Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 713–720

SHARPNESS OF FALCONER'S $\frac{d+1}{2}$ ESTIMATE

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Abstract. In the paper introducing the celebrated Falconer distance problem, Falconer proved that the Lebesgue measure of the distance set is positive, provided that the Hausdorff dimension of the underlying set is greater than $\frac{d+1}{2}$. His result is based on the estimate

$$\mu \times \mu\{(x,y)\colon 1 \le |x-y| \le 1+\epsilon\} \lesssim \epsilon, \tag{0.1}$$

where μ is a Borel measure satisfying the energy estimate $I_s(\mu) = \int\int |x-y|^{-s}\,d\mu(x)\,d\mu(y) < \infty$ for $s > \frac{d+1}{2}$. An example due to Mattila [12, Remark 4.5], [11] shows in two dimensions that for no $s < \frac{3}{2}$ does $I_s(\mu) < \infty$ imply (0.1). His construction can be extended to three dimensions. Mattila's example readily applies to the case when the Euclidean norm in (0.1) is replaced by a norm generated by a convex body with a smooth boundary and non-vanishing Gaussian curvature. In this paper we prove, for all $d \ge 2$, that for no $s < \frac{d+1}{2}$ does $I_s(\mu) < \infty$ imply (0.1) or the analogous estimate where the Euclidean norm is replaced by the norm generated by a particular convex body B with a smooth boundary and everywhere non-vanishing curvature. Our construction is based on a combinatorial construction due to Valtr [15].

1. Introduction

The classical Falconer distance conjecture [5] says that if the Hausdorff dimension of a compact set $E \subset \mathbf{R}^d$, $d \ge 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of

$$\Delta(E) = \{|x-y|\colon x, y \in E\}$$

is positive. Here $|\cdot|$ denotes the Euclidean distance. The problem was introduced by Falconer in [5] where he proves that the Lebesgue measure of $\Delta(E)$, denoted by $\mathcal{L}^1(\Delta(E))$, is indeed positive if the Hausdorff dimension of E, denoted by $\dim_{\mathcal{H}}(E)$, is greater than $\frac{d+1}{2}$. Since then, due to efforts of Bourgain [2], Erdoğan [4], Mattila [12, 13], Wolff [16] and others, the exponent has been improved, with the best current result due to Wolff in two dimensions [16] and Erdoğan in higher dimensions [4]. They proved that $\mathcal{L}^1(\Delta(E)) > 0$ provided that $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$. See also [14] where the authors prove that if $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, then $\Delta(E)$ contains an interval.

Falconer's $\frac{d+1}{2}$ exponent follows from the following key estimate. Suppose that μ is Borel measure on E such that

$$I_s(\mu) = \iint |x-y|^{-s}\,d\mu(x)\,d\mu(y) < \infty$$

for some $s \ge \frac{d+1}{2}$. Then

$$\mu \times \mu\{(x,y)\colon 1 \le |x-y| \le 1+\epsilon\} \lesssim \epsilon, \tag{1.1}$$

doi:10.5186/aasfm.2016.4145
2010 Mathematics Subject Classification: Primary 28A75, 51A20, 52C10.
Key words: Geometric measure theory, discrete geometric combinatorics.
This work was partially supported by the NSA Grant H98230-15-1-0319.

where here and throughout, $X \lesssim Y$ means that there exists a uniform $C > 0$ such that $X \le CY$.

This estimate follows by Plancherel and the fact that if σ denotes the Lebesgue measure on the unit sphere, then

$$(1.2) \qquad |\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

This implies, in particular, that (1.1) still holds if the Euclidean distance $|\cdot|$ is replaced by $\|\cdot\|_B$, where B is a symmetric convex body with a smooth boundary and everywhere non-vanishing Gaussian curvature. This is because the estimate (1.2) still holds if σ is replaced by σ_B, the Lebesgue measure on ∂B. To be precise, under these assumptions on B, the estimate

$$(1.3) \qquad \mu \times \mu\{(x,y)\colon 1 \le \|x-y\|_B \le 1+\epsilon\} \lesssim \epsilon$$

holds provided that $I_s(\mu) < \infty$ with $s > \frac{d+1}{2}$.

