Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 803–811

C^1-EMBEDDINGS BETWEEN GRAPH-DIRECTED SETS

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Abstract. For graph-directed sets, we obtain that a C^1-embedding implies an affine embedding. We only pose the open set condition for the image sets. We can apply our result to self-similar sets with overlaps, for example all λ-Cantor sets.

1. Introduction

There are many works devoted to the bilipschitz embedding between fractals. For example, Mattila and Saaranen [9] investigate the bilipschitz embedding between Ahlfors–David regular sets, Llorente and Mattila [7] study the bilipschitz embedding between subsets of self-conformal fractals, Deng, Wen, Xiong and Xi[1] obtain the bilipschitz embedding for self-similar sets.

Feng, Huang and Rao [3] recently established the following relation between C^1-embeddings and affine embeddings for self-similar sets:

Theorem 1.1 of [3]. Let E and F be self-similar sets. Suppose that the open set condition holds for F and

$$\dim_H E = \dim_S E, \tag{1.1}$$

where $\dim_S(\cdot)$ is the self-similarity dimension. If there is a C^1-embedding from E to F, then E can be embedded into F affinely.

Here are several minor comments for the conditions of the theorem.

1° Under the assumption (1.1), the IFS of E does not contain complete overlaps.
2° The theorem requires the C^1-embedding globally, but by the self-similarity, it seems a local C^1-embedding will be enough.
3° As we will see later, some overlapping self-similar sets can be viewed as the attractors of graph-directed IFSs satisfying the open set condition. And we are led to discuss the embeddings between graph-directed sets.

The main result of this note is Theorem 1 below which generalizes that [3, Theorem 1.1] from the three points mentioned above.

We started with some basic definitions and notations which will be used later.

For two compact sets $K \subset \mathbf{R}^{m_1}$ and $K' \subset \mathbf{R}^{m_2}$, we say that an embedding f from K to K' is *affine* if there exists $a \in \mathbf{R}^{m_2}$ and a nondegenerate $(m_2 \times m_1)$-matrix

doi:10.5186/aasfm.2016.4153
2010 Mathematics Subject Classification: Primary 28A80.
Key words: Fractal, graph-directed sets, embedding.
Lifeng Xi is the corresponding author. The work is supported by NSFC (Nos. 11371329, 11071224, 11471124, 11301346, 11271223, 11431007), NSF of Zhejiang Province (Nos. LR13A010001).

M such that $f(x) = Mx + a$ for all $x \in K$. Here M is said to be nondegenerate, if $\text{rank}(M) = m_1 \le m_2$, i.e., $Mx = 0$ for $x \in \mathbf{R}^{m_1}$ if and only if $x = 0$. We say that an embedding g from K to K' is $\mathbf{C^1}$, if there is an extension $\bar{g}$ of g such that $\bar{g} \in C^1(O, \mathbf{R}^{m_2})$ for an open neighbourhood O of K and the Jacobian $D\bar{g}_x$ is nondegenerate at each $x \in K$.

Recall the graph directed construction [10] as follows. Given a directed graph $G = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$, the graph directed sets $\{K_i(\subset \mathbf{R}^m)\}_{i\in\mathcal{V}}$ on G with contracting similitudes $\{S_e\colon \mathbf{R}^m \to \mathbf{R}^m\}_{e\in\mathcal{E}}$ are non-empty compact sets satisfying

$$K_i = \bigcup\nolimits_{j\in\mathcal{V}} \bigcup\nolimits_{e\in\mathcal{E}_{i,j}} S_e(K_j) \quad \text{for all } i \in \mathcal{V}, \tag{1.2}$$

where $\mathcal{E}_{i,j}$ is the collection of directed edges from i to j, and S_e has similarity ratio r_e for any $e \in \mathcal{E}$. In particular, if $\mathcal{V}$ is a singleton, we obtain a self-similar set. We say that the open set condition (OSC) holds for graph-directed sets $\{K_i\}_{i\in\mathcal{V}}$, if there are non-empty and bounded open sets $\{O_i\}_{i\in\mathcal{V}}$ satisfying the *disjoint* union

$$\bigcup\nolimits_{j\in\mathcal{V}} \bigcup\nolimits_{e\in\mathcal{E}_{i,j}} S_e(O_j) \subset O_i \quad \text{for all } i \in \mathcal{V}. \tag{1.3}$$

A path $\mathbf{e} = e_1e_2\cdots e_k$ is said to be *admissible*, if the ending vertex of e_i is exactly the starting vertex of e_{i+1} for every i. Throughout the paper, when we talk about a path, we always mean an admissible one. If for any vertices $i \ne j \in \mathcal{V}$, there is a path from i to j, we will say that the *transitivity condition* holds for $\{K_i\}_{i\in\mathcal{V}}$. For more characterizations of graph-directed sets, please see [11]–[16].

