PARTIAL REGULARITY RESULTS FOR ASYMPTOTIC QUASICONVEX FUNCTIONALS WITH GENERAL GROWTH

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Abstract. We prove partial regularity for minimizers of vectorial integrals of the Calculus of Variations, with general growth condition, imposing quasiconvexity assumptions only in an asymptotic sense.

1. Introduction

In this paper we study variational integrals of the type

$$\mathcal{F}(u) := \int_\Omega f(Du)\, dx \quad \text{for } u\colon \Omega \to \mathbf{R}^N$$

where Ω is an open bounded set in $\mathbf{R}^n$, $n \geq 2$, $N \geq 1$. Here $f\colon \mathbf{R}^{Nn} \to \mathbf{R}$ is a continuous function satisfying a φ-growth condition:

$$|f(z)| \leq C(1 + \varphi(|z|)), \quad \forall z \in \mathbf{R}^{Nn},$$

where C is a positive constant and φ is a given N-function (see Definition 2.1).

Some examples of N-functions are the following:

$$\varphi(t) = t^p, \quad 1 < p < \infty,$$
$$\varphi(t) = t^p \log^\alpha(1+t), \quad p > 1 \text{ and } \alpha > 0.$$

If φ is an N-function satisfying the Δ_2-condition (see Section 2), by $L^\varphi(\Omega)$ and $W^{1,\varphi}(\Omega)$ we denote the classical Orlicz and Orlicz–Sobolev spaces, i.e. $u \in L^\varphi(\Omega)$ if and only if $\int_\Omega \varphi(|u|)\, dx < \infty$ and $u \in W^{1,\varphi}(\Omega)$ if and only if $u, Du \in L^\varphi(\Omega)$. The Luxembourg norm is defined as follows:

$$\|u\|_{L^\varphi(\Omega)} = \inf\left\{\lambda > 0\colon \int_\Omega \varphi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}.$$

With this norm $L^\varphi(\Omega)$ is a Banach space.

If there is no misunderstanding, we will write $\|u\|_\varphi$. Moreover, we denote by $W^{1,\varphi}_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ functions with respect to the norm

$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|Du\|_\varphi$$

and by $W^{-1,\varphi}(\Omega)$ its dual.

We will consider the following definition of a minimizer of $\mathcal{F}$.

Definition 1.1. A map $u \in W^{1,\varphi}(\Omega, \mathbf{R}^N)$ is a $W^{1,\varphi}$-minimizer of $\mathcal{F}$ in Ω if

$$\mathcal{F}(u) \leq \mathcal{F}(u + \xi)$$

for every $\xi \in W_0^{1,\varphi}(\Omega, \mathbf{R}^N)$.

Let us recall the notion of quasiconvexity introduced by Morrey [32]:

Definition 1.2. (Quasiconvexity) A continuous function $f \colon \mathbf{R}^{Nn} \to \mathbf{R}$ is said to be quasiconvex if and only if

$$\fint_{\mathcal{B}_1} f(z + D\xi) \, dx \geq f(z)$$

holds for every $z \in \mathbf{R}^{Nn}$ and every smooth function $\xi \colon \mathcal{B}_1 \to \mathbf{R}^N$ with compact support in the unit ball $\mathcal{B}_1$ in $\mathbf{R}^n$.

Let us note that by scaling and translating, the unit ball in the definition above may be replaced by an arbitrary ball in $\mathbf{R}^n$.

The quasiconvexity was originally introduced for proving the lower semicontinuity and the existence of minimizers of variational integrals of the Calculus of Variations. In fact, assuming a power growth condition on f, quasiconvexity is a necessary and sufficient condition for the sequential lower semicontinuity on $W^{1,p}(\Omega, \mathbf{R}^N)$, $p > 1$ (see [1] and [31]). In the regularity theory a stronger definition, the strict quasiconvexity, is needed, a notion which has nowadays become a common condition in the vectorial Calculus of Variations (see [24, 2, 7]).

In order to treat the general growth case, we consider the notion of strictly $W^{1,\varphi}$-quasiconvexity introduced in [14] (see also [6]).

Definition 1.3. (Strict $W^{1,\varphi}$-quasiconvexity) A continuous function $f \colon \mathbf{R}^{Nn} \to \mathbf{R}$ is said to be strictly $W^{1,\varphi}$-quasiconvex if there exists a positive constant $k > 0$ such that

$$\fint_{\mathcal{B}_1} f(z + D\xi) \, dx \geq f(z) + k \fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|) \, dx$$

for all $\xi \in C_0^1(\mathcal{B}_1)$, for all $z \in \mathbf{R}^{Nn}$, where $\varphi_a(t) \sim t^2 \varphi''(a + t)$ for $a, t \geq 0$.

A precise definition of φ_a is given in Section 2.

In this paper we will exhibit an adequate notion of strict $W^{1,\varphi}$-quasiconvexity at infinity which we will call $W^{1,\varphi}$-asymptotic quasiconvexity (see Definition 2.5).

We note that in recent years a growing literature has considered the subject of asymptotic regular problems: regularity theory for integrands with a particular structure near infinity has been investigated first in [9] and subsequently in [29, 35, 10, 17, 30, 33, 26, 27, 16, 18].

We deal with the problem wondering if, when you *localize at infinity* the natural assumptions to have regularity, this regularity breaks down or not. It is the same question faced in [5] and [4], where you do not require a global strict convexity or quasiconvexity assumption: all the hypotheses are localized in some point z_0 and you obtain that minimizers are Hölder continuous near points where the integrand function is "close" to the value z_0.

Thus, after establishing several characterizations of the notion of $W^{1,\varphi}$-asymptotic quasiconvexity (see Theorem 4.1) we will prove the following result.

Theorem 1.1. *Let f satisfy $(\mathcal{H}1)$–$(\mathcal{H}5)$ (see Section 2.3). Let $z_0 \in \mathbf{R}^n$ with $|z_0| > M + 1$ so that (7.2) holds in $\mathcal{B}_\rho(x_0)$, let $u \in W^{1,\varphi}(\Omega, \mathbf{R}^N)$ be a minimizer of $\mathcal{F}$, and $V(z) = \sqrt{\frac{\varphi'(|z|)}{|z|}}z$. If for some $x_0 \in \Omega$*

$$(1.1) \qquad \lim_{r \to 0} \fint_{\mathcal{B}_r(x_0)} |V(Du) - V(z_0)|^2 \, dx = 0,$$

then in a neighborhood of x_0 the minimizer u is $C^{1,\overline{\alpha}}$ for some $\overline{\alpha} < 1$.

In order to achieve this regularity result, we have to prove an *excess decay estimate*, where the excess function is defined by

$$\mathbf{E}(\mathcal{B}_R(x_0), u) = \fint_{\mathcal{B}_R(x_0)} |V(Du) - (V(Du))_{\mathcal{B}_R(x_0)}|^2 \, dx.$$

In the power case the main idea is to use a blow-up argument based strongly on the homogeneity of $\varphi(t) = t^p$. Here we have to face with the lack of the homogeneity since the general growth condition. Thus one makes use of the so-called $\mathcal{A}$-harmonic approximation proved in [14] (see also [37, 19, 20, 21, 22] for the power case). Such tool allows us to compare the solutions of our problem with the solution of the regular one in terms of the closeness of the gradient.

Moreover, we will prove that minimizers of $\mathcal{F}$ are Lipschitz continuous on an open and dense subset of Ω. More precisely we define the set of regular points $\mathcal{R}(u)$ by

$$\mathcal{R}(u) = \{x \in \Omega \colon u \text{ is Lipschitz near } x\},$$

following that $\mathcal{R}(u) \subset \Omega$ is open.

Corollary 1.1. *Assume that f satisfies $(\mathcal{H}1)$–$(\mathcal{H}5)$. Then, for every minimizer $u \in W^{1,\varphi}(\Omega, \mathbf{R}^N)$ of $\mathcal{F}$, the regular set $\mathcal{R}(u)$ is dense in Ω.*

We remark that a counterexample of [36] shows that it is not possible to establish regularity outside a negligible set (which would be the natural thing in the vectorial regularity theory). So, our regularity result generalizes the ones given in [36] and [8] for integrands with a power growth condition which become strictly convex and strictly quasiconvex near infinity, respectively.

2. Definitions and assumptions

To simplify the presentation, the letters c, C will denote generic positive constants, which may change from line to line, but does not depend on the crucial quantities. For $v \in L^1_{loc}(\mathbf{R})$ and a ball $\mathcal{B}_r(x_0) \subset \mathbf{R}^n$ we define

$$(v)_{\mathcal{B}_r(x_0)} := \fint_{\mathcal{B}_r(x_0)} v(x) \, dx := \frac{1}{|\mathcal{B}_r(x_0)|} \int_{\mathcal{B}_r(x_0)} v(x) \, dx$$

where $|\mathcal{B}_r(x_0)|$ is the n-dimensional Lebesgue measure of $\mathcal{B}_r(x_0)$. When it is clear from the context we shall omit the center as follows: $\mathcal{B}_r \equiv \mathcal{B}_r(x_0)$.

2.1. N-functions. The following definitions and results are standard in the context of N-functions (see [34]).

Definition 2.1. A real function $\varphi \colon [0, \infty) \to [0, \infty)$ is said to be an N-function if $\varphi(0) = 0$ and there exists a right continuous nondecreasing derivative φ' satisfying $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$ and $\lim_{t \to \infty} \varphi'(t) = \infty$. Especially φ is convex.

Definition 2.2. We say that φ satisfies the Δ_2-condition (we shall write $\varphi \in \Delta_2$) if there exists a constant $c > 0$ such that

$$\varphi(2t) \leq c\,\varphi(t) \quad \text{for all } t \geq 0.$$

We denote the smallest possible constant by $\Delta_2(\varphi)$.

We shall say that two real functions φ_1 and φ_2 are *equivalent* and write $\varphi_1 \sim \varphi_2$ if there exist constants $c_1, c_2 > 0$ such that $c_1\varphi_1(t) \leq \varphi_2(t) \leq c_2\varphi_1(t)$ if $t \geq 0$. Since $\varphi(t) \leq \varphi(2t)$ the Δ_2-condition implies $\varphi(2t) \sim \varphi(t)$. Moreover, if φ is a function satisfying the Δ_2-condition, then $\varphi(t) \sim \varphi(at)$ uniformly in $t \geq 0$ for any fixed $a > 1$. Let us also note that, if φ satisfies the Δ_2-condition, then any N-function which is equivalent to φ satisfies this condition too.

By $(\varphi')^{-1}\colon [0, \infty) \to [0, \infty)$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{s \geq 0\colon \varphi'(s) \leq t\}.$$

If φ' is strictly increasing, then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^*\colon [0, \infty) \to [0, \infty)$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds$$

is again an N-function and for $t > 0$ it results $(\varphi^*)'(t) = (\varphi')^{-1}(t)$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s \geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. Examples of such complementary pairs are

$$\varphi(t) = \frac{t^p}{p}(\ln t)^{\gamma_1}(\ln \ln t)^{\gamma_2}\cdots(\ln \ln \cdots \ln t)^{\gamma_n},$$

$$\varphi^*(t) = \frac{t^q}{q}[(\ln t)^{-\gamma_1}(\ln \ln t)^{-\gamma_2}\cdots(\ln \ln \cdots \ln t)^{-\gamma_n}]^{q-1}$$

with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and γ_i are arbitrary numbers.

If φ, φ^* satisfy the Δ_2-condition we will write that $\Delta_2(\varphi, \varphi^*) < \infty$. Assume that $\Delta_2(\varphi, \varphi^*) < \infty$. Then for all $\delta > 0$ there exists c_δ depending only on $\Delta_2(\varphi, \varphi^*)$ such that for all $s, t \geq 0$ it holds that

$$t\,s \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(s).$$

This inequality is called Young's inequality. For all $t \geq 0$

$$t \leq \varphi^{-1}(t)(\varphi^*)^{-1}(t) \leq 2t,$$

$$\frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\varphi'(t),$$

$$\varphi\Big(\frac{\varphi^*(t)}{t}\Big) \leq \varphi^*(t) \leq \varphi\Big(\frac{2\varphi^*(t)}{t}\Big).$$

Therefore, uniformly in $t \geq 0$,

(2.1) $\varphi(t) \sim t\varphi'(t), \quad \varphi^*(\varphi'(t)) \sim \varphi(t),$

where constants depend only on $\Delta_2(\varphi, \varphi^*)$.