A consequence of this more general version of (1.1) is that $\mathcal{L}^1(\Delta_B(E)) > 0$ whenever $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, where

$$\Delta_B(E) = \{\|x-y\|_B\colon x, y \in E\}.$$

See, for example, [1], [6], [7], [8] and [9] for the description of this generalization of the Falconer distance problem and its connections with other interesting problems in geometric measure theory and other areas.

An example due to Mattila (see [11]) shows in two dimensions that for no $s < \frac{3}{2}$ does

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y) < \infty$$

imply (1.3). Mattila's construction can be generalized to three dimensions (see Section 3 below). However, in dimensions four and higher, his method does not seem to apply. It is important to note that in any dimension, an example due to Falconer [5] shows that for no $s < \frac{d}{2}$ does $I_s(\mu) < \infty$ imply that the estimate (1.3) holds. We record these calculations for the reader's convenience in the Section 3 below.

In this paper we construct a measure in all dimensions which shows that for no $s < \frac{d+1}{2}$ does $I_s(\mu) < \infty$ imply that (1.3) holds. More precisely, we have the following result.

Theorem 1.1. *There exists a symmetric convex body B with a smooth boundary and non-vanishing Gaussian curvature, such that for any $s < \frac{d+1}{2}$, there exists a Borel measure μ_s, such that $I_s(\mu) \approx 1$ and*

$$(1.4) \qquad \limsup_{\epsilon \to 0}\ \epsilon^{-1} \mu_s \times \mu_s\{(x,y)\colon 1 \le \|x-y\|_B \le 1+\epsilon\} = \infty.$$

Remark 1.2. The proof will show that ϵ^{-1} in (1.4) may be replaced by $\epsilon^{-\frac{2s}{d+1}-\gamma}$ for any $\gamma > 0$. We also note that we only need to establish (1.4) with $s \ge \frac{d}{2}$ since if $s < \frac{d}{2}$, the example due to Falconer [5], mentioned above, does the job.

Another way of stating the conclusion of Theorem 1.1 is that for no $s < \frac{d+1}{2}$ does $I_s(\mu) < \infty$ imply that the distance measure is in $L^\infty(\mathbf{R})$. The distance measure ν is defined by the relation

$$\int g(t)\, d\nu(t) = \iint g(\|x-y\|_B)\, d\mu(x)\, d\mu(y).$$

1.1. Structure of the paper. Theorem 1.1 is proved in Section 2 below. The idea is to make a construction for a specific convex body obtained by glueing the upper and lower hemispheres of the paraboloid. In the Subsection 2.1 we describe the combinatorial construction used in the proof of Theorem 1.1. In Subsection 2.2 we use the combinatorial construction from Section 2.1 to complete the proof of Theorem 1.1. In Section 3 we describe the aforementioned example due to Mattila and generalize it to three dimensions.

2. Proof of the main result

2.1. Combinatorial underpinnings. The proof of Theorem 1.1 uses a generalization of the two-dimensional construction due to Pavel Valtr (see [3, 15]). A similar construction can also be found in [10] in a slightly different context. Let

$$P_n = \left\{ \left(\frac{i_1}{n}, \frac{i_2}{n}, \ldots, \frac{i_{d-1}}{n}, \frac{i_d}{n^2} \right) : 0 \le i_j \le n-1, \text{ for } 1 \le j \le d-1, \text{ and } 1 \le i_d \le n^2 \right\}.$$

