Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 973–1004

INFINITELY MANY SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING DOUBLE CRITICAL TERMS AND BOUNDARY GEOMETRY

Chunhua Wang and Chang-Lin Xiang

School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences
Central China Normal University, Wuhan 430079, P. R. China; chunhuawang@mail.ccnu.edu.cn

Yangtze University, School of Information and Mathematics, Jingzhou 434023, P. R. China
and University of Jyväskylä, Department of Mathematics and Statistics
P. O. Box 35, FI-40014 University of Jyväskylä, Finland; Xiang_math@126.com

Abstract. In this paper, we study the following problem

$$\begin{cases} -\Delta_p u = \mu|u|^{\frac{Np}{N-p}-2}u + \frac{|u|^{\frac{(N-s)p}{N-p}-2}u}{|x|^s} + a(x)|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $1 < p < N$, $0 < s < p$, $\mu \geq 0$ are constants, Δ_p is the p-Laplacian operator, $\Omega \subset \mathbf{R}^N$ is a C^2 bounded domain with $0 \in \bar{\Omega}$ and $a \in C^1(\bar{\Omega})$. By an approximation argument, we prove that if $N > p^2 + p, a(0) > 0$ and Ω satisfies some geometry conditions if $0 \in \partial\Omega$, for example, all the principle curvatures of $\partial\Omega$ at 0 are negative, then the above problem has infinitely many solutions.

1. Introduction and main results

In this paper, we study the following quasilinear elliptic problem

$$(1.1)\qquad \begin{cases} -\Delta_p u = \mu|u|^{p^*-2}u + \frac{|u|^{p^*(s)-2}u}{|x|^s} + a(x)|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $1 < p < N$, $0 < s < p$, $p^* = Np/(N-p)$, $p^*(s) = (N-s)p/(N-p)$ and $\mu \geq 0$ are constants,

$$\Delta_p u = \sum_{i=1}^N \partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i}u), \quad \nabla u = (\partial_{x_1}u, \cdots, \partial_{x_N}u)$$

is the p-Laplacian operator, $\Omega \subset \mathbf{R}^N$ is a bounded C^2 domain with $0 \in \bar{\Omega}$ and $a \in C^1(\bar{\Omega})$. Equation (1.1) is known as the Euler–Lagrange equation of the energy functional $I\colon W_0^{1,p}(\Omega) \to \mathbf{R}$ defined as

$$(1.2)\qquad I(u) = \frac{1}{p}\int_\Omega \big(|\nabla u|^p - a(x)|u|^p\big)dx - \frac{\mu}{p^*}\int_\Omega |u|^{p^*}dx - \frac{1}{p^*(s)}\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}dx$$

doi:10.5186/aasfm.2016.4161
2010 Mathematics Subject Classification: Primary 35J60, 35B33.
Key words: Quasilinear elliptic equations, double critical terms, boundary geometry condition, infinitely many solutions, approximation argument.
The first-named author is partially supported by NSFC (No. 11301204; No. 11371159; No. 11571130) and self-determined research funds of Central China Normal University from the colleges basic research and operation of MOE(CCNU14A05036). The second-named author is financially supported by the Academy of Finland, project 259224.

for $u \in W_0^{1,p}(\Omega)$. All of the integrals in energy functional I are well defined, due to the Sobolev inequality

$$\left(\int_\Omega |\varphi|^{p^*}\,dx\right)^{p/p^*} \le C\int_\Omega |\nabla\varphi|^p\,dx, \quad \forall\,\varphi \in W_0^{1,p}(\Omega),$$

where $C = C(N,p) > 0$, and due to the Caffarelli–Kohn–Nirenberg inequality (see [5])

$$\left(\int_\Omega \frac{|\varphi|^{p^*(s)}}{|x|^s}\,dx\right)^{p/p^*(s)} \le C'\int_\Omega |\nabla\varphi|^p\,dx, \quad \forall\,\varphi \in W_0^{1,p}(\Omega),$$

where $C' = C'(N,p,s) > 0$.

Equations of type (1.1) has been studied extensively in the literature. A prototype of equation (1.1) is the following semilinear equation

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a constant. As one of their main results, Brézis and Nirenberg [4] proved that equation (1.3) has a positive solution if and only if $0 < \lambda < \lambda_1$ when $N \ge 4$, or $\lambda^* < \lambda < \lambda_1$ when $N = 3$, where λ_1 is the first eigenvalue of the Laplacian in the domain Ω with respect to zero Dirichlet boundary condition and $\lambda^* \in (0, \lambda_1)$ is a constant. For more existence results, we refer to e.g. [2, 3, 8, 9, 13, 15, 30] on semilinear problems and [6, 11, 12, 14, 16, 20, 21, 26, 34] on quasilinear problems.

A natural question arises from the results of Brézis and Nirenberg [4] is whether equation (1.3) has infinitely many solutions. This problem has been answered affirmatively by Devillanova and Solimini [15] for all $\lambda > 0$, under the assumption that $N > 6$, see also the references therein for more related results.

Since we are interested in quasilinear elliptic equations, let us first consider equation (1.1) without the Hardy term $|x|^{-s}|u|^{p^*(s)-2}u$. Then equation (1.1) is reduced to

$$\begin{cases} -\Delta_p u = \mu|u|^{p^*-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\mu > 0$ is assumed. By the same idea of Devillanova and Solimini [15], Cao, Peng and Yan [6] proved the existence of infinitely many solutions to equation (1.4) under the assumption that $N > p^2 + p$ and $a \equiv \text{constant} > 0$. We remark that their results can be extended directly to equation (1.4) in the case when $N > p^2 + p$, $\mu > 0$ and $a \in C^1(\bar\Omega)$ with $a(0) > 0$.

Recently, some attention is paid to elliptic problems with double critical terms together with boundary geometry conditions on the domain. For instance, among other problems, Hsia, Lin and Wadade [23] considered the following equation

$$\begin{cases} -\Delta u = \mu|u|^{2^*-2}u + \dfrac{|u|^{2^*(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\mu > 0$. Note that equation (1.5) is the special case of equation (1.1) when $p = 2$ and $a \equiv 0$. Assuming that $0 \in \partial\Omega$ and the mean curvature of $\partial\Omega$ at 0 is negative, Hsia, Lin and Wadade [23] proved the existence of positive solutions to equation (1.5) for all $0 < s < 2$ when $N \ge 4$, and for $0 < s < 1$ when $N = 3$. For more results in this respect, we refer to e.g. [10, 18, 19, 24].

As to the existence of infinitely many solutions to equations of type (1.1) with double critical terms, to our best knowledge, the first result was obtained by Yan and Yang [33] in the semilinear case ($p = 2$). Note that $|x|^{-s}$ is unbounded in Ω since we assume that $0 \in \bar{\Omega}$. This brings extra difficulties and requires careful analysis for Yan and Yang to study their problem. Under the assumptions that $N > 6$, $a(0) > 0$ and that $\Omega \in C^3$ satisfies the following geometry condition:

$$\text{all the principle curvatures of } \partial\Omega \text{ at } 0 \text{ are negative when } 0 \in \partial\Omega, \tag{1.6}$$

they proved the existence of infinitely many solutions for equation (1.1) with $p = 2$, see [33, Theorem 1.2]. In this paper, our aim is to extend the results of Yan and Yang [33] to the quasilinear setting. That is, we consider equation (1.1) for all $1 < p < N$. We will use the same idea as in Yan and Yang [33], which was originally from Devillanova and Solimini [15]. But in the quasilinear setting, it is expected that there are much more complexity that will be encountered than that of Yan and Yang [33]. In the following, we first illustrate the idea that will be used in this paper, and then give the main results of this paper. See also [6, 7, 33] for more applications of the same idea.

Note that the functional I defined by (1.2) does not satisfy the Palais–Smale condition at large energy level. So it is impossible to apply the mountain pass lemma [1] directly to obtain the existence of infinitely many solutions for equation (1.1). Thus, to derive approximation solutions to equation (1.1), we turn to the following perturbed problem:

$$\begin{cases} -\Delta_p u = \mu |u|^{p^*-2-\epsilon} u + \dfrac{|u|^{p^*(s)-2-\epsilon} u}{|x|^s} + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $p^*(s) - 1 > \epsilon > 0$ is a constant which will tend to zero in the end. The functional corresponding to equation (1.7) is given by

$$\begin{aligned} I_\epsilon(u) = {} & \frac{1}{p}\int_\Omega \big(|\nabla u|^p - a(x)|u|^p\big)\,dx - \frac{\mu}{p^* - \epsilon}\int_\Omega |u|^{p^*-\epsilon}\,dx \\ & - \frac{1}{p^*(s) - \epsilon}\int_\Omega \frac{|u|^{p^*(s)-\epsilon}}{|x|^s}\,dx \end{aligned} \tag{1.8}$$

for $u \in W_0^{1,p}(\Omega)$. Now I_ϵ is an even functional and satisfies the Palais–Smale condition in all energy levels. It follows from the symmetric mountain pass lemma [1, 28] that equation (1.7) has infinitely many solutions. See also [17, 20, 27, 29]. Precisely, for fixed ϵ, $\epsilon > 0$, there are positive numbers $c_{l,\epsilon}$ and critical points $u_{l,\epsilon}$, $l = 1, 2, \cdots$, such that

$$I_\epsilon(u_{l,\epsilon}) = c_{l,\epsilon} \to \infty \quad \text{as } l \to \infty.$$

Moreover, for each fixed $l \geq 1$, the sequence $\{c_{l,\epsilon}\}_{\epsilon>0}$ is bounded and thus can be assumed to converge to a finite limit c_l as $\epsilon \to 0$. To obtain the existence of infinitely many solutions for equation (1.1), the first step is to investigate whether $u_{l,\epsilon}$ converges strongly in $W_0^{1,p}(\Omega)$ as $\epsilon \to 0$ for fixed l. That is, we need to study the compactness of the set of solutions for equation (1.7) for all $\epsilon > 0$ small. If $u_{l,\epsilon}$ is proven to converge to some u_l strongly in $W_0^{1,p}(\Omega)$, then u_l is a solution to equation (1.1) and $I(u_l) = c_l$. The next step is to investigate whether we obtained infinitely many different critical values of $\{c_l\}_l$. This step will be disposed via index theory in case $\{c_l\}_l$ is a finite set, see e.g. [6, 7, 15, 17, 27, 33]. Both steps being confirmed implies that equation (1.1) admits infinitely many solutions.

Now it is time to present our assumptions in this paper. Throughout the paper, we use $\|\cdot\|$ to denote the norm of $W_0^{1,p}(\Omega)$. We assume that Ω is a bounded C^2 domain satisfying the following geometry condition:

$$x \cdot \nu \leq 0 \text{ in a neighborhood of } 0 \text{ in } \partial\Omega \text{ when } 0 \in \partial\Omega, \tag{1.9}$$

where ν is the outward unit normal of $\partial\Omega$. Examples of domains that satisfy (1.9) will be given in the below. Our main results are the following theorems.

Theorem 1.1. *Suppose that $a(0) > 0$ and $\Omega \in C^2$ satisfies the condition (1.9). If $N > p^2 + p$, then for any u_n $(n = 1, 2, \cdots)$, which is a solution to equation (1.7) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant C independent of n, u_n converges strongly in $W_0^{1,p}(\Omega)$ up to a subsequence as $n \to \infty$.*

Combining Theorem 1.1 together with index theory, we obtain the following existence result for equation (1.1).