Definition 2.3. We say that an N-function φ is of type (p_0, p_1) with $1 \leq p_0 \leq p_1 < \infty$ if

(2.2) $\varphi(st) \leq C \max\{s^{p_0}, s^{p_1}\}\varphi(t) \quad \forall s, t \geq 0.$

The following Lemma can be found in [14] (see Lemma 5).

Lemma 2.1. *Let φ be an N-function with $\varphi \in \Delta_2$ together with its conjugate. Then φ is of type (p_0, p_1) with $1 < p_0 < p_1 < \infty$ where p_0 and p_1 and the constant C depend only on $\Delta_2(\varphi, \varphi^*)$.*

Throughout the paper we will assume that φ satisfies the following assumption.

Assumption 2.1. Let φ be an N-function such that φ is $C^1([0, +\infty))$ and $C^2(0, +\infty)$. Further assume that

$$(2.3) \qquad \varphi'(t) \sim t\varphi''(t).$$

We remark that under this assumption $\Delta_2(\varphi, \varphi^*) < \infty$ will be automatically satisfied, where $\Delta_2(\varphi, \varphi^*)$ depends only on the characteristics of φ.

We consider a family of N-functions $\{\varphi_a\}_{a \geq 0}$ setting, for $t \geq 0$,

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\, ds \quad \text{with} \quad \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}.$$

The following lemma can be found in [12] (see Lemma 23 and Lemma 26).

Lemma 2.2. *Let φ be an N-function with $\varphi \in \Delta_2$ together with its conjugate. Then for all $a \geq 0$ the function φ_a is an N-function and $\{\varphi_a\}_{a \geq 0}$ and $\{(\varphi_a)^*\}_{a \geq 0} \sim \{\varphi^*_{\varphi'(a)}\}_{a \geq 0}$ satisfy the Δ_2 condition uniformly in $a \geq 0$.*

Let us observe that by the previous lemma $\varphi_a(t) \sim t\varphi_a'(t)$. Moreover, for $t \geq a$ we have $\varphi_a(t) \sim \varphi(t)$ and for $t \leq a$ we have $\varphi_a(t) \sim t^2\varphi''(a)$. This implies that $\varphi_a(st) \leq cs^2\varphi_a(t)$ for all $s \in [0, 1]$, $a \geq 0$ and $t \in [0, a]$.

For given φ we define the associated N-function ψ by

$$\psi'(t) = \sqrt{t\varphi'(t)}.$$

Note that

$$\psi''(t) = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\sqrt{\frac{\varphi'(t)}{t}} = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\frac{\psi'(t)}{t}.$$

It is shown in [12] (see Lemma 25) that if φ satisfies Assumption 2.1 then also φ^*, ψ and ψ^* satisfy Assumption 2.1 and $\psi''(t) \sim \sqrt{\varphi''(t)}$.

We define $A, V \colon \mathbf{R}^{Nn} \to \mathbf{R}^{Nn}$ in the following way:

$$(2.4) \qquad A(z) = D\Phi(z), \quad V(z) = D\Psi(z),$$

where $\Phi(z) := \varphi(|z|)$ and $\Psi(z) := \psi(|z|)$. About the functions A and V, the following three lemmas can be found in [12] (see Lemma 21, Lemma 24, and Lemma 3, respectively).

Lemma 2.3. *Let φ satisfying Assumption 2.1, then $A(z) = \varphi'(|z|)\frac{z}{|z|}$ for $z \neq 0$, $A(0) = 0$ and A satisfies*

$$|A(z_1) - A(z_2)| \leq c\varphi''(|z_1| + |z_2|)|z_1 - z_2|,$$
$$(A(z_1) - A(z_2), z_1 - z_2) \geq C\varphi''(|z_1| + |z_2|)|z_1 - z_2|^2,$$

for $z_1, z_2 \in \mathbf{R}^{Nn}$.

The same conclusions of Lemma 2.3 holds with A and φ replaced by V and ψ.

Lemma 2.4. *Let φ satisfy Assumption 2.1. Then, uniformly in $z_1, z_2 \in \mathbf{R}^n$, $|z_1| + |z_2| > 0$*

$$\varphi''(|z_1| + |z_2|)|z_1 - z_2| \sim \varphi'_{|z_1|}(|z_1 - z_2|),$$
$$\varphi''(|z_1| + |z_2|)|z_1 - z_2|^2 \sim \varphi_{|z_1|}(|z_1 - z_2|).$$

Lemma 2.5. *Let φ satisfy Assumption 2.1 and let A and V be defined by (2.4). Then, uniformly in $z_1, z_2 \in \mathbf{R}^{Nn}$,*

$$(A(z_1) - A(z_2), z_1 - z_2) \sim |V(z_1) - V(z_2)|^2 \sim \varphi_{|z_1|}(|z_1 - z_2|),$$

and

$$|A(z_1) - A(z_2)| \sim \varphi'_{|z_1|}(|z_1 - z_2|).$$

Moreover

$$(A(z_1), z_1) \sim |V(z_1)|^2 \sim \varphi(|z_1|), \quad |A(z_1)| \sim \varphi'(|z_1|),$$

uniformly in $z_1 \in \mathbf{R}^{Nn}$.

2.2. Asymptotic $W^{1,\varphi}$-quasiconvexity. Before introducing the notion of asymptotic $W^{1,\varphi}$-quasiconvexity, let us consider a uniform version of the strict $W^{1,\varphi}$-quasiconvexity.

Definition 2.4. (Uniform strict $W^{1,\varphi}$-quasiconvexity) A continuous function $f\colon \mathbf{R}^{Nn} \to \mathbf{R}$ is said to be uniformly strictly $W^{1,\varphi}$-quasiconvex if there exists a positive constant $k > 0$ such that

$$(2.5) \qquad \fint_{\mathcal{B}_1} f(z + D\xi) \, dx \geq f(z) + k \fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|) \, dx$$

for all $\xi \in C_c^1(\mathcal{B}_1)$, for all $z \in \mathbf{R}^{Nn}$, where $\varphi_a(t) \sim t^2 \varphi''(a + t)$ for $a, t \geq 0$.

Definition 2.5. (Asymptotic $W^{1,\varphi}$-quasiconvexity) A function $f\colon \mathbf{R}^{Nn} \to \mathbf{R}$ is asymptotically $W^{1,\varphi}$-quasiconvex if there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that

$$f(z) = g(z) \quad \text{for } |z| > M.$$

Considering an N-function satisfying Assumption 2.1, we will work with the following set of hypotheses.

2.3. Assumptions. Let $f\colon \mathbf{R}^{Nn} \to \mathbf{R}$ be such that

($\mathcal{H}1$) $f \in C^1(\mathbf{R}^{Nn}) \cap C^2(\mathbf{R}^{Nn} \setminus \{0\})$;
($\mathcal{H}2$) $\forall z \in \mathbf{R}^{Nn}$, $|f(z)| \leq C(1 + \varphi(|z|))$;
($\mathcal{H}3$) f is asymptotically $W^{1,\varphi}$-quasiconvex;
($\mathcal{H}4$) $\forall z \in \mathbf{R}^{Nn} \setminus \{0\}$, $|D^2 f(z)| \leq C \varphi''(|z|)$;
($\mathcal{H}5$) $\forall z_1, z_2 \in \mathbf{R}^{Nn}$ such that $|z_1| \leq \frac{1}{2}|z_2|$ it holds

$$|D^2 f(z_2) - D^2 f(z_2 + z_1)| \leq C \varphi''(|z_2|)|z_2|^{-\beta}|z_1|^{\beta}$$

for some $\beta \in (0, 1]$.

Remark 2.1. Due to hypothesis ($\mathcal{H}2$), $\mathcal{F}$ is well defined on the Sobolev–Orlicz space $W^{1,\varphi}(\Omega, \mathbf{R}^N)$.

Let us also observe that Assumption ($\mathcal{H}5$), that is a Hölder continuity of $D^2 f$ away from zero, has been used to show everywhere regularity of radial functionals with φ-growth (see [15]). We will use it in Lemma 6.2 below.

3. Technical lemmas

For $z_1, z_2 \in \mathbf{R}^{Nn}$, $\theta \in [0,1]$ we define $z_\theta = z_1 + \theta(z_2 - z_1)$. The following fact can be found in [3] (see Lemma 2.1).

Lemma 3.1. *Let $\beta > -1$, then uniformly in $z_1, z_2 \in \mathbf{R}^{Nn}$ with $|z_1| + |z_2| > 0$, it holds:*

$$\int_0^1 |z_\theta|^\beta \, d\theta \sim (|z_1| + |z_2|)^\beta.$$

Next result is a slight generalization of Lemma 20 in [12].

Lemma 3.2. *Let φ be an N-function with $\Delta_2(\{\varphi, \varphi*\}) < \infty$; then, uniformly in $z_1, z_2 \in \mathbf{R}^{Nn}$ with $|z_1| + |z_2| > 0$, and in $\mu \geq 0$, it holds*

$$\frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|} \sim \int_0^1 \frac{\varphi'(\mu + |z_\theta|)}{\mu + |z_\theta|} d\theta.$$

From the previous lemmas we derive the following one.

Lemma 3.3. *Let φ be an N-function satisfying Assumption 2.1. Then, uniformly in $z_1, z_2 \in \mathbf{R}^{Nn}$ with $|z_1| + |z_2| > 0$, and in $\mu \geq 0$, it holds*

$$\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt \sim \varphi''(\mu + |z_1| + |z_2|).$$

Proof. Using $\varphi'(t) \sim t\varphi''(t)$, applying twice Lemma 3.2, and taking into account that $\mu + |z_1| + |z_1 + z_2| \sim \mu + |z_1| + |z_2|$ and $\varphi'(2t) \sim \varphi'(t)$, we obtain

$$
\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt \leq c \int_0^1 \int_0^1 t\frac{\varphi'(\mu + |z_1 + stz_2|)}{\mu + |z_1 + stz_2|} \, ds dt
$$
$$
\leq c\frac{\varphi'(\mu + |z_1| + |z_1| + |z_1 + z_2|)}{\mu + |z_1| + |z_1| + |z_1 + z_2|}
$$
$$
\leq c\frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|}
$$
$$
\leq c\varphi''(\mu + |z_1| + |z_2|).
$$

Similarly, for the other inequality, we have

$$
\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt \geq c \int_0^1 \int_0^1 t\frac{\varphi'(\mu + |z_1 + stz_2|)}{\mu + |z_1 + stz_2|} \, ds \, dt
$$
$$
\geq c \int_0^1 t\frac{\varphi'(\mu + |z_1| + |z_1 + tz_2|)}{\mu + |z_1| + |z_1 + tz_2|} \, dt
$$
$$
\geq \frac{c}{(\mu + |z_1| + |z_2|)^2} \int_0^1 \varphi(\mu + |z_1| + |z_1 + tz_2|) \, t \, dt,
$$

where, in the last line, we used that $\varphi(t) \sim t\varphi'(t)$. Due to the Jensen inequality, we go ahead and we obtain

$$
\int_0^1 \int_0^1 t\varphi''(\mu + |z_1 + stz_2|) \, ds \, dt \geq \frac{c}{(\mu + |z_1| + |z_2|)^2} \varphi\left(\int_0^1 (\mu + |z_1| + |z_1 + tz_2|) \, t \, dt\right)
$$
$$
\geq \frac{c}{(\mu + |z_1| + |z_2|)^2} \varphi(\mu + |z_1| + |z_2|)
$$
$$
\geq c\frac{\varphi'(\mu + |z_1| + |z_2|)}{\mu + |z_1| + |z_2|} \geq c\,\varphi''(\mu + |z_1| + |z_2|),
$$

thanks also to the equivalence between $\varphi(2t)$ and $\varphi(t)$, $\varphi(t)$ and $t\varphi'(t)$, and $\varphi'(t)$ and $t\varphi''(t)$. $\qquad\square$

Remark 3.1. From the previous lemma we easily deduce that

$$\int_0^1 \int_0^1 t\varphi''(\sqrt{1 + |z_1 + stz_2|^2})\, ds\, dt \sim \varphi''(1 + |z_1| + |z_2|),$$

since $\varphi'(t) \sim t\varphi''(t)$, φ' is increasing and $\varphi'(2t) \sim \varphi'(t)$.

