1}{2}\sup_{\eta\in\Gamma}\log l_\eta(X_0) - \log\rho_0, \frac{1}{2}\sup_{\eta\in\Gamma} l_\eta(X_0) + \frac{1}{2}\log 2\}$, the proposition follows. □

5. Actions of mapping class group

The mapping class group $MCG(S_{g,n})$ consists of the isotopy classes of orientation-preserving self homeomorphisms of $S_{g,n}$. In this section, we study the actions of mapping class group $MCG(S_{g,n})$ on the metric space $(T_{g,n}, d_h^\Gamma)$. The action is defined as following. For $g \in MCG(S_{g,n})$ and $(X, f) \in T_{g,n}$, $g\circ(X, f)$ is defined as the marked hyperbolic surface $(X, f\circ g^{-1})$.

Denote by $PMCG(S_{g,n})$ the subgroup of $MCG(S_{g,n})$ consisting of elements that fix each puncture individually. It is well known that $PMCG(S_{g,n})$ can be generated by finitely many Dehn twists about nonseparating simple closed curves, where a nonseparating simple closed curve α is a closed curve such that $S_{g,n}\backslash\alpha$ is connected (see [2, Chap. 5]).

The lemma below describes the changes of $l_{\eta_i}(X)$ under a Dehn twist.

Lemma 5.1. *Assume that Δ_{123} is an ideal triangle on $X \in T_{g,n}$ with three sides η_1, η_2, η_3, and that g is a positive Dehn twist of $S_{g,n}$ along an essential simple closed curve α. Denote by Δ'_{123}, $\eta'_1, \eta'_2, \eta'_3$, the images of Δ_{123} , η_1, η_2, η_3, respectively, under the action of g. Then, there is a constant C depending on the length $l_\alpha(X)$, the reference height ρ_0 and the isotopy classes of α,η_1, η_2, η_3 such that*

$$\frac{1}{C} \leq \frac{l_{\eta'_{i+1}}(X) + l_{\eta'_{i-1}}(X) - l_{\eta'_i}(X)}{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)} \leq C, \quad i = 1, 2, 3.$$

Proof. Note that

$$|l_{\eta_i'}(X) - l_{\eta_i}(X)| \le i(\eta_i, \alpha) l_\alpha(X), \quad i = 1, 2, 3,$$
$$i(\eta_i', \alpha) = i(\eta_i, \alpha), \quad i = 1, 2, 3.$$

On the other hand, from (10),

$$l_{\eta_{i+1}'}(X) + l_{\eta_{i-1}'}(X) - l_{\eta_i'}(X) > 2\rho_0, \quad i = 1, 2, 3;$$
$$l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X) > 2\rho_0, \quad j = i-1, i, i+1.$$

Hence

$$\begin{aligned}\frac{l_{\eta_{i+1}'}(X) + l_{\eta_{i-1}'}(X) - l_{\eta_i'}(X)}{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)} &\le 1 + \frac{[i(\eta_{i+1}, \alpha) + i(\eta_i, \alpha) + i(\eta_{i-1}, \alpha)] l_\alpha(X)}{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)} \\ &\le 1 + \frac{[i(\eta_{i+1}, \alpha) + i(\eta_i, \alpha) + i(\eta_{i-1}, \alpha)] l_\alpha(X)}{2\rho_0}.\end{aligned}$$

Interchange η_i with η_i', we get the inverse inequality

$$\frac{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)}{l_{\eta_{i+1}'}(X) + l_{\eta_{i-1}'}(X) - l_{\eta_i'}(X)} \le 1 + \frac{[i(\eta_{i+1}, \alpha) + i(\eta_i, \alpha) + i(\eta_{i-1}, \alpha)] l_\alpha(X)}{2\rho_0}. \qquad \square$$

As an application, we have the following.

Corollary 5.2. *Let Γ be a preferred triangulation of $S_{g,n}$. Let g be a positive Dehn twist of $S_{g,n}$ along an essential simple closed curve α. Set $M_{\alpha,l} \triangleq \{X \in T_{g,n} : l_\alpha(X) \le l\}$. Then there is a constant $C_{\Gamma,\alpha,\rho,l}$ depending on Γ, the isotopy class of α, the reference height ρ_0 and l such that*

$$|d_h^\Gamma(X, Y) - d_h^\Gamma(gX, gY)| \le C_{\Gamma,\alpha,\rho_0,l}, \quad \text{for any } X, Y \in M_{\alpha,l}.$$

Proof. It follows from Lemma 5.1 that

$$\begin{aligned}\frac{1}{C_{\Gamma,\alpha,\rho}} \frac{l_{\eta_{i+1}}(Y) + l_{\eta_{i-1}}(Y) - l_{\eta_i}(Y)}{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)} &\le \frac{l_{\eta_{i+1}}(gY) + l_{\eta_{i-1}}(gY) - l_{\eta_i}(gY)}{l_{\eta_{i+1}}(gX) + l_{\eta_{i-1}}(gX) - l_{\eta_i}(gX)} \\ &\le C_{\Gamma,\alpha,\rho_0} \frac{l_{\eta_{i+1}}(Y) + l_{\eta_{i-1}}(Y) - l_{\eta_i}(Y)}{l_{\eta_{i+1}}(X) + l_{\eta_{i-1}}(X) - l_{\eta_i}(X)},\end{aligned}$$

where $C_{\Gamma,\alpha,\rho_0} = \left[1 + (l/\rho_0) \sum_{i=1}^{6g-5+2n} i(\eta_i, \alpha)\right] \left[1 + (l/\rho_0) \sum_{i=1}^{6g-5+2n} i(\eta_i, \alpha)\right]$. $\square$

We do not know whether or not the action of $MCG(S_{g,n})$ on $(T_{g,n}, d_h^\Gamma)$ is quasi-isometric. But for any given $X, Y \in T_{g,n}$, we have the following asymptotic behaviour.

Proposition 5.3. *Let Γ be a preferred triangulation, and $g \in MCG(S_{g,n})$ be a positive Dehn twist about a simple closed curve α. For any given $X, Y \in T_{g,n}$, there is a positive number $C_{X,Y}$ depending on X, Y such that*

$$\lim_{n \to \infty} d_h^\Gamma(g^n X, g^n Y) = C_{X,Y}.$$

Moreover, for any $X \in T_{g,n}$,

$$\lim_{n \to \infty} d_h^\Gamma(g^n X, g^{n+1} X) = 0.$$

Proof. Note that a Dehn twist is also an earthquake map, i.e. $g = \mathcal{E}_\alpha^{l_\alpha(X)}$. The remaining discussion is similar to the proof of the Main Theorem. $\square$

It follows immediately from Proposition 5.3 that $(T_{g,n}, d_h^\Gamma)$ is not $MCG(S_{g,n})$ invariant. More precisely, we have the following corollary.

Corollary 5.4. *Let Γ be a preferred triangulation, and $g \in MCG(S_{g,n})$ be a positive Dehn twist about a simple closed curve α. Then the action of g on $T_{g,n}$ is not isometric. In particular, $(T_{g,n}, d_h^\Gamma)$ is not $MCG(S_{g,n})$ invariant.*

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Received 30 September 2015 • Accepted 15 January 2016