Theorem 1.2. *Suppose that $a(0) > 0$ and $\Omega \in C^2$ satisfies the condition (1.9). If $N > p^2 + p$, then equation (1.1) has infinitely many solutions.*

Note that our assumption (1.9) on the boundary geometry of Ω when $0 \in \partial\Omega$ is slightly different from (1.6) of Yan and Yang [33]. In fact, our assumption (1.9) is slightly weaker than (1.6). Indeed, suppose that $\Omega \in C^3$ and $0 \in \partial\Omega$ such that (1.6) is satisfied. Then we can choose a coordinate system such that

$$\Omega \cap B_\delta(0) = \{x \colon x_N > \varphi(x')\} \quad \text{and} \quad \partial\Omega \cap B_\delta(0) = \{x \colon x_N = \varphi(x')\}, \tag{1.10}$$

where $\delta > 0$ is a small constant and $\varphi \colon \{x' \in \mathbf{R}^{N-1} \colon |x'| < \delta\} \to \mathbf{R}$ is a C^3 function that has the following expansion at $x' = 0$:

$$\varphi(x') = -\sum_{j=1}^{N-1} \alpha_j x_j^2 + O(|x'|^3) \quad \text{for } |x'| \text{ small enough}$$

with constants $\alpha_j > 0$, $j = 1, \cdots, N-1$. Then the outward unit normal ν of $\partial\Omega$ in $B_\delta(0)$ is given by

$$\nu(x) = \frac{(\partial_{x_1}\varphi(x'), \cdots, \partial_{x_{N-1}}\varphi(x'), -1)}{\sqrt{1 + \sum_{j=1}^{N-1} |\partial_{x_j}\varphi(x')|^2}} \quad \text{for } x \in \partial\Omega \cap B_\delta(0).$$

Thus

$$x \cdot \nu(x) = \frac{-\sum_{j=1}^{N-1} \alpha_j x_j^2 + O(|x'|^3)}{\sqrt{1 + \sum_{j=1}^{N-1} |\partial_{x_j}\varphi(x')|^2}} \quad \text{for } x \in \partial\Omega \cap B_\delta(0),$$

which implies that

$$x \cdot \nu(x) \leq 0 \quad \text{for } x \in \partial\Omega \cap B_{\delta'}(0),$$

for some $0 < \delta' < \delta$ small enough. That is, (1.9) is satisfied. Thus we conclude that assumption (1.6) is slightly stronger than our assumption (1.9).

On the other hand, assumption (1.9) does allow more possibilities than that of (1.6). For instance, consider the case when $\partial\Omega$ has a piece of concave boundary close to 0 if $0 \in \partial\Omega$. Precisely, let $\varphi \in C^1$ be such that (1.10) holds, and

$$0 = \varphi(0) \leq \varphi(x') + \sum_{j=1}^{N-1} \partial_{x_j}\varphi(x')(0 - x_j)$$

for x' small enough. Then we have

$$x \cdot \nu(x) = -\frac{\varphi(x') + \sum_{j=1}^{N-1} \partial_{x_j}\varphi(x')(0 - x_j)}{\sqrt{1 + \sum_{j=1}^{N-1} |\partial_{x_j}\varphi(x')|^2}} \leq 0$$

for x' small enough. That is, (1.9) is satisfied. In particular, if Ω has a piece of flat boundary in a neighborhood of 0 when $0 \in \partial\Omega$, then (1.9) is satisfied, while in such case all the principle curvatures of $\partial\Omega$ vanish in a neighborhood of 0, which is against (1.6).

Our paper is organized as follows. In Section 2 we establish some integral estimates. In Section 3 we establish estimates for solutions of equation (1.7) in the region which is close to but also suitably away from the blow up point. We prove Theorems 1.1 and 1.2 in Section 4. In order to give a clear line of our framework, we will list some necessary estimates on solutions of quasilinear equation with Hardy potential in Appendix A, a decay estimate for solutions of equations with critical Sobolev growth in Appendix B, some estimates on solutions of p-Laplacian equation by Wolff potential in Appendix C, and a global compactness result for the solution u_n of equation (1.7) in Appendix D, respectively.

Our notations are standard. $B_R(x)$ is the open ball in $\mathbf{R}^N$ centered at x with radius R. We write

$$\fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx,$$

whenever E is a measurable set with $0 < |E| < \infty$, the n-dimensional Lebesgue measure of E. Let D be an arbitrary domain in $\mathbf{R}^N$. We denote by $C_0^\infty(D)$ the space of smooth functions with compact support in D. For any $1 \leq r \leq \infty$, $L^r(D)$ is the Banach space of Lebesgue measurable functions u such that the norm

$$\|u\|_{r,D} = \begin{cases} \left(\int_D |u|^r\right)^{\frac{1}{r}} & \text{if } 1 \leq r < \infty \\ \operatorname{ess\,sup}_D |u| & \text{if } r = \infty \end{cases}$$

is finite. The local space $L^r_{\text{loc}}(D)$ consists of functions belonging to $L^r(D')$ for all $D' \subset\subset D$. We also denote $d\mu_s = |x|^{-s}\,dx$ and $\|v\|_{q,\mu_s} = \left(\int |v|^q\,d\mu_s\right)^{1/q}$ when there is no confusion on the domain of the integral. A function u belongs to the Sobolev space $W^{1,r}(D)$ if $u \in L^r(D)$ and its first order weak partial derivatives also belong to $L^r(D)$. We endow $W^{1,r}(D)$ with the norm

$$\|u\|_{1,r,D} = \|u\|_{r,D} + \|\nabla u\|_{r,D}.$$

The local space $W^{1,r}_{\text{loc}}(D)$ consists of functions belonging to $W^{1,r}(D')$ for all open $D' \subset\subset D$. We recall that $W_0^{1,r}(D)$ is the completion of $C_0^\infty(D)$ in the norm $\|\cdot\|_{1,r,D}$. For the properties of the Sobolev functions, we refer to the monograph [35].

2. Integral estimates

Let u_n, $n = 1, 2, \ldots$, be a solution of equation (1.7) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some constant C independent of n. In this section we deduce some integral estimates for u_n. For any function u, we define

$$\rho_{x,\lambda}(u) = \lambda^{\frac{N-p}{p}} u(\lambda(\cdot - x)) \tag{2.1}$$

for any $\lambda > 0$ and $x \in \mathbf{R}^N$. By Proposition D.1, u_n can be decomposed as

$$u_n = u_0 + \sum_{j=1}^{m} \rho_{x_{n,j},\lambda_{n,j}}(U_j) + \omega_n.$$

Here $x_{n,j} = 0$ for $j = k+1, \ldots, m$.

To prove that u_n converges strongly in $W_0^{1,p}(\Omega)$, we only need to show that the bubbles $\rho_{x_{n,j},\lambda_{n,j}}(U_j)$ will not appear in the decomposition of u_n. Among all the bubbles, we can choose one bubble such that this bubble has the slowest concentration rate. That is, the corresponding λ is the lowest order infinity among all the λ appearing in the bubbles. For simplicity, we denote by λ_n the slowest concentration rate and by x_n the corresponding concentration point throughout the paper.

For any $q > 1$, denote

$$\|u\|_{*,q} = \Big(\int_\Omega |u|^q dx\Big)^{\frac{1}{q}} + \Big(\int_\Omega |u|^{\frac{(N-s)q}{N}} d\mu_s\Big)^{\frac{N}{(N-s)q}}$$

and $q' = q/(q-1)$. Here we write $d\mu_s = |x|^{-s}dx$. For any $p^*/p' < p_2 < p^* < p_1$, $\alpha > 0$ and $\lambda \geq 1$, consider the following relation

$$\begin{cases} \|u_1\|_{*,p_1} \leq \alpha, \\ \|u_2\|_{*,p_2} \leq \alpha\lambda^{\frac{N}{p^*} - \frac{N}{p_2}}, \end{cases} \tag{2.2}$$

and define

$$\|u\|_{*,p_1,p_2,\lambda} = \inf \alpha, \tag{2.3}$$

where the infimum is taken over all $\alpha > 0$ for which there exist u_1, u_2 such that $|u| \leq u_1 + u_2$ and (2.2) holds. Our main result in this section is the following estimate.

Proposition 2.1. *Let u_n, $n = 1, 2, \ldots$, be a solution of equation (1.7) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \leq C$ for some positive constant C independent of n. Then for any $p_1, p_2 \in (p^*/p', \infty)$, $p_2 < p^* < p_1$, there exists a constant $C = C(p_1, p_2) > 0$, independent of n, such that*

$$\|u_n\|_{*,p_1,p_2,\lambda_n} \leq C$$

for all n. Here λ_n is the slowest concentration rate of u_n.

Several lemmas are needed to prove Proposition 2.1. In the rest of this section, let us fix a bounded domain D such that $\Omega \subset\subset D$ and set $r = \frac{1}{3}\mathrm{dist}(\Omega, \partial D)$.

Lemma 2.2. *Let $w \in W_0^{1,p}(D)$, $w \geq 0$, be the solution of*

$$\begin{cases} -\Delta_p w = \big(a_1(x) + \frac{a_2(x)}{|x|^s}\big)v^{p-1} & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \tag{2.4}$$

where $a_1, a_2, v \in L^\infty(D)$ are nonnegative functions in D. Then for any $p_1, p_2 \in (p^*/p', \infty)$, $p_2 < p^* < p_1$, there is a constant $C = C(p_1, p_2) > 0$, such that for any $\lambda \geq 1$,

$$\|w\|_{*,p_1,p_2,\lambda} \leq C \left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \|v\|_{*,p_1,p_2,\lambda}. \tag{2.5}$$

Proof. Let α, $\alpha > \|v\|_{*,p_1,p_2,\lambda}$, be an arbitrary constant. Then by the definition of $\|v\|_{*,p_1,p_2,\lambda}$, there exist v_1, v_2 such that $|v| \leq v_1 + v_2$ and (2.2) holds with $u_i = v_i$, $i = 1, 2$.

Let $w_i \in W_0^{1,p}(D)$, $w_i \geq 0$, $i = 1, 2$, be the solution of equation (2.4) with $v = 2v_i$. Then Corollary A.2 implies that

$$\|w_i\|_{*,p_i} \leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \|v_i\|_{*,p_i}. \tag{2.6}$$

Let $\tilde{w} \in W_0^{1,p}(D)$, $\tilde{w} \geq 0$, be the solution of equation

$$\begin{cases} -\Delta_p w = \left(a_1(x) + \frac{a_2(x)}{|x|^s}\right)\left((2v_1)^{p-1} + (2v_2)^{p-1}\right) & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

Applying Corollary A.2 gives us

$$\begin{aligned} \|\tilde{w}\|_{*,p_2} &\leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \left\|\left((2v_1)^{p-1} + (2v_2)^{p-1}\right)^{\frac{1}{p-1}}\right\|_{*,p_2} \\ &\leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \left(\|v_1\|_{*,p_2} + \|v_2\|_{*,p_2}\right) \\ &\leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \alpha. \end{aligned} \tag{2.7}$$

Thus for any $x \in \Omega$, we have

$$\inf_{B_r(x)} \tilde{w} \leq \left(⨍_{B_r(x)} \tilde{w}^{p_2} dy\right)^{\frac{1}{p_2}} \leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \alpha. \tag{2.8}$$

Note that $v^{p-1} \leq (2v_1)^{p-1} + (2v_2)^{p-1}$. Thus $w \leq \tilde{w}$ by comparison principle. Applying Proposition C.1 gives us

$$w(x) \leq \tilde{w}(x) \leq C \inf_{B_r(x)} \tilde{w} + Cw_1(x) + Cw_2(x), \quad \forall\, x \in \Omega.$$

Let $\tilde{w}_1(x) = C \inf_{B_r(x)} \tilde{w} + Cw_1(x)$ and $\tilde{w}_2(x) = Cw_2(x)$ for $x \in \Omega$. Then $w \leq \tilde{w}_1 + \tilde{w}_2$ in Ω. By (2.6) and (2.8), we have that

$$\|\tilde{w}_1\|_{*,p_1} \leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \alpha,$$

and that

$$\|\tilde{w}_2\|_{*,p_2} \leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \alpha \lambda^{\frac{N}{p^*} - \frac{N}{p_2}}.$$

Hence by definition (2.3), we obtain that

$$\|w\|_{*,p_1,p_2,\lambda} \leq C\left(\|a_1\|_{\frac{N}{p}} + \|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}} \alpha.$$

Since $\alpha > \|v\|_{*,p_1,p_2,\lambda}$ is arbitrary, we obtain (2.5). The proof of Lemma 2.2 is completed. □

We also have the following result which will be used in the proof of Proposition 2.1.