The following version of the Sobolev–Poincaré inequality can be found in [12] (Theorem 7):

Theorem 3.1. *Let φ be an N-function with $\Delta_2(\varphi, \varphi^*) < \infty$. Then there exist $\alpha \in (0,1)$ and $k > 0$ such that, if $\mathcal{B} \subset \mathbf{R}^n$ is a ball of radius R and $u \in W^{1,\varphi}(\mathcal{B}, \mathbf{R}^N)$, then*

$$\fint_{\mathcal{B}} \varphi\left(\frac{|u - (u)_{\mathcal{B}}|}{R}\right) dx \le k \left(\fint_{\mathcal{B}} \varphi^\alpha(|Du|)\, dx\right)^{\frac{1}{\alpha}}.$$

The following two lemmas will be useful later.

Lemma 3.4. *Let φ satisfy Assumption 2.1 and p_0, p_1 be as in Lemma 2.1. Then for each $\eta \in (0,1]$ it holds*

$$\varphi_{|a|}(t) \le C\eta^{1-\bar{p}'}\varphi_{|b|}(t) + \eta|V(a) - V(b)|^2,$$
$$(\varphi_{|a|})^*(t) \le C\eta^{1-\bar{q}}(\varphi_{|b|})^*(t) + \eta|V(a) - V(b)|^2$$

for all $a, b \in \mathbf{R}^n$, $t \ge 0$ and $\bar{p} = \min\{p_0, 2\}$, $\bar{q} = \max\{p_1, 2\}$. The constants depend only on the characteristics of φ.

For the proof see Lemma 2.5 in [13].

Lemma 3.5. *Let φ be an N-function satisfying Assumption 2.1 and let us consider the function $z \in \mathbf{R}^{Nn} \mapsto \varphi(\sqrt{1 + |z|^2})$. Then, uniformly in $y, z \in \mathbf{R}^{Nn}$ it holds*

$$(D^2\varphi(\sqrt{1 + |z + y|^2})y, y) \sim \varphi''(\sqrt{1 + |z + y|^2})|y|^2.$$

Proof. We can see that

$$D\varphi(\sqrt{1 + |z + y|^2}) = \varphi'(\sqrt{1 + |z + y|^2})\frac{z + y}{\sqrt{1 + |z + y|^2}},$$

and

$$D^2\varphi(\sqrt{1 + |z + y|^2}) = \varphi''(\sqrt{1 + |z + y|^2})\frac{z + y}{\sqrt{1 + |z + y|^2}} \otimes \frac{z + y}{\sqrt{1 + |z + y|^2}}$$
$$+ \frac{\varphi'(\sqrt{1 + |z + y|^2})}{\sqrt{1 + |z + y|^2}}\left[\mathbf{I} - \frac{z + y}{\sqrt{1 + |z + y|^2}} \otimes \frac{z + y}{\sqrt{1 + |z + y|^2}}\right],$$

where $\mathbf{I} \in \mathbf{R}^{Nn}$ is the identity matrix. Therefore

$$(D^2\varphi(\sqrt{1 + |z + y|^2})y, y) = \varphi''(\sqrt{1 + |z + y|^2})\frac{|(z + y, y)|^2}{1 + |z + y|^2}$$
$$+ \frac{\varphi'(\sqrt{1 + |z + y|^2})}{\sqrt{1 + |z + y|^2}}\left[|y|^2 - \frac{|(z + y, y)|^2}{1 + |z + y|^2}\right].$$

Using Assumption 2.1 and the fact that $\dfrac{|(z+y,y)|^2}{1+|z+y|^2} \leq |y|^2$ we deduce

$$(D^2\varphi(\sqrt{1+|z+y|^2})y,y) \leq \varphi''(\sqrt{1+|z+y|^2})\frac{|(z+y,y)|^2}{1+|z+y|^2}$$
$$+ C\varphi''(\sqrt{1+|z+y|^2})\left[|y|^2 - \frac{|(z+y,y)|^2}{1+|z+y|^2}\right]$$
$$\leq C\varphi''(\sqrt{1+|z+y|^2})|y|^2.$$

Similarly,

$$(D^2\varphi(\sqrt{1+|z+y|^2})y,y)$$
$$\geq \varphi''(\sqrt{1+|z+y|^2})\frac{|(z+y,y)|^2}{1+|z+y|^2} + C\varphi''(\sqrt{1+|z+y|^2})\left[|y|^2 - \frac{|(z+y,y)|^2}{1+|z+y|^2}\right]$$
$$= C\varphi''(\sqrt{1+|z+y|^2})|y|^2 + (1-C)\varphi''(\sqrt{1+|z+y|^2})\frac{|(z+y,y)|^2}{1+|z+y|^2}$$
$$\geq C\varphi''(\sqrt{1+|z+y|^2})|y|^2. \qquad\qquad \square$$

4. Characterization of asymptotic $W^{1,\varphi}$-quasiconvexity

In this section we will establish some characterizations of asymptotic $W^{1,\varphi}$-quasiconvexity.

Theorem 4.1. *Each of the following assertions is equivalent to the asymptotic $W^{1,\varphi}$-quasiconvexity of a function $f\colon \mathbf{R}^{Nn} \to \mathbf{R}$:*

(i) *If f is C^2 outside a large ball, then there exists a uniformly strictly $W^{1,\varphi}$-quasiconvex function g which is C^2 outside a large ball with*

$$(4.1) \qquad\qquad \lim_{|z|\to\infty} \frac{|D^2f(z) - D^2g(z)|}{\varphi''(|z|)} = 0.$$

(ii) *If f is locally bounded from below, then there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that*

$$f(z) = g(z) \quad \text{for } |z| > M$$

and

$$g \leq f \quad \text{on } \mathbf{R}^{Nn}.$$

(iii) *If f is locally bounded from above, then there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that*

$$f(z) = g(z) \quad \text{for } |z| > M$$

and

$$g \geq f \quad \text{on } \mathbf{R}^{Nn}.$$

(iv) *If f satisfies ($\mathcal{H}2$), then there exist positive constants M, k, L such that*

$$(4.2) \qquad\qquad \fint_{\mathcal{B}_1} f(z + D\xi)\,dx \geq f(z) + k\fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\,dx$$

for $|z| > M$ and $\xi \in C_c^\infty(\mathcal{B}_1, \mathbf{R}^N)$, and

$$(4.3) \qquad\qquad |f(z_2) - f(z_1)| \leq L|z_1 - z_2|\varphi'(1 + |z_1| + |z_2|)$$

for all $|z_1|, |z_2| > M$.

Proof. The proof stands on four steps.

Step 1: We want to prove that f asymptotically $W^{1,\varphi}$-quasiconvex is equivalent to (i). Let us show that (i) implies the asymptotic $W^{1,\varphi}$-quasiconvexity of f, the other implication being evidently true.

Let g be as in (i). We may assume that f, g are $C^2(\mathbf{R}^{Nn}\backslash\overline{\mathcal{B}}_{\frac{1}{2}})$ and taking $h = f - g$ we have that $h \in C^2(\mathbf{R}^{Nn} \setminus \overline{\mathcal{B}}_{\frac{1}{2}})$. In particular, by (4.1) it holds

$$(4.4) \qquad \lim_{|z|\to\infty} \frac{|D^2h(z)|}{\varphi''(|z|)} = 0.$$

Our aim is to prove that

$$(4.5) \qquad \lim_{|z|\to\infty} \frac{|Dh(z)|}{\varphi'(|z|)} = 0,$$

and

$$(4.6) \qquad \lim_{|z|\to\infty} \frac{|h(z)|}{\varphi(|z|)} = 0.$$

Let us consider $|z| > 1$. Take $\overline{z} := \dfrac{z}{|z|}$, then $|\overline{z}| = 1$ and

$$
\begin{aligned}
\frac{|Dh(z)|}{\varphi'(|z|)} &\le \frac{1}{\varphi'(|z|)} \left[\int_0^1 |D^2h(\overline{z} + t(z - \overline{z}))||z - \overline{z}|\, dt + |Dh(\overline{z})| \right] \\
&= \int_0^{\frac{1}{\sqrt{|z|}}} \frac{|D^2h(\overline{z} + t(z - \overline{z}))|}{\varphi''(|\overline{z} + t(z - \overline{z})|)} \frac{\varphi''(|\overline{z} + t(z - \overline{z})|)}{\varphi'(|z|)}|z - \overline{z}|\, dt \\
&\quad + \int_{\frac{1}{\sqrt{|z|}}}^1 \frac{|D^2h(\overline{z} + t(z - \overline{z}))|}{\varphi''(|\overline{z} + t(z - \overline{z})|)} \frac{\varphi''(|\overline{z} + t(z - \overline{z})|)}{\varphi'(|z|)}|z - \overline{z}|\, dt + \frac{|Dh(\overline{z})|}{\varphi'(|z|)} \\
&= \mathcal{I} + \mathcal{II} + \mathcal{III}.
\end{aligned}
$$

Estimate for $\mathcal{I}$:

$$
\begin{aligned}
\mathcal{I} &\le \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \int_0^{\frac{1}{\sqrt{|z|}}} \frac{\varphi''(|\overline{z} + t(z - \overline{z})|)}{\varphi'(|z|)}|z - \overline{z}|\, dt \\
&\le \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \int_0^{\frac{1}{\sqrt{|z|}}} \varphi''(1 + t(|z| - 1))(|z| - 1)\, dt \\
&= \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \left[\varphi'(1 + t(|z| - 1)) \right]_0^{\frac{1}{\sqrt{|z|}}} \\
&\le \sup_{|y|>1} \frac{|D^2h(y)|}{\varphi''(|y|)} \frac{1}{\varphi'(|z|)} \varphi'\left(1 + \frac{|z| - 1}{\sqrt{|z|}} \right).
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
\frac{\varphi'\left(1 + \frac{|z|-1}{\sqrt{|z|}} \right)}{\varphi'(|z|)} &\le \frac{\varphi'(1 + \sqrt{|z|})}{\varphi'(|z|)} \le c\frac{\varphi(1 + \sqrt{|z|})}{1 + \sqrt{|z|}} \frac{|z|}{\varphi(|z|)} \\
&\le c\frac{\varphi(1 + \sqrt{|z|})}{\varphi(|z|)} \sqrt{|z|} \le c\frac{\varphi(\sqrt{|z|})}{\varphi(|z|)} \sqrt{|z|}
\end{aligned}
$$

and using Lemma 2.1 we can find $p_0 > 1$ and $C > 0$ such that

$$\varphi(\sqrt{|z|}) = \varphi\left(\frac{|z|}{\sqrt{|z|}}\right) \leq C\left(\frac{1}{\sqrt{|z|}}\right)^{p_0}\varphi(|z|).$$

Then we obtain

$$(4.7) \qquad \frac{\varphi'\left(1 + \frac{|z|-1}{\sqrt{|z|}}\right)}{\varphi'(|z|)} \leq C\frac{\varphi(\sqrt{|z|})}{\varphi(|z|)}\sqrt{|z|} \leq C\frac{\sqrt{|z|}}{(\sqrt{|z|})^{p_0}} \to 0 \quad \text{as } |z| \to \infty.$$

At this point, using (4.4) and (4.7), we can conclude that $\mathcal{I} \to 0$ as $|z| \to +\infty$.

Now we estimate $\mathcal{II}$:

$$\mathcal{II} \leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)} \int_{\frac{1}{\sqrt{|z|}}}^{1} \frac{\varphi''(|\overline{z} + t(z - \overline{z})|)}{\varphi'(|z|)}|z - \overline{z}|\,dt$$

$$\leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)}\frac{1}{\varphi'(|z|)} \int_{\frac{1}{\sqrt{|z|}}}^{1} \varphi''(1 + t(|z| - 1))(|z| - 1)\,dt$$

$$\leq \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)}\frac{1}{\varphi'(|z|)}\Big[\varphi'(1 + t(|z| - 1))\Big]_{\frac{1}{\sqrt{|z|}}}^{1}$$

$$= \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)}\frac{1}{\varphi'(|z|)}\left[\varphi'(|z|) - \varphi'\left(1 + \frac{|z| - 1}{\sqrt{|z|}}\right)\right]$$

$$= \sup_{|y|>\sqrt{|z|}} \frac{|D^2h(y)|}{\varphi''(|y|)}\left[1 - \frac{\varphi'\left(1 + \frac{|z|-1}{\sqrt{|z|}}\right)}{\varphi'(|z|)}\right] \to 0 \quad \text{as } |z| \to \infty$$

where we used (4.4) and (4.7) to conclude. Finally,

$$\mathcal{III} \leq \frac{1}{\varphi'(|z|)}\max_{|y|=1}|Dh(y)| \to 0 \quad \text{as } |z| \to \infty.$$

Analogously we also obtain (4.6). We can see that if $\frac{|D^ih(z)|}{\varphi^{(i)}(|z|)} \to 0$ as $|z| \to \infty$ for $i = 0, 1, 2$, then $\frac{|D^ih(z)|}{\varphi^{(i)}(1+|z|)} \to 0$ as $|z| \to \infty$.