Notice that in each of the first $d-1$ coordinates, there are n evenly distributed points, but in the last dimension, there are n^2 evenly distributed points. Now, let

$$H = \{(t_1, t_2, \ldots, t_{d-1}, t_1^2 + t_2^2 + \cdots + t_{d-1}^2) \in \mathbf{R}^d \colon t_1, t_2, \ldots, t_{d-1} \in \mathbf{R}\}$$

and define

$$L_H = \{H + p \colon p \in P_n\}.$$

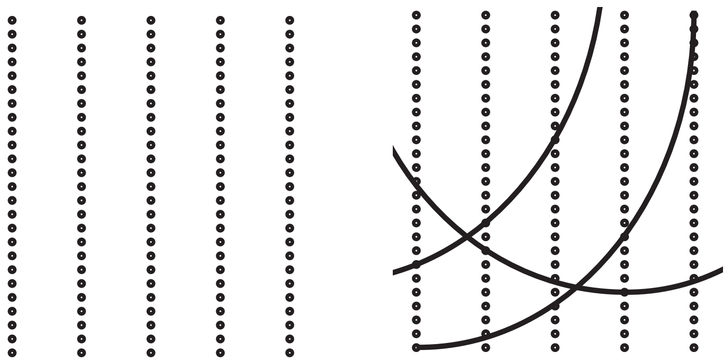

Figure 1. On the left, we see a picture of the set P_5, on the right, we see it again with a few parabolic arcs, which intersect a point in each column.

Let $N = n^{d+1}$. By construction, $\#P_n = \#L_H = N$. Also by construction, each element of L_H is incident to about $n^{d-1} \approx N^{\frac{d-1}{d+1}}$ elements of P. Thus the total number of incidences between P and L_H is

$$\approx N^{1+\frac{d-1}{d+1}} = N^{\frac{2d}{d+1}} = N^{2-\frac{2}{d+1}}.$$

2.1.1. Construction of the norm. With this construction in hand, it is easy enough to flip the paraboloid upside down and glue it to another copy. Explicitly, let

$$\begin{aligned} B_U = \Big\{ (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d \colon x_i \in [-1,1], \text{ for } 1 \le i \le d-1, \\ \text{and } x_d = 1 - \left(x_1^2 + x_2^2 + \cdots + x_{d-1}^2\right) \Big\}, \end{aligned}$$

and

$$\begin{aligned} B_L = \Big\{ (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d \colon x_i \in [-1,1], \text{ for } 1 \le i \le d-1, \\ \text{and } x_d = -1 + x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \Big\}. \end{aligned}$$

Now, let

$$B' = \left(B_U \cap \left\{(x_1, x_2, \ldots, x_d) \in \mathbf{R}^d \colon x_d \geq 0\right\}\right) \\ \cup \left(B_L \cap \left\{(x_1, x_2, \ldots, x_d) \in \mathbf{R}^d \colon x_d \leq 0\right\}\right).$$

Finally, define B to be a convex body B', with the ridge at the transition between B_U and B_L smoothed.

Let L denote N copies of ∂B, each translated by an element of P_n. Now we have a symmetric convex body $B \subset \mathbf{R}^d$ with a smooth boundary and everywhere non-vanishing curvature, a point set P_n of size N and a set L of translates of ∂B, of size $\approx N$, such that the number of incidences between P_n and L is $\approx N^{2-\frac{2}{d+1}}$.

The reader may be aware of the fact that in dimensions four and higher, a more dramatic combinatorial example is available.

2.1.2. Lenz construction. (See e.g. [3]) More precisely, choose $N/2$ points evenly spaced on the circle

$$\{(\cos(\theta), \sin(\theta), 0, 0) \colon \theta \in [0, 2\pi)\}$$

and $N/2$ points evenly spaced on the circle

$$\{(0, 0, \cos(\phi), \sin(\phi)) \colon \phi \in [0, 2\pi)\}.$$

Let K_N be the union of the two point sets. It is not hard to check that all the distances between the points on one circle and the points on the other circle are equal to $\sqrt{2}$. It follows that the number of incidences between the points of K_N and the circles of radius $\sqrt{2}$ centered at the points of K_N is $\approx N^2$, which is about as bad as it can be and much larger than the $N^{2-\frac{2}{d+1}}$ obtained in the generalization of Valtr's example above. However, this construction will not help in the continuous setting due to certain peculiarities of the Hausdorff dimension.