Lemma 2.3. *Let $w \in W_0^{1,p}(D)$, $w \geq 0$, be the solution of*

$$\begin{cases} -\Delta_p w = 2\mu v^{p^*-1} + \frac{2v^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \tag{2.9}$$

where $v \geq 0$ is a bounded function and $A \geq 0$ is a constant. Then for any $p_1, p_2 \in (p^* - 1, \frac{N}{p}(p^* - 1))$, $p_2 < p^* < p_1$, and for any $\lambda \geq 1$, there exists a constant

$C = C(p_1, p_2) > 0$, such that

$$\|w\|_{*,q_1,q_2,\lambda} \le C\|v\|_{*,p_1,p_2,\lambda}^{\frac{p^*-1}{p-1}} + C, \tag{2.10}$$

where q_1, q_2 are given by

$$q_1 = \frac{(p-1)N\hat{p}_1}{N - p\hat{p}_1} \quad \text{with } \hat{p}_1 = \frac{Np_1}{(p^*(s)-1)N + sp_1},$$

and

$$q_2 = \frac{(p-1)N\hat{p}_2}{N - p\hat{p}_2} \quad \text{with } \hat{p}_2 = \frac{p_2}{p^*-1}.$$

Proof. Let α, $\alpha > \|v\|_{*,p_1,p_2,\lambda}$, be an arbitrary constant. Then by the definition of $\|v\|_{*,p_1,p_2,\lambda}$, there exist v_1, v_2 such that $|v| \le v_1 + v_2$ and (2.2) holds with $u_i = v_i$, $i = 1, 2$.

Let $w_1 \in W_0^{1,p}(D)$, $w_1 \ge 0$, be the solution of equation (2.9) with $v = 2v_1$. Let

$$\hat{p}_1 = \min\left\{\frac{p_1}{p^*-1}, \frac{Np_1}{(p^*(s)-1)\,N + sp_1}\right\}$$

such that $(p^*-1)\hat{p}_1 \le p_1$ and $(p^*(s)-1)\frac{(N-s)\hat{p}_1}{N-s\hat{p}_1} \le \frac{(N-s)p_1}{N}$. By the assumptions on the parameters N, p, s and p_1, we obtain that

$$\hat{p}_1 = \frac{Np_1}{(p^*(s)-1)\,N + sp_1} \in \left(1, \frac{N}{p}\right).$$

Then applying Proposition A.1 gives us

$$\begin{aligned}\|w_1\|_{*,q_1} &\le C\left(\|v_1^{p^*-1}\|_{\hat{p}_1} + \|v_1^{p^*(s)-1} + A\|_{\frac{(N-s)\hat{p}_1}{N-s\hat{p}_1},\mu_s}\right)^{\frac{1}{p-1}} \\ &\le C\left(\|v_1\|_{p_1}^{p^*-1} + \|v_1\|_{\frac{(N-s)p_1}{N},\mu_s}^{p^*(s)-1} + 1\right)^{\frac{1}{p-1}} \le C\alpha^{\frac{p^*-1}{p-1}} + C,\end{aligned}$$

where $q_1 = (p-1)N\hat{p}_1/(N - p\hat{p}_1)$.

Similarly, let $w_2 \in W_0^{1,p}(D)$, $w_2 \ge 0$, be the solution of equation

$$\begin{cases} -\Delta_p w_2 = 2\mu v^{p^*-1} + \frac{2v^{p^*(s)-1}}{|x|^s} & \text{in } D, \\ w_2 = 0 & \text{on } \partial D. \end{cases}$$

Let

$$\hat{p}_2 = \min\left\{\frac{p_2}{p^*-1}, \frac{Np_2}{(p^*(s)-1)N + sp_2}\right\}$$

such that $\frac{(N-s)\hat{p}_2}{N-s\hat{p}_2} \le \frac{(N-s)p_2}{N}$ and $(p^*-1)\hat{p}_2 \le p_2$. Then by the assumptions on the parameters N, p, s and p_2, we obtain that

$$\hat{p}_2 = \frac{p_2}{p^*-1} \in (1, N/p).$$

Applying Proposition A.1 as above, we obtain that

$$\|w_2\|_{*,q_2} \le \left(C\alpha^{\frac{p^*-1}{p-1}} + C\right)\lambda^{\frac{N}{p^*} - \frac{N}{q_2}},$$

where $q_2 = (p-1)N\hat{p}_2/(N - p\hat{p}_2)$. To obtain the above estimate, we used the equality that

$$\left(\frac{N}{p^*} - \frac{N}{p_2}\right)\frac{p^*-1}{p-1} = \frac{N}{p^*} - \frac{N}{q_2}.$$

Let $\tilde{w} \in W_0^{1,p}(D)$, $\tilde{w} \geq 0$, be the solution of equation

$$\begin{cases} -\Delta_p \tilde{w} = 2\mu\left((2v_1)^{p^*-1} + (2v_2)^{p^*-1}\right) + 2\frac{(2v_1)^{p^*(s)-1}+(2v_2)^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} & \text{in } D, \\ \tilde{w} = 0 & \text{on } \partial D. \end{cases}$$

Estimate as that of (2.7). We obtain that

$$\|\tilde{w}\|_{*,q_2} \leq C\alpha^{\frac{p^*-1}{p-1}} + C,$$

which implies that

$$\inf_{B_r(x)} \tilde{w} \leq \left(\fint_{B_r(x)} \tilde{w}^{q_2}\, dy\right)^{\frac{1}{q_2}} \leq C\alpha^{\frac{p^*-1}{p-1}} + C, \quad \forall\, x \in \Omega.$$

Note that $w \leq \tilde{w}$ in Ω. Argue as that of Lemma 2.2. We prove Lemma 2.3. This completes the proof. □

Now define $u_n = 0$ in $D\backslash\Omega$. It is easy to see that

$$\left|\mu|u|^{p^*-2-\epsilon}u + \frac{|u|^{p^*(s)-2-\epsilon}u}{|x|^s} + a(x)|u|^{p-2}u\right| \leq 2\mu|u|^{p^*-1} + \frac{2|u|^{p^*(s)-1} + A}{|x|^s}$$

for sufficiently large constant $A > 0$. Let $w_n \in W_0^{1,p}(D)$, $w_n \geq 0$, satisfy

$$\begin{cases} -\Delta_p w_n = 2\mu|u_n|^{p^*-1} + \frac{2|u_n|^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} & \text{in } D, \\ w_n = 0 & \text{on } \partial D. \end{cases} \tag{2.11}$$

Then by comparison principle,

$$|u_n| \leq w_n \quad \text{in } D. \tag{2.12}$$

Moreover, note that $\|u_n\| \leq C$. Multiply both sides of equation (2.11) by w_n and then integrate on the domain D. We easily obtain that

$$\|w_n\|_{p^*} + \|w_n\|_{p^*(s),\mu_s} \leq C, \tag{2.13}$$

where $C > 0$ is independent of n.

To prove Proposition 2.1, it is enough to prove the estimate of Proposition 2.1 for w_n. We have the following result.

Lemma 2.4. *There exist $p_1, p_2 \in (p^*/p', \infty)$, $p_2 < p^* < p_1$, and constant $C = C(p_1, p_2) > 0$, independent of n, such that*

$$\|w_n\|_{*,p_1,p_2,\lambda_n} \leq C. \tag{2.14}$$

Proof. By Proposition D.1, u_n can be decomposed as

$$u_n = u_0 + \sum_{j=1}^{k} \rho_{x_{n,j},\lambda_{n,j}}(U_j) + \sum_{j=k+1}^{m} \rho_{0,\lambda_{n,j}}(U_j) + \omega_n.$$

Write $x_{n,j} = 0$ for $j = k+1, \ldots, m$. In the following proof, we denote

$$u_{n,0} = u_0, \quad u_{n,1} = \sum_{j=1}^{m} \rho_{x_{n,j},\lambda_{n,j}}(U_j), \quad \text{and } u_{n,2} = \omega_n.$$

By (2.12), we have

$$2\mu|u_n|^{p^*-1} + \frac{2|u_n|^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} \leq C\sum_{i=0}^{2}\left(|u_{n,i}|^{p^*-p} + \frac{|u_{n,i}|^{p^*(s)-p}}{|x|^s}\right) w_n^{p-1} + \frac{A}{|x|^s}.$$

Let $\tilde{w}_n \in W_0^{1,p}(D)$, $\tilde{w}_n \geq 0$, satisfy

$$\begin{cases} -\Delta_p \tilde{w}_n = C \sum_{i=0}^{2} \left(|u_{n,i}|^{p^*-p} + \frac{|u_{n,i}|^{p^*(s)-p}}{|x|^s} \right) w_n^{p-1} + \frac{A}{|x|^s} & \text{in } D, \\ \tilde{w}_n = 0 & \text{on } \partial D. \end{cases} \tag{2.15}$$

Comparison principle implies that

$$w_n \leq \tilde{w}_n \quad \text{in } D.$$

Multiply both sides of equation (2.15) by $\tilde{w}_n$. By (2.13) we obtain that

$$\|\tilde{w}_n\|_{p^*} + \|\tilde{w}_n\|_{p^*(s),\mu_s} \leq C. \tag{2.16}$$

Thus we have

$$\inf_{B_r(x)} \tilde{w}_n \leq C, \quad \forall\, x \in \Omega. \tag{2.17}$$

Now let $w_i \in W_0^{1,p}(D)$, $w_i \geq 0$, $i = 0, 1, 2$, be the solution of equation

$$\begin{cases} -\Delta_p w = C \left(|u_{n,i}|^{p^*-p} + \frac{|u_{n,i}|^{p^*(s)-p}}{|x|^s} \right) w_n^{p-1} + \frac{A\delta_{i0}}{|x|^s} & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

respectively, where $\delta_{00} = 1$ and $\delta_{10} = \delta_{20} = 0$. Then by Proposition C.1 and (2.17), we obtain that

$$\tilde{w}_n(x) \leq C + Cw_0(x) + Cw_1(x) + Cw_2(x), \quad \forall\, x \in \Omega. \tag{2.18}$$

In the following we estimate w_i, $i = 0, 1, 2$, term by term.

First, we use Proposition A.1 to estimate w_0. Since $0 < s < p$, we can choose $q \geq 1$ such that

$$\frac{s}{N} + \frac{p-1}{p^*} < \frac{1}{q} < \frac{p}{N} + \frac{p-1}{p^*} = \frac{p^*-1}{p^*}$$

and that

$$q < N/p.$$

Then

$$\frac{(p-1)Nq}{N-pq} > p^* \quad \text{and} \quad \frac{(p-1)(N-s)q}{N-sq} < p^*(s).$$

Let $p_1 = \frac{(p-1)Nq}{N-pq}$. Applying Proposition A.1 to w_0 gives us

$$\begin{aligned} \|w_0\|_{*,p_1} &\leq C \left(\left\| |u_{n,0}|^{p^*-p} w_n^{p-1} \right\|_q + \left\| |u_{n,0}|^{p^*(s)-p} w_n^{p-1} + A \right\|_{\frac{(N-s)q}{N-sq},\mu_s} \right)^{\frac{1}{p-1}} \\ &\leq C \left(\left\| w_n^{p-1} \right\|_q + \left\| w_n^{p-1} \right\|_{\frac{(N-s)q}{N-sq},\mu_s} + 1 \right)^{\frac{1}{p-1}} \\ &\leq C \left(\|w_n\|_{(p-1)q} + \|w_n\|_{\frac{(p-1)(N-s)q}{N-sq},\mu_s} + 1 \right) \\ &\leq C \left(\|w_n\|_{p^*} + \|w_n\|_{p^*(s),\mu_s} + 1 \right) \leq C. \end{aligned} \tag{2.19}$$

Here in the second inequality we used the boundedness of $u_{n,0} = u_0$ and in the last inequality we used (2.13). So this gives estimate for w_0.

Next, we use Corollary A.3 to estimate w_1. We will choose $p_2 < p^*$, p_2 close to p^* enough such that

$$\|w_1\|_{*,p_2} \leq C\lambda_n^{\frac{N}{p^*} - \frac{N}{p_2}}. \tag{2.20}$$

Indeed, applying Corollary A.3 to w_1 gives us that

$$\|w_1\|_{*,p_2} \leq C \left(\left\| |u_{n,1}|^{p^*-p} \right\|_{r_1} + \left\| |u_{n,1}|^{p^*(s)-p} \right\|_{r_2,\mu_s} \right)^{\frac{1}{p-1}} \|w_n\|_{*,p^*},$$

where r_1, r_2 are defined as in (A.4), that is,

$$\frac{1}{r_1} = (p-1)\left(\frac{1}{p_2} - \frac{1}{p^*}\right) + \frac{p}{N} \quad \text{and} \quad \frac{1}{r_2} = (p-1)\left(\frac{N}{(N-s)p_2} - \frac{1}{p^*(s)}\right) + \frac{p-s}{N-s}.$$

By (2.13), we have

$$\|w_1\|_{*,p_2} \leq C \left(\left\| |u_{n,1}|^{p^*-p} \right\|_{r_1} + \left\| |u_{n,1}|^{p^*(s)-p} \right\|_{r_2,\mu_s} \right)^{\frac{1}{p-1}}. \tag{2.21}$$

We only need to estimate $\left\| |u_{n,1}|^{p^*-p} \right\|_{r_1}$ and $\left\| |u_{n,1}|^{p^*(s)-p} \right\|_{r_2,\mu_s}$.