Taking into account (4.4),(4.5) and (4.6), fixed $\nu > 0$, that we will choose later, there exists $M >> 1$ such that if $|z| > M$, then

$$|D^2h(z)| \leq \nu\varphi''(1 + |z|), \quad |Dh(z)| \leq \nu\varphi'(1 + |z|), \quad |h(z)| \leq \nu\varphi(1 + |z|).$$

Let us consider a cut-off function η defined by

$$\begin{cases} 0 \leq \eta \leq 1 & \text{if } 1 < |x| \leq 2, \\ \eta = 1 & \text{if } |x| > 2, \\ \eta = 0 & \text{if } |x| \leq 1. \end{cases}$$

Set

$$\alpha := \max\left\{\sup_{\mathbf{R}^{Nn}}|D\eta|, \sup_{\mathbf{R}^{Nn}}|D^2\eta|\right\}$$

and let us consider $\eta_M(z) = \eta(\frac{z}{M+1})$. Then we have

$$|D\eta_M| \leq \frac{\alpha}{M + 1} \quad \text{and} \quad |D^2\eta_M| \leq \frac{\alpha}{(M + 1)^2}.$$

Let $\Phi := \eta_M h$; then for $M \leq |z| \leq 2M$ we have

$$|D^2\Phi(z)| \leq |D^2\eta_M(z)||h(z)| + 2|D\eta_M(z)||Dh(z)| + |\eta_M(z)||D^2h(z)|.$$

Taking into account the previous estimates, (2.1), (2.3) and $M \leq |z| \leq 2M$, we have

$$|D^2\Phi(z)| \leq \frac{\nu\alpha}{(M+1)^2}\varphi(1+|z|) + \frac{2\nu\alpha}{(M+1)}\varphi'(1+|z|) + \nu\varphi''(1+|z|)$$

$$\leq \left[\frac{\nu\alpha}{(M+1)^2}(1+|z|)^2 + \frac{2\nu\alpha}{(M+1)}(1+|z|) + \nu\right]\varphi''(1+|z|) = \lambda\nu\varphi''(1+|z|).$$

In particular, we can conclude that

(4.8) $$|D^2\Phi(z)| \leq \lambda\nu\varphi''(1+|z|) \quad \forall z \in \mathbf{R}^{Nn}.$$

Let $\xi \in C_c^\infty(\mathcal{B}_1)$; we can write

$$\Phi(z+D\xi) = \Phi(z) + (D\Phi(z), D\xi) + \int_0^1 (1-t)(D^2\Phi(z+tD\xi)D\xi, D\xi)\,dt.$$

Integrating over $\mathcal{B}_1$ and by using (4.8), (2.3), Lemma 3.2, the fact

(4.9) $$1 + |z| + |D\xi + z| \sim 1 + |z| + |D\xi|$$

and $\varphi_a(t) \sim \varphi''(a+t)t^2$ we get

$$\fint_{\mathcal{B}_1} \Phi(z+D\xi)\,dx = \fint_{\mathcal{B}_1}\Phi(z)\,dx + \fint_{\mathcal{B}_1}(D\Phi(z), D\xi)\,dx$$

$$+ \fint_{\mathcal{B}_1}\int_0^1 (1-t)(D^2\Phi(z+tD\xi))D\xi, D\xi)\,dt\,dx$$

$$\geq \Phi(z) - \fint_{\mathcal{B}_1}\int_0^1 (1-t)|D^2\Phi(z+tD\xi))||D\xi|^2\,dt\,dx$$

$$\geq \Phi(z) - \lambda\nu\fint_{\mathcal{B}_1}\int_0^1 (1-t)\varphi''(1+|z+tD\xi|)|D\xi|^2\,dt\,dx$$

(4.10) $$\geq \Phi(z) - \lambda\nu c\fint_{\mathcal{B}_1}\int_0^1 \frac{\varphi'(1+|z+tD\xi|)}{1+|z+tD\xi|}|D\xi|^2\,dt\,dx$$

$$\geq \Phi(z) - \lambda\nu c\fint_{\mathcal{B}_1} \frac{\varphi'(1+|z|+|D\xi+z|)}{1+|z|+|D\xi+z|}|D\xi|^2\,dt\,dx$$

$$\geq \Phi(z) - \lambda\nu c\fint_{\mathcal{B}_1} \frac{\varphi'(1+|z|+|D\xi|)}{1+|z|+|D\xi|}|D\xi|^2\,dt\,dx$$

$$\geq \Phi(z) - \lambda\nu c\fint_{\mathcal{B}_1} \varphi''(1+|z|+|D\xi|)|D\xi|^2\,dx$$

$$\geq \Phi(z) - \lambda\nu c\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx.$$

Let us take $G := g + \Phi$ with g uniformly strictly $W^{1,\varphi}$-quasiconvex with constant $k > 0$ and Φ satisfying (4.10). Consequently,

$$\fint_{\mathcal{B}_1} G(z+D\xi)\,dx \geq G(z) + (k - \lambda\nu c)\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx$$

$$= G(z) + \tilde{k}\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\,dx$$

where $\tilde{k} > 0$ if we choose $\nu < \frac{k}{\lambda c}$. Thus G is uniformly strictly $W^{1,\varphi}$-quasiconvex with constant $\tilde{k} > 0$ and $G(z) = f(z)$ for $|z| > 2(M+1)$. This proves the asymptotic quasiconvexity of f.

Step 2: We want to prove that f asymptotically $W^{1,\varphi}$- quasiconvex is equivalent to (ii), and it suffices to prove that asymptotic $W^{1,\varphi}$-quasiconvexity of f implies (ii). Assume f asymptotic $W^{1,\varphi}$-quasiconvex, i.e., there exist a positive constant M and a uniformly strictly $W^{1,\varphi}$-quasiconvex function g such that $f(z) = g(z)$ for $|z| > M$. Now g is locally bounded and f is locally bounded from below, so we have that

$$\alpha := \sup_{|z| \leq M} [g(z) - f(z)] < \infty.$$

Let $R > M$ and η be a $C_c^\infty(\mathcal{B}_R)$ function, non-negative on $\mathbf{R}^{Nn}$ and such that

(4.11) $\qquad |D^2\eta(z)| \leq \nu\varphi''(1 + |z|)$ on $\mathbf{R}^{Nn}$ and $\eta(z) \geq \alpha$ for $|z| \leq M$

where ν will be chosen later. Let $\xi \in C_c^\infty(\mathcal{B}_1)$; then we can write

$$\eta(z + D\xi) = \eta(z) + (D\eta(z), D\xi) + \int_0^1 (1-t)(D^2\eta(z + tD\xi)D\xi, D\xi)\, dt.$$

Integrating over $\mathcal{B}_1$ it holds, by (4.11),

$$\fint_{\mathcal{B}_1} \eta(z + D\xi)\, dx = \fint_{\mathcal{B}_1} \eta(z)\, dx + \fint_{\mathcal{B}_1} (D\eta(z), D\xi)\, dx$$

$$+ \fint_{\mathcal{B}_1} \int_0^1 (1-t)(D^2\eta(z + tD\xi)D\xi, D\xi)\, dt\, dx$$

(4.12)
$$\leq \eta(z) + \fint_{\mathcal{B}_1} \int_0^1 (1-t)|D^2\eta(z + tD\xi)||D\xi|^2\, dt\, dx$$

$$\leq \eta(z) + \nu\fint_{\mathcal{B}_1} \int_0^1 \varphi''(1 + |z + tD\xi|)|D\xi|^2\, dt\, dx$$

$$\leq \eta(z) + \nu c\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx,$$

where we used, as before, $\varphi'(t) \sim t\varphi''(t)$, Lemma 3.2, (4.9) and $\varphi_a(t) \sim \varphi''(a + t)t^2$.

Now taking $G = g - \eta$, with g and η satisfying (2.5) and (4.12), we have

$$\fint_{\mathcal{B}_1} G(z + D\xi)dx \geq G(z) + \tilde{k}\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx$$

where $\tilde{k} > 0$ if we choose $\nu = \frac{k}{2c}$. This means that G is uniformly strictly $W^{1,\varphi}$-quasiconvex. But $\eta(z) \geq \alpha \geq g(z) - f(z)$ and $\eta(z) = g(z) - G(z)$, so $G(z) \leq f(z)$ for $|z| \leq M$.

Step 3: The proof is similar to the previous one.

Step 4: Assume that f is a Borel measurable function, satisfying ($\mathcal{H}2$). Since quasiconvex functions are locally Lipschitz (see [25]), we can see that (ii) implies (iv). So it suffices to show that a function satisfying (iv) is asymptotically $W^{1,\varphi}$-quasiconvex.

Assume that f satisfies (iv) and consider the function

$$F(z) := f(z) - \varepsilon\varphi(\sqrt{1 + |z|^2})$$

for $z \in \mathbf{R}^{Nn}$. Here $\varepsilon > 0$ will be chosen later appropriately. Now we prove that F satisfies (4.2) and (4.3).

Let $\xi \in C_c^\infty(\mathcal{B}_1)$. Since f satisfies (4.2), we can write

$$\fint_{\mathcal{B}_1} F(z + D\xi)\, dx = \fint_{\mathcal{B}_1} f(z + D\xi)\, dx - \varepsilon \fint_{\mathcal{B}_1} \varphi(\sqrt{1 + |z + D\xi|^2})\, dx$$

$$(4.13) \quad \geq f(z) + k \fint_{\mathcal{B}_1} \varphi_{|z|}(D\xi)\, dx - \varepsilon \fint_{\mathcal{B}_1} \varphi(\sqrt{1 + |z + D\xi|^2})\, dx$$

$$= F(z) + k \fint_{\mathcal{B}_1} \varphi_{|z|}(D\xi)\, dx - \varepsilon \fint_{\mathcal{B}_1} [\varphi(\sqrt{1 + |z + D\xi|^2}) - \varphi(\sqrt{1 + |z|^2})]\, dx.$$

Note that

$$\varphi(\sqrt{1 + |z + D\xi|^2}) = \varphi(\sqrt{1 + |z|^2}) + (D\varphi(\sqrt{1 + |z|^2}), D\xi)$$

$$+ \int_0^1 \int_0^1 t(D^2\varphi(\sqrt{1 + |z + stD\xi|^2})D\xi, D\xi)\, ds\, dt.$$

Thus integrating over $\mathcal{B}_1$ and applying Lemma 3.5, Remark 3.1 and $\varphi''(a+t)t^2 \sim \varphi_a(t)$ for $a, t \geq 0$, it follows that

$$\fint_{\mathcal{B}_1} [\varphi(\sqrt{1 + |z + D\xi|^2}) - \varphi(\sqrt{1 + |z|^2})]\, dx = \fint_{\mathcal{B}_1} (D\varphi(\sqrt{1 + |z|^2}), D\xi)\, dx$$

$$+ \fint_{\mathcal{B}_1} \int_0^1 \int_0^1 t(D^2\varphi(\sqrt{1 + |z + stD\xi|^2})D\xi, D\xi)\, ds\, dt\, dx$$

$$(4.14) \quad \leq C \fint_{\mathcal{B}_1} \int_0^1 \int_0^1 t\varphi''(\sqrt{1 + |z + stD\xi|^2})|D\xi|^2\, ds\, dt\, dx$$

$$\leq C \fint_{\mathcal{B}_1} \varphi''(1 + |z| + |D\xi|)|D\xi|^2\, dx \leq C \fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx$$

$$\leq C \fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\, dx$$

for $|z|$ sufficiently large. Using together (4.13) and (4.14), and choosing ε small enough, we have

$$\fint_{\mathcal{B}_1} F(z + D\xi)\, dx \geq F(z) + K \fint_{\mathcal{B}_1} \varphi_{|z|}(|D\xi|)\, dx.$$

Moreover, taking into account that f satisfies (4.3) we deduce, for $|z_1|, |z_2| > M$,

$$|F(z_2) - F(z_1)| \leq |f(z_2) - f(z_1)| + \varepsilon|\varphi(\sqrt{1 + |z_1|^2}) - \varphi(\sqrt{1 + |z_2|^2})|$$

$$\leq L|z_2 - z_1|\varphi'(1 + |z_1| + |z_2|) + \varepsilon|\varphi(\sqrt{1 + |z_1|^2}) - \varphi(\sqrt{1 + |z_2|^2})|$$

$$\leq (L + c)|z_2 - z_1|\varphi'(1 + |z_1| + |z_2|).$$

Next we let

$$G(z) := \inf \left\{ \fint_{\mathcal{B}_1} F(z + D\xi)\, dx \colon \xi \in C_c^\infty(\mathcal{B}_1, \mathbf{R}^N) \right\}$$

for $z \in \mathbf{R}^{Nn}$. With this definition we have that $G(z) \leq F(z)$ on $\mathbf{R}^{Nn}$ and $G(z) = F(z)$ for $|z| > M$. Now our aim is to prove that G is locally bounded from below.

