Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 721–744

SOME NORM INEQUALITIES IN MUSIELAK–ORLICZ SPACES

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Abstract. Our aim in this paper is to establish various norm inequalities in Musielak–Orlicz spaces. We give a generalization of a result due to Cruz-Uribe, Fiorenza, Martell and Pérez and apply it to obtain norm inequalities for classical operators as well as an Olsen inequality in Musielak–Orlicz spaces.

1. Introduction

There has been a considerable amount of studies on the variable exponent Lebesgue spaces $L^{p(\cdot)}$; see [5, 7] etc. for exhaustive account of this direction of research. In those studies, various kinds of norm inequalites were discussed, including those which show the boundedness of important operators. Cruz-Uribe, Fiorenza, Martell and Pérez [6] gave a method to obtain $L^{p(\cdot)}$-norm inequalities from $L^{p_0}(w)$-norm inequalities with a constant exponent p_0 and weights w. In fact, they proved [6, Theorem 1.3]:

Theorem A. *Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions on $\mathbf{R}^N$. Suppose that*

$$\int_{\mathbf{R}^N} f(x)^{p_0} w(x)\, dx \le C_0 \int_{\mathbf{R}^N} g(x)^{p_0} w(x)\, dx \tag{1.1}$$

for some $p_0 > 0$, for all $(f, g) \in \mathcal{F}$ and for all A_1-weights w with a constant C_0 depending only on p_0 and the A_1-constant of w. Let $p(\cdot)$ be a variable exponent such that

$$1 \le p^- = \operatorname{ess\,inf}_{x\in\mathbf{R}^N} p(x) \le p^+ = \operatorname{ess\,sup}_{x\in\mathbf{R}^n} p(x) < \infty.$$

If $p_0 < p^-$ and the Hardy–Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbf{R}^N)$, then there is a constant $C > 0$ such that

$$\|f\|_{p(\cdot)} \le C\|g\|_{p(\cdot)}$$

for all $(f, g) \in \mathcal{F}$ with $g \in L^{p(\cdot)}(\mathbf{R}^N)$.

In the present paper, we call this theorem CFMP-theorem.

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doi:10.5186/aasfm.2016.4148
2010 Mathematics Subject Classification: Primary 26A33, 42B25, 42B35, 41A17.
Key words: Musielak–Orlicz space, Riesz potential, weighted inequality, singular integral operator, Kerman–Sawyer inequality, Olsen inequality.
The third author was partially supported by Grant-in-Aid for Scientific Research (C), No. 24540174, Japan Society for the Promotion of Science.

Variable exponent Lebesgue spaces are special cases of Musielak–Orlicz spaces, which were first considered by Nakano as modulared function spaces in [25] and then developed by Musielak as generalized Orlicz spaces in [22]. Our main aim in this paper is to extend Theorem A to Musielak–Orlicz spaces $L^{\Phi}(\mathbf{R}^N)$ defined by a general function $\Phi(x,t)$ satisfying certain conditions (Theorem 5.2). See Section 2 for the definition of Φ and $L^{\Phi}(\mathbf{R}^N)$.

Many types of norm inequalities depend on the boundedness of the Hardy–Littlewood maximal operator M. The boundedness of M on $L^{\Phi}(\mathbf{R}^N)$ was established in [19, Corollary 4.4]; we give its improvement in Section 3 of the present paper. The proof of Theorem A also depends on the boundedness of M on the dual space of $L^{p(\cdot)}(\mathbf{R}^N)$.

In Section 4, we study properties of the complementary function of Φ and look for conditions on Φ that assure the boundedness of M on the dual space of $L^{\Phi}(\mathbf{R}^N)$. We follow [6] for the proof of our generalization of Theorem A, Theorem 5.2, and applications of extrapolation theorems to obtain vector-valued inequalities in $L^{\Phi}(\mathbf{R}^N)$. As applications of Theorem 5.2, we prove L^{Φ}-norm inequalities for classical operators such as sharp maximal operators and singular integral operators in Section 6. We shall also show the L^{Φ}-version of Kerman–Sawyer inequality.

Using the vector-valued inequality, in Section 7 we shall establish a decomposition result for functions in Musielak–Orlicz spaces as an extension of [23] and [24] for the case of Lebesgue spaces with variable exponents and Orlicz spaces. See [4, 13, 14, 26, 27, 37] for related results. As an application of the decomposition result, we obtain an Olsen inequality in the final section. By an Olsen inequality, or a trace inequality, we mean an inequality of type

$$\|g \cdot I_\alpha f\|_X \le C\|g\|_Y \cdot \|f\|_Z \tag{1.2}$$

for some Banach function spaces X, Y and Z, where $I_\alpha f$ is the Riesz potential (of order α) of f. There is a vast amount of literatures on Olsen inequalities [11, 12, 28, 29, 30, 32, 33, 34, 36]. We shall show that (1.2) holds with $X = Z = L^{\Phi}(\mathbf{R}^N)$ and a certain Morrey space Y.

Throughout this paper, let C denote various constants independent of the variables in question, and $C(a, b, \ldots)$ a constant that depends on $a, b, \ldots$.

2. Preliminaries

We consider a function

$$\Phi(x,t) = t\phi(x,t)\colon \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ3):

(Φ1) $\phi(\,\cdot\,,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

(Φ3) $\phi(x,\cdot)$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2\phi(x,at) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } t > 0 \text{ and } a > 1.$$

Let $\bar{\phi}(x,t) = \sup_{0\le s\le t}\phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in \mathbf{R}^N$ and $t \ge 0$. Then $\bar{\phi}(x,\cdot)$ is continuous nondecreasing, $\overline{\Phi}(x,\cdot)$ is convex and

$$\Phi(x,t/2) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t) \tag{2.1}$$

for all $x \in \mathbf{R}^N$ and $t \ge 0$. Given $\Phi(x,t)$ as above, the associated Musielak–Orlicz space

$$L^\Phi(\mathbf{R}^N) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^N) \colon \int_{\mathbf{R}^N} \Phi\big(y,|f(y)|/\lambda\big)\,dy < \infty \text{ for some } \lambda > 0 \right\}$$

is a Banach space with respect to the norm (cf. [22])

$$\|f\|_\Phi = \|f\|_{L^\Phi(\mathbf{R}^N)} = \inf\left\{ \lambda > 0\colon \int_{\mathbf{R}^N} \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \le 1 \right\}.$$

We shall also consider the following conditions: Let $\varepsilon \ge 0$, $\nu > 0$ and $\omega > 0$.

(Φ3; ε) $t \mapsto t^{-\varepsilon}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_{2,\varepsilon} \ge 1$ such that

$$\phi(x,t) \le A_{2,\varepsilon} a^{-\varepsilon}\phi(x,at) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } t > 0 \text{ and } a > 1;$$

(Φ4) $\phi(x,\cdot)$ satisfies the uniform doubling condition, namely there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0;$$

(Φ5; ν) For every $\gamma > 0$, there exists a constant $B_{\gamma,\nu} \ge 1$ such that

$$\Phi(x,t) \le B_{\gamma,\nu}\Phi(y,t)$$

whenever $|x-y| \le \gamma t^{-\nu}$ and $t \ge 1$;

(Φ6; ω) there exist a function g on $\mathbf{R}^N$ and a constant $B_\infty \ge 1$ such that $0 \le g(x) < 1$ for all $x \in \mathbf{R}^N$, $g^\omega \in L^1(\mathbf{R}^N)$ and

$$B_\infty^{-1}\Phi(x,t) \le \Phi(x',t) \le B_\infty\Phi(x,t)$$

whenever $|x'| \ge |x|$ and $g(x) \le t \le 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \le p^- = \inf_{x\in\mathbf{R}^N} p(x) \le \sup_{x\in\mathbf{R}^N} p(x) = p^+ < \infty$

and

(Q1) $-\infty < q_j^- = \inf_{x\in\mathbf{R}^N} q_j(x) \le \sup_{x\in\mathbf{R}^N} q_j(x) = q_j^+ < \infty$

for all $j = 1,\dots,k$. Set $L_c(t) = \log(c+t)$ for $c \ge e$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^k (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if there is a constant $K \ge 0$ such that $K(p(x)-1) + q_j(x) \ge 0$ for all $x \in \mathbf{R}^N$ and $j = 1,\dots,k$; in particular if $p^- > 1$ or $q_j^- \ge 0$ for all $j = 1,\dots,k$. If $p^- > 1$, then $\Phi(x,t)$ satisfies (Φ3; ε) for $0 < \varepsilon < p^- - 1$.

Moreover, we see that $\Phi(x,t)$ satisfies $(\Phi5;\nu)$ for every $\nu > 0$ if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p \ge 0$ and

(Q2) $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L_e^{(j+1)}(1/|x-y|)}$$

with constants $C_{q_j} \ge 0$, $j = 1, \dots k$.

Finally, we see that $\Phi(x,t)$ satisfies $(\Phi6;\omega)$ for every $\omega > 0$ with $g(x) = 1/(1 + |x|)^{(N+1)/\omega}$ if $p(\cdot)$ is log-Hölder continuous at ∞, namely if it satisfies

(P3) $|p(x) - p(x')| \le \dfrac{C_{p,\infty}}{L_e(|x|)}$ whenever $|x'| \ge |x|$ with a constant $C_{p,\infty} \ge 0$.

Note that $(\Phi3;0) = (\Phi3)$. If $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, then it satisfies $(\Phi3;\varepsilon')$ for $0 \le \varepsilon' \le \varepsilon$. If $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, then

$$\bar{\phi}(x,t) \le A_{2,\varepsilon} a^{-\varepsilon} \bar{\phi}(x,at) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } t > 0 \text{ and } a > 1$$

and

$$\overline{\Phi}(x,t) \le A_{2,\varepsilon} a^{-1-\varepsilon} \overline{\Phi}(x,at) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } t > 0 \text{ and } a > 1.$$

If $\Phi(x,t)$ satisfies $(\Phi5;\nu)$, then it satisfies $(\Phi5;\nu')$ for all $\nu' \ge \nu$; if $\Phi(x,t)$ satisfies $(\Phi6;\omega)$, then it satisfies $(\Phi6;\omega')$ for all $\omega' \ge \omega$.

The following example shows that if $0 < \nu' < \nu$ and $0 < \omega' < \omega$, then there exists $\Phi(x,t)$ satisfying (Φj), $j = 1,2,3,4$ such that it satisfies $(\Phi5;\nu)$ and $(\Phi6;\omega)$, while it does not satisfy $(\Phi5;\nu')$ nor $(\Phi6;\omega')$.

Example 2.2. For $p \ge 1$, $q > 0$ and $r > 0$, set

$$\Phi(x,t) = \begin{cases} t^p \max\big(1,\, t^q \min(1,|x|)\big) & \text{if } t \ge 1, \\ t^p \max\big(t,\, \min(1/2, |x|^{-N/r})\big) & \text{if } t < 1. \end{cases}$$

This $\Phi(x,t)$ satisfies (Φj), $j = 1,2,3,4$; it satisfies $(\Phi3; p-1)$. We shall show:

(a) $\Phi(x,t)$ satisfies $(\Phi5;\nu)$ if and only if $\nu \ge q$;

(b) $\Phi(x,t)$ satisfies $(\Phi6;\omega)$ if $\omega > r$ but does not satisfy $(\Phi6;\omega)$ if $\omega < r$.