Given a λ-Cantor set $E_\lambda = E_\lambda/3 \cup (E_\lambda/3 + \lambda/3) \cup (E_\lambda/3 + 2/3)$, Hochman [4] proved the Furstenberg's conjecture that $\dim_H E_\lambda = 1$ for every $\lambda \notin \mathbf{Q}$. When $\lambda \in \mathbf{Q}$, Kenyon [5], Lagarias and Wang [6], Rao and Wen [14] proved that $\dim_H E_\lambda = 1$ for any $\lambda = p/q \in \mathbf{Q}$ with $(p, q) = 1$ and $p \equiv q \not\equiv 0 (\text{mod}\, 3)$. In these two cases, we have

$$\dim_H E_\lambda = \dim_S E_\lambda.$$

Using *graph-directed sets*, Rao and Wen [14] proved that if $\lambda = p/q \in \mathbf{Q}$ with $(p, q) = 1$ and $p \not\equiv q (\text{mod}\, 3)$, then

$$\dim_H E_\lambda < \dim_S E_\lambda \tag{1.4}$$

and there are graph directed sets $\{E_\lambda^{(1)}(= E_\lambda), \cdots, E_\lambda^{(k)}\}$ satisfying the OSC. In particular, for any $\lambda = 2/3^n$ with $n \ge 1$,

$$\dim_H E_{2/3^n} = \log_3 \frac{3 + \sqrt{5}}{2} < \dim_S E_{2/3^n} = 1$$

and the *transitivity condition* holds for corresponding graph-directed sets $\{E_\lambda^{(1)}(= E_\lambda), E_\lambda^{(2)}, \cdots, E_\lambda^{(2^n)}\}$.

Remark 1. Let $\alpha^{-1} > 1$ be a P.V. number, for example, α^{-1} is $\frac{1+\sqrt{5}}{2}$, $\sqrt{2}+1$ or a positive integer greater than 1. An interesting fact is that we can obtain certain graph-directed sets satisfying the OSC from the IFS $\{\alpha^{p_i}x + b_i\}_{i=1}^m$ with $p_i \in \mathbf{N}$ and $b_i \in \mathbf{Q}$ for all i. For details, we refer to see [14, 8, 11, 17].

Now we state our main result.

Theorem 1. *Suppose $\{K_i\}_i$ and $\{E_j\}_j$ are graph directed sets, the OSC holds for $\{K_i\}_i$ and the transitivity condition holds for $\{E_j\}_j$. If there is a C^1-embedding from E_{j_0} to K_{i_0} for some i_0 and j_0, then there exists an index i such that there is an affine embedding from E_j to K_i for every j.*

Since any self-similar set has graph-directed construction satisfying the transitivity condition, we have

Theorem 2. *Let E and F be self-similar sets. Suppose $F_1 = F \subset \mathbf{R}^m$ and $\{F_1, \cdots, F_k\}$ are graph directed sets satisfying the OSC such that*

$$F_i = \bigcup_{j=1}^{t(i)} S_{i,j}(F) \tag{1.5}$$

$S_{i,j}\colon \mathbf{R}^m \to \mathbf{R}^m$ is the similarity for each (i,j). If there is a C^1-embedding from E to F, then E can be embedded into F affinely.

Taking $k = 1$ in Theorem 2, we have the following corollary which is Theorem 1.1 of [3] without the assumption (1.1).

Corollary 1. *Let E and F be self-similar sets and assume that the OSC holds for F. If there is a C^1-embedding from E to F, then E can be embedded into F affinely.*

Remark 2. (1) In Theorem 2, we pose no additional conditions for E, such as (1.1). Then we can take $E = E_\lambda$ in Theorem 2 where $\lambda = p/q \in \mathbf{Q}$ with $p + q \equiv 0 (\mathrm{mod}\, 3)$ and $pq \not\equiv 0 (\mathrm{mod}\, 3)$, in this case, we have (1.4).