Fix $z \in \mathbf{R}^{Nn}$ such that $|z| \leq M + 1$ and take $\overline{z} \in \mathbf{R}^{Nn}$ such that $|\overline{z}| = 2(M + 1)$. We have

$$\fint_{\mathcal{B}_1} F(z + D\xi) \, dx = \fint_{\mathcal{B}_1} [F(z + D\xi) - F(\overline{z} + D\xi)] \, dx + \fint_{\mathcal{B}_1} F(\overline{z} + D\xi) \, dx = I + II$$

Since F satisfies (4.2), we get

$$II = \fint_{\mathcal{B}_1} F(\overline{z} + D\xi) \, dx \geq F(\overline{z}) + k \fint_{\mathcal{B}_1} \varphi_{|\overline{z}|}(|D\xi|) \, dx.$$

Now we estimate I:

$$I = \frac{1}{|\mathcal{B}_1|} \left[\int_{\{|D\xi| \leq 3(M+1)\}} [F(z + D\xi) - F(\overline{z} + D\xi)] \, dx \right.$$
$$\left. + \int_{\{|D\xi| > 3(M+1)\}} [F(z + D\xi) - F(\overline{z} + D\xi)] \, dx \right] = \frac{1}{|\mathcal{B}_1|}[I_1 + I_2].$$

To estimate I_1 we use the fact that F is locally bounded: $I_1 \geq \tilde{C}$. Regarding I_2 we take into account that F satisfies (4.3), then we apply Young's inequality, $\varphi^*(\varphi'(t)) \sim \varphi(t)$ and the Δ_2 condition to deduce

$$I_2 = \int_{\{|D\xi| > 3(M+1)\}} [F(z + D\xi) - F(\overline{z} + D\xi)] \, dx \Big]$$
$$\geq -L \int_{\{|D\xi| > 3(M+1)\}} |z - \overline{z}| \varphi'(1 + |z + D\xi| + |\overline{z} + D\xi|) \, dx$$
$$\geq -L\delta c \int_{\{|D\xi| > 3(M+1)\}} \varphi(1 + |z + D\xi| + |\overline{z} + D\xi|) \, dx$$
$$- LC_\delta \int_{\{|D\xi| > 3(M+1)\}} \varphi(|z - \overline{z}|) \, dx$$
$$\geq -L\delta c \int_{\{|D\xi| > 3(M+1)\}} \varphi(1 + |\overline{z}| + |D\xi|) \, dx - C_\delta$$
$$\geq -L\delta c \int_{\{|D\xi| > 3(M+1)\}} \varphi_{1+|\overline{z}|}(|D\xi|) \, dx - C_\delta$$

where in the last inequality we used $\varphi_{1+|\overline{z}|}(|D\xi|) \sim \varphi(1+|\overline{z}|+|D\xi|)$ since $|D\xi| > 1+|\overline{z}|$.

Putting together estimates on I_1, I_2 and II, taking into account that $\varphi_{|\overline{z}|}(t) \sim \varphi_{1+|\overline{z}|}(t)$ and choosing δ suitably we have

$$\fint_{\mathcal{B}_1} F(z + D\xi) \, dx \geq -C\delta \int_{\{|D\xi| > 3(M+1)\}} \varphi_{|\overline{z}|}(|D\xi|) \, dx + F(\overline{z})$$
$$+ k \fint_{\mathcal{B}_1} \varphi_{|\overline{z}|}(|D\xi|) \, dx - C \geq -C.$$

So we get $G(z) \geq -C$ for $|z| \leq M + 1$. Moreover for $|z| > M + 1$ we gain

$$G(z) = f(z) - \varepsilon\varphi(\sqrt{1 + |z|^2}) \geq -C(1 + \varphi(|z|))$$

and this proves the local boundedness of G from below. By Dacorogna's formula[1] we have that G coincides with the quasiconvex envelope QF of F, and thus it is quasiconvex.

Finally we can prove that

$$g(z) = G(z) + \varepsilon\varphi(\sqrt{1 + |z|^2}) \quad \text{for } z \in \mathbf{R}^{Nn}$$

is a uniformly strictly $W^{1,\varphi}$-quasiconvex function. By the quasiconvexity of G we get

$$\fint_{\mathcal{B}_1} g(z + D\xi)\, dx = \fint_{\mathcal{B}_1} G(z + D\xi)\, dx + \varepsilon\fint_{\mathcal{B}_1} \varphi(\sqrt{1 + |z + D\xi|^2})\, dx$$

$$\geq G(z) + \varepsilon\varphi(\sqrt{1 + |z|^2}) + \varepsilon\fint_{\mathcal{B}_1} [\varphi(\sqrt{1 + |z + D\xi|^2}) - \varphi(\sqrt{1 + |z|^2})]\, dx$$

$$= g(z) + \varepsilon\fint_{\mathcal{B}_1} [\varphi(\sqrt{1 + |z + D\xi|^2}) - \varphi(\sqrt{1 + |z|^2})]\, dx.$$

Using Lemma 3.5, Remark 3.1 and $\varphi_a(t) \sim \varphi''(a + t)t^2$ it holds

$$\fint_{\mathcal{B}_1} [\varphi(\sqrt{1 + |z + D\xi|^2}) - \varphi(\sqrt{1 + |z|^2})]\, dx$$

$$= \fint_{\mathcal{B}_1} (D\varphi(\sqrt{1 + |z|^2}), D\xi)\, dx + \fint_{\mathcal{B}_1}\int_0^1\int_0^1 t(D^2\varphi(\sqrt{1 + |z + stD\xi|^2})D\xi, D\xi)\, ds\, dt\, dx$$

$$= \fint_{\mathcal{B}_1}\int_0^1\int_0^1 t(D^2\varphi(\sqrt{1 + |z + stD\xi|^2})D\xi, D\xi)\, ds\, dt\, dx$$

$$\geq C\fint_{\mathcal{B}_1}\int_0^1\int_0^1 t\varphi''(\sqrt{1 + |z + stD\xi|^2})|D\xi|^2\, ds\, dt\, dx$$

$$\geq C\fint_{\mathcal{B}_1} \varphi''(1 + |z| + |D\xi|)|D\xi|^2\, dx \geq C\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx.$$

We deduce that g is uniformly strictly $W^{1,\varphi}$-quasiconvex, i.e.

$$\fint_{\mathcal{B}_1} g(z + D\xi)\, dx \geq g(z) + \varepsilon c\fint_{\mathcal{B}_1} \varphi_{1+|z|}(|D\xi|)\, dx.$$

Moreover we have that $g(z) = f(z)$ for $|z| > M + 1$. This proves that f is asymptotically quasiconvex. $\qquad\square$

5. Caccioppoli estimate

The starting point for the investigation of the regularity properties of weak solutions is a Caccioppoli-type inequality. We need the following Lemma (see [14, Lemma 10]):

Lemma 5.1. *Let ψ be an N-function with $\psi \in \Delta_2$, let $r > 0$ and let $h \in L^\psi(\mathcal{B}_{2r}(x_0))$. Further, let $f: [\frac{r}{2}, r] \to [0, \infty)$ be a bounded function such that for all $\frac{r}{2} < s < t < r$*

$$f(s) \leq \theta f(t) + A\int_{\mathcal{B}_t(x_0)} \psi\left(\frac{|h(y)|}{t - s}\right)\, dy,$$

---

[1]In [11] Theorem 5 it is assumed that there exists a quasiconvex function from below F, and the verification of this hypothesis is not immediate in our situation. However, we may still apply the Theorem since the missing hypothesis is only needed to conclude that G is locally bounded from below. Moreover, by (2.2) we can say that $\varphi(|z|) \leq c(1 + |z|^{p_1})$, $p_1 > 1$.

where $A > 0$ and $\theta \in [0, 1)$. Then

$$f\left(\frac{r}{2}\right) \leq C(\theta, \Delta_2(\psi))A \int_{\mathcal{B}_r(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy.$$

Theorem 5.1. Let $u \in W_{loc}^{1,\varphi}(\Omega)$ be a minimizer of $\mathcal{F}$ and let $\mathcal{B}_R$ be a ball such that $\mathcal{B}_{2R} \Subset \Omega$. Then

$$\int_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|) \, dx \leq c \int_{\mathcal{B}_{2R}} \varphi_{|z|}\left(\frac{|u - q|}{R}\right) dx$$

for all $z \in \mathbf{R}^{Nn}$ with $|z| > M$ and all linear polynomials q on $\mathbf{R}^n$ with values in $\mathbf{R}^N$ such that $Dq = z$.

Proof. Let $0 < s < t$ and consider $\mathcal{B}_s \subset \mathcal{B}_t \subset \Omega$. Let $\eta \in C_c^\infty(\mathcal{B}_t)$ be a standard cut-off function between $\mathcal{B}_s$ and $\mathcal{B}_t$, such that $|D\eta| \leq \frac{c}{t-s}$. Define $\xi = \eta(u - q)$ and $\zeta = (1 - \eta)(u - q)$; then $D\xi + D\zeta = Du - z$. Consider

$$\mathcal{I} := \int_{\mathcal{B}_t} [f(z + D\xi) - f(z)] \, dx.$$

By hypothesis f is asymptotically $W^{1,\varphi}$-quasiconvex, and by Theorem 4.1 we know that f satisfies (iv), so for $|z| > M$ we have

(5.1)
$$\mathcal{I} \geq k \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx.$$

Moreover,

$$\mathcal{I} = \int_{\mathcal{B}_t} [f(z + D\xi) - f(Du) + f(Du) - f(Du - D\xi) + f(Du - D\xi) - f(z)] \, dx$$

$$= \int_{\mathcal{B}_t} [f(z + D\xi) - f(z + D\xi + D\zeta)] \, dt + \int_{\mathcal{B}_t} [f(Du) - f(Du - D\xi)] \, dt$$

$$+ \int_{\mathcal{B}_t} [f(z + D\zeta) - f(z)] \, dx = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

Note that $\mathcal{I}_2 \leq 0$ since u is a minimizer. Let us concentrate on $\mathcal{I}_1$:

$$\mathcal{I}_1 = -\int_{\mathcal{B}_t} \int_0^1 Df(z + D\xi + \theta D\zeta)D\zeta \, d\theta \, dx.$$

Analogously concerning $\mathcal{I}_3$, we have

$$\mathcal{I}_3 = \int_{\mathcal{B}_t} \int_0^1 Df(z + \theta D\zeta)D\zeta \, d\theta \, dx.$$

Thus we obtain that

$$\mathcal{I}_1 + \mathcal{I}_3 = \int_{\mathcal{B}_t} \int_0^1 [Df(z + \theta D\zeta) - Df(z + D\xi + \theta D\zeta)]D\zeta \, d\theta \, dx$$

$$= \int_{\mathcal{B}_t} \int_0^1 [Df(z + \theta D\zeta) - Df(z) + Df(z) - Df(z + D\xi + \theta D\zeta)]D\zeta \, d\theta \, dx$$

$$= \int_{\mathcal{B}_t} \int_0^1 [Df(z + \theta D\zeta) - Df(z)]D\zeta \, d\theta \, dx$$

$$- \int_{\mathcal{B}_t} \int_0^1 [Df(z + D\xi + \theta D\zeta) - Df(z)]D\zeta \, d\theta \, dx$$

from which

$$\mathcal{I}_1 + \mathcal{I}_3 \leq \int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta|\, d\theta\, dx$$

$$+ \int_{\mathcal{B}_t} \int_0^1 |Df(z + D\xi + \theta D\zeta) - Df(z)||D\zeta|\, d\theta\, dx.$$