Proof of (a). Let $t \ge 1$ and $|x - y| \le \gamma t^{-\nu}$. If $\nu \ge q$, then

$$\min(1,|x|) \le \min(1,|y|) + |x-y| \le \min(1,|y|) + \gamma t^{-\nu} \le \min(1,|y|) + \gamma t^{-q},$$

so that

$$\max\big(1, t^q \min(1,|x|)\big) \le \max\big(1, t^q \min(1,|y|)\big) + \gamma,$$

which implies

$$\Phi(x,t) \le t^p \max\big(1, t^q \min(1,|y|)\big) + \gamma t^p \le (1+\gamma)\Phi(y,t).$$

Hence $\Phi(x,t)$ satisfies $(\Phi5;\nu)$ if $\nu \ge q$.

Next, suppose $\nu < q$. Let $e_1 = (1,0,\dots,0)$. Since $\Phi(0,t) = t^p$ and $\Phi(t^{-\nu}e_1, t) = t^p \max(1, t^{q-\nu}) = t^{p+q-\nu}$,

$$\frac{\Phi(t^{-\nu}e_1,t)}{\Phi(0,t)} \to \infty \quad (t \to \infty).$$

This shows that $\Phi(x,t)$ does not satisfy $(\Phi5;\nu)$.

Proof of (b). First, let $\omega > r$. Take

$$g_r(x) = \min\bigl(1/2, |x|^{-N/r}\bigr) \quad (x \in \mathbf{R}^N).$$

Then $0 < g_r(x) \le 1/2$ for all $x \in \mathbf{R}^N$ and $g_r^\omega \in L^1(\mathbf{R}^N)$. If $g_r(x) \le t < 1$ and $|x'| \ge |x|$, then $g_r(x') \le g_r(x) \le t$, so that $\Phi(x,t) = \Phi(x',t) = t^{p+1}$. Hence $\Phi(x,t)$ satisfies $(\Phi6;\omega)$ if $\omega > r$.

Next, assume that $\omega < r$ and suppose that there exists a function g on $\mathbf{R}^N$ such that $0 \le g(x) < 1$ for all $x \in \mathbf{R}^N$ and

$$\Phi(x,t) \le B\Phi(x',t) \quad \text{whenever } |x'| \ge |x| \text{ and } g(x) \le t < 1 \tag{2.2}$$

with a constant $B \ge 1$. We claim that there exists $R > 1$ such that

$$g(x) \ge |x|^{-N/\omega} \quad \text{for } |x| \ge R. \tag{2.3}$$

Suppose on the contrary that there exists a sequence $\{x_n\}$ such that $|x_n| \to \infty$ and $g(x_n) < |x_n|^{-N/\omega}$ for all n. We may assume $|x_n| \ge 2^{r/N}$. Then

$$\Phi(x_n, |x_n|^{-N/\omega}) = |x_n|^{-pN/\omega} \max(|x_n|^{-N/\omega}, |x_n|^{-N/r}) = |x_n|^{-(p/\omega+1/r)N}.$$

If we take $x_n' \in \mathbf{R}^N$ such that $|x_n'| = |x_n|^{r/\omega}$ $(> |x_n|)$, then

$$\Phi(x_n', |x_n|^{-N/\omega}) = |x_n|^{-pN/\omega} \max(|x_n|^{-N/\omega}, |x_n'|^{-N/r}) = |x_n|^{-(p/\omega+1/\omega)N}.$$

Hence,

$$\frac{\Phi(x_n, |x_n|^{-N/\omega})}{\Phi(x_n', |x_n|^{-N/\omega})} = |x_n|^{(1/\omega-1/r)N} \to \infty \quad (n \to \infty),$$

which contradicts (2.2). Thus, (2.3) holds, and hence $g^\omega \notin L^1(\mathbf{R}^N)$, which means that $\Phi(x,t)$ does not satisfy $(\Phi6;\omega)$ if $\omega < r$.

3. Boundedness of the maximal operator

For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, its maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

As the boundedness of the maximal operator M on $L^\Phi(\mathbf{R}^N)$, we give the following theorem, which is an improvement of [19, Corollary 4.4] by relaxing assumptions on $\Phi(x,t)$ in [19]. In fact, we shall show our result by assuming $(\Phi5;\nu)$ and $(\Phi6;\omega)$ below instead of $(\Phi5)$ and $(\Phi6)$ in [19]. Further, the result is proved without $(\Phi4)$ which is assumed in [19].

Theorem 3.1. *Suppose that $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < (1+\varepsilon)/N$ and $\omega \le 1+\varepsilon$. Then the maximal operator M is bounded from $L^\Phi(\mathbf{R}^N)$ into itself, namely*

$$\|Mf\|_\Phi \le C_M \|f\|_\Phi$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

We prove this theorem by modifying the proof of [19, Theorem 4.1].

For a nonnegative $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $x \in \mathbf{R}^N$ and $r > 0$, let

$$I(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \Phi\bigl(y, f(y)\bigr)\, dy.$$

Lemma 3.2. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon_1)$ and $(\Phi5;\nu)$ for $\varepsilon_1>0$ and $\nu>0$ satisfying $\nu\le(1+\varepsilon_1)/N$. Then, given $L\ge 1$, there exist constants $C_1=C(L)\ge 2$ and $C_2>0$ such that*

$$\Phi\big(x,I(f;x,r)/C_1\big)\le C_2J(f;x,r)$$

for all $x\in\mathbf{R}^N$, $r>0$ and for all nonnegative $f\in L^1_{\rm loc}(\mathbf{R}^N)$ such that $f(y)\ge 1$ or $f(y)=0$ for each $y\in\mathbf{R}^N$ and

$$\int_{\mathbf{R}^N}\Phi(y,f(y))\,dy\le L. \tag{3.1}$$

Proof. Given f as in the statement of the lemma, $x\in\mathbf{R}^N$ and $r>0$, set $I=I(f;x,r)$ and $J=J(f;x,r)$. Note that (3.1) implies $J\le L|B(0,1)|^{-1}r^{-N}$. By $(\Phi2)$ and $(\Phi3)$, $\Phi\big(y,f(y)\big)\ge(A_1A_2)^{-1}f(y)$, since $f(y)\ge 1$ or $f(y)=0$. Hence $I\le A_1A_2J$. Thus, if $J\le 1$, then

$$\Phi(x,I/C_1)\le A_2J\Phi(x,1)\le A_1A_2J$$

whenever $C_1\ge A_1A_2$.

Next, suppose $J>1$. Since $\Phi(x,t)\to\infty$ as $t\to\infty$, there exists $K\ge 1$ such that

$$\Phi(x,K)=\Phi(x,1)J$$

by $(\Phi1)$ and the mean value theorem. With this K, we have

$$\int_{B(x,r)}f(y)\,dy\le K|B(x,r)|+A_2\int_{B(x,r)}f(y)\frac{\phi\big(y,f(y)\big)}{\phi(y,K)}\,dy.$$

Since $K>1$, by $(\Phi3;\varepsilon_1)$ we have

$$\Phi(x,1)J=\Phi(x,K)\ge A_{2,\varepsilon_1}^{-1}K^{1+\varepsilon_1}\Phi(x,1),$$

so that $J\ge A_{2,\varepsilon_1}^{-1}K^{1+\varepsilon_1}$, which implies

$$K^{1+\varepsilon_1}\le A_{2,\varepsilon_1}J\le A_{2,\varepsilon_1}L|B(0,1)|^{-1}r^{-N},$$

or $r\le\gamma K^{-(1+\varepsilon_1)/N}$ with $\gamma=(A_{2,\varepsilon_1}L|B(0,1)|^{-1})^{1/N}$. Thus, if $|y-x|\le r$, then $|y-x|\le\gamma K^{-(1+\varepsilon_1)/N}\le\gamma K^{-\nu}$. Hence, by $(\Phi5;\nu)$ there is $\beta>0$, independent of f, x, r, such that

$$\phi(x,K)\le\beta\phi(y,K)\quad\text{for all } y\in B(x,r).$$

Thus, we have

$$\begin{aligned}\int_{B(x,r)}f(y)\,dy&\le K|B(x,r)|+\frac{A_2\beta}{\phi(x,K)}\int_{B(x,r)}f(y)\phi\big(y,f(y)\big)\,dy\\&=K|B(x,r)|+A_2\beta|B(x,r)|\frac{J}{\phi(x,K)}\\&=K|B(x,r)|\left(1+\frac{A_2\beta}{\phi(x,1)}\right)\le K|B(x,r)|\left(1+A_1A_2\beta\right).\end{aligned}$$

Therefore

$$I\le(1+A_1A_2\beta)K,$$

so that by $(\Phi2)$ and $(\Phi3)$

$$\Phi(x,I/C_1)\le A_2\Phi(x,K)\le A_1A_2J$$

whenever $C_1\ge 1+A_1A_2\beta$. □

The next lemma can be shown in the same way as [19, Lemma 3.2]; note that the value of ω is irrelevant in this lemma.

Lemma 3.3. *Suppose $\Phi(x,t)$ satisfies $(\Phi6;\omega)$ for some $\omega > 0$. Then there exists a constant $C_3 > 0$ such that*

$$\Phi\big(x, I(f;x,r)/2\big) \le C_3 \left\{J(f;x,r) + \Phi(x,g(x))\right\}$$

for all $x \in \mathbf{R}^N$, $r > 0$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that $g(y) \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$, where g is the function appearing in $(\Phi6;\omega)$.

Proof of Theorem 3.1. Choose $p_0 \in (1, 1+\varepsilon)$ such that $p_0 \le (1+\varepsilon)/(N\nu)$ and consider the function

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\Phi_0(x,t)$ satisfies the conditions $(\Phi1)$, $(\Phi2)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ with the same g. Since

$$\Phi_0(x,t) = t\phi_0(x,t) \quad \text{with} \quad \phi_0(x,t) = \left[t^{1-p_0}\phi(x,t)\right]^{1/p_0},$$

condition $(\Phi3;\varepsilon)$ implies that $\Phi_0(x,t)$ satisfies $(\Phi3;\varepsilon_1)$ with $\varepsilon_1 = (1+\varepsilon)/p_0 - 1 > 0$. Note that

$$\frac{1+\varepsilon_1}{N} = \frac{1+\varepsilon}{p_0 N} \ge \nu. \tag{3.2}$$

Let $f \ge 0$ and $\|f\|_\Phi \le 1/2$. Let $f_1 = f\chi_{\{x:f(x)\ge 1\}}$, $f_2 = f\chi_{\{x:g(x)\le f(x)<1\}}$ with g in $(\Phi6;\omega)$ and $f_3 = f - f_1 - f_2$, where χ_E is the characteristic function of E. Since $\Phi(x,t) \ge (A_1A_2)^{-1}$ for $t \ge 1$ by $(\Phi3)$,

$$\Phi_0(x,t) \le (A_1A_2)^{1-1/p_0}\Phi(x,t) \le (A_1A_2)^{1-1/p_0}\overline{\Phi}(x,2t)$$

if $t \ge 1$. Hence

$$\int_{\mathbf{R}^N} \Phi_0(y, f_1(y))\,dy \le (A_1A_2)^{1-1/p_0}.$$

In view of (3.2), we can apply Lemma 3.2 to Φ_0, f_1 and $L = (A_1A_2)^{1-1/p_0}$, and we have

$$\Phi_0\big(x, Mf_1(x)/C_1\big) \le C_2 M\Phi_0\big(\cdot, f_1(\cdot)\big)(x),$$

so that

$$\Phi\big(x, Mf_1(x)/C_1\big) \le C_2^{p_0}\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} \tag{3.3}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Next, applying Lemma 3.3 to Φ_0 and f_2, we have

$$\Phi_0(x, Mf_2(x)/2) \le C\left[M\Phi_0\big(\cdot, f_2(\cdot)\big)(x) + \Phi_0\big(x, g(x)\big)\right].$$