(2) If $\alpha^{-1} > 1$ is a P.V. number and $F = \bigcup_{i=1}^m (\alpha^{p_i} F + b_i)$ with $p_i \in \mathbf{N}$ and $b_i \in \mathbf{Q}$ for all i, then there are graph directed sets $\{F_1(= F), \cdots, F_k\}$ satisfying the OSC such that

$$F_i = \bigcup_{j=1}^{t(i)} (\alpha^{q_{i,j}} F + c_{i,j})$$

where $q_{i,j} \in \mathbf{Z}$ and $c_{i,j} \in \mathbf{R}$, that means (1.5) holds in Theorem 2. We also obtain graph-directed sets satisfying (1.5) for λ-Cantor set $E_{p/q}$ with $p + q \not\equiv 0 (\mathrm{mod}\, 3)$ and $pq \not\equiv 0 (\mathrm{mod}\, 3)$ [14].

(3) In Theorem 1.1 of [3], the C^1-embedding $g \in C^1(\mathbf{R}^m, \mathbf{R}^m)$ is a C^1-diffeomorphism on $\mathbf{R}^m$. Our result only need $\bar{g} \in C^1(O, \mathbf{R}^{m_2})$ for an open neighbourhood O of a compact set E.

The paper is organized as follows. In Section 2 we give some preliminaries, including graph-directed construction and nearly affine mappings. In Section 3, using Arzela–Ascoli theorem and Baire category theorem, we prove the main theorems. To avoid the notational confusion, we draw a figure to illustrate the proof.

2. Preliminaries

For subsets A, B of $\mathbf{R}^m$, let $\mathrm{dist}(x, A) = \inf_{y \in A} |x - y|$, $\mathrm{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|$ and $|A|$ the diameter of A. For a given Euclidean space, let $B(x, r)$ be the open ball centered at x with radius $r > 0$, and $\bar{B}(x, r)$ its closure.

2.1. Graph-directed construction. Let $\{K_i\}_{i \in \mathcal{V}}$ be the graph directed sets on $G = (\mathcal{V}, \mathcal{E})$ with contracting similitudes $\{S_e\}_{e \in \mathcal{E}}$. For any path $\mathbf{e} = e_1 e_2 \cdots e_k$, we denote its length $|\mathbf{e}| = k$. For $\mathbf{e} = e_1 \cdots e_k$ and $\mathbf{e}' = e_1 \cdots e_k e_{k+1} \cdots e_{k+m}$, we denote by $\mathbf{e} \prec \mathbf{e}'$. Then we give a *partial order*. Suppose $\mathbf{e} = e_1 e_2 \cdots e_k$ is a path from vertex i to vertex j. Then $S_{\mathbf{e}} = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ is a contracting similitude from K_j to K_i, with ratio $r_{\mathbf{e}} = r_{e_1} r_{e_2} \cdots r_{e_k}$. Write $K_{\mathbf{e}} = S_{\mathbf{e}}(K_j)$. If the OSC holds as in (1.3), then $K_j \subset \bar{O}_j$ for all $j \in \mathcal{V}$. For any path $\mathbf{e}$ from vertex i to vertex j, we also have $O_{\mathbf{e}}$ with its closure $\bar{O}_{\mathbf{e}}$. Then $K_{\mathbf{e}} = S_{\mathbf{e}}(K_j) \subset S_{\mathbf{e}}(\bar{O}_j) = \bar{O}_{\mathbf{e}}$.

Claim 1. *If the OSC holds as in* (1.3), *then for every path* $\mathbf{e}$ *we have* $K_{\mathbf{e}} \subset \bar{O}_{\mathbf{e}}$.

Claim 2. *When the transitivity condition holds, then for all pair* $(i,j) \in \mathcal{V} \times \mathcal{V}$, $y \in K_i$ *and* $\varepsilon > 0$, *there exists a similitude* S *such that* $S(K_j) \subset B(y,\varepsilon) \cap K_i$.

Proof. Given $y \in K_i$, there exists an infinite path $\mathbf{e}^* = e_1 \cdots e_k \cdots$ such that

$$\bigcap_{k \geq 1} K_{\mathbf{e}(k)} = \{y\}$$

where $\mathbf{e}(k) = e_1 \cdots e_k$. Then there is an index $j' \in \mathcal{V}$ and an infinite sequence $k_1 < \cdots < k_n < k_{n+1} < \cdots$ of integers such that e_{k_i} is ending at the vertex $j' \in \mathcal{V}$ for all i. By the transitivity condition, there is a path $\mathbf{e}'$ from j' to j. Therefore, we obtain that

$$y \in K_{\mathbf{e}(k_i)} \quad \text{and} \quad S_{\mathbf{e}(k_i)} S_{\mathbf{e}'}(K_j) \subset K_{\mathbf{e}(k_i)} \quad \text{with} \ \ |K_{\mathbf{e}(k_i)}| \to 0 \ \text{ as } \ i \to \infty.$$