By using hypothesis ($\mathcal{H}4$) and Lemma 3.2 we have

$$\int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta|\, d\theta\, dx$$

$$\leq \int_{\mathcal{B}_t} \int_0^1 \int_0^1 |D^2 f(tz + (1-t)(z + \theta D\zeta)|\, |\theta D\zeta|\, |D\zeta|\, dt\, d\theta\, dx$$

$$\leq c \int_{\mathcal{B}_t} \int_0^1 \int_0^1 \varphi''(|tz + (1-t)(z + \theta D\zeta)|)\, |D\zeta|^2\, dt\, d\theta\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi''(2|z| + |z + D\zeta|)\, |D\zeta|^2\, dx$$

$$\leq c \int_{\mathcal{B}_t} \frac{\varphi'(2|z| + |z + D\zeta|)}{2|z| + |z + D\zeta|} |D\zeta|^2\, dx.$$

Taking into account the Δ_2 condition for φ' and $\varphi_a(t) \sim \varphi''(a + t)t^2$, it follows that

$$\int_{\mathcal{B}_t} \int_0^1 |Df(z + \theta D\zeta) - Df(z)||D\zeta|\, d\theta\, dx \leq c \int_{\mathcal{B}_t} \frac{\varphi'(|z| + |D\zeta|)}{|z| + |D\zeta|} |D\zeta|^2\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi''(|z| + |D\zeta|)|D\zeta|^2\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|)\, dx.$$

Analogously we can deduce

$$\int_{\mathcal{B}_t} \int_0^1 |Df(z + D\xi + \theta D\zeta) - Df(z)||D\zeta|\, d\theta\, dx$$

$$\leq \int_{\mathcal{B}_t} \int_0^1 \int_0^1 |D^2 f(t(z + D\xi + \theta D\zeta) + (1-t)z)|\, |D\xi + \theta D\zeta|\, |D\zeta|\, dt\, d\theta\, dx$$

$$\leq c \int_{\mathcal{B}_t} \int_0^1 \int_0^1 \varphi''(|t(z + D\xi + \theta D\zeta) + (1-t)z|)\, |D\xi + \theta D\zeta|\, |D\zeta|\, dt\, d\theta\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi''(|z| + |D\xi| + |D\zeta|)\, (|D\xi| + |D\zeta|)\, |D\zeta|\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi'_{|z|}(|D\xi| + |D\zeta|)\, |D\zeta|\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi'_{|z|}(|D\xi|)\, |D\zeta|\, dx + c \int_{\mathcal{B}_t} \varphi'_{|z|}(|D\zeta|)\, |D\zeta|\, dx$$

$$\leq c \int_{\mathcal{B}_t} \varphi'_{|z|}(|D\xi|)\, |D\zeta|\, dx + c \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|)\, dx$$

where in the last line we used the equivalence $\varphi'_a(t) \sim t\varphi''(a + t)$ and the fact that

$$\varphi'_{|z|}(|D\xi| + |D\zeta|) \leq c\varphi'_{|z|}(|D\xi|) + c\varphi'_{|z|}(|D\zeta|).$$

Applying Young's inequality for φ_a we have

$$\mathcal{I}_1 + \mathcal{I}_3 \le c \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) \, dx + c \int_{\mathcal{B}_t} \varphi'_{|z|}(|D\xi|) \, |D\zeta| \, dx$$

$$\le c \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) \, dx + c\delta \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx + C_\delta \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) \, dx$$

$$\le C'_\delta \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) \, dx + c\delta \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx.$$

Taking into account (5.1) and choosing δ such that $k - c\delta > 0$ we conclude

$$\int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx \le C \int_{\mathcal{B}_t} \varphi_{|z|}(|D\zeta|) \, dx.$$

Now, by the definition of ζ we have $D\zeta = (1 - \eta)(Du - z) - D\eta(u - q)$ and we can note that $D\zeta = 0$ in $\mathcal{B}_s$. Moreover using the convexity of $\varphi_{|z|}$ and the fact that $|D\eta| \le \frac{c}{t-s}$, we have

$$\varphi_{|z|}(|D\zeta|) \le \varphi_{|z|}\left((1 - \eta)|Du - z| + \frac{c}{t-s}|u - q|\right)$$

$$\le c\varphi_{|z|}(|Du - z|) + c\varphi_{|z|}\left(\frac{|u - q|}{t - s}\right).$$

Hence

$$\int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx \le C \int_{\mathcal{B}_t \setminus \mathcal{B}_s} \varphi_{|z|}(|D\zeta|) \, dx$$

$$\le c \int_{\mathcal{B}_t \setminus \mathcal{B}_s} \varphi_{|z|}(|Du - z|) \, dx + c \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u - q|}{t - s}\right) \, dx.$$

Thus we have

$$\int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|) \, dx = \int_{\mathcal{B}_s} \varphi_{|z|}(|D\xi|) \, dx \le \int_{\mathcal{B}_t} \varphi_{|z|}(|D\xi|) \, dx$$

$$\le c \int_{\mathcal{B}_t \setminus \mathcal{B}_s} \varphi_{|z|}(|Du - z|) \, dx + c \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u - q|}{t - s}\right) \, dx.$$

We fill the hole by adding to both sides the term $c \int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|) \, dx$ and we divide by $c + 1$, thus obtaining

$$\int_{\mathcal{B}_s} \varphi_{|z|}(|Du - z|) \, dx \le \frac{c}{c+1} \int_{\mathcal{B}_t} \varphi_{|z|}(|Du - z|) \, dx + C \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u - q|}{t - s}\right) dx$$

$$= \lambda \int_{\mathcal{B}_t} \varphi_{|z|}(|Du - z|) \, dx + \alpha \int_{\mathcal{B}_t} \varphi_{|z|}\left(\frac{|u - q|}{t - s}\right) dx$$

where $\lambda := \frac{c}{c+1} < 1$ and $\alpha > 0$. Now we can apply Lemma 5.1 to get the desired result. $\qquad\square$

An immediate consequence of the previous result is the following:

Corollary 5.1. *There exists $\alpha \in (0, 1)$ such that for all minimizers $u \in W^{1,\varphi}(\Omega)$ of $\mathcal{F}$, all balls $\mathcal{B}_R$ with $\mathcal{B}_{2R} \Subset \Omega$, and all $z \in \mathbf{R}^{Nn}$ with $|z| > M$*

$$\fint_{\mathcal{B}_R} |V(Du) - V(z)|^2 \, dx \le c \left(\fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^{2\alpha} \, dx\right)^{\frac{1}{\alpha}}$$

Proof. By using Lemma 2.5, applying Theorem 5.1 with q such that $(u-q)_{\mathcal{B}_{2R}} = 0$ and Theorem 3.1 we have

$$\fint_{\mathcal{B}_R} |V(Du) - V(z)|^2 dx \le c \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|) \, dx \le c \fint_{\mathcal{B}_{2R}} \varphi_{|z|} \left(\frac{|u - q|}{R} \right) dx$$

$$\le c \left(\fint_{\mathcal{B}_{2R}} \varphi_{|z|}^{\alpha}(|Du - z|) \, dx \right)^{\frac{1}{\alpha}}$$

$$\le c \left(\fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^{2\alpha} \, dx \right)^{\frac{1}{\alpha}}. \qquad \square$$

Using Gehring's Lemma we deduce the following result.

Corollary 5.2. *There exists $s > 1$ such that for all minimizers $u \in W^{1,\varphi}(\Omega)$ of $\mathcal{F}$, all balls $\mathcal{B}_R$ with $\mathcal{B}_{2R} \Subset \Omega$, and all $z \in \mathbf{R}^{Nn}$ with $|z| > M$*

$$\left(\fint_{\mathcal{B}_R} |V(Du) - V(z)|^{2s} \, dx \right)^{\frac{1}{s}} \le c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 \, dx$$

6. Almost $\mathcal{A}$-harmonicity

In this section we recall a generalization of the $\mathcal{A}$-harmonic approximation Lemma in Orlicz space (see [14]). We say that $\mathcal{A} = (\mathcal{A}_{ij}^{\alpha\beta})_{\substack{i,j=1,\cdots,N \\ \alpha,\beta=1,\cdots,n}}$ is strongly elliptic in the sense of Legendre–Hadamard if

$$\mathcal{A}(a \otimes b, a \otimes b) \ge k_{\mathcal{A}} |a|^2 |b|^2$$

holds for all $a \in \mathbf{R}^N, b \in \mathbf{R}^n$ for some constant $k_{\mathcal{A}} > 0$. We say that a Sobolev function w on $\mathcal{B}_R$ is $\mathcal{A}$-harmonic if

$$-\operatorname{div}(\mathcal{A}Dw) = 0$$

is satisfied in the sense of distributions. Given a function $u \in W^{1,2}(\mathcal{B}_R)$, we want to find a function h that is $\mathcal{A}$-harmonic and is close to u. In particular, we are looking for a function $h \in W^{1,2}(\mathcal{B}_R)$ such that

$$\begin{cases} -\operatorname{div}(\mathcal{A}Dh) = 0 & \text{in } \mathcal{B}_R, \\ h = u & \text{on } \partial\mathcal{B}_R. \end{cases}$$

Let $w := h - u$. Then w satisfies

(6.1)
$$\begin{cases} -\operatorname{div}(\mathcal{A}Dw) = -\operatorname{div}(\mathcal{A}Du) & \text{in } \mathcal{B}_R, \\ w = 0 & \text{on } \partial\mathcal{B}_R. \end{cases}$$

We recall Theorem 14 in [14]:

Theorem 6.1. *Let $\mathcal{B}_R \Subset \Omega$ and let $\tilde{\mathcal{B}} \subset \Omega$ denote either $\mathcal{B}_R$ or $\mathcal{B}_{2R}$. Let $\mathcal{A}$ be strongly elliptic in the sense of Legendre–Hadamard. Let ψ be an N-function with $\Delta_2(\psi, \psi^*) < \infty$ and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ depending on $n, N, k_{\mathcal{A}}, |\mathcal{A}|, \Delta_2(\psi, \psi^*)$ and s such that the following holds: let $u \in W^{1,\psi}(\tilde{\mathcal{B}})$ be almost $\mathcal{A}$-harmonic on $\mathcal{B}_R$ in the sense that*

$$\left| \fint_{\mathcal{B}_R} (\mathcal{A} Du, D\xi) \, dx \right| \le \delta \fint_{\tilde{\mathcal{B}}} |Du| \, dx \| D\xi \|_{L^\infty(\mathcal{B}_R)}$$

for all $\xi \in C_0^\infty(\mathcal{B}_R)$. Then the unique solution $w \in W_0^{1,\psi}(\mathcal{B}_R)$ of (6.1) satisfies

$$\fint_{\mathcal{B}_R} \psi\left(\frac{|w|}{R}\right) dx + \fint_{\mathcal{B}_R} \psi(|Dw|) \, dx \leq \varepsilon \left[\left(\fint_{\mathcal{B}_R} \psi^s(|Du|) \, dx \right)^{\frac{1}{s}} + \fint_{\tilde{\mathcal{B}}} \psi(|Du|) \, dx \right].$$

The following results can be found in [14].

Lemma 6.1. *Let $\mathcal{B}_R \subset \mathbf{R}^n$ be a ball and let $u \in W^{1,\varphi}(\mathcal{B}_R)$. Then*

$$\fint_{\mathcal{B}_R} |V(Du) - (V(Du))_{\mathcal{B}_R}|^2 \, dx \sim \fint_{\mathcal{B}_R} |V(Du) - V((Du)_{\mathcal{B}_R})|^2 \, dx.$$

Lemma 6.2. *Let $z := (Du)_{\mathcal{B}_{2R}}$. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $u \in W^{1,\varphi}(\Omega)$ minimizer of $\mathcal{F}$ and every $\mathcal{B}_R$ such that $\mathcal{B}_{2R} \Subset \Omega$, and for*

$$\fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 \, dx \leq \delta \fint_{\mathcal{B}_{2R}} |V(Du)|^2 \, dx$$

it holds

(6.2) $$\left| \fint_{\mathcal{B}_R} D^2 f(z)(Du - z, D\xi) \, dx \right| \leq \varepsilon \varphi''(|z|) \fint_{\mathcal{B}_{2R}} |Du - z| \, dx \| D\xi \|_{L^\infty(\mathcal{B}_R)},$$

for every $\xi \in C_c^\infty(\mathcal{B}_R)$.

7. Excess decay estimate

Following the ideas in [3] we will prove the following Lemma.