Noting that $\Phi_0(x,g(x)) \le Cg(x)^{(1+\varepsilon)/p_0}$ by $(\Phi3;\varepsilon)$, we have

$$\Phi\big(x, Mf_2(x)/2\big) \le C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + g(x)^{1+\varepsilon}\right\} \tag{3.4}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Since $0 \le f_3 \le g \le 1$, $0 \le Mf_3 \le Mg \le 1$. Hence we have

$$\Phi\big(x, Mf_3(x)\big) \le A_2\Phi_0(x, Mg(x))^{p_0} \le C[Mg(x)]^{1+\varepsilon} \tag{3.5}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Combining (3.3), (3.4) and (3.5), and noting that $g(x) \le Mg(x)$ for a.e. $x \in \mathbf{R}^N$, we obtain

$$\Phi\big(x, Mf(x)/(C_1+3)\big) \le C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + [Mg(x)]^{1+\varepsilon}\right\} \tag{3.6}$$

for a.e. $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Since M is bounded on $L^{p_0}(\mathbf{R}^N)$ and on $L^{1+\varepsilon}(\mathbf{R}^N)$, there exists a constant $C_4 \geq 1$ such that

$$\begin{aligned}
&\int_{\mathbf{R}^N} \Phi\big(x, Mf(x)/(C_1+3)\big)\,dx \\
&\leq C\left\{\int_{\mathbf{R}^N}\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(y)\Big]^{p_0} dy + \int_{\mathbf{R}^N}[Mg(y)]^{1+\varepsilon}\,dy\right\} \\
&\leq C\left\{\int_{\mathbf{R}^N}\Phi_0\big(x, f(x)\big)^{p_0} dx + \int_{\mathbf{R}^N} g(x)^{1+\varepsilon}\,dx\right\} \\
&\leq C\left\{\int_{\mathbf{R}^N}\Phi\big(x, f(x)\big)\,dx + \int_{\mathbf{R}^N} g(x)^{\omega}\,dx\right\} \leq C_4,
\end{aligned}$$

so that

$$\int_{\mathbf{R}^N} \overline{\Phi}\big(y, Mf(y)/(A_2(C_1+3)C_4)\big)\,dy \leq 1.$$

This completes the proof of the theorem. □

4. Properties of the complementary function

Hereafter, we assume that $\Phi(x,t)$ further satisfies

($\Phi3^*$) $\lim_{t\to\infty}\phi(x,t)=\infty$ and $\lim_{t\to 0+}\phi(x,t)=0$ for every $x\in\mathbf{R}^N$.

Note that this condition implies the same condition with $\bar\phi$ in place of ϕ. Also, note that if $\Phi(x,t)$ satisfies ($\Phi3;\varepsilon$) for some $\varepsilon>0$, then it satisfies ($\Phi3^*$).

Under this asumption, we consider the complementary function $\Phi^*(x,s)$ of $\overline{\Phi}(x,t)$: set

$$\phi^*(x,s) = \sup\{t\geq 0\colon \bar\phi(x,t)\leq s\}$$

and

$$\Phi^*(x,s) = \int_0^s \phi^*(x,r)\,dr$$

for $x\in\mathbf{R}^N$ and $s\geq 0$. Note that $\Phi^*(x,\cdot)$ is nonnegative, convex and $\Phi^*(x,0)=0$; $\Phi^*(x,t)$ satisfies ($\Phi1$) and ($\Phi3$).

Furthermore, we have

Proposition 4.1. (1) *If $\Phi(x,t)$ satisfies ($\Phi3;\varepsilon$) for some $\varepsilon>0$, then $\Phi^*(x,t)$ satisfies ($\Phi2$) and ($\Phi4$).*

(2) *Define $\varepsilon^* = (\log 2)/(\log A_3)$ where $A_3>1$ is a constant appearing in ($\Phi4$). If $\Phi(x,t)$ satisfies ($\Phi4$), then $\Phi^*(x,t)$ satisfies ($\Phi3;\varepsilon^*$).*

Proof. (1) Suppose $\Phi(x,t)$ satisfies ($\Phi3;\varepsilon$) for some $\varepsilon>0$. Then $\bar\phi(x,t)\to\infty$ as $t\to\infty$ and $\bar\phi(x,t)\to 0$ as $t\to 0$, both uniformly in $x\in\mathbf{R}^N$. It then follows that

$$0 < \inf_{x\in R^N}\phi^*(x,1) \leq \sup_{x\in R^N}\phi^*(x,1) < \infty$$

and

$$0 < \inf_{x\in R^N}\Phi^*(x,1) \leq \sup_{x\in R^N}\Phi^*(x,1) < \infty,$$

which is ($\Phi2$) for $\Phi^*(x,t)$.

Since

$$\bar\phi(x,(2A_{2,\varepsilon})^{-1/\varepsilon}t) \leq A_{2,\varepsilon}(2A_{2,\varepsilon})^{-1}\bar\phi(x,t) = \frac{1}{2}\bar\phi(x,t)$$

by $(\Phi3;\varepsilon)$,

$$\begin{aligned}\phi^*(x,2s) &= \sup\{t \ge 0 \colon \bar\phi(x,t) \le 2s\}\\ &\le \sup\{t \ge 0 \colon \bar\phi(x,(2A_{2,\varepsilon})^{-1/\varepsilon}t) \le s\} = (2A_{2,\varepsilon})^{1/\varepsilon}\phi^*(x,s),\end{aligned}$$

which implies

$$\Phi^*(x,2s) \le 2A_3^*\Phi^*(x,s),$$

where $A_3^* = (2A_{2,\varepsilon})^{1/\varepsilon}$. Thus $\Phi^*(x,t)$ satisfies $(\Phi4)$.

(2) First, we show

$$\bar\phi(x,t) \le s \;\Rightarrow\; \bar\phi(x,A_3^{-\varepsilon^*}a^{\varepsilon^*}t) \le as \tag{4.1}$$

for $t \ge 0$ and $a > 1$. Let $\bar\phi(x,t) \le s$. If $1 < a \le A_3$, then

$$\bar\phi(x,A_3^{-\varepsilon^*}a^{\varepsilon^*}t) \le \bar\phi(x,t) \le s \le as.$$

If $a \ge A_3$, then using $(\Phi4)$ we have

$$\bar\phi(x,A_3^{-\varepsilon^*}a^{\varepsilon^*}t) \le A_3(A_3^{-\varepsilon^*}a^{\varepsilon^*})^{1/\varepsilon^*}\bar\phi(x,t) = a\bar\phi(x,t) \le as.$$

Thus (4.1) holds.

Now (4.1) implies

$$\phi^*(x,s) \le A_3^{\varepsilon^*}a^{-\varepsilon^*}\phi^*(x,as)$$

for $s > 0$ and $a > 1$, which in turn implies

$$\Phi^*(x,s) \le A_3^{\varepsilon^*}a^{-1-\varepsilon^*}\Phi^*(x,as)$$

whenever $s > 0$ and $a > 1$. This means that $\Phi^*(x,t)$ satisfies $(\Phi3;\varepsilon^*)$. □

Lemma 4.2. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$ for some $\varepsilon > 0$. Define*

$$\eta(x,t) = \begin{cases}\overline{\Phi}(x,t)/t & (t>0),\\ 0 & (t=0),\end{cases}\qquad \eta^*(x,s) = \begin{cases}\Phi^*(x,s)/s & (s>0),\\ 0 & (s=0)\end{cases}$$

for $x \in \mathbf{R}^N$, $t \ge 0$ and $s \ge 0$. Then there is a constant $A_4 \ge 1$ such that

$$A_4^{-1}t \le \eta^*(x,\eta(x,t)) \le A_4 t$$

for $x \in \mathbf{R}^N$ and $t \ge 0$.

Proof. First, we note that

$$t \le \phi^*(x,\bar\phi(x,t)) \le A_{2,\varepsilon}^{1/\varepsilon}t \tag{4.2}$$

for all $x \in \mathbf{R}^N$ and $t > 0$. In fact, the first inequality is obvious from the definition of ϕ^*. Suppose $\bar\phi(x,at) \le \bar\phi(x,t)$ with $a \ge 1$. Then, by $(\Phi3;\varepsilon)$,

$$\bar\phi(x,t) \ge \bar\phi(x,at) \ge A_{2,\varepsilon}^{-1}a^{\varepsilon}\bar\phi(x,t),$$

so that $a \le A_{2,\varepsilon}^{1/\varepsilon}$. This shows the second inequality of (4.2).

Since $\phi^*(x,\cdot)$ and $\bar\phi(x,\cdot)$ are non-decreasing, so are $\eta^*(x,\cdot)$ and $\eta(x,\cdot)$; and $\eta^*(x,s) \le \phi^*(x,s)$ as well as $\eta(x,t) \le \bar\phi(x,t)$. Hence, by (4.2), we have

$$\eta^*(x,\eta(x,t)) \le \phi^*(x,\bar\phi(x,t)) \le A_{2,\varepsilon}^{1/\varepsilon}t.$$

On the other hand,

$$\eta(x,t) \ge \frac{1}{t}\int_{t/2}^{t}\bar\phi(x,r)\,dr \ge \frac{1}{2}\bar\phi(x,t/2)$$

and, similarly, $\eta^*(x,s) \ge (1/2)\phi^*(x,s/2)$. Hence

$$\eta^*(x,\eta(x,t)) \ge \eta^*\left(x,\frac{1}{2}\bar\phi(x,t/2)\right) \ge \frac{1}{2}\phi^*\left(x,\frac{1}{4}\bar\phi(x,t/2)\right).$$

Thus, by Proposition 4.1 (1) and (4.2), we have

$$\eta^*(x,\eta(x,t)) \ge \frac{1}{2(A_3^*)^2}\phi^*(x,\bar\phi(x,t/2)) \ge \frac{1}{4(A_3^*)^2}t. \qquad \square$$

Proposition 4.3. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$ and $(\Phi4)$ for some $\varepsilon > 0$.*

(1) *If $\Phi(x,t)$ satisfies $(\Phi5;\nu)$, then $\Phi^*(x,s)$ satisfies $(\Phi5;\nu/\varepsilon)$.*

(2) *If $\Phi(x,t)$ satisfies $(\Phi6;\omega)$, then $\Phi^*(x,s)$ satisfies $(\Phi6;\omega/\varepsilon)$.*

Proof. (1) Let $\gamma' > 0$ and $|x-y| \le \gamma' t^{-\nu/\varepsilon}$. First, we consider the case $t \ge A_1A_2$. Since $\eta(x,1) \le A_1A_2$, there is $s \ge 1$ such that $t = \eta(x,s)$. Since

$$\eta(x,s) \ge A_{2,\varepsilon}^{-1}s^\varepsilon\eta(x,1) \ge (2A_1A_{2,\varepsilon}A_3)^{-1}s^\varepsilon,$$

we have

$$|x-y| \le \gamma'(2A_1A_{2,\varepsilon}A_3)^{\nu/\varepsilon}s^{-\nu}.$$

Hence, by $(\Phi5;\nu)$ and (2.1),

$$\bar B^{-1}\eta(x,s) \le \eta(y,s) \le \bar B\eta(x,s)$$

with $\bar B = 2A_2A_3B_{\gamma,\nu}$, $\gamma = \gamma'(2A_1A_{2,\varepsilon}A_3)^{\nu/\varepsilon}$. By Proposition 4.1 (1), there is a constant $B' \ge 1$ such that $\eta^*(z,\bar Bt) \le B'\eta^*(z,t)$ for all $z \in \mathbf{R}^N$ and $t > 0$. Then, using Lemma 4.2 twice, we have

$$\begin{aligned}\eta^*(y,t) &= \eta^*(y,\eta(x,s)) \le \eta^*(y,\bar B\eta(y,s))\\ &\le B'\eta^*(y,\eta(y,s)) \le A_4B's \le A_4^2B'\eta^*(x,t)\end{aligned}$$

and

$$\begin{aligned}\eta^*(x,t) &= \eta^*(x,\eta(x,s)) \le A_4s \le A_4^2\eta^*(y,\eta(y,s))\\ &\le A_4^2\eta^*(y,\bar B\eta(x,s)) \le A_4^2B'\eta^*(y,\eta(x,s)) = A_4^2B'\eta^*(y,t).\end{aligned}$$

Thus

$$B''^{-1}\eta^*(x,t) \le \eta^*(y,t) \le B''\eta^*(x,t) \quad \text{if } t \ge A_1A_2 \tag{4.3}$$

with $B'' = A_4^2B'$.