Take i large enough with $|K_{\mathbf{e}(k_i)}| < \varepsilon$, then $S = S_{\mathbf{e}(k_i)} S_{\mathbf{e}'}$ is the contracting similitude required. □

Denote by $\Omega(x,\varepsilon)$ the collection of all paths with their copies, whose diameters are comparable to ε, intersecting the closed ball $\bar{B}(x,\varepsilon)$, i.e.,

$$\Omega(x,\varepsilon) = \{\mathbf{e} \colon \bar{B}(x,\varepsilon) \cap K_{\mathbf{e}} \neq \varnothing \text{ and } (\min_{e \in \mathcal{E}} r_e) \cdot \varepsilon \leq |K_{\mathbf{e}}| \leq \varepsilon\}.$$

The following lemma is natural from the OSC, we give its proof for self-containedness.

Lemma 1. *If the OSC holds as in (1.3), then there exists an integer* N_0 *such that for any* $\varepsilon > 0$ *and* $x \in K_i$,

$$\#\Omega(x,\varepsilon) \leq N_0.$$

Proof. Suppose $x \in K_i$ and $B(y_j, r^*) \subset O_j$ for all j with some small $r^* > 0$. Write $O_{\mathbf{e}} = S_{\mathbf{e}}(O_j)$ and $B_{\mathbf{e}} = S_{\mathbf{e}}(B(y_j, r^*))$ if the path $\mathbf{e}$ is ending at j.

For any path $\mathbf{e} = e_1 e_2 \cdots e_{k-1} e_k$, denote $\mathbf{e}^- = e_1 e_2 \cdots e_{k-1}$. Let

$$\Omega^*(x,\varepsilon) = \{\mathbf{e} \colon |K_{\mathbf{e}}| \leq \varepsilon, |K_{\mathbf{e}^-}| > \varepsilon\} (\subset \Omega(x,\varepsilon)).$$

Let N is an integer satisfying $(\max_{e \in \mathcal{E}} r_e)^N < \min_{e \in \mathcal{E}} r_e$, we have

$$\sup_{\mathbf{e} \in \Omega^*(x,\varepsilon)} \#\{\mathbf{e}' \in \Omega(x,\varepsilon) \colon \mathbf{e} \prec \mathbf{e}'\} \leq (\#\mathcal{E})^N$$

Therefore, we obtain

$$\#\Omega(x,\varepsilon) \leq (\#\mathcal{E})^N \cdot \#\Omega^*(x,\varepsilon).$$

Using the OSC, we find that $O_{\mathbf{e}} \cap O_{\mathbf{e}'} = \varnothing$ whenever $\mathbf{e} \neq \mathbf{e}' \in \Omega^*(x,\varepsilon)$. Now,

$$B_{\mathbf{e}} \subset O_{\mathbf{e}} \quad \text{and} \quad B_{\mathbf{e}} \cap B_{\mathbf{e}'} = \varnothing \quad \text{for} \ \ \mathbf{e} \neq \mathbf{e}' \in \Omega^*(x,\varepsilon).$$

Denote by $R_{\mathbf{e}}$ the radius of $B_{\mathbf{e}}$. If $\mathbf{e}$ is ending at j, then $R_{\mathbf{e}} = r^* r_{\mathbf{e}}$ and

$$\frac{\min_{e \in \mathcal{E}} r_e}{\max_{i \in \mathcal{V}} |K_i|} \varepsilon \leq r_{\mathbf{e}} = \frac{|K_{\mathbf{e}}|}{|K_j|} \leq \frac{\varepsilon}{\min_{i \in \mathcal{V}} |K_i|}, \tag{2.1}$$

Hence $\frac{r^* \min_{e \in \mathcal{E}} r_e}{\max_{i \in \mathcal{V}} |K_i|} \varepsilon \leq R_{\mathbf{e}} \leq \frac{r^*}{\min_{i \in \mathcal{V}} |K_i|} \varepsilon$.