For all $1 \leq j \leq m$, we have

$$\int_{\mathbf{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*-p)r_1}\, dy = \lambda_{n,j}^{pr_1-N} \int_{\mathbf{R}^N} |U_j|^{(p^*-p)r_1}\, dy.$$

By Proposition B.1, for all $1 \leq j \leq m$,

$$|U_j(y)| \leq \frac{C}{1+|y|^{\frac{N-p}{p-1}}}, \quad \forall\, y \in \mathbf{R}^N.$$

Since $\frac{N-p}{p-1}(p^*-p)r_1 \to \frac{pN}{p-1}$ as $p_2 \to p^*$, we can choose p_2 close to p^* enough such that $\frac{N-p}{p-1}(p^*-p)r_1 > N$. Then

$$\int_{\mathbf{R}^N} |U_j|^{(p^*-p)r_1}\, dy < \infty.$$

Thus for all $1 \leq j \leq m$,

$$\int_{\mathbf{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*-p)r_1}\, dy \leq C\lambda_{n,j}^{pr_1-N}.$$

Therefore

$$\begin{aligned} \left\| |u_{n,1}|^{p^*-p} \right\|_{r_1}^{\frac{1}{p-1}} &= \|u_{n,1}\|_{(p^*-p)r_1}^{\frac{p^*-p}{p-1}} \leq C \sum_{j=1}^{m} \left\| \rho_{x_{n,j},\lambda_{n,j}}(U_j) \right\|_{(p^*-p)r_1}^{\frac{p^*-p}{p-1}} \\ &\leq C \sum_{j=1}^{m} \lambda_{n,j}^{\frac{pr_1-N}{(p^*-p)r_1}\cdot\frac{p^*-p}{p-1}} \leq C\lambda_n^{\frac{N}{p^*}-\frac{N}{p_2}}. \end{aligned} \tag{2.22}$$

We used the equality

$$\frac{pr_1-N}{(p^*-p)r_1} \cdot \frac{p^*-p}{p-1} = \frac{N}{p^*} - \frac{N}{p_2}$$

in the last inequality of (2.22). This gives estimate for $\left\| |u_{n,1}|^{p^*-p} \right\|_{r_1}$.

We can also choose p_2 close to p^* enough such that for all $1 \leq j \leq m$,

$$\int_{\mathbf{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*(s)-p)r_2}\, d\mu_s \leq C\lambda_{n,j}^{(p-s)r_2-N+s}.$$

Indeed, we have

$$\int_{\mathbf{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*(s)-p)r_2}\, d\mu_s = \lambda_{n,j}^{(p-s)r_2-N+s} \int_{\mathbf{R}^N} \frac{|U_j(y)|^{(p^*(s)-p)r_2}}{|y+\lambda_{n,j}x_{n,j}|^s}\, dy.$$

Write $y_{n,j} = -\lambda_{n,j} x_{n,j}$. Let

$$I_1 = \int_{B_1(y_{n,j})} \frac{|U_j(y)|^{(p^*(s)-p)r_2}}{|y-y_{n,j}|^s}\,dy \quad \text{and} \quad I_2 = \int_{\mathbf{R}^N \setminus B_1(y_{n,j})} \frac{|U_j(y)|^{(p^*(s)-p)r_2}}{|y-y_{n,j}|^s}\,dy.$$

Since U_j is bounded and $0 < s < N$, we have

$$I_1 \le C.$$

Let $\delta > 0$ be a number to be determined. By Hölder's inequality, we have

$$\begin{aligned} I_2 &\le \left(\int_{\mathbf{R}^N \setminus B_1(y_{n,j})} \frac{1}{|y-y_{n,j}|^{N+\delta}}\,dy \right)^{\frac{s}{N+\delta}} \left(\int_{\mathbf{R}^N \setminus B_1(y_{n,j})} |U_j(y)|^{\frac{(p^*(s)-p)r_2(N+\delta)}{N+\delta-s}}\,dy \right)^{\frac{N+\delta-s}{N+\delta}} \\ &\le C_\delta \left(\int_{\mathbf{R}^N} |U_j(y)|^{\frac{(p^*(s)-p)r_2(N+\delta)}{N+\delta-s}}\,dy \right)^{\frac{N+\delta-s}{N+\delta}}. \end{aligned}$$

Since

$$\frac{N-p}{p-1} \frac{(p^*(s)-p)\,r_2(N+\delta)}{N+\delta-s} \to \frac{p(N-s)(N+\delta)}{(p-1)(N+\delta-s)} \quad \text{as } p_2 \to p^*,$$

and

$$\frac{p(N-s)(N+\delta)}{(p-1)(N+\delta-s)} > N \quad \text{for } \delta > 0 \text{ small enough},$$

we can choose p_2 close to p^* enough and $\delta > 0$ small enough such that

$$\frac{N-p}{p-1} \frac{(p^*(s)-p)\,r_2(N+\delta)}{N+\delta-s} > N.$$

Then

$$\int_{\mathbf{R}^N} |U_j(y)|^{\frac{(p^*(s)-p)r_2(N+\delta)}{N+\delta-s}}\,dy < \infty,$$

and we obtain

$$I_2 \le C.$$

Combining the estimates of I_1 and I_2 yields

$$\int_{\mathbf{R}^N} |\rho_{x_{n,j},\lambda_{n,j}}(U_j)|^{(p^*(s)-p)r_2} d\mu_s \le C \lambda_{n,j}^{(p-s)r_2-N+s}.$$

Hence we have

$$\begin{aligned} \big\| |u_{n,1}|^{p^*(s)-p} \big\|_{r_2,\mu_s}^{\frac{1}{p-1}} &= \|u_{n,1}\|_{(p^*(s)-p)r_2,\mu_s}^{\frac{p^*(s)-p}{p-1}} \le C \sum_{j=1}^m \lambda_{n,j}^{\frac{(p-s)r_2-N+s}{(p^*(s)-p)r_2} \cdot \frac{p^*(s)-p}{p-1}} \\ &\le C \lambda_n^{\frac{N}{p^*}-\frac{N}{p_2}}. \end{aligned} \tag{2.23}$$

In the above inequality we used the equality

$$\frac{(p-s)r_2 - N + s}{(p^*(s)-p)\,r_2} \cdot \frac{p^*(s)-p}{p-1} = \frac{N}{p^*} - \frac{N}{p_2}.$$

Combining (2.21)–(2.23) gives (2.20).

Finally, we use Lemma 2.2 to estimate w_2. By Lemma 2.2, we have

$$\begin{aligned} \|w_2\|_{*,p_1,p_2,\lambda_n} &\le C \left(\big\| |u_{n,2}|^{p^*-p} \big\|_{\frac{N}{p}} + \big\| |u_{n,2}|^{p^*(s)-p} \big\|_{\frac{N-s}{p-s},\mu_s} \right)^{\frac{1}{p-1}} \|w_n\|_{*,p_1,p_2,\lambda_n} \\ &\le \frac{1}{2C'} \|w_n\|_{*,p_1,p_2,\lambda_n}, \end{aligned} \tag{2.24}$$

since $\omega_n \to 0$ in $W_0^{1,p}(\Omega)$, where the constant C' is given by (2.18).

Now combining (2.13), (2.18)–(2.20) and (2.24), we obtain that

$$\begin{aligned}\|\tilde{w}_n\|_{*,p_1,p_2,\lambda_n} &\le C + C\|w_0\|_{*,p_1,p_2,\lambda_n} + C\|w_1\|_{*,p_1,p_2,\lambda_n} + C\|w_2\|_{*,p_1,p_2,\lambda_n}\\ &\le C + C\|w_0\|_{*,p_1} + C\|w_1\|_{*,p_2} + \frac{1}{2}\|w_n\|_{*,p_1,p_2,\lambda_n}\\ &\le C + \frac{1}{2}\|\tilde{w}_n\|_{*,p_1,p_2,\lambda_n},\end{aligned}$$

which completes the proof. □

Now we can prove Proposition 2.1.

Proof of Proposition 2.1. Recall that w_n is a solution to equation (2.11). It is standard to prove Proposition 2.1 by Lemma 2.2 and Lemma 2.3. See details in e.g. [6]. This finishes the proof of Proposition 2.1. □

3. Estimates on safe regions

Since the number of the bubbles of u_n is finite, by Proposition D.1 we can always find a constant $\bar{C} > 0$, independent of n, such that the region

$$\mathcal{A}_n^1 = \Big(B_{(\bar{C}+5)\lambda_n^{-\frac{1}{p}}}(x_n)\backslash B_{\bar{C}\lambda_n^{-\frac{1}{p}}}(x_n)\Big)\cap\Omega$$

does not contain any concentration point of u_n for any n. We call this region a safe region for u_n. Also denote

$$\mathcal{A}_n^2 = \Big(B_{(\bar{C}+4)\lambda_n^{-\frac{1}{p}}}(x_n)\backslash B_{(\bar{C}+1)\lambda_n^{-\frac{1}{p}}}(x_n)\Big)\cap\Omega.$$

In this section, we prove the following result.

Proposition 3.1. *Let u_n, $n = 1, 2, \cdots$, be a solution of equation (1.7) with $\epsilon = \epsilon_n \to 0$, satisfying $\|u_n\| \le C$ for some positive constant C independent of n. Then for any $q \ge p$, there is a constant $C > 0$ independent of n, such that*

$$\int_{\mathcal{A}_n^2} |u_n|^q\,dx \le C\lambda_n^{-\frac{N}{p}}.$$

In order to prove Proposition 3.1, we need the following lemma.

Lemma 3.2. *Let D be a bounded domain with $\Omega \subset\subset D$ and let w_n be the solution of equation (2.11). Then there exist a number $\gamma > p-1$ and a constant $C > 0$ independent of n, such that*

$$\left(\frac{1}{r^N}\int_{B_r(y)\cap\Omega} w_n^\gamma\,dx\right)^{\frac{1}{\gamma}} \le C, \quad \forall y \in \Omega, \tag{3.1}$$

for all $r \ge \bar{C}\lambda_n^{-\frac{1}{p}}$.

Proof. We will combine Proposition 2.1 and Proposition C.2 to prove Lemma 3.2. Since w_n is the solution of equation (2.11), applying Proposition C.2 gives us a number $\gamma \in (p-1, (p-1)N/(N-p+1))$ and a constant $C = C(N, p, \gamma)$ such that

$$\begin{aligned}\left(\frac{1}{r^N}\int_{B_r(y)\cap\Omega} w_n^\gamma\right)^{\frac{1}{\gamma}} &\le C + C\int_r^R\left(\int_{B_t(y)}\left(2\mu|u_n|^{p^*-1} + \frac{2|u_n|^{p^*(s)-1}}{|x|^s} + \frac{A}{|x|^s}\right)\right)^{\frac{1}{p-1}}\frac{dt}{t^{\frac{N-1}{p-1}}}\\ &\le C + C\int_r^R\left(\frac{1}{t^{N-p}}\int_{B_t(y)}\left(|u_n|^{p^*-1} + \frac{|u_n|^{p^*(s)-1}}{|x|^s}\right)\right)^{\frac{1}{p-1}}\frac{dt}{t},\end{aligned}$$

for all $0<r<R$, where $R=\mathrm{dist}(\Omega,\partial D)$. Let

$$I_1=\int_r^R\left(\frac{1}{t^{N-p}}\int_{B_t(y)}|u_n|^{p^*-1}dx\right)^{\frac{1}{p-1}}\frac{dt}{t}$$

and

$$I_2=\int_r^R\left(\frac{1}{t^{N-p}}\int_{B_t(y)}\frac{|u_n|^{p^*(s)-1}}{|x|^s}\,dx\right)^{\frac{1}{p-1}}\frac{dt}{t}$$

such that

$$\left(\frac{1}{r^N}\int_{B_r(y)\cap\Omega}w_n^{\gamma}\,dx\right)^{\frac{1}{\gamma}}\le C+CI_1+CI_2. \tag{3.2}$$

We now estimate I_1 and I_2 for $r\ge \bar{C}\lambda_n^{-1/p}$.