Next, let $C_1 = \inf_{z\in\mathbf{R}^N}\eta^*(z,1)$ and $C_2 = \sup_{z\in R^N}\eta^*(z,A_1A_2)$. Then $C_1 > 0$ and $C_2 < \infty$ by Proposition 4.1 (1). Then

$$C_1C_2^{-1}\eta^*(x,t) \le \eta^*(y,t) \le C_1^{-1}C_2\eta^*(x,t) \quad \text{for } 1 \le t \le A_1A_2. \tag{4.4}$$

Now, (4.3) and (4.4) show that $\Phi^*(x,t)$ satisfies $(\Phi5;\nu/\varepsilon)$.

(2) Let $g(x)$ be the function appearing in $(\Phi6;\omega)$ for $\Phi(x,t)$. Set

$$g^*(x) = \min\left(\frac{1}{2A_1A_3},\ \eta(x,g(x))\right).$$

Then, $0 \le g^*(x) \le 1/2 < 1$ and

$$g^*(x) \le \eta(x,g(x)) \le A_1A_2A_{2,\varepsilon}g(x)^\varepsilon,$$

which implies $(g^*)^{\omega/\varepsilon} \in L^1(\mathbf{R}^N)$. We want to show that there exists a constant $B'_\infty \ge 1$ such that

$$(B'_\infty)^{-1}\eta^*(x,t) \le \eta^*(x',t) \le B'_\infty\eta^*(x,t) \tag{4.5}$$

whenever $|x'| \ge |x|$ and $g^*(x) \le t \le 1$.

First, suppose $g^*(x) < t < 1/(2A_1A_3)$. Then $g^*(x) = \eta(x, g(x))$. Take $s > 0$ such that $\eta(x, s) = t$. Then $\eta(x, g(x)) < \eta(x, s) < \eta(x, 1)$, which implies $g(x) < s < 1$. Thus, by $(\Phi6; \omega)$ and (2.1)

$$\bar{B}_\infty^{-1}\eta(x, s) \le \eta(x', s) \le \bar{B}_\infty \eta(x, s)$$

or,

$$\bar{B}_\infty^{-1} t \le \eta(x', s) \le \bar{B}_\infty t$$

whenever $|x'| \ge |x|$, where $\bar{B}_\infty = 2A_2A_3B_\infty$. Again by Proposition 4.1 (1), there is a constant $B^* \ge 1$ such that $\eta^*(z, \bar{B}_\infty t) \le B^*\eta^*(z, t)$ for all $z \in \mathbf{R}^N$ and $t > 0$. Then, by Lemma 4.2, we have

$$\begin{aligned}\eta^*(x, t) = \eta^*(x, \eta(x, s)) &\le A_4 s \le A_4^2 \eta^*(x', \eta(x', s))\\ &\le A_4^2\eta^*(x', \bar{B}_\infty t) \le A_4^2 B^*\eta^*(x', t)\end{aligned}$$

and

$$\begin{aligned}\eta^*(x', t) \le \eta^*(x', \bar{B}_\infty \eta(x', s)) &\le B^*\eta^*(x', \eta(x', s))\\ &\le A_4B^*s \le A_4^2B^*\eta^*(x, \eta(x, s)) = A_4^2B^*\eta^*(x, t).\end{aligned}$$

Thus, we have shown that (4.5) holds for $g^*(x) < t < 1/(2A_1A_3)$ and $|x'| \ge |x|$ with $B'_\infty = A_4^2B^*$. By continuity, this holds also for $t = g^*(x)$.

Next, let $C'_1 = \inf_{z\in\mathbf{R}^N} \eta^*(z, 1/(2A_1A_3))$ and $C'_2 = \sup_{z\in R^N} \eta^*(z, 1)$. Then $C'_1 > 0$, $C'_2 < \infty$ and

$$C'_1(C'_2)^{-1}\eta^*(x, t) \le \eta^*(x', t) \le (C'_1)^{-1}C'_2\eta^*(x, t)$$

for $1/(2A_1A_3) \le t \le 1$ and $|x'| \ge |x|$, which shows (4.5) for $1/(2A_1A_3) \le t \le 1$ with $B'_\infty = (C'_1)^{-1}C'_2$. □

Proposition 4.4. (1) *For* $f \in L^\Phi(\mathbf{R}^N)$,

$$\|f\|_\Phi \le \sup\left\{\left|\int_{\mathbf{R}^N} f(x)g(x)\,dx\right| : g \in L^{\Phi^*}(\mathbf{R}^N),\ \|g\|_{\Phi^*} \le 1\right\} \le 2\|f\|_\Phi. \tag{4.6}$$

(2) *If a measurable function f satisfies*

$$\sup\left\{\int_{\mathbf{R}^N} |f(x)g(x)|\,dx \colon g \in L^{\Phi^*}(\mathbf{R}^N),\ \|g\|_{\Phi^*} \le 1\right\} < \infty,$$

then $f \in L^\Phi(\mathbf{R}^N)$.

Proof. (1) This assertion is proved in [22, Theorem 13.11].

(2) Given a measurable function f, set $E_n = \{x \in \mathbf{R}^N : |x| \le n,\ |f(x)| \le n\}$ and $f_n = f\,\chi_{E_n}$ for $n \in \mathbf{N}$. Then $f_n \in L^\Phi(\mathbf{R}^N)$ and by (4.6), we have

$$\|f_n\|_\Phi \le \sup\left\{\int_{\mathbf{R}^N} |f(x)g(x)|\,dx \colon g \in L^{\Phi^*}(\mathbf{R}^N),\ \|g\|_{\Phi^*} \le 1\right\} < \infty.$$

By the monotone convergence theorem, we conclude that $f \in L^\Phi(\mathbf{R}^N)$. □

5. A generalization of a theorem of CFMP

In this section, we give a generalization of a result due to Cruz-Uribe, Fiorenza, Martell and Pérez [6, Theorem 1.3]. Before we state the theorem, we prepare the following lemma, which is easily verified:

Lemma 5.1. *For $\theta > 0$, set*

$$\Phi_\theta(x,t) = \overline{\Phi}(x, t^{1/\theta}). \tag{5.1}$$

Then:

(1) *$\Phi_\theta(x,t)$ also satisfies ($\Phi1$) and ($\Phi2$);*
(2) *if $\Phi(x,t)$ satisfies ($\Phi3;\varepsilon$) and $\theta \le 1+\varepsilon$, then $\Phi_\theta(x,t)$ satisfies ($\Phi3;(1+\varepsilon-\theta)/\theta$);*
(3) *if $\Phi(x,t)$ satisfies ($\Phi4$), then $\Phi_\theta(x,t)$ also satisfies ($\Phi4$): for*

$$\phi_\theta(x,t) = \begin{cases} \Phi_\theta(x,t)/t & (t>0), \\ 0 & (t=0), \end{cases}$$

we have

$$\phi_\theta(x,2t) \le A_{3,\theta}\phi_\theta(x,t)$$

with $A_{3,\theta} = 2^{1/\theta-1}A_3^{j(\theta)}$, where $j(\theta)$ is the integer such that

$$j(\theta) - 1 < 1/\theta \le j(\theta); \tag{5.2}$$

(4) *if $\Phi(x,t)$ satisfies ($\Phi5;\nu$), then $\Phi_\theta(x,t)$ satisfies ($\Phi5;\nu/\theta$); if $\Phi(x,t)$ satisfies ($\Phi6;\omega$), then $\Phi_\theta(x,t)$ satisfies ($\Phi6;\omega/\theta$).*

Further, if $\theta \le 1+\varepsilon$, then

$$\|f\|_{\Phi_\theta} = \||f|^{1/\theta}\|_\Phi^\theta \tag{5.3}$$

for $f \in L^{\Phi_\theta}(\mathbf{R}^N)$.

Theorem 5.2. *Suppose $\Phi(x,t)$ satisfies ($\Phi3;\varepsilon$), ($\Phi4$), ($\Phi5;\nu$) and ($\Phi6;\omega$) for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ and let $0 < p_0 < 1+\varepsilon$. Assume that*

$$\nu < \frac{(1+\varepsilon-p_0)(1+\varepsilon^*(p_0))}{N} \quad \text{and} \quad \omega \le (1+\varepsilon-p_0)(1+\varepsilon^*(p_0)), \tag{5.4}$$

where, defining $j(p_0)$ by (5.2), we write

$$\varepsilon^*(p_0) = (\log 2)/(\log A_{3,p_0}) = (\log 2)/(\log(2^{1/p_0-1}A_3^{j(p_0)})). \tag{5.5}$$

Let $\mathcal{F}$ be a family of ordered pairs (f,g) of nonnegative measurable functions on $\mathbf{R}^N$. If

$$\int_{\mathbf{R}^N} f(x)^{p_0} w(x)\,dx \le C_0 \int_{\mathbf{R}^N} g(x)^{p_0} w(x)\,dx \tag{5.6}$$

for all $(f,g) \in \mathcal{F}$ and for all A_1-weights w with a constant C_0 depending only on p_0 and the A_1-constant of w, then there is a constant $C > 0$ such that

$$\|f\|_\Phi \le C\|g\|_\Phi$$

for all $(f,g) \in \mathcal{F}$ with $g \in L^\Phi(\mathbf{R}^N)$.