Notice that $K_{\mathbf{e}} \subset \bar{O}_{\mathbf{e}}$ (Claim 1) and $\bar{B}(x,\varepsilon) \cap K_{\mathbf{e}} \neq \varnothing$, we obtain that

$$B_{\mathbf{e}} \subset \bar{O}_{\mathbf{e}} \subset \bar{B}\left(x, \varepsilon + r_{\mathbf{e}} \max_{i \in \mathcal{V}} |O_i|\right) \subset \bar{B}\left(x, \varepsilon + \frac{\max_{i \in \mathcal{V}} |O_i|)}{\min_{i \in \mathcal{V}} |K_i|} \varepsilon\right).$$

Write $c_1 = 1 + \frac{\max_{i\in\mathcal{V}}|O_i|}{\min_{i\in\mathcal{V}}|K_i|}$ and $c_2 = \frac{r^*\min_{e\in\mathcal{E}} r_e}{\max_{i\in\mathcal{V}}|K_i|}$. Now $\{B_{\mathbf{e}}\}_{\mathbf{e}\in\Omega^*(x,\varepsilon)}$ are pairwise disjoint open balls in $\bar{B}(x, c_1\varepsilon)$ and the radius $R_{\mathbf{e}} \geq c_2\varepsilon$ for each $\mathbf{e}$. We obtain

$$(\#\Omega^*(x,\varepsilon))\mathcal{L}(B(x,c_2\varepsilon)) \leq \sum_{\mathbf{e}\in\Omega^*(x,\varepsilon)} \mathcal{L}(B_{\mathbf{e}}) = \mathcal{L}\left(\bigcup_{\mathbf{e}\in\Omega^*(x,\varepsilon)} B_{\mathbf{e}}\right) \leq \mathcal{L}(B(x,c_1\varepsilon)),$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbf{R}^m$. Therefore, we have

$$\#\Omega^*(x,\varepsilon) \leq (\frac{c_1}{c_2})^n.$$

The lemma follows. □

2.2. Bilipschitz and nearly affine mapping. Suppose $K \subset \mathbf{R}^{m_1}$ and $K' \subset \mathbf{R}^{m_2}$. Given an embedding $f\colon K \to K'$, we denote

$$U(f) = \sup_{x\neq y\in K} \frac{|f(x)-f(y)|}{|x-y|} \quad \text{and} \quad L(f) = \inf_{x\neq y\in K} \frac{|f(x)-f(y)|}{|x-y|}.$$

It is clear that

$$U(f\circ g) \leq U(f)U(g) \quad \text{and} \quad L(f\circ g) \geq L(f)L(g),$$

and

$$U(S) = L(S) = r$$

for any similitude S with ratio r. For nondegenerate matrix (linear transformation) $M\colon \mathbf{R}^{m_1} \to \mathbf{R}^{m_2}$, we have $L(M) > 0$. Throughout the paper, when we say that the mapping $f(x) = Mx + a$ is affine, we mean that the matrix M is nondegenerate. Hence $L(f) > 0$ for any affine mapping f.

Claim 3. *If $g\colon K \to K'$ is a C^1-embedding, then g is a bilipschitz mapping.*

Proof. Suppose $\bar{g} \in C^1(O, \mathbf{R}^{m_2})$ for some open neighbourhood O of K with $\bar{g}|_K = g$. We can take a small number $r \in (0, \mathrm{dist}(K, \mathbf{R}^{m_1}\backslash O)/2)$ such that

$$0 < \inf_{\mathrm{dist}(y,K)\leq r} L(D\bar{g}_y) \leq \sup_{\mathrm{dist}(y,K)\leq r} U(D\bar{g}_y) < \infty.$$

We obtain finitely many open balls $\{B(z_i,r)\}_{i=1}^p$ centered at K such that $K \subset \cup_{i=1}^p B(z_i,r)$. Take δ be the Lebesgue constant of the open covering $\{B(z_i,r)\}_{i=1}^p$. Notice that the C^1-embedding g is a continuous embedding, we only need to estimate $\frac{|g(x)-g(x')|}{|x-x'|}$ for $x, x' \in K$ with $0 < |x-x'| < \delta$.

We can verify that $L(g) > 0$. In fact, whenever $0 < |x-x'| < \delta$, there exists an index $i \leq p$ such that $x, x' \in B(z_i, r)$. Therefore, we obtain a point $\xi \in B(z_i,r)$ in line segment between x and x' such that

$$\frac{|g(x)-g(x')|}{|x-x'|} = \frac{|D\bar{g}_\xi(x-x')|}{|x-x'|} \geq \inf_{\mathrm{dist}(y,K)\leq r} L(D\bar{g}_y).$$