By Proposition 2.1, $\|u_n\|_{*,p_1,p_2,\lambda}\le C$ for any $p_1,p_2\in(p^*/p',\infty)$, $p_2<p^*<p_1$. Let $p_1>p^*$ be a number to be determined and $p_2=p^*-1$. There exist $u_{n,1}$, $u_{n,2}$ with $|u_n|\le u_{n,1}+u_{n,2}$ such that $\|u_{n,1}\|_{*,p_1}\le C$ and $\|u_{n,2}\|_{*,p_2}\le C\lambda_n^{\frac{N}{p^*}-\frac{N}{p_2}}$. Then

$$\int_{B_t(y)}|u_{n,1}|^{p^*-1}\,dx\le C\left(\int_{B_t(y)}|u_{n,1}|^{p_1}\,dx\right)^{\frac{p^*-1}{p_1}}|B_t(y)|^{1-\frac{p^*-1}{p_1}}\le Ct^{\left(1-\frac{p^*-1}{p_1}\right)N},$$

and

$$\int_{B_t(y)}|u_{n,2}|^{p^*-1}\,dx=\int_{B_t(y)}|u_{n,2}|^{p_2}\,dx\le C\lambda_n^{\left(\frac{N}{p^*}-\frac{N}{p_2}\right)p_2}=C\lambda_n^{\frac{p-N}{p}}.$$

Thus

$$\begin{aligned}\int_{B_t(y)}|u_n|^{p^*-1}\,dx&\le C\int_{B_t(y)}|u_{n,1}|^{p^*-1}\,dx+C\int_{B_t(y)}|u_{n,2}|^{p^*-1}\,dx\\&\le Ct^{\left(1-\frac{p^*-1}{p_1}\right)N}+C\lambda_n^{\frac{p-N}{p}}.\end{aligned}$$

Since $\frac{N}{p-1}\left(1-\frac{p^*-1}{p_1}\right)+\frac{p-N}{p-1}\to\frac{p}{p-1}$ as $p_1\to\infty$, we can choose p_1 large enough such that $\frac{N}{p-1}\left(1-\frac{p^*-1}{p_1}\right)+\frac{p-N}{p-1}>0$. Then

$$\int_0^R t^{\left(1-\frac{p^*-1}{p_1}\right)\frac{N}{p-1}+\frac{p-N}{p-1}}\frac{dt}{t}<C.$$

Note also that for $r\ge\bar{C}\lambda_n^{-1/p}$, we have

$$\int_r^\infty t^{\frac{p-N}{p-1}}\frac{dt}{t}\le C\lambda_n^{\frac{N-p}{p(p-1)}}.$$

Therefore

$$\begin{aligned}I_1&\le\int_r^R\left(Ct^{\left(1-\frac{p^*-1}{p_1}\right)N}+C\lambda_n^{\frac{p-N}{p}}\right)^{\frac{1}{p-1}}t^{\frac{p-N}{p-1}}\frac{dt}{t}\\&\le C\int_0^R t^{\frac{N}{p-1}\left(1-\frac{p^*-1}{p_1}\right)+\frac{p-N}{p-1}}\frac{dt}{t}+C\lambda_n^{\frac{p-N}{p(p-1)}}\int_r^\infty t^{\frac{p-N}{p-1}}\frac{dt}{t}\le C.\end{aligned} \tag{3.3}$$

This gives estimate for I_1.

Next we estimate I_2. Let $\bar{p}_1 > p^*$ be a number to be determined and $\bar{p}_2 = N(p^*(s)-1)/(N-s)$. There exist $\bar{u}_{n,1}$, $\bar{u}_{n,2}$ with $|u_n| \le \bar{u}_{n,1} + \bar{u}_{n,2}$ such that $\|\bar{u}_{n,1}\|_{*,\bar{p}_1} \le C$ and $\|\bar{u}_{n,2}\|_{*,\bar{p}_2} \le C\lambda_n^{\frac{N}{p^*}-\frac{N}{\bar{p}_2}}$. Then

$$\begin{aligned}\int_{B_t(y)} |\bar{u}_{n,1}|^{p^*(s)-1}\,d\mu_s &\le \left(\int_{B_t(y)} |\bar{u}_{n,1}|^{\frac{N-s}{N}\bar{p}_1}\,d\mu_s\right)^{\frac{(p^*(s)-1)N}{(N-s)\bar{p}_1}} \left(\int_{B_t(y)} d\mu_s\right)^{1-\frac{(p^*(s)-1)N}{(N-s)\bar{p}_1}}\\ &\le Ct^{N-s-\frac{(p^*(s)-1)N}{\bar{p}_1}},\end{aligned}$$

and

$$\int_{B_t(y)} |\bar{u}_{n,2}|^{p^*(s)-1}\,d\mu_s = \int_{B_t(y)} |\bar{u}_{n,2}|^{\frac{N-s}{N}\bar{p}_2}\,d\mu_s \le C\lambda_n^{\frac{p-N}{p}}.$$

Arguing as above yields

$$I_2 \le C, \tag{3.4}$$

if we choose $\bar{p}_1$ large enough. This gives estimate for I_2.

By (3.2)–(3.4), we obtain (3.1). The proof of Lemma 3.2 is complete. □

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Let $\gamma > p-1$ be as in Lemma 3.2. Since $|u_n| \le w_n$, we have

$$\int_{B_{\lambda_n^{-1/p}}(y)} |u_n|^\gamma\,dx \le C\lambda_n^{-\frac{N}{p}}, \quad \forall y \in \mathcal{A}_n^2. \tag{3.5}$$

Let $v_n(x) = u_n(\lambda_n^{-\frac{1}{p}}x)$, $x \in \Omega_n = \{x; \lambda_n^{-\frac{1}{p}}x \in \Omega\}$. Then v_n is a solution to equation

$$\begin{cases} -\Delta_p v_n = \lambda_n^{-1}\left(\mu|v_n|^{p^*-p-\epsilon_n} + \dfrac{\lambda_n^{\frac{s}{p}}|v_n|^{p^*(s)-p-\epsilon_n}}{|x|^s} + a\left(\lambda_n^{-\frac{1}{p}}x\right)\right)|v_n|^{p-2}v_n & x \in \Omega_n,\\ v_n = 0 & \text{on } \partial\Omega_n.\end{cases}$$

Let $z = \lambda_n^{\frac{1}{p}}y$, $y \in \mathcal{A}_n^2$. Since $B_{\lambda_n^{-1/p}}(y)$ does not contain any concentration point of u_n, we can deduce that

$$\begin{aligned}\int_{B_1(z)} \left|\lambda_n^{-1}\big(\mu|v_n|^{p^*-p-\epsilon_n} + a(\lambda_n^{-\frac{1}{p}}x)\big)\right|^{\frac{N}{p}}dx &\le C\int_{B_1(z)} |\lambda_n^{-1}(|v_n|^{p^*-p}+1)|^{\frac{N}{p}}\,dx\\ &\le C\int_{B_{\lambda_n^{-\frac{1}{p}}}(y)} |u_n|^{p^*}\,dx + C\lambda_n^{-\frac{N}{p}} \to 0,\end{aligned}$$

and that

$$\begin{aligned}\int_{B_1(z)} \frac{|\lambda_n^{-1}\lambda_n^{\frac{s}{p}}|v_n|^{p^*(s)-p-\epsilon_n}|^{\frac{N-s}{p-s}}}{|x|^s}\,dx &\le C\int_{B_1(z)} \frac{|\lambda_n^{\frac{s-p}{p}}(|v_n|^{p^*(s)-p}+1)|^{\frac{N-s}{p-s}}}{|x|^s}\,dx\\ &\le C\int_{B_{\lambda_n^{-\frac{1}{p}}}(y)} \frac{|u_n|^{p^*(s)}}{|x|^s}\,dx + C\lambda_n^{-\frac{N-s}{p}} \to 0,\end{aligned}$$

as $n \to \infty$.

Thus for any $q > p^*$, we obtain by Lemma A.4 and (3.5) that,

$$\|v_n\|_{q,B_{1/2}(z)} \le C\left(\int_{B_1(z)} |v_n|^\gamma\,dx\right)^{\frac{1}{\gamma}} = C\left(\fint_{B_{\lambda_n^{-1/p}}(y)} |u_n|^\gamma\,dx\right)^{\frac{1}{\gamma}} \le C.$$

Equivalently, we arrive at

$$\int_{B_{\frac{1}{2}\lambda_n^{-1/p}}(y)} |u_n|^q\,dx \le C\lambda_n^{-\frac{N}{p}}, \quad \forall y \in \mathcal{A}_n^2.$$

Now by a covering argument we proves Proposition 3.1 in the case when $q > p^*$.

For $p \le q \le p^*$, we apply Hölder's inequality to obtain that

$$\left(\fint_{\mathcal{A}_n^2} |u_n|^q\,dx\right)^{\frac{1}{q}} \le \left(\fint_{\mathcal{A}_n^2} |u_n|^{2p^*}\,dx\right)^{\frac{1}{2p^*}} \le C.$$

The proof of Proposition 3.1 is complete. □

Let

$$\mathcal{A}_n^3 = \left(B_{(\bar{C}+3)\lambda_n^{-\frac{1}{p}}}(x_n)\backslash B_{(\bar{C}+2)\lambda_n^{-\frac{1}{p}}}(x_n)\right) \cap \Omega.$$

In the end of this section, we prove the following gradient estimate for u_n.

Proposition 3.3. *We have*

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^p\,dx \le C\int_{\mathcal{A}_n^2}\left(|u_n|^{p^*} + \frac{|u_n|^{p^*(s)}}{|x|^s} + 1\right)dx + C\lambda_n\int_{\mathcal{A}_n^2} |u_n|^p\,dx. \tag{3.6}$$

In particular, we have

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^p\,dx \le C\lambda_n^{\frac{p-N}{p}}. \tag{3.7}$$

Proof. Let $\phi \in C_0^\infty(\mathcal{A}_n^2)$ be a cut-off function with $\phi = 1$ in $\mathcal{A}_n^3$, $0 \le \phi \le 1$ and $|\nabla\phi| \le C\lambda_n^{\frac{1}{p}}$. Multiply the equation of u_n by $\phi^p u_n$ and integrate on the domain $\mathcal{A}_n^2$. We obtain that

$$\int_{\mathcal{A}_n^2} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(\phi^p u_n)\,dx = \int_{\mathcal{A}_n^2}\left(\mu|u_n|^{p^*-\epsilon_n} + \frac{|u_n|^{p^*(s)-\epsilon_n}}{|x|^s} + a|u_n|^p\right)\phi^p\,dx.$$

Then we have

$$\int_{\mathcal{A}_n^2} |\nabla u_n|^p\phi^p\,dx \le C\int_{\mathcal{A}_n^2} |\nabla\phi|^p|u|_n^p\,dx + C\int_{\mathcal{A}_n^2}\left(|u_n|^{p^*} + \frac{|u_n|^{p^*(s)}}{|x|^s} + |u_n|^p\right)\phi^p\,dx.$$

(3.6) follows easily from above inequality.