Proof. By Lemma 5.1, $\Phi_{p_0}(x,t)$ satisfies ($\Phi1$), ($\Phi2$), ($\Phi4$) with constant $A_{3,p_0} = 2^{1/p_0-1}A_3^{j(p_0)}$, ($\Phi3;(1+\varepsilon-p_0)/p_0$), ($\Phi5;\nu/p_0$) and ($\Phi6;\omega/p_0$). Let $\Psi(x,t) = {\Phi_{p_0}}^*(x,t)$. Then, by Propositions 4.1 and 4.3, $\Psi(x,t)$ satisfies ($\Phi1$), ($\Phi2$), ($\Phi4$), ($\Phi3;\varepsilon^*(p_0)$), ($\Phi5;\nu/(1+\varepsilon-p_0)$) and ($\Phi6;\omega/(1+\varepsilon-p_0)$). Therefore, the maximal operator M is bounded on $L^\Psi(\mathbf{R}^N)$ by (5.4) and Theorem 3.1, namely there is a constant $A > 0$ such that

$$\|Mh\|_\Psi \le A\|h\|_\Psi \quad \text{for all } h \in L^\Psi(\mathbf{R}^N).$$

Let $j = 0, 1, \ldots$. Denote by M^j the j-fold composition of M, where it is understood that $M^0 f = |f|$. Consider an operator $T\colon L^\Psi(\mathbf{R}^N) \to L^\Psi(\mathbf{R}^N)$ defined by

$$Th = \sum_{j=0}^{\infty} \frac{M^j h}{2^j A^j}, \qquad h \in L^\Psi(\mathbf{R}^N).$$

Note that $\|Th\|_\Psi \le 2\|h\|_\Psi$.

Now, let $h \in L^\Psi(\mathbf{R}^N)$ and $h \ge 0$. Then $Th \ge h$ and $M(Th) \le 2ATh$. A direct calculation shows that Th is an A_1-weight with A_1-constant less than or equal to $2A$ (see, e.g., [9, Lemma 5.1]). Therefore by our assumption there is a constant C_0 independent of h such that

$$\int_{\mathbf{R}^N} f(x)^{p_0} h(x)\,dx \le \int_{\mathbf{R}^N} f(x)^{p_0} Th(x)\,dx \le C_0 \int_{\mathbf{R}^N} g(x)^{p_0} Th(x)\,dx$$

for all $(f, g) \in \mathcal{F}$.

Thus, if $(f, g) \in \mathcal{F}$ with $g \in L^\Phi(\mathbf{R}^N)$, then, by Proposition 4.4 (applied to Φ_{p_0}), we have

$$\int_{\mathbf{R}^N} f(x)^{p_0} h(x)\,dx \le 2C_0 \|g^{p_0}\|_{\Phi_{p_0}} \|Th\|_\Psi \le 4C_0 \|g^{p_0}\|_{\Phi_{p_0}} \|h\|_\Psi$$

for all $h \in L^\Psi(\mathbf{R}^N)$ with $h \ge 0$. Therefore, by Proposition 4.4 again, $f^{p_0} \in L^{\Phi_{p_0}}(\mathbf{R}^N)$ and $\|f^{p_0}\|_{\Phi_{p_0}} \le 4C_0 \|g^{p_0}\|_{\Phi_{p_0}}$. By (5.3), $f \in L^\Phi(\mathbf{R}^N)$ and

$$\|f\|_\Phi \le (4C_0)^{1/p_0} \|g\|_\Phi. \qquad \square$$

Remark 5.3. If $p_0 \ge 1$, then $\varepsilon^*(p_0) \ge \varepsilon^*(1) = \varepsilon^*$.

By using two types of extrapolation theorems as in [6], we obtain the following corollaries. Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions on $\mathbf{R}^N$.

Corollary 5.4. (cf. [6, Corollary 1.10]) *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ with $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying*

$$\nu < (1+\varepsilon)/N \tag{5.7}$$

and

$$\omega < 1 + \varepsilon. \tag{5.8}$$

Let $0 < p_0 < \infty$. If (5.6) holds for all $(f, g) \in \mathcal{F}$ and for all A_∞-weights w with a constant C_0 depending only on p_0 and the A_∞-constant of w, then there is a constant $C > 0$ such that

$$\|f\|_\Phi \le C\|g\|_\Phi$$

for all $(f, g) \in \mathcal{F}$ with $g \in L^\Phi(\mathbf{R}^N)$. Furthermore,

$$\Big\|\Big(\sum_j (f_j)^q\Big)^{1/q}\Big\|_\Phi \le C \Big\|\Big(\sum_j (g_j)^q\Big)^{1/q}\Big\|_\Phi$$

for every $0 < q < \infty$ and $\big\{(f_j, g_j)\big\} \subset \mathcal{F}$.

Proof. By an extrapolation theorem [6, Theorem 6.1], for every $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbf{R}^N} f(x)^p w(x)\,dx \le C \int_{\mathbf{R}^N} g(x)^p w(x)\,dx, \quad (f, g) \in \mathcal{F}$$

and, for every $0 < p,\ q < \infty$ and $w \in A_\infty$,

$$\int_{\mathbf{R}^N} \Big(\sum_j f_j(x)^q\Big)^{p/q} w(x)\, dx \le C \int_{\mathbf{R}^N} \Big(\sum_j g_j(x)^q\Big)^{p/q} w(x)\, dx.$$

Choosing $p_1 > 0$ satisfying $\nu < (1+\varepsilon-p_1)/N$ and $\omega \le 1+\varepsilon-p_1$ and applying Theorem 5.2 with this p_1 in place of p_0, we obtain the first assertion. The second assertion can be derived by applying Theorem 5.2 to the family

$$\mathcal{F}_q = \Big\{\Big(\Big(\sum_j (f_j)^q\Big)^{1/q}, \Big(\sum_j (g_j)^q\Big)^{1/q}\Big) : \big\{(f_j, g_j)\big\}_j \subset \mathcal{F}\Big\}. \qquad \square$$

Corollary 5.5. (cf. [6, Corollary 1.11]) *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon > 0$, $\nu > 0$ *and* $\omega > 0$ *satisfying*

$$\nu < \varepsilon(1+\varepsilon^*)/N \tag{5.9}$$

and

$$\omega < \varepsilon(1+\varepsilon^*) \tag{5.10}$$

for ε^* *given in Proposition 4.1. Let* $\mathcal{F}$ *be a family of ordered pairs* (f,g) *of nonnegative measurable functions on* $\mathbf{R}^N$. *Let* $1 < p_0 < \infty$. *If* (5.6) *holds for all* $(f,g) \in \mathcal{F}$ *and for all* A_{p_0}*-weights* w *with a constant* C_0 *depending only on* p_0 *and the* A_{p_0}*-constant of* w, *then there is a constant* $C > 0$ *such that*

$$\|f\|_\Phi \le C\|g\|_\Phi \tag{5.11}$$

for all $(f,g) \in \mathcal{F}$ *with* $g \in L^\Phi(\mathbf{R}^N)$. *Furthermore,*

$$\Big\|\Big(\sum_j (f_j)^q\Big)^{1/q}\Big\|_\Phi \le C\Big\|\Big(\sum_j (g_j)^q\Big)^{1/q}\Big\|_\Phi$$

for every $1 < q < \infty$ *and* $\big\{(f_j, g_j)\big\} \subset \mathcal{F}$.

Proof. By the extrapolation theorem [6, Theorem 6.2], for every $1 < p < \infty$ and $w \in A_p$,

$$\int_{\mathbf{R}^N} f(x)^p w(x)\, dx \le C \int_{\mathbf{R}^N} g(x)^p w(x)\, dx, \quad (f,g) \in \mathcal{F}$$

and, for every $1 < p,\ q < \infty$ and $w \in A_p$,

$$\int_{\mathbf{R}^N} \Big(\sum_j f_j(x)^q\Big)^{p/q} w(x)\, dx \le C \int_{\mathbf{R}^N} \Big(\sum_j g_j(x)^q\Big)^{p/q} w(x)\, dx.$$

Choosing $1 < p_1 < 1+\varepsilon$ satisfying $\nu < (1+\varepsilon-p_1)(1+\varepsilon^*)/N$ and $\omega \le (1+\varepsilon-p_1)(1+\varepsilon^*)$ and applying Theorem 5.2 with this p_1 in place of p_0, we obtain the first assertion. The second assertion follows from the same arguments as in the previous corollary. $\square$

Remark 5.6. Assumptions (5.7) and (5.8) are weaker than (5.9) and (5.10), respectively. In fact, we see that

$$(A_1 A_{2,\varepsilon})^{-1} t^\varepsilon \le \phi(x,t) \le A_1 A_2 A_3 t^{1/\varepsilon^*}$$

for $t \ge 1$, which implies $\varepsilon\varepsilon^* \le 1$, so that $\varepsilon(1+\varepsilon^*) \le 1+\varepsilon$.

From Corollary 5.5 with the pairs $(Mf, |f|)$, we obtain vector-valued inequalities for M on $L^\Phi(\mathbf{R}^N)$. Recall that $\varepsilon^*(p_0)$ is defined by (5.5).

Corollary 5.7. *Suppose that* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon>0$, $\nu>0$ *and* $\omega>0$. *Let* $q>1$.

(1) *If* $\nu<\varepsilon(1+\varepsilon^*)/N$ *and* $\omega<\varepsilon(1+\varepsilon^*)$, *then*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^q\right)^{1/q}\right\|_\Phi \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\right\|_\Phi$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ *of measurable functions.*

(2) *If* $\nu<(1+\varepsilon-1/q)(1+\varepsilon^*(1/q))/N$ *and* $\omega<(1+\varepsilon-1/q)(1+\varepsilon^*(1/q))$, *then*

$$\left\|\sum_{j=1}^{\infty}(Mf_j)^q\right\|_\Phi \le C\left\|\sum_{j=1}^{\infty}|f_j|^q\right\|_\Phi$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ *of measurable functions .*

Proof. (1) This is a direct consequence of Corollary 5.5 applied to the family $\mathcal{F}=\{(Mf,|f|)\}$; see [1].

(2) By Lemma 5.1, $\Phi_{1/q}(x,t)\ \big(=\overline{\Phi}(x,t^q)\big)$ satisfies $(\Phi3;\varepsilon')$, $(\Phi4)$, $(\Phi5;\nu')$ and $(\Phi6;\omega')$ with

$$\varepsilon'=q(1+\varepsilon)-1,\quad \nu'=q\nu \quad\text{and}\quad \omega'=q\omega.$$

By assumption,

$$\varepsilon'>0,\quad 0<\nu'<\varepsilon'(1+\varepsilon^*(1/q))/N,\quad 0<\omega'<\varepsilon'(1+\varepsilon^*(1/q)).$$

Hence, by Corollary 5.5, we have

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^q\right)^{1/q}\right\|_{\Phi_{1/q}} \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\right\|_{\Phi_{1/q}},$$

which implies the required inequality in view of Lemma 5.1. □

6. Some applications of CFMP-theorem

6.1. Sharp maximal function. For $f\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, the sharp maximal function $M^{\#}f$ is defined by

$$M^{\#}f(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|\,dy,$$

where $f_B=(1/|B|)\int_B f(x)\,dx$ for a ball B.

Since $0\le M^{\#}f\le 2Mf$, by Theorem 3.1, we have

Proposition 6.1. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon>0$, $\nu>0$ *and* $\omega>0$ *satisfying* $\nu<(1+\varepsilon)/N$ *and* $\omega\le 1+\varepsilon$. *Then*

$$\|M^{\#}f\|_\Phi\le 2C_M\|f\|_\Phi$$

for all $f\in L^\Phi(\mathbf{R}^N)$.