In the same way, we can obtain $U(g) < \infty$. □

Let K, K' be compact sets as above and $c > 0$ is fixed. We say that a sequence

$$\{f_t\colon K \to K'\}_{t=1}^\infty \subset \{f\colon c^{-1} \leq L(f) \leq U(f) \leq c\}$$

is *nearly affine*, if there is a sequence $\{A_t\}_{t=1}^\infty$ of affine mappings satisfying

$$\lim_{t\to\infty} \sup_{x\in K} |f_t(x) - A_t(x)| = 0.$$

Using Arzela–Ascoli theorem, we have

Lemma 2. *If $\{f_t\}_{t=1}^\infty$ is nearly affine, then there is an affine mapping A and a subsequence $\{f_{t_i}\}_i$ of $\{f_t\}_t$ such that*

$$\lim_{i\to\infty} f_{t_i}(x) = A(x) \quad \text{uniformly on } x \in K.$$

Fix positive constants M, $N \in \mathbf{N}$ and $c \geq 1$. Suppose E, $\{B_{i,j}\}_{1\leq i<\infty, 1\leq j\leq N}$ and $\{C_i\}_{i=1}^M$ are compact subsets of Euclidean spaces. We assume that for all $i \geq 1$,

$$E = B_{i,1} \cup \cdots \cup B_{i,N}, \tag{2.2}$$

and for every j, there is a family $\{f_{i,j}\}_{i=1}^\infty$ of nearly affine mappings and an index set $\{\alpha(i,j)\}_{i=1}^\infty$ with $1 \leq \alpha(i,j) \leq M$ for all i such that

$$f_{i,j}(B_{i,j}) \subset C_{\alpha(i,j)} \tag{2.3}$$

and

$$f_{i,j} \in \{f\colon c^{-1} \leq L(f) \leq U(f) \leq c\}. \tag{2.4}$$

Here $B_{i,j}$ may be empty set. Using Arzela–Ascoli theorem again, we have

Lemma 3. *Suppose (2.2)–(2.4) hold. Then there is an integer $N^* \leq N$ and a family of affine mappings $\{A_j\}_{j=1}^{N^*}$, non-empty compact subsets $\{B_j\}_{j=1}^{N^*}$ and index set $\{\alpha(j)\}_{j=1}^{N^*}$ such that*

$$E = B_1 \cup \cdots \cup B_{N^*},$$

and $A_j(B_j) \subset C_{\alpha(j)}$ satisfying $A_j \in \{f\colon c^{-1} \leq L(f) \leq U(f) \leq c\}$.

Given compact subsets E and $B_1, \cdots, B_{N^*}$ of some Euclidean space, if

$$E = B_1 \cup \cdots \cup B_{N^*},$$

using Baire category theorem, we have

Lemma 4. *Suppose (2.2)–(2.4) hold. Then there exist an integer j with $1 \leq j \leq N^*$ and an open ball $B(x,r)$ with $x \in E$ such that*

$$E \cap B(x,r) \subset B_j.$$

3. Proof of Theorems 1 and 2

Suppose $\{E_j\}_{j\in\mathcal{U}}$ are graph-directed sets on the graph $(\mathcal{U},\mathcal{D})$ with vertex set $\mathcal{U}$ and the edge set $\mathcal{D}$ satisfying

$$E_j = \bigcup\nolimits_{j'\in\mathcal{U}} \bigcup\nolimits_{d\in\mathcal{D}_{j,j'}} T_d(E_{j'}) \text{ for all } j \in \mathcal{U}, \tag{3.1}$$

where $\mathcal{D}_{j_1,j_2} = \{d\colon \text{edge } d \text{ from } j_1 \text{ to } j_2\}$ and T_d is the contracting similitude with respect to edge d. Write $E_{d_1\cdots d_k} = T_{d_1} \circ \cdots \circ T_{d_k}(E_j)$, where the path $d_1 \cdots d_k$ is ending at vertex j.

Given $j \in \mathcal{U}$, using Claim 2, we obtain an affine embedding

$$f_j\colon E_j \to E_{j_0}.$$

To prove Theorem 1, we only need to verify

Proposition 1. *There exists $i \in \mathcal{V}$ such that E_{j_0} can be embedded to K_i affinely.*