2.2. Using combinatorial information to construct the needed measures. Let $\frac{d}{2} \leq s < \frac{d+1}{2}$. There is no point going below $\frac{d}{2}$ because the lattice-based construction in [5] shows that (1.1) cannot hold in that regime. Partition $[0,1]^d$ into lattice cubes of side-lengths ϵ where $\epsilon^{-s} = N$ for some large integer N. Let $n = N^{\frac{1}{d+1}}$. Put P_n in the unit cube and select any lattice cube which contains a point of P_n. Let Q_n denote the set of centers of the selected lattice cubes.

Now, we define L_ϵ to be the union of the ϵ-neighborhoods of the elements of L. That is, for every translate of ∂B by an element, $p \in P_n$, let l_p denote the locus of points that are within ϵ of the translate of ∂B by p. Then

$$L_\epsilon = \bigcup_{p \in P_n} l_p.$$

Lemma 2.1. *Let μ_s denote the Lebesgue measure on the union of the selected cubes above, normalized so that*

$$\int d\mu_s(x) = 1.$$

More precisely,

$$d\mu_s(x) = \epsilon^{s-d} \sum_{p \in P_n} \chi_{R_\epsilon(p)}(x)\, dx,$$

where $R_\epsilon(p)$ denotes the cube of side-length ϵ centered at p. Then

$$I_s(\mu_s) \approx 1.$$

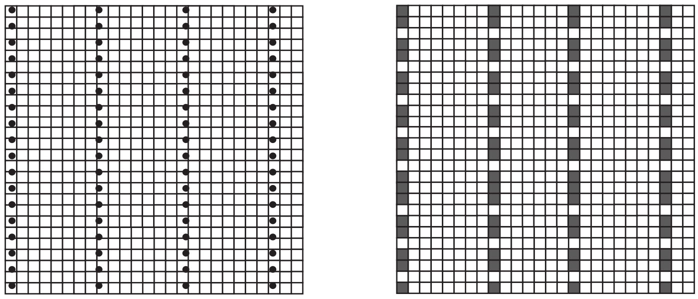

Figure 2. On the left, we have P_5, in the partitioned unit cube. On the right, we filled in every cube which contained a point. Notice that there are gaps in the columns corresponding to cubes which did not contain any points.

Proof. To prove the lemma, observe that

$$\begin{aligned} I_s(\mu) &= \int\int |x-y|^{-s}\,d\mu_s(x)\,d\mu_s(y) \\ &= \epsilon^{2(s-d)} \sum_{p,q\in P_n} \int\int |x-y|^{-s}\chi_{R_\epsilon(p)}(x)\chi_{R_\epsilon(q)}(y)\,dx\,dy \\ &= \epsilon^{2(s-d)} \sum_{p\in P_n} \int\int |x-y|^{-s}\chi_{R_\epsilon(p)}(x)\chi_{R_\epsilon(p)}(y)\,dx\,dy \\ &\quad + \epsilon^{2(s-d)} \sum_{p\neq q\in P_n} \int\int |x-y|^{-s}\chi_{R_\epsilon(p)}(x)\chi_{R_\epsilon(q)}(y)\,dx\,dy = I + II. \end{aligned}$$

We have

$$I = \epsilon^{2(s-d)} \sum_{p\in P_n} \int_{R_\epsilon(p)}\int_{R_\epsilon(p)} |x-y|^{-s}\,dx\,dy.$$

Making the change of variables $X = x-y, Y = y$, we see that

$$I \lesssim \epsilon^{2(s-d)} \sum_{p\in P_n} \epsilon^d \int_{|X|\leq\sqrt{d}\epsilon} |X|^{-s}\,dX \lesssim \epsilon^{2(s-d)}\cdot\epsilon^d\cdot\epsilon^{d-s}\sum_{p\in P_N} 1 = \epsilon^s\cdot N \lesssim 1.$$