The following inequality is known (cf. [16]): for $0<p<\infty$,

$$\int_{\mathbf{R}^N}[Mf(x)]^p w(x)\,dx\le C\int_{\mathbf{R}^N}[M^{\#}f(x)]^p w(x)\,dx$$

for all $f\in L^\infty_{\mathrm{c}}(\mathbf{R}^N)$ (= the space of L^∞-functions with compact support) and $w\in A_\infty$.

Thus, applying Corollary 5.4 to $\mathcal{F}=\{(Mf,M^{\#}f)\colon f\in L^\infty_{\mathrm{c}}(\mathbf{R}^N)\}$, we have

Proposition 6.2. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon>0$, $\nu>0$ and $\omega>0$ satisfying $\nu<(1+\varepsilon)/N$ and $\omega<1+\varepsilon$. Then*

$$\|Mf\|_\Phi \le C\|M^\# f\|_\Phi$$

for all $f\in L^\infty_{\rm c}(\mathbf{R}^N)$.

In view of Propositions 6.1 and 6.2, we can state:

Corollary 6.3. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon>0$, $\nu>0$ and $\omega>0$ satisfying $\nu<(1+\varepsilon)/N$ and $\omega<1+\varepsilon$. Then*

$$C^{-1}\|f\|_\Phi \le \|M^\# f\|_\Phi \le C\|f\|_\Phi$$

for $f\in L^\Phi(\mathbf{R}^N)$.

6.2. Singular integral operators. We consider a singular integral operator T associated to a standard kernel $k(x,y)$ (see, e.g., [7, Section 6.3]). By $C^\infty_{\rm c}(\mathbf{R}^N)$ we denote the set of all compactly supported C^∞-functions in $\mathbf{R}^N$.

Recall the following result due to Alvarez and Pérez [3]:

Lemma 6.4. *Let T be a singular integral operator associated to a standard kernel and suppose T extends to a bounded operator from $L^1(\mathbf{R}^N)$ to w-$L^1(\mathbf{R}^N)$. Then, for $0<\theta<1$ there exists a constant $C(\theta)>0$ such that*

$$M^\#(|Tf|^\theta)(x) \le C(\theta)[Mf(x)]^\theta$$

for all $f\in C^\infty_{\rm c}(\mathbf{R}^N)$ and $x\in\mathbf{R}^N$.

Theorem 6.5. *Let T be a singular integral operator associated to a standard kernel and suppose T extends to a bounded operator from $L^1(\mathbf{R}^N)$ to w-$L^1(\mathbf{R}^N)$. Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon>0$, $\nu>0$ and $\omega>0$ satisfying $\nu<(1+\varepsilon)/N$ and $\omega<1+\varepsilon$. Then T, defined initially on $C^\infty_{\rm c}(\mathbf{R}^N)$, can be extended to a bounded operator from $L^\Phi(\mathbf{R}^N)$ into itself.*

Proof. Let $0<\theta<1$. By Lemma 5.1, $\Phi_\theta(x,t)=\overline{\Phi}(x,t^{1/\theta})$ satisfies $(\Phi3;\varepsilon')$ with $\varepsilon'=(1+\varepsilon-\theta)/\theta$, $(\Phi5;\nu/(\theta N))$ and $(\Phi6;\omega/\theta)$. Note that $(1+\varepsilon)/\theta=1+\varepsilon'$.

Let $f\in C^\infty_{\rm c}(\mathbf{R}^N)$. Then, using Proposition 6.2, the above lemma and then Theorem 3.1, we obtain

$$\begin{aligned}\|Tf\|_\Phi &= \Big(\big\||Tf|^\theta\big\|_{\Phi_\theta}\Big)^{1/\theta} \le C_\#\big\|M^\#(|Tf|^\theta)\big\|_{\Phi_\theta}^{1/\theta}\\ &= C_\#\big\|[M^\#(|Tf|^\theta)]^{1/\theta}\big\|_\Phi \le C_\# C(\theta)^{1/\theta}\|Mf\|_\Phi \le C\|f\|_\Phi.\end{aligned}$$

Since $C^\infty_{\rm c}(\mathbf{R}^N)$ is dense in $L^\Phi(\mathbf{R}^N)$ (cf. [20]), we obtain the required assertion. □

Remark 6.6. If K is a locally integrable function on $\mathbf{R}^N\setminus\{0\}$ such that its Fourier transform is bounded and

$$|K(x)|\le\frac{C}{|x|^N},\quad |\nabla K(x)|\le\frac{C}{|x|^{N+1}},\quad x\ne 0.$$

Then, for the singular integral operator T_K defined by $T_Kf=K*f$ and for $1<p<\infty$, there exists a constant $C>0$ such that

$$\int_{\mathbf{R}^N}|T_Kf(x)|^p w(x)\,dx \le C\int_{\mathbf{R}^N}|f(x)|^p w(x)\,dx$$

for all $w\in A_p$ and $f\in L^p(\mathbf{R}^N;w)$ (see, e.g., [9, Theorem 3.1, p. 411]). Therefore, by Corollary 5.5, we have

Proposition 6.7. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon > 0$, $\nu > 0$ *and* $\omega > 0$ *satisfying* $\nu < \varepsilon(1+\varepsilon^*)/N$ *and* $\omega < \varepsilon(1+\varepsilon^*)$. *Then*

$$\|T_K f\|_\Phi \le C\|f\|_\Phi$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

6.3. Kerman–Sawyer inequality. In this subsection, let $k(r)$ be a non-negative nonincreasing lower semi-continuous function on $(0,\infty)$ such that

$$\int_0^1 k(r)r^{N-1}\,dr < \infty$$

and there is $R_0 > 0$ such that $k(r)$ is positive and satisfies the doubling condition on $(0,R_0)$, i.e., $k(r) \le C_d k(2r)$ for $0 < r < R_0/2$.

Set $k(0) = \liminf_{r\to 0+} k(r)$. ($k(0)$ may be ∞.) With an abuse of notation, we write $k(x) = k(|x|)$ for $x \in \mathbf{R}^N$. Let $\bar{k}(r) = r^{-N}\int_0^r k(t)t^{N-1}\,dt$. The k-maximal function of a non-negative measure μ is defined by

$$M_k\mu(x) = \sup_{r>0} \bar{k}(r)\mu(B(x,r)).$$

Kerman and Sawyer [17, Theorem 2.2] showed that

$$C^{-1}\|M_k\mu\|_p \le \|k*\mu\|_p \le C\|M_k\mu\|_p \tag{6.1}$$

for $1 < p < \infty$. The left inequality follows from

$$M_k\mu(x) \le CM(k*\mu)(x) \quad \text{for all } x \in \mathbf{R}^N,$$

which was proved in [2, Lemma 4.3.1] and [17, Theorem 2.2(a)], and the boundedness of the maximal operator M. This inequality, together with our Theorem 3.1, gives

Proposition 6.8. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon > 0$, $\nu > 0$ *and* $\omega > 0$ *satisfying* $\nu < (1+\varepsilon)/N$ *and* $\omega \le 1+\varepsilon$. *Then*

$$\|M_k\mu\|_\Phi \le C\|k*\mu\|_\Phi.$$

As a weighted version of the right inequality in (6.1), we have

Lemma 6.9. *Let* $1 < p < \infty$ *and* $w \in A_p$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} [(k*\mu)(x)]^p w(x)\,dx \le C\int_{\mathbf{R}^N} [(M_k\mu)(x)]^p w(x)\,dx$$

for all nonnegative measures μ *on* $\mathbf{R}^N$.

This lemma is essentially proved in [15, Section 5] and [18, Proposition 1']. By using the method given in the proof of [35, Theorem 3.1.2] and modifying the proof of [2, Part II, Theorem 4.3.1] to the weighted case, we can prove this lemma.

By this lemma and Corollary 5.5, we have

Theorem 6.10. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ *and* $(\Phi6;\omega)$ *for* $\varepsilon > 0$, $\nu > 0$ *and* $\omega > 0$ *satisfying* $\nu < \varepsilon(1+\varepsilon^*)/N$ *and* $\omega < \varepsilon(1+\varepsilon^*)$. *Then there exists a constant* $C > 0$ *such that*

$$\|k*\mu\|_\Phi \le C\|M_k\mu\|_\Phi$$

for all nonnegative measures μ *on* $\mathbf{R}^N$ *such that* $M_k\mu \in L^\Phi(\mathbf{R}^N)$.

7. Decomposition for Musielak–Orlicz spaces

In this section, we give a decomposition theorem for functions in $L^{\Phi}(\mathbf{R}^N)$.

Theorem 7.1. *Let $d \in \mathbf{N} \cup \{0\}$. Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < (1+\varepsilon)/N$ and $\omega < 1+\varepsilon$. Then every $f \in L^{\Phi}(\mathbf{R}^N)$ has a decomposion*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

with $a_j \in L^{\infty}(\mathbf{R}^N)$ and $\lambda_j \in [0,\infty)$, $j = 1, 2, \ldots$, such that

$$|a_j| \le \chi_{Q_j} \quad \text{for a cube } Q_j \text{ for each } j,$$

$$\int_{\mathbf{R}^N} a_j(x) x^{\beta}\, dx = 0 \quad \text{for all multi-indices } \beta \text{ with } |\beta| \le d, \text{ for each } j$$

and

$$\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\Phi} \le C\|f\|_{\Phi}$$

with a constant $C > 0$ independent of f.

To prove this theorem, we introduce the grand maximal operator which is originally used in the definition of the Hardy space $H^p(\mathbf{R}^N)$ with $0 < p < \infty$. The grand maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) = \sup\{|t^{-N}\varphi(t^{-1}\,\cdot\,) * f(x)| \colon t > 0, \varphi \in \mathcal{F}_L\} \tag{7.1}$$

for $f \in \mathcal{S}'(\mathbf{R}^N)$ and $x \in \mathbf{R}^N$, where L is a fixed large integer and

$$\mathcal{F}_L = \left\{ \varphi \in \mathcal{S}(\mathbf{R}^N) \;:\; \sup_{x \in \mathbf{R}^N} (1+|x|)^L |\partial^{\beta}\varphi(x)| \le 1 \text{ for all } \beta \text{ with } |\beta| \le L \right\}.$$

Here and below, we suppose $L \ge N+1$.

Lemma 7.2. *Suppose that Φ satisfies $(\Phi3;\varepsilon)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < (1+\varepsilon)/N$ and $\omega \le 1+\varepsilon$. If $f \in L^{\Phi}(\mathbf{R}^N)$, then $\mathcal{M}f \in L^{\Phi}(\mathbf{R}^N)$ and*

$$\|\mathcal{M}f\|_{\Phi} \le C\|f\|_{\Phi}. \tag{7.2}$$

Proof. In view of Theorem 3.1, it is enough to show

$$\mathcal{M}f(x) \le CMf(x) \tag{7.3}$$

for $f \in \mathcal{S}'(\mathbf{R}^N) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Let $h_t(x) = t^{-N}(1+|x|/t)^{-L}$ for $t > 0$. Then,

$$|h_t * f(x)| \le \|h_t\|_1 Mf(x) = \|h_1\|_1 Mf(x)$$

(cf., e.g., [8, Proposition 2.7] or [10, Theorem 2.1.10]). Since $|\varphi| \le h_1$ if $\varphi \in \mathcal{F}_L$,

$$|t^{-N}\varphi(t^{-1}\,\cdot\,) * f(x)| \le h_t * |f|(x) \le \|h_1\|_1 Mf(x)$$

for all $\varphi \in \mathcal{F}_L$, $t > 0$ and $x \in \mathbf{R}^N$, which yields (7.3) with $C = \|h_1\|_1 < \infty$. □

We shall use the following lemma. We refer to [31, Chap. III, §2] for the proof.