Let $q > p^*(s)$. By Proposition 3.1, we have

$$\begin{aligned}\int_{\mathcal{A}_n^2} \frac{\phi^p|u_n|^{p^*(s)}}{|x|^s}\,dx &\le \left(\int_{\mathcal{A}_n^2} \phi^p|u_n|^q\,dx\right)^{\frac{p^*(s)}{q}}\left(\int_{\mathcal{A}_n^2} \phi^p|x|^{-\frac{sq}{q-p^*(s)}}\,dx\right)^{\frac{q-p^*(s)}{q}} \\ &\le C\lambda_n^{-\frac{p^*(s)N}{pq}}\lambda_n^{-\frac{1}{p}\left(N-\frac{sq}{q-p^*(s)}\right)\left(\frac{q-p^*(s)}{q}\right)} = C\lambda_n^{\frac{s-N}{p}}.\end{aligned} \tag{3.8}$$

Now from (3.8), (3.6) and Proposition 3.1, we obtain that

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^p\,dx \le C\lambda_n^{-\frac{N}{p}} + C\lambda_n^{\frac{p-N}{p}} + C\lambda_n^{\frac{s-N}{p}} \le C\lambda_n^{\frac{p-N}{p}}.$$

This proves (3.7). The proof of Proposition 3.3 is complete. □

4. Proofs of main results

In this section we prove Theorem 1.1 and Theorem 1.2. For notational simplicity, we write $p_n = p^* - \epsilon_n$ and $p_n(s) = p^*(s) - \epsilon_n$. Choose $t_n \in [\bar{C}+2, \bar{C}+3]$ such that

$$\begin{aligned}&\int_{\partial B_{t_n\lambda_n^{-\frac{1}{p}}}(x_n)} \left(\mu|u_n|^{p_n} + |u_n|^p + \lambda_n^{-1}|\nabla u_n|^p + \lambda_n^{-\frac{s}{p}}\frac{|u_n|^{p_n(s)}}{|x|^s}\right) d\sigma \\ &\le C\lambda_n^{\frac{1}{p}} \int_{\mathcal{A}_n^3} \left(\mu|u_n|^{p_n} + |u_n|^p + \lambda_n^{-1}|\nabla u_n|^p + \lambda_n^{-\frac{s}{p}}\frac{|u_n|^{p_n(s)}}{|x|^s}\right) dx.\end{aligned} \tag{4.1}$$

By Proposition 3.1, (3.7) and (3.8), we obtain that

$$\int_{\partial B_{t_n\lambda_n^{-\frac{1}{p}}}(x_n)} \left(\mu|u_n|^{p_n} + |u_n|^p + \lambda_n^{-1}|\nabla u_n|^p + \lambda_n^{-\frac{s}{p}}\frac{|u_n|^{p_n(s)}}{|x|^s}\right) d\sigma \le C\lambda_n^{\frac{1-N}{p}}. \tag{4.2}$$

We also have the following Pohozaev identity for u_n on $B_n = B_{t_n\lambda_n^{-\frac{1}{p}}}(x_n) \cap \Omega$

$$\begin{aligned}&\left(\frac{N}{p_n} - \frac{N-p}{p}\right)\mu\int_{B_n}|u_n|^{p_n}\,dx + \int_{B_n}\left[a(x) - \frac{1}{p}\nabla a(x)\cdot(x-x_0)\right]|u_n|^p\,dx \\ &+ \left(\frac{N-s}{p_n(s)} - \frac{N-p}{p}\right)\int_{B_n}\frac{|u_n|^{p_n(s)}}{|x|^s}\,dx + \frac{s}{p_n(s)}\int_{B_n}\frac{|u_n|^{p_n(s)}}{|x|^{2+s}}(x_0\cdot x)\,dx \\ &= \frac{N-p}{p}\int_{\partial B_n}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial\nu}u_n d\sigma + \int_{\partial B_n}|\nabla u_n|^{p-2}\nabla u_n\cdot(x-x_0)\frac{\partial u_n}{\partial\nu}\,d\sigma \\ &- \frac{1}{p}\int_{\partial B_n}|\nabla u_n|^p(x-x_0)\cdot\nu\,d\sigma \\ &+ \int_{\partial B_n}\left(\frac{1}{p_n}|u_n|^{p_n} + \frac{1}{p_n(s)}\frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p}a(x)|u_n|^p\right)(x-x_0)\cdot\nu\,d\sigma,\end{aligned}$$

where ν is the outward unit normal to ∂B_n and $x_0 \in \mathbf{R}^N$. Since $p_n < p^*$ and $p_n(s) < p^*(s)$, we have $\frac{N}{p_n} - \frac{N-p}{p} > 0$ and $\frac{N-s}{p_n(s)} - \frac{N-p}{p} > 0$. Thus we deduce from above the following inequality that

$$\begin{aligned}&\int_{B_n}\left[a(x) - \frac{1}{p}\nabla a(x)\cdot(x-x_0)\right]|u_n|^p\,dx + \frac{s}{p_n(s)}\int_{B_n}\frac{|u_n|^{p_n(s)}}{|x|^{2+s}}(x_0\cdot x)\,dx \\ &\le \frac{N-p}{p}\int_{\partial B_n}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial\nu}u_n\,d\sigma + \int_{\partial B_n}|\nabla u_n|^{p-2}\nabla u_n\cdot(x-x_0)\frac{\partial u_n}{\partial\nu}\,d\sigma \\ &- \frac{1}{p}\int_{\partial B_n}|\nabla u_n|^p(x-x_0)\cdot\nu\,d\sigma \\ &+ \int_{\partial B_n}\left(\frac{1}{p_n}|u_n|^{p_n} + \frac{1}{p_n(s)}\frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p}a(x)|u_n|^p\right)(x-x_0)\cdot\nu\,d\sigma.\end{aligned} \tag{4.3}$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Since $\{x_n\} \subset \Omega$ is a bounded sequence, we may assume that $x_n \to x^* \in \bar{\Omega}$ as $n \to \infty$. We have two cases:

Case 1. $x^* = 0$;

Case 2. $x^* \neq 0$.

In Case 1, choose $x_0 = 0$ in (4.3). Then we obtain that

$$
\begin{aligned}
&\int_{B_n} \Big[a(x) - \frac{1}{p}\nabla a(x)\cdot x\Big]|u_n|^p\,dx \\
&\leq \frac{N-p}{p}\int_{\partial B_n} |\nabla u_n|^{p-2}\frac{\partial u_n}{\partial \nu}u_n\,d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2}\nabla u_n\cdot x\frac{\partial u_n}{\partial \nu}\,d\sigma \\
&\quad - \frac{1}{p}\int_{\partial B_n} |\nabla u_n|^p x\cdot\nu\,d\sigma \\
&\quad + \int_{\partial B_n}\left(\frac{1}{p_n}|u_n|^{p_n} + \frac{1}{p_n(s)}\frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p}a(x)|u_n|^p\right)x\cdot\nu\,d\sigma.
\end{aligned}
\tag{4.4}
$$

Decompose ∂B_n into $\partial B_n = \partial_i B_n \cup \partial_e B_n$, where $\partial_i B_n = \partial B_n \cap \Omega$ and $\partial_e B_n = \partial B_n \cap \partial\Omega$. Consider the case $0 \in \partial\Omega$. Note that $u_n = 0$ on $\partial\Omega$. Thus (4.4) implies that

$$
\begin{aligned}
L_1 &:= \int_{B_n} \Big[a(x) - \frac{1}{p}\nabla a(x)\cdot x\Big]|u_n|^p\,dx - \left(1-\frac{1}{p}\right)\int_{\partial_e B_n}|\nabla u_n|^p x\cdot\nu\,d\sigma \\
&\leq \frac{N-p}{p}\int_{\partial_i B_n} |\nabla u_n|^{p-2}\frac{\partial u_n}{\partial \nu}u_n\,d\sigma + \int_{\partial_i B_n} |\nabla u_n|^{p-2}\nabla u_n\cdot x\frac{\partial u_n}{\partial \nu}\,d\sigma \\
&\quad - \frac{1}{p}\int_{\partial_i B_n} |\nabla u_n|^p x\cdot\nu\,d\sigma \\
&\quad + \int_{\partial_i B_n}\left(\frac{1}{p_n}|u_n|^{p_n} + \frac{1}{p_n(s)}\frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p}a(x)|u_n|^p\right)x\cdot\nu\,d\sigma =: R_1.
\end{aligned}
\tag{4.5}
$$

By assumption (1.9), we have that

$$\int_{\partial_e B_n} |\nabla u_n|^p x\cdot\nu\,d\sigma \leq 0.$$

Recall that we assume $a(0) > 0$. Thus (4.5) gives us

$$L_1 \geq \frac{1}{2}a(0)\int_{B_n}|u_n|^p\,dx. \tag{4.6}$$

On the other hand, since $|x| \leq C\lambda_n^{-1/p}$ for $x \in \partial B_n$, we have by (4.2) that

$$
\begin{aligned}
R_1 &\leq C\lambda_n^{-\frac{1}{p}}\int_{\partial_i B_n}\left(|u_n|^{p_n} + |u_n|^p + |\nabla u_n|^p + \frac{|u_n|^{p_n(s)}}{|x|^s}\right)d\sigma \\
&\quad + C\int_{\partial_i B_n}|\nabla u_n|^{p-1}|u_n|\,d\sigma \leq C\lambda_n^{\frac{p-N}{p}}.
\end{aligned}
\tag{4.7}
$$

Thus combining (4.5), (4.6) and (4.7) yields that

$$\int_{B_n}|u_n|^p\,dx \leq C\lambda_n^{\frac{p-N}{p}}. \tag{4.8}$$

Now we can follow the argument of [6] to obtain that

$$\int_{B_n}|u_n|^p\,dx \geq C'\lambda_n^{-p}.$$

Therefore we arrive at

$$\lambda_n^{-p} \leq C\lambda_n^{\frac{p-N}{p}}. \tag{4.9}$$

Since $\lambda_n \to \infty$, (4.9) can not happen under the assumption that

$$N > p^2 + p.$$

Now we consider the case $0 \in \Omega$. Then $\partial_e B_n = \emptyset$. So (4.5) holds with $\int_{\partial_e B_n} |\nabla u_n|^p x \cdot \nu d\sigma = 0$. By the same argument, we obtain (4.9). We reach a contradiction.

In Case 2, we have two possibilities: $B_{t_n \lambda_n^{-\frac{1}{p}}}(x_n) \subset\subset \Omega$ or $B_{t_n \lambda_n^{-\frac{1}{p}}}(x_n) \cap (\mathbf{R}^N \backslash \Omega) \neq \emptyset$.

Suppose that $B_{t_n \lambda_n^{-\frac{1}{p}}}(x_n) \subset\subset \Omega$. Then $B_n = B_{t_n \lambda_n^{-\frac{1}{p}}}(x_n)$. Take $x_0 = x_n$ in (4.3). We obtain that

(4.10)

$$\begin{aligned} L_2 := & \frac{s}{p_n(s)} \int_{B_n} \frac{|u_n|^{p_n(s)}}{|x|^{2+s}} (x_n \cdot x)\, dx \\ \leq & - \int_{B_n} \Big[a(x) - \frac{1}{p} \nabla a(x) \cdot (x - x_n)\Big] |u_n|^p \, dx \\ & + \frac{N-p}{p} \int_{\partial B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n \, d\sigma + \int_{\partial B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_n) \frac{\partial u_n}{\partial \nu} \, d\sigma \\ & - \frac{1}{p} \int_{\partial B_n} |\nabla u_n|^p (x - x_n) \cdot \nu \, d\sigma \\ & + \int_{\partial B_n} \left(\frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right) (x - x_n) \cdot \nu \, d\sigma =: R_2. \end{aligned}$$

Since $x_n \to x^*$, we have $x_n \cdot x \geq \frac{1}{2}|x_n|^2 \geq \frac{1}{4}|x^*|^2 > 0$ for $x \in B_n$. Thus

$$L_2 \geq C \int_{B_n} |u_n|^{p_n(s)} \, dx.$$

Again, applying the same argument as that of [6] gives us that

$$L_2 \geq C \int_{B_n} |u_n|^{p_n(s)} \, dx \geq C' \lambda_n^{-N + p_n(s) \frac{N-p}{p}}. \tag{4.11}$$

On the other hand, by the same argument as that of (4.7), we obtain that

$$R_2 \leq C \lambda_n^{\frac{p-N}{p}} + C \int_{B_n} |u_n|^p \, dx, \tag{4.12}$$

in which the assumption $a \in C^1(\bar{\Omega})$ was used. We claim that

$$\int_{B_n} |u_n|^p \, dx \leq C \lambda_n^{-p}. \tag{4.13}$$

Indeed, let $p_1 > p^*$ such that $\frac{N}{p}(1 - \frac{p}{p_1}) > p$. This is possible since $N > p^2 + p$. Also, let $p_2 = p$. Then we have $p^*/p' < p_2 < p^*$. By Proposition 2.1, there exist $v_i \geq 0, \ i = 1, 2$, such that $|u_n| \leq v_1 + v_2$ and

$$\|v_1\|_{*,p_1} \leq C, \quad \|v_2\|_{*,p} \leq C \lambda_n^{\frac{N-p}{p} - \frac{N}{p}} = C \lambda_n^{-1}.$$

Hence by Hölder's inequality, we deduce that

$$\int_{B_n} |u_n|^p \, dx \leq 2^{p-1} \int_{B_n} |v_1|^p \, dx + 2^{p-1} \int_{B_n} |v_2|^p \, dx \leq C \lambda_n^{-\frac{N}{p}(1 - \frac{p}{p_1})} + C \lambda_n^{-p} \leq C \lambda_n^{-p}.$$

This gives (4.13). Now combining (4.11)-(4.13) yields

$$\lambda_n^{-N + \frac{N-p}{p} p_n(s)} \leq C \lambda_n^{-p} + C \lambda_n^{\frac{p-N}{p}} \leq C \lambda_n^{-p}, \tag{4.14}$$

since $N > p^2 + p$. That is,

$$\lambda_n^{p-s-\frac{N-p}{p}\epsilon_n} \le C,$$

which is impossible since $s < p$. We reach a contradiction.