On the other hand,

$$II \approx \sum_{p\neq q\in P_n} |p-q|^{-s}\epsilon^{2s} = N^{-2}\sum_{p\neq q\in P_n} |p-q|^{-s}.$$

We have

$$p = (p', p_d) = \left(\frac{i_1}{n},\dots,\frac{i_{d-1}}{n},\frac{i_d}{n^2}\right)$$

and

$$q = (q', q_d) = \left(\frac{j_1}{n},\dots,\frac{j_{d-1}}{n},\frac{j_d}{n^2}\right).$$

Let $i' = (i_1,\dots,i_{d-1})$ and $j' = (j_1,\dots,j_{d-1})$. Thus we must consider

$$N^{-2} \sum_{i\neq j;|i'|,|j'|\leq n; i_d,j_d\leq n^2} \left|\left|\frac{i'-j'}{n}\right| + \left|\frac{i_d-j_d}{n^2}\right|\right|^{-s}.$$

Replacing the sum by the integral, we obtain

$$N^{-2}\iint\dots\int_{\substack{|i_1|,|j_1|,\dots,|i_{d-1}|,|j_{d-1}|\le n\\ i_d,j_d\le n^2}}\left|\left|\frac{i'-j'}{n}\right|+\left|\frac{i_d-j_d}{n^2}\right|\right|^{-s}di'\,dj'\,di_d\,dj_d,$$

which, by a change of variables, $u'=(i'/n), u_d=(i_d/n^2), v'=(j'/n)$, and $v_d=(j_d/n^2)$, with similarly named coordinates, becomes

$$=\iint\dots\int_{\substack{u\neq v\\ |u_1|,|v_1|,\dots,|u_{d-1}|,|v_{d-1}|\le 1\\ u_d,v_d\le 1}}|u-v|^{-s}\,du'\,dv'\,du_d\,dv_d\lesssim 1.$$

This completes the proof of Lemma 2.1. □

We are now ready to complete the argument in the case of the paraboloid. We have

$$\mu_s\times\mu_s\{(x,y)\colon 1\le\|x-y\|_B\le 1+\epsilon\}$$

is $\approx C\epsilon^{2s}$ times the number of incidences between the elements of Q_n and L_ϵ, where Q_n and L_ϵ are constructed in the beginning of this section. Invoking our generalization of Valtr's construction from Section 2.1 above, we see that

$$\mu_s\times\mu_s\{(x,y)\colon 1\le\|x-y\|_B\le 1+\epsilon\}\approx\epsilon^{2s}\cdot N^{2-\frac{2}{d+1}}\approx N^{-\frac{2}{d+1}}.$$

This quantity is much greater than $\epsilon=N^{-\frac{1}{s}}$ when $s<\frac{d+1}{2}$. This completes the proof of Theorem 1.1.

3. Mattila's construction

In this section we describe Mattila's construction from [11] and its generalization to three dimensions.

First, we review the method of constructing a Cantor set of a given Hausdorff dimension, (see [13]). If we want a Cantor set, $\mathcal{C}_\alpha$, of Hausdorff dimension $0<\alpha<1$, we need to find the $0<\lambda<1/2$ which satisfies $\alpha=\log 2/\log(1/\lambda)$. Start with the unit segment, then remove the interval $(\frac{1}{2}-\lambda/2,\frac{1}{2}+\lambda/2)$. Next, remove the middle λ proportion of each of the remaining subintervals, and so on. The classic "middle-thirds" Cantor set would be generated with $\lambda=1/3$.

To construct the two-dimensional example, $\mathcal{M}_2(\alpha)$, we let $F=(\mathcal{C}_\alpha)\cup(\mathcal{C}_\alpha-1)$. Then define $\mathcal{M}_2(\alpha)=F\times[0,1]$. Define the measure μ to be $(\mathcal{H}^\alpha|F)\times(\mathcal{L}^1|[0,1])$, where $\mathcal{H}^\alpha$ is the α-dimensional Hausdorff measure.