Lemma 7.3. *Let $f \in \mathcal{S}'(\mathbf{R}^N) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $d \in \mathbf{N} \cup \{0\}$ and $r > 0$. Set $\mathcal{O} = \{y \in \mathbf{R}^N \colon \mathcal{M}f(y) > r\}$ and consider a collection of cubes $\{Q_k^*\}$ which has the bounded intersection property and for which $\mathcal{O} = \bigcup_k Q_k^*$. (Such a collection can be obtained* *via the Whitney decomposition of $\mathcal{O}$.) Then, f is expressed as $f = g + \sum_k b_k$ with $b_k,\ g \in \mathcal{S}'(\mathbf{R}^N) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that b_k is supported in Q_k^* and*

$$\int_{\mathbf{R}^N} b_k(x) x^\beta \, dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \le d$$

for each k. Furthermore,

$$|g(x)| \le Cr \quad \text{for all } x \in \mathbf{R}^N \tag{7.4}$$

and

$$\mathcal{M} b_k \le C \left(\mathcal{M} f \cdot \chi_{Q_k^*} + r \cdot \frac{{\ell_k}^{N+d+1}}{|\,\cdot - x_k|^{N+d+1}} \chi_{\mathbf{R}^N \setminus Q_k^*} \right) \tag{7.5}$$

with constants C depending only on N, where x_k and ℓ_k denote the center and the side-length of Q_k^*, respectively.

Remark 7.4. We have the following pointwise estimate from [10, Example 2.1.8]:

$$\left(\frac{\ell_k}{\ell_k + |x - x_k|} \right)^N \le C M \chi_{Q_j^*}(x)$$

with $C > 0$ independent of k, so that (7.5) implies

$$\mathcal{M} b_k \le C \left(\mathcal{M} f \cdot \chi_{Q_k^*} + r \big(M \chi_{Q_k^*} \big)^{(N+d+1)/N} \right). \tag{7.6}$$

Proof of Theorem 7.1. Choose $d_1 \ge d$ ($d_1 \in \mathbf{N}$) such that

$$\nu \le \frac{1 + \varepsilon - 1/q}{N} \quad \text{and} \quad \omega \le 1 + \varepsilon - 1/q$$

for $q = (N + d_1 - 1)/N$. For each $j \in \mathbf{Z}$, let

$$\mathcal{O}_j = \{ x \in \mathbf{R}^N : \mathcal{M} f(x) > 2^j \}.$$

By the previous lemma and remark, we find collections of cubes $\{Q_{j,k}^*\}_{k \in K_j}$ having the bounded intersection property such that $\bigcup_{k \in K_j} Q_{j,k}^* = \mathcal{O}_j$; and we have a decomposition

$$f = g_j + b_j, \quad b_j = \sum_{k \in K_j} b_{j,k}$$

with $b_{j,k},\ g_j \in \mathcal{S}'(\mathbf{R}^N) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N)$ such that $b_{j,k}$ supported in $Q_{j,k}^*$,

$$\int_{\mathbf{R}^N} b_{j,k}(x) x^\beta \, dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \le d_1,$$

$|g_j(x)| \le C 2^j$ and

$$\mathcal{M} b_{j,k} \le C \left(\mathcal{M} f \cdot \chi_{Q_{j,k}^*} + 2^j \big(M \chi_{Q_{j,k}^*} \big)^{1/q} \right). \tag{7.7}$$

Since $g_j \to 0$ uniformly as $j \to -\infty$, $g_j \to 0$ in $\mathcal{S}'$ as $j \to -\infty$. On the other hand, by (7.7) we have

$$\begin{aligned} \|b_j\|_\Phi &\le \Big\| \sum_k \mathcal{M} b_{j,k} \Big\|_\Phi \le C \Big\| \mathcal{M} f \cdot \chi_{\mathcal{O}_j} + \sum_k 2^j (M \chi_{Q_{j,k}^*})^q \Big\|_\Phi \\ &\le C \left\| \mathcal{M} f \cdot \chi_{\mathcal{O}_j} \right\|_\Phi + C \Big\| 2^j \sum_k (M \chi_{Q_{j,k}^*})^q \Big\|_\Phi . \end{aligned}$$

Now, by Corollary 5.7 (2), we see

$$\left\| 2^j \sum_k (M\chi_{Q^*_{j,k}})^q \right\|_\Phi \le C \left\| 2^j \sum_k \chi_{Q^*_{j,k}} \right\|_\Phi \le C \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_\Phi .$$

Hence,

$$\|b_j\|_\Phi \le C \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_\Phi \to 0$$

as $j \to \infty$, which implies that $b_j \to 0$ in the sense of distributions as $j \to \infty$.

Therefore

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j), \tag{7.8}$$

with the sum converging in the sense of distributions. Going through the same arguments as in [31, pp. 108–109], we have functions $\{A_{j,k}\}_{k \in K_j}$ such that

$$g_{j+1} - g_j = \sum_{k \in K_j} A_{j,k}, \tag{7.9}$$

$|A_{j,k}| \le C_0 2^j \chi_{Q^*_{j,k}}$ for some universal constant C_0 and $\int_{\mathbf{R}^N} A_{j,k}(x) x^\beta \, dx = 0$ for all β with $|\beta| \le d$.

Let us set

$$a_{j,k} = \frac{A_{j,k}}{C_0 2^j}, \quad \lambda_{j,k} = C_0 2^j, \quad \text{for } k \in K_j,\ j \in \mathbf{Z}.$$

Then

$$|a_{j,k}| \le \chi_{Q^*_{j,k}}, \quad \int_{\mathbf{R}^N} a_{j,k}(x) x^\beta \, dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \le d$$

and

$$f = \sum_{j \in \mathbf{Z}} \sum_{k \in K_j} \lambda_{j,k} a_{j,k},$$

where the summation is convergent in the sense of distributions.

What remains to show is the estimate

$$\left\| \sum_{j \in \mathbf{Z}} \sum_{k \in K_j} \lambda_{j,k} \chi_{Q^*_{j,k}} \right\|_\Phi \le C \|f\|_\Phi . \tag{7.10}$$

By the bounded intersection property, $\sum_{k \in K_j} \chi_{Q^*_{j,k}} \le C \chi_{\mathcal{O}_j}$, and by the definition of $\mathcal{O}_j$, $\sum_{j \in \mathbf{Z}} 2^j \chi_{\mathcal{O}_j} \le 2\mathcal{M}f$. Hence

$$\left\| \sum_{j \in \mathbf{Z}} \sum_{k \in K_j} \lambda_{j,k} \chi_{Q^*_{j,k}} \right\|_\Phi = C_0 \left\| \sum_{j \in \mathbf{Z}} \sum_{k \in K_j} 2^j \chi_{Q^*_{j,k}} \right\|_\Phi \le C \|\mathcal{M}f\|_\Phi .$$

Required inequality (7.10) now follows from Lemma 7.2. □

8. An application to Olsen inequality

For $0 < q \le p < \infty$, recall that the Morrey space $\mathcal{M}^p_q(\mathbf{R}^N)$ is defined by

$$\mathcal{M}^p_q(\mathbf{R}^N) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^N) \colon \|f\|_{\mathcal{M}^p_q} < \infty \right\},$$

where

$$\|f\|_{\mathcal{M}^p_q} = \sup_{Q\colon \text{cube}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^N$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^N} (1+|y|)^{\alpha-N} |f(y)|\, dy < \infty \tag{8.1}$$

(see [21, Theorem 1.1, Chapter 2]), which is a necessary and sufficient condition for the integral defining $I_\alpha f(x)$ to converge for almost all $x \in \mathbf{R}^N$.

Theorem 8.1. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < \varepsilon(1+\varepsilon^*)/N$ and $\omega < \varepsilon(1+\varepsilon^*)$ for ε^* given in Proposition 4.1. Let*

$$0 < \alpha < \frac{N\varepsilon^*}{1+\varepsilon^*} \quad \text{and} \quad 1 + \frac{1}{\varepsilon^*} < u \le \frac{N}{\alpha}. \tag{8.2}$$

Then

$$\|g \cdot I_\alpha f\|_\Phi \le C \|g\|_{\mathcal{M}^{N/\alpha}_u} \|f\|_\Phi$$

for all $f \in L^\Phi(\mathbf{R}^N)$ satisfying (8.1) and $g \in \mathcal{M}^{N/\alpha}_u(\mathbf{R}^N)$.

To prove Theorem 8.1, we need the following lemmas:

Lemma 8.2. [14, Lemma 4.2] *Let $d \in \mathbf{N} \cup \{0\}$. Suppose h is an L^∞-function supported on a cube Q. Assume in addition that $\int_{\mathbf{R}^N} x^\beta h(x)\, dx = 0$ for all β with $|\beta| \le d$. Then,*

$$|I_\alpha h(x)| \le C_{\alpha,d} \|h\|_\infty \ell(Q)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(N+d+1-\alpha)}} \chi_{2^k Q}(x) \quad (x \in \mathbf{R}^N), \tag{8.3}$$

where $\ell(Q)$ denotes the side length of Q.

Lemma 8.3. *Suppose Φ satisfies $(\Phi3;\varepsilon)$, $(\Phi4)$, $(\Phi5;\nu)$ and $(\Phi6;\omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < \varepsilon(1+\varepsilon^*)/N$ and $\omega \le \varepsilon(1+\varepsilon^*)$. Let $\{Q_j\}_{j=1}^\infty$ be a sequence of cubes, $\{a_j\}_{j=1}^\infty$ be a sequence of non-negative functions in $L^u(\mathbf{R}^N)$ for $u > 1 + 1/\varepsilon^*$ and let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of non-negative numbers. If* $\operatorname{supp} a_j \subset Q_j$, *$\|a_j\|_u \le |Q_j|^{1/u}$ for each j and $\left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_\Phi < \infty$, then $\sum_{j=1}^\infty \lambda_j a_j \in L^\Phi(\mathbf{R}^N)$ and*

$$\left\| \sum_{j=1}^\infty \lambda_j a_j \right\|_\Phi \le C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_\Phi.$$

Proof of Lemma 8.3. Consider $g \in L^{\Phi^*}(\mathbf{R}^N)$ with $\|g\|_{\Phi^*} \le 1$ and set

$$\Lambda(g) = \int_{\mathbf{R}^N} |g(x)| \sum_{j=1}^\infty \lambda_j a_j(x)\, dx.$$