Lemma 7.1. *Let $z_0 \in \mathbf{R}^n$ such that $|z_0| > 1$. Let $f \in C^2(B_{2\sigma}(z_0))$ be strictly $W^{1,\varphi}$-quasiconvex at z_0, that is,*

(7.1) $$\int_{\mathcal{B}} [f(z_0 + D\xi) - f(z_0)] \, dx \geq k \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|) \, dx$$

holds for all $\xi \in C_c^1(\mathcal{B}, \mathbf{R}^N)$. Then, there exists $\rho > 0$ such that for all $z \in \mathcal{B}_\rho(z_0)$

(7.2) $$\int_{\mathcal{B}} [f(z + D\xi) - f(z)] \, dx \geq \frac{k}{2} \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|) \, dx$$

holds for all $\xi \in C_c^1(\mathcal{B}, \mathbf{R}^N)$.

Proof. Let

$$\omega_\rho := \sup\{|D^2 f(z_1) - D^2 f(z_2)| : z_1, z_2 \in \mathcal{B}_\sigma(z_0), \ |z_1 - z_2| < \rho\}$$

and fix z such that $|z - z_0| < \rho < \frac{\sigma}{2}$. For $\eta \in \mathbf{R}^{Nn}$, define

$$G(\eta) = f(z + \eta) - f(z_0 + \eta).$$

By using (7.1) we have

$$\int_{\mathcal{B}} [f(z + D\xi) - f(z)] \, dx$$

$$= \int_{\mathcal{B}} [f(z_0 + D\xi) - f(z_0)] \, dx + \int_{\mathcal{B}} [f(z + D\xi) - f(z_0 + D\xi) + f(z_0) - f(z)] \, dx$$

$$\geq k \int_{\mathcal{B}} \varphi_{|z_0|}(|D\xi|) \, dx + \int_{\mathcal{B}} [G(D\xi) - G(0) - (DG(0), D\xi)] \, dx.$$

Now we split $\mathcal{B}$ as

$$\mathbf{X} = \left\{ x \in \mathcal{B} : |D\xi| \leq \frac{\sigma}{2} \right\} \quad \text{and} \quad \mathbf{Y} = \left\{ x \in \mathcal{B} : |D\xi| > \frac{\sigma}{2} \right\}.$$

Let us observe that

$$G(D\xi) - G(0) - (DG(0), D\xi) = \frac{1}{2}(D^2 G(\theta D\xi)D\xi, D\xi)$$

with $\theta \in (0,1)$. Moreover if $x \in \mathbf{X}$ then $|D\xi| \leq \frac{\sigma}{2}$, so $z + D\xi \in \mathcal{B}_\sigma(z_0)$. Hence

$$\int_{\mathbf{X}} [G(D\xi) - G(0) - DG(0)D\xi]\, dx = \frac{1}{2} \int_{\mathbf{X}} (D^2 G(\theta D\xi)D\xi, D\xi)\, dx$$

$$\geq -\frac{1}{2} \int_{\mathbf{X}} |D^2 f(z + \theta D\xi) - D^2 f(z_0 + \theta D\xi)||D\xi|^2\, dx \geq -\frac{\omega_\rho}{2} \int_{\mathbf{X}} |D\xi|^2\, dx$$

$$\geq -\frac{c\,\omega_\rho}{2} \int_{\mathbf{X}} \varphi''(|z_0| + |D\xi|)|D\xi|^2\, dx \geq -\frac{c\,\omega_\rho}{2} \int_{\mathbf{X}} \varphi_{|z_0|}(|D\xi|)\, dx$$

where we used the fact that on $\mathbf{X}$ we have

$$\varphi''(|z_0| + |D\xi|) \geq c\frac{\varphi'(|z_0| + |D\xi|)}{|z_0| + |D\xi|} \geq c\frac{\varphi'(1)}{|z_0| + \frac{\sigma}{2}} > 0.$$

Let us define $H(z, x) = f(z + D\xi(x)) - f(z) - (Df(z), D\xi(x))$ so that

$$\int_{\mathbf{Y}} [H(z, x) - H(z_0, x)]\, dx = \int_{\mathbf{Y}} [G(D\xi) - G(0) - (DG(0), D\xi)]\, dx.$$

We can see that

$$\int_{\mathbf{Y}} |H(z, x) - H(z_0, x)|\, dx \leq \int_{\mathbf{Y}} |z - z_0||D_z H(\tau, x)|\, dx$$

$$\leq \rho \left[\int_{\mathbf{Y}} |Df(\tau + D\xi) - Df(\tau)|\, dx + \int_{\mathbf{Y}} |D^2 f(\tau)||D\xi|\, dx \right] = \rho[\mathcal{I} + \mathcal{II}].$$

Now we estimate $\mathcal{I}$. We use hypothesis $(\mathcal{H}4)$, Lemma 3.2 and the fact that $|\tau| + |D\xi + \tau| \sim |\tau| + |D\xi|$ to get

$$\mathcal{I} \leq \int_{\mathbf{Y}} \int_0^1 |D^2 f(\tau + tD\xi)||D\xi|\, dt\, dx \leq c \int_{\mathbf{Y}} \int_0^1 \varphi''(|\tau + tD\xi|)|D\xi|\, dt\, dx$$

$$\leq c \int_{\mathbf{Y}} \frac{\varphi'(|\tau| + |D\xi|)}{|\tau| + |D\xi|}|D\xi|\, dx \leq c_\sigma \int_{\mathbf{Y}} \varphi'(|z_0| + |D\xi|)|D\xi|\, dx$$

where in the last inequality we used $|\tau| + |D\xi| \leq |z| + |z_0| + |D\xi| \leq \rho + 2|z_0| + |D\xi| < c(|z_0| + |D\xi|)$ as well as $|\tau| + |D\xi| > 1 + \frac{\sigma}{2} =: c_\sigma$ on $\mathbf{Y}$, if ρ is small enough.

Analogously, we estimate $\mathcal{II}$:

$$\mathcal{II} \leq c \int_{\mathbf{Y}} \varphi''(|\tau|)|D\xi|\, dx \leq c \int_{\mathbf{Y}} \varphi'(|z_0| + |D\xi|)|D\xi|\, dx$$

since $|\tau| \leq |z| + |z_0| \leq c(|z_0| + |D\xi|)$. On the other hand, since on $\mathbf{Y}$

$$\varphi'(|z_0| + |D\xi|)|D\xi| \leq c\varphi''(|z_0| + |D\xi|)(|z_0| + |D\xi|)|D\xi|$$

$$\leq c(|z_0|, \sigma)\varphi''(|z_0| + |D\xi|)|D\xi|^2 \leq c(|z_0|, \sigma)\varphi_{|z_0|}(|D\xi|),$$

we can say that

$$\int_{\mathbf{Y}} [G(D\xi) - G(0) - (DG(0), D\xi)]\, dx \geq -\tilde{c}\rho \int_{\mathbf{Y}} \varphi_{|z_0|}(|D\xi|)\, dx,$$

where $\tilde{c}$ depends on the characteristics of φ, σ and $|z_0|$. Choosing ρ such that $\frac{c\,\omega_\rho}{2} + \tilde{c}\rho < \frac{k}{2}$ we have the result. $\qquad\square$

In the sequel we assume that $z_0 \in \mathbf{R}^n$, with $|z_0| > M + 1$, so that (7.2) holds in $\mathcal{B}_\rho(z_0)$ with $\rho < 1$.

We define the excess function

$$\mathbf{E}(\mathcal{B}_R(x_0), u) = \fint_{\mathcal{B}_R(x_0)} |V(Du) - (V(Du))_{\mathcal{B}_R(x_0)}|^2 \, dx.$$

The main ingredient to prove our regularity result is the following decay estimate:

Proposition 7.1. *For all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \varphi) > 0$ and $\beta \in (0,1)$, such that, if u is a minimizer and if for some ball $\mathcal{B}_R(x_0)$ with $\mathcal{B}_{2R}(x_0) \Subset \Omega$ the following estimates*

$$(7.3) \qquad \mathbf{E}(\mathcal{B}_{2R}(x_0), u) \le \delta \fint_{\mathcal{B}_{2R}(x_0)} |V(Du)|^2 \, dx, \quad |(Du)_{\mathcal{B}_{2R}(x_0)} - z_0| < \rho$$

hold true, then for every $\tau \in (0, \frac{1}{2}]$

$$\mathbf{E}(\mathcal{B}_{\tau R}(x_0), u) \le C\tau^\beta (\varepsilon\tau^{-n-1} + 1) \, \mathbf{E}(\mathcal{B}_{2R}(x_0), u),$$

where $C = C(\varphi, n)$ and it is independent of ε.

Proof. Let q be a linear function such that $(u-q)_{\mathcal{B}_{2R}} = 0$ and $z := Dq = (Du)_{\mathcal{B}_{2R}}$. Let $w := u - q$. Fix $\varepsilon > 0$ and δ as in Lemma 6.2, then w is almost $\mathcal{A}$-harmonic with $\mathcal{A} = \frac{D^2 f(z)}{\varphi''(|z|)}$. Let us observe that by Lemma 7.1 such $\mathcal{A}$ is strongly elliptic in the sense of Legendre–Hadamard, since for every $a \in \mathbf{R}^N$ and $b \in \mathbf{R}^n$,

$$\frac{D^2 f(z)}{\varphi''(|z|)}(a \otimes b, a \otimes b) \ge \frac{\varphi''(|z_0|)}{\varphi''(|z|)}|a|^2|b|^2 \ge c|a|^2|b|^2$$

for $z_0 \in \mathbf{R}^n$ with $|z_0| > 1$ and z such that $|z - z_0| < \rho$, where c depends on z_0, ρ and φ.

Let h be the $\mathcal{A}$-harmonic approximation of w with $h = w$ on $\partial\mathcal{B}_R$. At this point we can apply Theorem 6.1 and conclude that, for $|z| > M$, h satisfies

$$\fint_{\mathcal{B}_R} \varphi_{|z|}(|Dw - Dh|) \, dx \le \varepsilon \left[\left(\fint_{\mathcal{B}_R} \varphi_{|z|}^s(|Du - z|) \, dx \right)^{\frac{1}{s}} + \fint_{\mathcal{B}_{2R}} \varphi_{|z|}(|Du - z|) \, dx \right]$$

where s is the same exponent of Corollary 5.2. Applying Lemma 2.5 and Corollary 5.2 we have

$$\left(\fint_{\mathcal{B}_R} \varphi_{|z|}^s(|Du - z|) \, dx \right)^{\frac{1}{s}} \le c \left(\fint_{\mathcal{B}_R} |V(Du) - V(z)|^{2s} \, dx \right)^{\frac{1}{s}}$$

$$\le c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 \, dx$$

from which, taking into account that $z = (Du)_{\mathcal{B}_{2R}}$ and using Lemma 6.1 we have

$$(7.4) \qquad \begin{aligned} \fint_{\mathcal{B}_R} \varphi_{|z|}(|Dw - Dh|) \, dx &\le \varepsilon c \fint_{\mathcal{B}_{2R}} |V(Du) - V(z)|^2 \, dx \\ &\le \varepsilon c \fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 \, dx \\ &= \varepsilon c \, \mathbf{E}(\mathcal{B}_{2R}, u). \end{aligned}$$

Now we want to compute $\mathbf{E}(\mathcal{B}_{\tau R}, u)$. Applying Lemma 6.1, Lemma 2.5 and Lemma 3.4 we get

$$\mathbf{E}(\mathcal{B}_{\tau R}, u) = \fint_{\mathcal{B}_{\tau R}} |V(Du) - (V(Du))_{\mathcal{B}_{\tau R}}|^2 \, dx \le c \fint_{\mathcal{B}_{\tau R}} |V(Du) - V((Dh)_{\mathcal{B}_{\tau R}} + z)|^2 \, dx$$

$$\le c \fint_{\mathcal{B}_{\tau R}} \varphi_{|(Dh)_{\mathcal{B}_{\tau R}}+z|}(|Du - (Dh)_{\mathcal{B}_{\tau R}} - z|) \, dx$$

$$= c \fint_{\mathcal{B}_{\tau R}} \varphi_{|(Dh)_{\mathcal{B}_{\tau R}}+z|}(|Dw - (Dh)_{\mathcal{B}_{\tau R}}|) \, dx$$

$$\le C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dw - (Dh)_{\mathcal{B}_{\tau R}}|) \, dx + \eta \fint_{\mathcal{B}_{\tau R}} |V((Dh)_{\mathcal{B}_{\tau R}} + z) - V(z)|^2 \, dx$$

$$= \mathcal{I} + \mathcal{II}.$$

Using Jensen's inequality, (7.4), the fact that

$$\sup_{\mathcal{B}_{\tau R}} |Dh - (Dh)_{\mathcal{B}_{\tau R}}| \le c \tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}| \, dx$$