It remains to consider $B_{t_n\lambda_n^{-\frac{1}{p}}}(x_n) \cap (\mathbf{R}^N \backslash \Omega) \neq \emptyset$. In this case, we take $x_0 \in \mathbf{R}^N \backslash \Omega$ in (4.3) such that $|x_0 - x_n| \le 2t_n\lambda_n^{-\frac{1}{p}}$ and $\nu \cdot (x - x_0) \le 0$ on $\partial\Omega \cap B_n$. With this x_0, we get from (4.3),

$$\begin{aligned}
&\frac{s}{p_n(s)} \int_{B_n} \frac{|u_n|^{p_n(s)}}{|x|^{2+s}} (x_n \cdot x)\, dx \\
&\le -\int_{B_n} \Big[a(x) - \frac{1}{p}\nabla a(x) \cdot (x - x_n)\Big] |u_n|^p\, dx \\
&\quad + \frac{N-p}{p} \int_{\partial_i B_n} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \nu} u_n\, d\sigma + \int_{\partial_i B_n} |\nabla u_n|^{p-2} \nabla u_n \cdot (x - x_n) \frac{\partial u_n}{\partial \nu}\, d\sigma \\
&\quad - \frac{1}{p} \int_{\partial_i B_n} |\nabla u_n|^p (x - x_n) \cdot \nu\, d\sigma \\
&\quad + \int_{\partial_i B_n} \left(\frac{1}{p_n} |u_n|^{p_n} + \frac{1}{p_n(s)} \frac{|u_n|^{p_n(s)}}{|x|^s} + \frac{1}{p} a(x) |u_n|^p \right) (x - x_n) \cdot \nu\, d\sigma.
\end{aligned}$$

Arguing as above, we find that (4.14) still holds. Thus we obtain a contradiction. The proof of Theorem 1.1 is complete. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. The proof is standard. We give a sketch of proof for the readers' convenience, and refer to Cao, Peng and Yan [6] for more details. First we prove that for each $k \ge 1$, there exists a bounded sequence of $\{u_{k,\epsilon_n}\}_n \subset W_0^{1,p}(\Omega)$ and $\{c_{k,\epsilon_n}\} \subset \mathbf{R}$ such that $I'_{\epsilon_n}(u_{k,\epsilon_n}) = 0$, $I_{\epsilon_n}(u_{k,\epsilon_n}) = c_{k,\epsilon_n}$ and $c_{k,\epsilon_n} \to c_k \in \mathbf{R}$ as $n \to \infty$. Indeed, this follows from a standard min-max argument as below (see e.g. Ghoussoub [17]).

For each k, define the $\mathbf{Z}_2$-homotopy class $\mathcal{F}_k$ by

$$\mathcal{F}_k = \{A \colon A \subset W_0^{1,p}(\Omega) \text{ is compact}, \mathbf{Z}_2\text{-invariant, and } \gamma(A) \ge k\},$$

where the genus $\gamma(A)$ is the smallest integer m, such that there exists an odd continuous mapping $\phi \in C(A, \mathbf{R}^m \backslash \{0\})$, and then define the min-max value (see property (I3) in page 134 of Ghoussoub [17])

$$c_{k,\epsilon} = \min_{A \in \mathcal{F}_k} \max_{u \in A} I_\epsilon(u)$$

for all $\epsilon > 0$. Since I_ϵ is an even functional that satisfies the Palais–Smale condition in all energy levels, we use Corollary 7.12 of Ghoussoub [17] to conclude that $c_{k,\epsilon}$ is a critical value of I_ϵ. Thus, there exists $u_{k,\epsilon}$ such that $I_\epsilon(u_{k,\epsilon}) = c_{k,\epsilon}$. By the same argument as in Cao, Peng and Yan [6], we can deduce that $c_{k,\epsilon}$ is bounded uniformly for ϵ small, and that $\{u_{k,\epsilon}\} \subset W_0^{1,p}(\Omega)$ is a bounded sequence as $\epsilon \to 0$. Hence we can apply Theorem 1.1 to infer that (up to a subsequence) $u_{k,\epsilon_n} \to u_k$ strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$, and $I'(u_k) = 0$ with $I(u_k) = c_k$.

Thus we get a sequence $\{c_k\}_k$ of critical values of I. Now, two cases may occur:

Case I: the set $\{c_k\}_k$ is infinite. In this case, it is obvious that I has infinitely many different critical points, and thus Theorem 1.2 holds;

Case II: for some $m \ge 1$, we have $c_k = c$ for all $k \ge m$.

We have to prove that Theorem 1.2 holds in the latter case as well. With no loss of generality, assume that c is an isolated critical value of I in the sense that, for some $\delta_0 > 0$, I has no critical value in $(c-\delta_0, c+\delta_0)\backslash\{c\}$. Fix such δ_0. Denote

$$K_c = \{u \in W_0^{1,p}(\Omega)\colon I(u) = c, I'(u) = 0\}.$$

Note that K_c is $\mathbf{Z}_2$-invariant since I is an odd functional, and K_c is also compact due to Theorem 1.1. The idea is to prove that K_c is an infinite set, from which Theorem 1.2 follows. Since any finite $\mathbf{Z}_2$-invariant set has genus one, it suffices to prove

$$\gamma(K_c) \geq 2. \tag{4.15}$$

We argue by contradiction. Suppose, on the contrary, that $\gamma(K_c) = 1$. Denote

$$\mathcal{K} = \{u \in W_0^{1,p}(\Omega)\colon \|u - K_c\| \leq \delta_1\},$$

where $0 < \delta_1 < \delta_0$ is so small that $\gamma(\mathcal{K}) = 1$. Such δ_1 exists due to the fact that K_c is compact and $\mathbf{Z}_2$-invariant (see page 132 of Ghoussoub [17]). For each $\epsilon > 0$, define

$$D_\epsilon = \left(I_\epsilon^{c+\delta_1}\backslash I_\epsilon^{c-\delta_1}\right)\backslash\mathcal{K},$$

where I_ϵ^t denotes the level set of I_ϵ given by

$$I_\epsilon^t = \{u \in W_0^{1,p}(\Omega)\colon I_\epsilon(u) < t\}.$$

We claim that I_ϵ has no critical point in D_ϵ for ϵ sufficiently small. Otherwise, there exist $\epsilon_n \to 0$ and $u_n \in D_{\epsilon_n}$ such that $I'_{\epsilon_n}(u_n) = 0$ and $I_{\epsilon_n}(u_n) \in [c-\delta_1, c+\delta_1)$. Then, applying Theorem 1.1 yields that (up to a subsequence) u_n converges to u strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Then we have $I'(u) = 0$ and $I(u) \in (c-\delta_0, c+\delta_0)$. Thus $u \in K_c \subset \mathcal{K}$ holds since c is assumed to be an isolated critical value of I. However, note also that $u_n \notin \mathcal{K}$ implies $u \notin \mathcal{K}$. We reach a contradiction. Hence, we conclude that for every $\epsilon > 0$ sufficiently small, there exists c_ϵ^* such that

$$\|I'_\epsilon(u)\| \geq c_\epsilon^* > 0 \quad \text{for } u \in D_\epsilon.$$

With the help of above lower bound, standard deformation techniques (pseudo-gradient flow) yield an odd homeomorphism $\eta\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ such that

$$\eta(I_\epsilon^{c+\delta_1}\backslash\mathcal{K}) \subset I_\epsilon^{c-\delta_1}.$$

See for example the proof of Theorem 1.9 of Rabinowitz [27]. Note that we need to replace the modified pseudo-gradient vector field V defined in page 150 of Rabinowitz [27] by V/c_ϵ^*.

Now we are ready to prove (4.15). Fix $k > m$. Since $c_{k,\epsilon}$ and $c_{k+1,\epsilon} \to c$ as $\epsilon \to 0$, we have

$$c_{k,\epsilon}, c_{k+1,\epsilon} \in (c - \delta_1/4, c + \delta_1/4)$$

for ϵ sufficiently small. By the definition of $c_{k+1,\epsilon}$, there exists $A \in \mathcal{F}_{k+1}$ such that

$$\max_A I_\epsilon < c_{k+1,\epsilon} + \delta_1/4 < c + \delta_1,$$

which implies that $A \subset I_\epsilon^{c+\delta_1}$. Then $\tilde{A} := \eta(A\backslash\mathcal{K}) \subset I_\epsilon^{c-\delta_1}$ holds. That is,

$$I_\epsilon(u) < c - \delta_1 \qquad \text{for } u \in \tilde{A}.$$

We claim that $\tilde{A} \in \mathcal{F}_k$. Indeed, since γ is subadditive, we deduce that $\gamma(A\backslash\mathcal{K}) \geq \gamma(A) - \gamma(\mathcal{K}) \geq k$ since we assume $\gamma(\mathcal{K}) = 1$. Thus $A\backslash\mathcal{K} \subset \mathcal{F}_k$ holds. Hence the supervariant of γ implies that $\tilde{A} \in \mathcal{F}_k$ (see Theorem 1.9 of Rabinowitz [27]). As a result,

$$c_{k,\epsilon} \leq \sup_{\tilde{A}} I_\epsilon \leq c - \delta_1.$$

This contradicts to $c_{k,\epsilon} \geq c - \delta_1/4$. Hence $\gamma(K_c) \geq 2$ and the proof of Theorem 1.2 is complete. □

Acknowledgments. Both authors would like to thank Xiao Zhong for many useful discussions. They would also like to thank the anonymous referees for valuable comments which helped to improve the manuscript.

Appendix A. Estimates for quasilinear problems with Hardy potential

In this section, we deduce some elementary estimates for solutions of a quasilinear elliptic problem involving a Hardy potential. Let D be a bounded domain in $\mathbf{R}^N$ and $0 \in D$. For any $0 \leq t < p$, write $d\mu_t = |x|^{-t}\,dx$ and $\|w\|_{q,\mu_t}^q = \int_D |w|^q\,d\mu_t$. We also use the notation $\|w\|_q = \|w\|_{q,\mu_0}$. Let us recall that

$$\|w\|_{*,q} = \|w\|_q + \|w\|_{\frac{(N-s)}{N}q,\mu_s}.$$

Proposition A.1. *For any $f_i \geq 0$ and $f_i \in L^\infty(D), i = 1, 2$, let $w \in W_0^{1,p}(D)$ be the solution of*

$$\begin{cases} -\Delta_p w = f_1(x) + \frac{f_2(x)}{|x|^s} & x \in D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

Then, for any $1 < q < N/p$, there exists $C = C(N, p, s, q) > 0$ such that

$$\|w\|_{*,\frac{(p-1)Nq}{N-pq}} \leq C\left(\|f_1\|_q + \|f_2\|_{\frac{(N-s)q}{N-sq},\mu_s}\right)^{\frac{1}{p-1}}.$$

Proof. By the maximum principle, we find that $w \geq 0$. We claim that if $r > 1/p'$, then

$$\|w\|_{*,p^*r}^{pr} \leq C\int_D \left(f_1 + \frac{f_2}{|x|^s}\right) w^{1+p(r-1)}\,dx, \tag{A.1}$$

for $C = C(N, p, s, r) > 0$.