Proof. By the transitivity condition, we can find a path $\mathbf{b}$ from j_0 to itself. Write

$$\mathbf{b}^n = \underbrace{\mathbf{b}\cdots\mathbf{b}}_{n}.$$

Suppose $x_0 \in E_{j_0}$ is the point with respect to $\mathbf{b}^\infty$, i.e.,

$$\{x_0\} = \bigcap_n E_{\mathbf{b}^n}.$$

Without loss of generality, we assume the diameter $|E_{j_0}| = 1$. Assume that $g\colon E_{j_0} \to K_{i_0}$ is the corresponding C^1-embedding, then by Claim 3, we have

$$c^{-1} \le L(g) \le U(g) \le c \tag{3.2}$$

for some constant $c > 0$. Let Dg_{x_0} be the Jacobian at point x_0. Fix an integer n. Consider the similitude $T_{\mathbf{b}^n}$ with ratio r_n. Then $|E_{\mathbf{b}^n}| = r_n|E_{j_0}| = r_n$. Then we have

$$T_{\mathbf{b}^n}\colon E_{j_0} \longrightarrow E_{\mathbf{b}^n} \quad \text{with} \quad E_{\mathbf{b}^n} \subset \bar{B}(x_0, r_n).$$

We also obtain a natural mapping

$$g|_{\bar{B}(x_0,r_n)}\colon \bar{B}(x_0, r_n) \longrightarrow \bar{B}(g(x_0), cr_n)$$

due to $U(g) \le c$.

Let $\Omega(x,\varepsilon)$ be defined as in Section 2. For any path $\mathbf{e}$ in $\Omega(g(x_0), cr_n)$, we have a natural mapping

$$(S_{\mathbf{e}})^{-1}\colon \bar{B}(g(x_0), cr_n) \cap K_{\mathbf{e}} \longrightarrow K_{\alpha(\mathbf{e})},$$

where $\alpha(\mathbf{e})$ is the ending vertex of $\mathbf{e}$. Therefore,

$$E_{j_0} = \bigcup_{\mathbf{e}\in\Omega(g(x_0),cr_n)} B_{\mathbf{e},n}, \tag{3.3}$$

where

$$B_{\mathbf{e},n} = (T_{\mathbf{b}^n})^{-1} g^{-1}(g(E_{\mathbf{b}^n}) \cap K_{\mathbf{e}}) \quad \text{and} \quad \#\Omega(g(x_0), cr_n) \le N_0,$$

where N_0 is defined in Lemma 1. Let

$$f_{\mathbf{e},n} = (S_{\mathbf{e}})^{-1} \circ g \circ T_{\mathbf{b}^n},$$

then

$$f_{\mathbf{e},n}(B_{\mathbf{e},n}) \subset K_{\alpha(\mathbf{e})}. \tag{3.4}$$

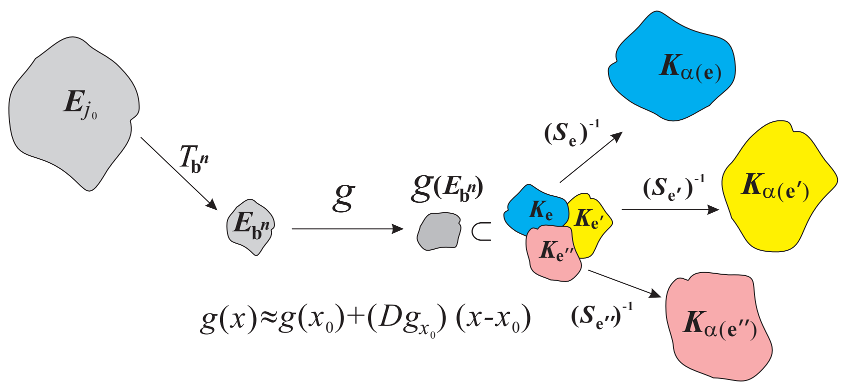

Figure 1. The case with $\#\Omega(g(x_0), cr_n) = 3$.

Now, we shall estimate $U(f_{\mathbf{e},n})$ and $L(f_{\mathbf{e},n})$. In fact, using (2.1), we have

$$(c_1)^{-1}(r_n)^{-1} \le (r_{\mathbf{e}})^{-1} \le c_1(r_n)^{-1}$$

for some constant $c_1 > 0$ depending on c and $\{K_i\}_i$. Therefore, we have

$$U((S_{\mathbf{e}})^{-1} \circ g \circ T_{\mathbf{b}^n}) \le U((S_{\mathbf{e}})^{-1})U(g)U(T_{\mathbf{b}^n}) = (r_{\mathbf{e}})^{-1}U(g)r_n \le cc_1.$$

In the same way, we have $L((S_{\mathbf{e}})^{-1} \circ g \circ T_{\mathbf{b}^n}) \geq (r_{\mathbf{e}})^{-1} L(g) r_n \leq (cc_1)^{-1}$. Now,

$$(cc_1)^{-1} \leq L(f_{\mathbf{e},n}) \leq U(f_{\mathbf{e},n}) \leq cc_1. \tag{3.5}$$