(see [28]), the convexity of φ, and the Δ_2-condition, we have

$$\mathcal{I} \le C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dw - Dh|) \, dx + C_\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|Dh - (Dh)_{\mathcal{B}_{\tau R}}|) \, dx$$

$$\le C_\eta \tau^{-n} \varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + C_\eta \varphi_{|z|} \left(\tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}| \, dx \right).$$

Taking into account that $\varphi_a(st) \le cs\varphi_a(t)$ for all $a \ge 0$, $s \in [0,1]$ and $t \ge 0$, using Jensen inequality and (7.4) we have

$$\varphi_{|z|} \left(\tau \fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}| \, dx \right) \le c \, \tau \varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh - (Dh)_{\mathcal{B}_R}| \, dx \right)$$

$$\le c \, \tau \varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh - Dw| \, dx + \fint_{\mathcal{B}_R} |Dw - (Dw)_{\mathcal{B}_R}| \, dx \right)$$

$$\le c \, \tau \varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh - Dw| \, dx \right) + c \, \tau \varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dw - (Dw)_{\mathcal{B}_R}| \, dx \right)$$

$$\le c \, \tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Dh - Dw|) \, dx + c \, \tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - (Du)_{\mathcal{B}_R}|) \, dx$$

$$\le c \, \tau \varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + c \, \tau \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - (Du)_{\mathcal{B}_R}|) \, dx$$

$$\le c \, \tau \varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + c \, \tau \, \mathbf{E}(\mathcal{B}_{2R}, u),$$

where in the last inequality we used

$$\fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - (Du)_{\mathcal{B}_R}|) \, dx \le c \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|) \, dx + c \fint_{\mathcal{B}_R} \varphi_{|z|}(|z - (Du)_{\mathcal{B}_R}|) \, dx$$

$$\le c \, \mathbf{E}(\mathcal{B}_{2R}, u) + c \, \varphi_{|z|} \left(\left| \fint_{\mathcal{B}_R} [Du - z] \, dx \right| \right)$$

$$\le c \, \mathbf{E}(\mathcal{B}_{2R}, u) + c \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|) \, dx$$

$$\le c \, \mathbf{E}(\mathcal{B}_{2R}, u).$$

So we have

$$\mathcal{I} \leq C_\eta \tau^{-n} \varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + C_\eta \tau \varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + C_\eta \, \tau \, \mathbf{E}(\mathcal{B}_{2R}, u).$$

Now we estimate $\mathcal{II}$; taking into account that

$$\sup_{\mathcal{B}_{\tau R}} |Dh| \leq \fint_{\mathcal{B}_R} |Dh| \, dx,$$

using Jensen's inequality, and (7.4) we obtain

$$\mathcal{II} \leq c\,\eta \fint_{\mathcal{B}_{\tau R}} \varphi_{|z|}(|(Dh)_{\mathcal{B}_{\tau R}}|) \, dx \leq c\,\eta\,\varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh| \, dx \right)$$

$$\leq c\,\eta\,\varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh - Dw| \, dx + \fint_{\mathcal{B}_R} |Dw| \, dx \right)$$

$$\leq c\,\eta\,\varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Dh - Dw| \, dx \right) + c\,\eta\,\varphi_{|z|} \left(\fint_{\mathcal{B}_R} |Du - z| \, dx \right)$$

$$\leq c\,\eta \fint_{\mathcal{B}_R} \varphi_{|z|}(|Dh - Dw|) \, dx + c\,\eta \fint_{\mathcal{B}_R} \varphi_{|z|}(|Du - z|) \, dx$$

$$\leq c\,\eta\,\varepsilon \, \mathbf{E}(\mathcal{B}_{2R}, u) + c\,\eta \, \mathbf{E}(\mathcal{B}_{2R}, u).$$

Putting together estimates for $\mathcal{I}$ and $\mathcal{II}$ we have

$$\mathbf{E}(\mathcal{B}_{\tau R}, u) \leq C \, \mathbf{E}(\mathcal{B}_{2R}, u)[C_\eta \, \tau^{-n} \, \varepsilon + C_\eta \, \tau \, \varepsilon + C_\eta \, \tau + \eta \, \varepsilon + \eta],$$

choosing $\eta = \tau^\alpha$, and consequently $C_\eta = \dfrac{1}{\tau^{\alpha(\bar{p}-1)}}$, with $\alpha < \dfrac{1}{\bar{p}-1}$, we have

$$\mathbf{E}(\mathcal{B}_{\tau R}, u) \leq C\tau^\beta (\varepsilon \tau^{-n-1} + 1) \, \mathbf{E}(\mathcal{B}_{2R}, u)$$

where $\beta = \min\{\alpha, 1 - \alpha(\bar{p}-1)\}$. $\qquad\qquad\square$

Proposition 7.2. *Let $\gamma \in (0,1)$. Then there exists δ that depends on γ and on the characteristics of φ such that: if for some ball $\mathcal{B}_R(x_0) \subset \Omega$*

$$(7.5) \qquad \mathbf{E}(\mathcal{B}_{2R}(x_0), u) \leq \delta \fint_{\mathcal{B}_{2R}(x_0)} |V(Du)|^2 \, dx, \quad |(Du)_{\mathcal{B}_{2R}(x_0)} - z_0| < \frac{\rho}{2}$$

hold, then for any $\rho \in (0,1]$,

$$(7.6) \qquad \mathbf{E}(\mathcal{B}_{\rho R}(x_0), u) \leq c\rho^{\gamma\beta} \, \mathbf{E}(\mathcal{B}_{2R}(x_0), u),$$

where c depends on the characteristics of φ.

Proof. Let $\Lambda(\varepsilon, \tau) = C\tau^\beta (\varepsilon \tau^{-n-1} + 1)$ where C depends on the characteristics of φ and on n. Let $\varepsilon = \varepsilon(\tau)$ such that

$$\Lambda(\varepsilon, \tau) \leq \min \left\{ \left(\frac{\tau}{2} \right)^{\gamma\beta}, \frac{1}{4} \right\}.$$

Let $\delta = \delta(\tau)$ such that Proposition 7.1 holds true and so small that are verified

$$(1 + \tau^{-\frac{n}{2}})\delta^{\frac{1}{2}} < \frac{1}{2} \quad \text{and} \quad c\frac{\delta^{\frac{1}{p}}}{\tau^{\frac{n}{p}}} < \frac{\rho}{2},$$

where c and p will be specified later.

With these choices we can prove that the inequalities in (7.3) hold when we replace $\mathcal{B}_{2R}$ with $\mathcal{B}_{\tau R}$, the first one being necessary to obtain the first inequality following exactly the lines of the proof of Proposition 28 in [23].

Concerning the second inequality we first observe that

$$|(Du)_{\mathcal{B}_{\tau R}} - z_0| < |(Du)_{\mathcal{B}_{\tau R}} - (Du)_{\mathcal{B}_{2R}}| + \frac{\rho}{2}.$$

Moreover, taking into account that φ is of type (p_0, p_1) and using Lemma 2.5, for some $p > 1$ we get

$$|(Du)_{\mathcal{B}_{\tau R}} - (Du)_{\mathcal{B}_{2R}}| \leq \fint_{\mathcal{B}_{\tau R}} |Du - (Du)_{\mathcal{B}_{2R}}| \, dx \leq \left(\fint_{\mathcal{B}_{\tau R}} |Du - (Du)_{\mathcal{B}_{2R}}|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq c \left(\fint_{\mathcal{B}_{\tau R}} \varphi_{|(Du)_{\mathcal{B}_{2R}}|} |Du - (Du)_{\mathcal{B}_{2R}}| \, dx \right)^{\frac{1}{p}}$$

$$\leq c \left(\fint_{\mathcal{B}_{\tau R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 \, dx \right)^{\frac{1}{p}}$$

$$\leq \frac{c}{\tau^{\frac{n}{p}}} \left(\fint_{\mathcal{B}_{2R}} |V(Du) - (V(Du))_{\mathcal{B}_{2R}}|^2 \, dx \right)^{\frac{1}{p}}$$

$$\leq \frac{c}{\tau^{\frac{n}{p}}} \delta^{\frac{1}{p}} \left(\fint_{\mathcal{B}_{2R}} |V(Du)|^2 \, dx \right)^{\frac{1}{p}} \leq c \frac{\delta^{\frac{1}{p}}}{\tau^{\frac{n}{p}}}$$

where in the last inequality we use that by Lemma 2.5 and Jensen inequality

$$
\begin{aligned}
\fint_{\mathcal{B}_{2R}(x)} |V(Du)|^2 \, dy &\sim \fint_{\mathcal{B}_{2R}(x)} \varphi(|Du|) \, dy \\
&\geq \varphi\left(\fint_{\mathcal{B}_{2R}(x)} |Du| \, dy \right) \geq \varphi(|(Du)_{\mathcal{B}_{2R}}|) \geq \varphi(M) > 0.
\end{aligned}
\tag{7.7}
$$

So, the smallness assumptions in (7.3) are satisfied for $\mathcal{B}_{\tau R}$. By induction we get

$$\mathbf{E}\left(\mathcal{B}_{(\frac{\tau}{2})^k 2R} \right) \leq \min\left\{ \left(\frac{\tau}{2}\right)^{\gamma \beta k}, \frac{1}{4^k} \right\} \mathbf{E}(\mathcal{B}_{2R})$$

which is the claim. $\square$

Proof of Theorem 1.1. By Jensen inequality and Lemma 2.5 we have

$$\varphi_{|z_0|}(|(Du)_{\mathcal{B}_r(x)} - z_0|) \leq \varphi_{|z_0|}\left(\fint_{\mathcal{B}_r(x)} |Du - z_0| \, dy \right) \leq \fint_{\mathcal{B}_r(x)} \varphi_{|z_0|}(|Du - z_0|) \, dy$$

$$\leq c \fint_{\mathcal{B}_r(x)} |V(Du) - V(z_0)|^2 \, dy$$

from which by (1.1) we can conclude that

$$|(Du)_{\mathcal{B}_{2R}(x)} - z_0| < \rho$$

for a suitable $R > 0$. Moreover by Lemma 2.5, Jensen's inequality, (7.7), and (1.1) we get

$$\mathbf{E}(\mathcal{B}_{2R}(x), u) \leq \fint_{\mathcal{B}_{2R}(x)} |V(Du) - V(z_0)|^2 \, dy \leq \delta \fint_{\mathcal{B}_{2R}(x)} |V(Du)|^2 \, dy.$$

Hence we have that the assumptions of Proposition 7.2 are verified in a neighborhood of x_0, say in $\mathcal{B}_s(x_0)$. Then by (7.6) we have

$$\mathbf{E}(\mathcal{B}_{\rho R}(x), u) \leq c \rho^{\gamma \beta} \mathbf{E}(\mathcal{B}_{2R}(x), u) \quad \forall x \in \mathcal{B}_s(x_0)$$

and by Campanato's characterization of Hölder continuity we deduce that $u \in C^{1,\overline{\alpha}}(\mathcal{B}_s(x_0))$ for some $\overline{\alpha} < 1$. $\qquad\square$

Let us recall that, for $u \in W^{1,\varphi}(\Omega, \mathbf{R}^N)$, the set of regular points $\mathcal{R}(u)$ is defined by

$$\mathcal{R}(u) = \{x \in \Omega \colon u \text{ is Lipschitz near } x\}.$$

Finally we prove Corollary 1.1.

Proof of Corollary 1.1. Using the characterization (iv) of Theorem 4.1 we can find $M > 0$ such that the assumptions of Theorem 1.1 are satisfied near every $z_0 \in \mathbf{R}^{Nn} \colon |z_0| > M$. By Theorem 1.1 we have that $u \in C^{1,\alpha}$ near every $x_0 \in \Omega$ that satisfies

$$\lim_{r \to 0} \fint_{\mathcal{B}_r(x_0)} |V(Du) - V(z_0)|^2 \, dx = 0$$

and these points x_0 belong to $\mathcal{R}(u)$.

By contradiction assume that some $x \in \Omega$ is not contained in $\overline{\mathcal{R}(u)}$; then in a neighborhood of x we cannot find x_0 as before. Thus, $V(Du)$ is essentially bounded by M on this neighborhood and u is Lipschitz near x. Consequently $x \in \mathcal{R}(u)$ and we have reached the desired contradiction. $\qquad\square$

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