By the Hölder inequality, we obtain

$$\begin{aligned}|\Lambda(g)| &\le \sum_{j=1}^{\infty} \lambda_j \left(\int_{Q_j} |g(x)|^{u'}\,dx \right)^{1/u'} \|a_j\|_u \le C \sum_{j=1}^{\infty} \lambda_j |Q_j| \left(\inf_{Q_j} M[|g|^{u'}] \right)^{1/u'} \\ &\le C \int_{\mathbf{R}^n} \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(x) \right) \left(M[|g|^{u'}](x) \right)^{1/u'} dx \\ &\le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\Phi} \left\| \left(M[|g|^{u'}] \right)^{1/u'} \right\|_{\Phi^*},\end{aligned}$$

where $1/u+1/u' = 1$. By Propositions 4.1 and 4.3, $\Phi^*(x,t)$ satisfies $(\Phi3;\varepsilon^*)$, $(\Phi5;\nu/\varepsilon)$ and $(\Phi6;\omega/\varepsilon)$. Hence, by Lemma 5.1, $\Phi^*_{u'}(x,t) = \Phi^*(x,t^{1/u'})$ satisfies $(\Phi3;(1+\varepsilon^*)/u'-1)$, $(\Phi5;\nu/(\varepsilon u'))$ and $(\Phi6;\omega/(\varepsilon u'))$. Note that $u' < 1+\varepsilon^*$. By our assumption $\nu/(\varepsilon u') < (1+\varepsilon^*)/N$ and $\omega/(\varepsilon u') \le 1+\varepsilon^*$. Thus, by Theorem 3.1, the maximal operator M is bounded on $L^{\Phi^*_{u'}}(\mathbf{R}^N)$. Hence

$$\left\| \left(M[|g|^{u'}] \right)^{1/u'} \right\|_{\Phi^*} = \left\| M[|g|^{u'}] \right\|_{\Phi^*_{u'}}^{1/u'} \le C \left\| |g|^{u'} \right\|_{\Phi^*_{u'}}^{1/u'} = C\|g\|_{\Phi^*} \le C,$$

which implies

$$|\Lambda(g)| \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\Phi}$$

for all $g \in L^{\Phi^*}(\mathbf{R}^N)$ with $\|g\|_{\Phi^*} \le 1$. Now the required conclusion follows from Proposition 4.4. □

Proof of Theorem 8.1. First note that $\nu < (1+\varepsilon)/N$ and $\omega < 1+\varepsilon$ by Remark 5.6. Choose $d \in \mathbf{N}$ so large that

$$\nu \le \frac{1}{N}\left(1+\varepsilon - \frac{N}{N+d}\right) \quad \text{and} \quad \omega \le 1+\varepsilon - \frac{N}{N+d}. \tag{8.4}$$

Let $f \in L^{\Phi}(\mathbf{R}^N)$ satisfy (8.1). We decompose f according to Theorem 7.1 with d chosen as above; $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where $a_j \in L^{\infty}(\mathbf{R}^N)$ and $\lambda_j \in [0,\infty)$, $j = 1,2,\ldots$, satisfying the conditions in Theorem 7.1 for cubes $\{Q_j\}_{j=1}^{\infty}$.

By Lemma 8.2,

$$|I_\alpha a_j| \le C \sum_{k=1}^{\infty} \frac{1}{2^{k(N+d+1)}} \ell(2^k Q_j)^{\alpha} \chi_{2^k Q_j},$$

so that

$$|g \cdot I_\alpha f| \le |g| \sum_{j=1}^{\infty} \lambda_j |I_\alpha a_j| \le C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j}{2^{k(N+d+1)}} \left(|g| \ell(2^k Q_j)^{\alpha} \chi_{2^k Q_j} \right).$$

Let

$$a_{j,k}(x) = \frac{\ell(2^k Q_j)^{\alpha}}{\|g\|_{\mathcal{M}_u^{N/\alpha}}} |g(x)| \chi_{2^k Q_j}(x).$$

Since

$$\|g \cdot \chi_{2^k Q_j}\|_u \le |2^k Q_j|^{-\alpha/N+1/u} \|g\|_{\mathcal{M}_u^{N/\alpha}} = \ell(2^k Q_j)^{-\alpha} |2^k Q_j|^{1/u} \|g\|_{\mathcal{M}_u^{N/\alpha}},$$

we see that $\|a_{j,k}\|_u \le |2^k Q_j|^{1/u}$. Hence, by Lemma 8.3, we have

$$\|g \cdot I_\alpha f\|_\Phi \le C\|g\|_{\mathcal{M}_u^{N/\alpha}} \left\| \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j}{2^{k(N+d+1)}} \cdot \chi_{2^k Q_j} \right\|_\Phi .$$

Observe that $\chi_{2^k Q_j} \le 2^{kN} M\chi_{Q_j}$. Hence

$$\begin{aligned}\|g \cdot I_\alpha f\|_\Phi &\le C\|g\|_{\mathcal{M}_u^{N/\alpha}} \left\| \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j}{2^{k(N+d+1)}} 2^{k(N+d)} \cdot (M\chi_{Q_j})^{(N+d)/N} \right\|_\Phi \\ &= C\|g\|_{\mathcal{M}_u^{N/\alpha}} \left\| \sum_{j=1}^\infty \lambda_j (M\chi_{Q_j})^{(N+d)/N} \right\|_\Phi .\end{aligned}$$

By Corollary 5.7 (2) and (8.4), we can remove the maximal operator M and we obtain

$$\|g \cdot I_\alpha f\|_\Phi \le C\|g\|_{\mathcal{M}_u^{N/\alpha}} \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_\Phi \le C\|g\|_{\mathcal{M}_u^{N/\alpha}} \|f\|_\Phi,$$

which is the required inequality. □

Acknowledgements. We would like to express our thanks to the referee for his/her kind comments and suggestions.

References

[1] Andersen, K. F., and R. T. John: Weighted inequalities for vecter-valued maximal functions and singular integrals. - Studia Math. 69, 1980, 19–31.

[2] Aikawa, H., and M. Essén, Potential theory - selected topics. - Lecture Notes in Math. 1633, Springer, 1996.

[3] Alvarez, J., and C. Pérez: Estimates with A_∞ weights for various singular integral operators. - Boll. Un. Mat. Ital. A(7) 8:1, 1994, 123–133.

[4] Batbold, T., and Y. Sawano: Decompositions for local Morrey spaces. - Eurasian Math. J. 5:3, 2014, 9–45.

[5] Cruz-Uribe, D., and A. Fiorenza: Variable Lebesgue spaces, foundations and harmonic analysis. - Appl. Numer. Harmon. Anal., Birkhauser/Springer, Heidelberg, 2013.

[6] Cruz-Uribe, D., A. Fiorenza, J. M. Martell, and C. Pérez: The boundedness of classical operators on variable L^p spaces. - Ann. Acad. Sci. Fenn. Math. 31, 2006, 239–264

[7] Diening, L., P. Harjulehto, P. Hästö, and M. Růžička: Lebesgue and Sobolev spaces with variable exponents. - Lecture Notes in Math. 2017, Springer, 2011.

[8] Duoandikoetxea, J.: Fourier analysis. - Grad. Stud. Math. 29, Amer. Math. Soc., Providence, RI, 2001.

[9] Garcia-Cuerva, J., and J. L. Rubio de Francia: Weighted norm inequalities and related topics. - North-Holland Math. Studies 116, 1985.

[10] Grafakos, L.: Classical Fourier analysis. - Grad. Texts in Math. 249, Springer, 2008.

[11] Gunawan, H., and Eridani: Fractional integrals and generalized Olsen inequalities. - Kyungpook Math. J. 49, 2009, 31–39.

[12] Gunawan, H., Y. Sawano, and I. Sihwaningrum: Fractional integral operators in nonhomogeneous spaces. - Bull. Aust. Math. Soc. 80:2, 2009, 324–334.

[13] Ho, K. P.: Atomic decomposition of Hardy–Morrey spaces with variable exponents. - Ann. Acad. Sci. Fenn. Math. 40:1, 2015, 31–62.

[14] Iida, T., Y. Sawano, and H. Tanaka: Atomic decomposition for Morrey spaces. - Z. Anal. Anwend. 33:2, 2014, 149–170.

[15] Jawerth, B., C. Pérez, and G. Welland: The positive cone in Triebel–Lizorkin spaces and relation among potential and maximal operators. - Contemp. Math. 107, 1990, 71–91.

[16] Journé, J.-L.: Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón. - Lecture Notes in Math. 994, Springer-Verlag, 1983.

[17] Kerman, R., and E. Sawyer: The trace inequality and eigenvalue estimates for Scrödinger operators. - Ann. Inst. Fourier (Grenoble) 36, 1986, 80–147.

[18] Maeda, F.-Y.: Variable exponent version of Hedberg–Wolff inequalities. - In: Potential Theory and its related Fields, RIMS Kôkyûroku 1669, 2009, 68–80.

[19] Maeda, F.-Y., Y. Mizuta, T. Ohno, and T. Shimomura: Boundedness of maximal operators and Sobolev's inequality on Musielak–Orlicz–Morrey spaces. - Bull. Sci. Math. 137, 2013, 76–96.

[20] Maeda, F.-Y., Y. Mizuta, T. Ohno, and T. Shimomura: Approximate identities and Young type inequalities in Musielak–Orlicz spaces. - Czechoslovak Math. J. 63, 138, 2013, 933–948.

[21] Mizuta, Y.: Potential theory in Euclidean spaces. - Gakkōtosho, Tokyo, 1996.

[22] Musielak, J.: Orlicz spaces and modular spaces. - Lecture Notes in Math. 1034, Springer-Verlag, 1983.

[23] Nakai, E., and Y. Sawano: Hardy spaces with variable exponents and generalized Campanato spaces. - J. Funct. Anal. 262, 2012, 3665–3748.

[24] Nakai, E., and Y. Sawano: Orlicz–Hardy spaces and their duals. - Sci. China Math. 57:5, 2014, 903–962.

[25] Nakano, H.: Modulared semi-ordered linear spaces. - Maruzen Co., Ltd., Tokyo, 1950.

[26] Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators. - Integral Equations Operator Theory 77:1, 2013, 123–148.

[27] Sawano, Y., D. I. Hakim, and H. Gunawan: Non-smooth atomic decomposition for generalized Orlicz–Morrey spaces. - Math. Nachr. 288:14-15, 2015, 1741–1775.

[28] Sawano, Y., S. Sugano, and H. Tanaka: Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. - Trans. Amer. Math. Soc. 363:12, 2011, 6481–6503.

[29] Sihwaningrum, I., H. P. Suryawan, and H. Gunawan: Fractional integral operators and Olsen inequalities on non-homogeneous spaces. - Aust. J. Math. Anal. Appl. 7:1, 2010, Art. 14, 1–6.

[30] Sihwaningrum, I., S. Maryani, and H. Gunawan: Weak type inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces. - Anal. Theory Appl. 28:1, 2012, 65–72.

[31] Stein, E. M.: Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals. - Princeton Univ. Press, Princeton, 1971.

[32] Sugano, S.: Some inequalities for generalized fractional integral operators on generalized Morrey spaces. - Math. Inequal. Appl. 14:4, 2011, 849–865.

[33] Sugano, S., and H. Tanaka: Boundedness of fractional integral operators on generalized Morrey spaces. - Sci. Math. Jpn. 58:3, 2003, 531–540.

[34] Tanaka, H.: Morrey spaces and fractional operators. - J. Aust. Math. Soc. 88:2, 2010, 247–259.

[35] Turesson, B. O.: Nonlinear potential theory and weighted Sobolev spaces. - Lecture Notes in Math. 1736, Springer-verlag, 2000.

[36] Utoyo, M. I., T. Nusantara, and B. S. Widodo: Fractional integral operator and Olsen inequality in the non-homogeneous classic Morrey space. - Int. J. Math. Anal. (Ruse) 6:29-32, 2012, 1501–1511.

[37] Zhuo, C., Y. Sawano, and D. Yang: Hardy spaces with variable exponents on RD-spaces and applications. - Dissertationes Math. (Rozprawy Mat.) (to appear).

Received 14 November 2014 • Revised received 17 September 2015 • Accepted 5 February 2016