We will show the family $\{f_{\mathbf{e},n}\}_n$ is nearly affine. In fact, we have

$$g(y) = g(x_0) + (Dg_{x_0})(y - x_0) + o(|y - x_0|).$$

Therefore,

$$f_{\mathbf{e},n}(x) = (S_{\mathbf{e}})^{-1} \circ g \circ T_{\mathbf{b}^n}(x) = A_{\mathbf{e},n}(x) + (S_{\mathbf{e}})^{-1} o(|T_{\mathbf{b}^n}(x) - x_0|)$$

where $A_{\mathbf{e},n}(x) = (S_{\mathbf{e}})^{-1}\,[g(x_0) + (Dg_{x_0})(T_{\mathbf{b}^n}(x) - x_0)]$ is affine. Since

$$\begin{aligned}|(S_{\mathbf{e}})^{-1}(T_{\mathbf{b}^n}(x) - x_0)| &\leq (r_{\mathbf{e}})^{-1} r_n \leq \frac{\max_{i\in\mathcal{V}} |K_i|}{|K_{\mathbf{e}}|} r_n \\ &\leq \frac{\max_{i\in\mathcal{V}} |K_i|}{(\min_{e\in\mathcal{E}} r_e)\cdot(cr_n)} r_n \leq \frac{\max_{i\in\mathcal{V}} |K_i|}{(\min_{e\in\mathcal{E}} r_e)\cdot c},\end{aligned}$$

we obtain that

$$\sup_{x\in E_{j_0}} |(S_{\mathbf{e}})^{-1} o(|T_{\mathbf{b}^n}(x) - x_0|)| \to 0 \quad \text{as} \quad n \to \infty,$$

i.e.,

$$\sup_{x\in E_{j_0}} |f_{\mathbf{e},n}(x) - A_{\mathbf{e},n}(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Then the family $\{f_{\mathbf{e},n}\}_n$ is nearly affine for fixed $\mathbf{e} \in \Omega(g(x_0), cr_n)$. Notice that $\#\Omega(g(x_0), cr_n) \leq N_0$.

Now we have (3.3)–(3.6) for the family $\{B_{\mathbf{e},n}\}_{\mathbf{e},n}$ of compact subsets and the family $\{f_{\mathbf{e},n}\}_{\mathbf{e},n}$ of nearly affine mappings. By Lemma 3 there is an integer $N^* \leq N_0$ and a family of affine mappings $\{A_j\}_{j=1}^{N^*}$, non-empty compact subsets $\{B_j\}_{j=1}^{N^*}$ and index set $\{\alpha(j)\}_{j=1}^{N^*} \subset \{\alpha(\mathbf{e})\}_{\mathbf{e}}$ such that

$$E_{j_0} = B_1 \cup \cdots \cup B_{N^*},$$

and $A_j(B_j) \subset K_{\alpha(j)}$ satisfying $A_j \in \{f \colon c^{-1} \leq L(f) \leq U(f) \leq c\}$.

Using Lemma 4, there exist an integer j with $1 \leq j \leq N^*$ and an open ball $B(x, r)$ with $x \in E_{j_0}$ such that

$$E_{j_0} \cap B(x, r) \subset B_j.$$

Applying Claim 2 to (j_0, j_0) and $B(x, r)$, we can find a similitude S such that

$$S(E_{j_0}) \subset E_{j_0} \cap B(x, r).$$

Then $S(E_{j_0}) \subset B_j$. Therefore, we have

$$(A_j \circ S)(E_{j_0}) \subset A_j(B_j) \subset K_{\alpha(j)},$$

then Proposition 1 follows. □

Proof of Theorem 2. By Theorem 1, we obtain that there exist F_i and an affine mapping A such that

$$A(E) \subset F_i = \bigcup_{j=1}^{t(i)} S_{i,j}(F).$$

Using Baire category theorem, we have $x \in A(E)$, $r > 0$ and j such that

$$A(E) \cap B(x, r) \subset S_{i,j}(F).$$

It follows from the self-similarity of E that there is a similitude S such that $S(E) \subset A^{-1}(B(x,r))$. Let $T(x) = S_{i,j}^{-1}(x)$, then

$$(T \circ A \circ S)(E) \subset F. \qquad \square$$

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Received 25 June 2015 • Accepted 4 March 2016