Pick a point $x=(x_1,x_2)\in\mathcal{M}_2(\alpha)$. Notice that if $x_1\in F$, either x_1+1 or x_1-1 is also in F. So there is an ϵ-annulus, with radius 1, centered at x, which contains a rectangle of width ϵ and length $\sqrt{\epsilon}$. This rectangle intersects $\mathcal{M}_2(\alpha)$ lengthwise. The measure of this intersection is $\epsilon^{1/2+\alpha}$. This follows easily from the fact that the circle has non-vanishing curvature. It follows that

$$\mu\{y\colon 1\le|x-y|\le 1+\epsilon\}\gtrsim\epsilon^{\alpha+1/2}$$

for every x. It follows that

$$\mu\times\mu\{(x,y)\colon 1\le|x-y|\le 1+\epsilon\}=\int\mu\{y\colon 1\le|x-y|\le 1+\epsilon\}\,d\mu(x)\gtrsim\epsilon^{\alpha+1/2}.$$

We conclude that

$$\mu\times\mu\{(x,y)\colon 1\le|x-y|\le 1+\epsilon\}\lesssim\epsilon$$

only if

$$\epsilon^{\alpha+\frac{1}{2}}\lesssim 1,$$

which can only hold if
$$\alpha \geq \frac{1}{2}.$$

Thus the estimate (1.3) does not in general hold for sets with Hausdorff dimension less than $\frac{3}{2}$. Letting α get arbitrarily small yields a family of counterexamples with Hausdorff dimensions arbitrarily close to 1 , below which there are already counterexamples. See, for example, [5]. Note that we worked in $[-1,1] \times [0,1]$ instead of $[0,1] \times [0,1]$, to allow the main point to shine.

To construct $\mathcal{M}_3(\delta)$, the three-dimensional example, we set
$$\mathcal{M}_3(\delta) = (\mathcal{C}_\alpha \cup (\mathcal{C}_\alpha - 1)) \times (\mathcal{C}_\alpha \cup (\mathcal{C}_\alpha - 1)) \times \mathcal{C}_\beta,$$
where $\alpha = 1 - \delta$, and $\beta = \delta/2$, and δ is determined later. We will set μ to be a product of the appropriate Hausdorff measures restricted to this set, much like the previous example. Notice that $\mathcal{M}_3(\delta)$ has a Hausdorff dimension of $2 - \frac{3}{2}\delta$, and for a given point $x \in \mathcal{M}_3(\delta)$, there is an $\epsilon^{\frac{1}{2}}$ by $\epsilon^{\frac{1}{2}}$ by ϵ box inside the annulus whose measure is
$$\epsilon^{\alpha/2} \cdot \epsilon^{\alpha/2} \cdot \epsilon^{\beta} = \epsilon^{1-\delta/2}.$$

Once again, we have used the fact that the sphere has non-vanishing Gaussian curvature, which implies, by elementary geometry, that the ϵ-annulus contains an $\epsilon^{\frac{1}{2}}$ by $\epsilon^{\frac{1}{2}}$ by ϵ box. It follows that
$$\mu\{y \colon 1 \leq |x-y| \leq 1+\epsilon\} \gtrsim \epsilon^{1-\delta/2}$$
for every $x \in \mathcal{M}_3(\delta)$, which means that
$$\mu \times \mu\{(x,y) \colon 1 \leq |x-y| \leq 1+\epsilon\} \gtrsim \epsilon^{1-\delta/2},$$
so (1.3) does not hold.

Thus we shown that for $s < 2 = \frac{d+1}{2}$ (when $d = 3$), $I_s(\mu) < \infty$ does not imply that (1.3) holds. Observe that both constructions in this section work for any convex B such that ∂B is smooth and has everywhere non-vanishing Gaussian curvature.

Acknowledgement. We would like to thank the anonymous referee for helpful comments.

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Received 7 January 2016 • Accepted 3 February 2016