First, we consider the case when $r \geq 1$. Since f_1, f_2 are bounded functions, it is standard to prove that $w \in L^\infty(D)$ by Moser's iteration method [25]. Then we can take a test function $\xi = w^{1+p(r-1)}$ so that

$$\frac{1 + p(r-1)}{r^p}\int_D |\nabla w^r|^p\,dx = \int_D \left(f_1 + \frac{f_2}{|x|^s}\right) w^{1+p(r-1)}\,dx. \tag{A.2}$$

Applying the Sobolev inequality and the Caffarelli–Kohn–Nirenberg inequality gives us

$$\|w\|_{p^*r}^{pr} + \|w\|_{p^*(s)r,\mu_s}^{pr} \leq C\int_D |\nabla w^r|^p\,dx$$

for $C = C(N, p, s, r) > 0$. That is,

$$\|w\|_{*,p^*r}^{pr} \leq C\int_D |\nabla w^r|^p\,dx. \tag{A.3}$$

Then, combining (A.2) and (A.3) yields (A.1).

Next, consider the case when $1/p' < r < 1$. Let $\epsilon > 0$. Define $\xi = w(w + \epsilon)^{p(r-1)}$. It is direct to verify that $\xi \in W_0^{1,p}(D)$ and

$$\nabla\xi = (w + \epsilon)^{p(r-1)}\nabla w + p(r-1)w(w + \epsilon)^{p(r-1)-1}\nabla w.$$

Take ξ as a test function. We have

$$\int_D |\nabla w|^{p-2}\nabla w\cdot\nabla\xi\,dx=\int_D\left(f_1+\frac{f_2}{|x|^s}\right)\xi\,dx.$$

A direct calculation gives that

$$\begin{aligned}\int_D |\nabla w|^{p-2}\nabla w\cdot\nabla\xi\,dx &\geq (1+p(r-1))\int_D (w+\epsilon)^{p(r-1)}|\nabla w|^p\,dx\\ &=\frac{1+p(r-1)}{r^p}\int_D |\nabla\left((w+\epsilon)^r-\epsilon^r\right)|^p\,dx\\ &\geq C\left(\|((w+\epsilon)^r-\epsilon^r)\|_{p^*}^p+\|((w+\epsilon)^r-\epsilon^r)\|_{p^*(s),\mu_s}^p\right),\end{aligned}$$

for $C=C(N,p,s,r)>0$.

Write $w_\epsilon=((w+\epsilon)^r-\epsilon^r)^{1/r}$. Then there exists $C>0$ such that

$$\int_D |\nabla w|^{p-2}\nabla w\cdot\nabla\xi\,dx\geq C\,\|w_\epsilon\|_{*,p^*r}^{pr}\,.$$

Thus

$$\|w_\epsilon\|_{*,p^*r}^{pr}\leq C\int_D\left(f_1+\frac{f_2}{|x|^s}\right)w(w+\epsilon)^{p(r-1)}\,dx\leq C\int_D\left(f_1+\frac{f_2}{|x|^s}\right)w^{1+p(r-1)}\,dx.$$

Letting $\epsilon\to0$, we obtain (A.1) in the case when $r\in(1/p',1)$.

To prove Proposition A.1, we apply Hölder's inequality to (A.1) and obtain that

$$\begin{aligned}\|w\|_{*,p^*r}^{pr}&\leq C\left(\|f_1\|_{\frac{p^*r}{p^*r-(1+p(r-1))}}+\|f_2\|_{\frac{p^*(s)r}{p^*(s)r-(1+p(r-1))},\mu_s}\right)\left(\|w\|_{p^*r}^{1+p(r-1)}+\|w\|_{p^*(s)r,\mu_s}^{1+p(r-1)}\right)\\ &\leq C\left(\|f_1\|_{\frac{p^*r}{p^*r-(1+p(r-1))}}+\|f_2\|_{\frac{p^*(s)r}{p^*(s)r-(1+p(r-1))},\mu_s}\right)\|w\|_{*,p^*r}^{1+p(r-1)},\end{aligned}$$

which implies that

$$\|w\|_{*,p^*r}\leq C\left(\|f_1\|_{\frac{p^*r}{p^*r-(1+p(r-1))}}+\|f_2\|_{\frac{p^*(s)r}{p^*(s)r-(1+p(r-1))},\mu_s}\right)^{\frac{1}{p-1}}.$$

Now for any $q\in(1,N/p)$. Let $r\in(1/p',\infty)$ be such that $q=\frac{p^*r}{p^*r-(1+p(r-1))}$. Then

$$\frac{p^*(s)r}{p^*(s)r-(1+p(r-1))}=\frac{(N-s)q}{N-sq}\quad\text{and}\quad p^*r=\frac{(p-1)Nq}{N-pq}.$$

Hence

$$\|w\|_{*,\frac{(p-1)Nq}{N-pq}}\leq C\left(\|f_1\|_q+\|f_2\|_{\frac{(N-s)q}{N-sq},\mu_s}\right)^{\frac{1}{p-1}}.$$

We finish the proof. □

As an application of Proposition A.1 we have the following corollary.

Corollary A.2. *Let $w\in W_0^{1,p}(D)$ be the solution of*

$$\begin{cases}-\Delta_p w=\left(a_1(x)+\dfrac{a_2(x)}{|x|^s}\right)v^{p-1} & x\in D,\\ w=0 & \text{on }\partial D,\end{cases}$$

where $a_1,a_2,v\in L^\infty(D)$ are nonnegative functions. Then for any $\infty>q>p^*/p'$, there holds

$$\|w\|_{*,q}\leq C\left(\|a_1\|_{\frac{N}{p}}+\|a_2\|_{\frac{N-s}{p-s},\mu_s}\right)^{\frac{1}{p-1}}\|v\|_{*,q}$$

for $C = C(N, p, s, q) > 0$.

Proof. Let $\infty > q > p^*/p'$ and define $r = Nq/(N(p-1)+pq)$. Then $1 < r < N/p$ and $q = (p-1)Nr/(N-pr)$.

By applying Proposition A.1 with $f_i = a_i v^{p-1}$, $i = 1, 2$, we obtain that

$$\|w\|_{*,q} \le C\left(\|f_1\|_r + \|f_2\|_{\frac{(N-s)r}{N-sr},\mu_s}\right)^{\frac{1}{p-1}},$$

for $C = C(N, p, s, q) > 0$. By Hölder's inequality and the definition of $\|\cdot\|_{*,q}$, we have that

$$\|f_1\|_r \le \|a_1\|_{\frac{N}{p}} \|v\|_q^{p-1} \le \|a_1\|_{\frac{N}{p}} \|v\|_{*,q}^{p-1}$$

and that

$$\|f_2\|_{\frac{(N-s)r}{N-sr},\mu_s} \le \|a_2\|_{\frac{N-s}{p-s},\mu_s} \|v\|^{p-1}_{\frac{(N-s)q}{N},\mu_s} \le \|a_2\|_{\frac{N-s}{p-s},\mu_s} \|v\|_{*,q}^{p-1}.$$

Combining the above inequalities gives Corollary A.2. □

We also have the following corollary.

Corollary A.3. *Let* $w \in W_0^{1,p}(D)$ *be the solution of*

$$\begin{cases} -\Delta_p w = \left(a_1(x) + \frac{a_2(x)}{|x|^s}\right) v^{p-1} & x \in D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

where $a_1, a_2, v \in L^\infty(D)$ *are nonnegative functions. Then for any* $p^* > p_2 > p^*/p'$, *there holds*

$$\|w\|_{*,p_2} \le C\left(\|a_1\|_{r_1} + \|a_2\|_{r_2,\mu_s}\right)^{\frac{1}{p-1}} \|v\|_{*,p^*}$$

for $C = C(N, p, s, p_2) > 0$, *where* r_1, r_2 *are defined by*

$$\frac{1}{r_1} = (p-1)\left(\frac{1}{p_2} - \frac{1}{p^*}\right) + \frac{p}{N}, \tag{A.4}$$

and

$$\frac{1}{r_2} = (p-1)\left(\frac{N}{(N-s)p_2} - \frac{1}{p^*(s)}\right) + \frac{p-s}{N-s}. \tag{A.5}$$

Proof. The proof is similar to that of Corollary A.2. By applying Proposition A.1 with $f_i = a_i v^{p-1}$, $i = 1, 2$, we obtain

$$\|w\|_{*,p_2} \le C\left(\|f_1\|_{\frac{Np_2}{(p-1)N+pp_2}} + \|f_2\|_{\frac{(N-s)p_2}{(p-1)N+(p-s)p_2},\mu_s}\right)^{\frac{1}{p-1}}.$$

Let r_1, r_2 be defined as in (A.4) (A.5). Applying Hölder's inequality gives us that

$$\|f_1\|_{\frac{Np_2}{(p-1)N+pp_2}} \le \|a_1\|_{r_1} \|v\|_{p^*}^{p-1},$$

and that

$$\|f_2\|_{\frac{(N-s)p_2}{(p-1)N+(p-s)p_2},\mu_s} \le \|a_2\|_{r_2,\mu_s} \|v\|^{p-1}_{p^*(s),\mu_s}.$$

Combining the above inequalities gives Corollary A.3. □

In the end of this section, we give the following result.

Lemma A.4. *Let* $w \in W^{1,p}_{loc}(\mathbf{R}^N)$, $w \ge 0$, *be a weak solution of the equation*

$$-\Delta_p w \le \left(a_1(x) + \frac{a_2(x)}{|x|^s}\right) w^{p-1}$$

in $\mathbf{R}^N$*, where* $a_1, a_2 \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ *are nonnegative functions. Then, for any unit ball* $B_1(y) \subset \mathbf{R}^N$ *and for any* $q > p^*$*, there is a small constant* $\delta = \delta(q) > 0$ *such that if*

$$\left(\int_{B_1(y)} a_1^{\frac{N}{p}}\,dx\right)^{\frac{p}{N}} + \left(\int_{B_1(y)} a_2^{\frac{N-s}{p-s}}\,d\mu_s\right)^{\frac{p-s}{N-s}} < \delta,$$

then for any $\gamma \in (0, p^*)$*, there holds*

$$\|w\|_{q,B_{1/2}(y)} \le C\|w\|_{\gamma,B_1(y)}$$

for $C = C(N,p,s,q,\gamma) > 0$*.*

Proof. For simplicity, we write $B_r = B_r(y)$ for $r > 0$ in the following proof. It is standard to show that $w \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ by Moser's iteration method [25]. For any $\eta \in C_0^\infty(B_1)$, take $\varphi = \eta^p w^{1+p(\tau-1)}$, $\tau \ge 1$, as a test function. We have

$$\int_{B_1} |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi\,dx \le \int_{B_1}\left(a_1(x) + \frac{a_2(x)}{|x|^s}\right)\eta^p w^{p\tau}\,dx. \tag{A.6}$$

First, a simple calculation gives us that

$$\int_{B_1} |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi\,dx \ge C\int_{B_1}|\nabla(\eta w^\tau)|^p\,dx - C\int_{B_1}|\nabla\eta|^p w^{p\tau}\,dx$$

for $C = C(N,p,\tau) > 0$. Second, by Hölder's inequality, we have

$$\int_{B_1} a_1(x)\eta^p w^{p\tau}\,dx \le \left(\int_{B_1} a_1^{\frac{N}{p}}\,dx\right)^{\frac{p}{N}}\left(\int_{B_1}(\eta w^\tau)^{p^*}\,dx\right)^{\frac{p}{p^*}},$$

and

$$\int_{B_1} \frac{a_2(x)}{|x|^s}\eta^p w^{p\tau}\,dx \le \left(\int_{B_1} a_2^{\frac{N-s}{p-s}}\,d\mu_s\right)^{\frac{p-s}{N-s}}\left(\int_{B_1}(\eta w^\tau)^{p^*(s)}\,d\mu_s\right)^{\frac{p}{p^*(s)}}.$$

Thus (A.6) implies that

$$\begin{aligned}\int_{B_1}|\nabla(\eta w^\tau)|^p\,dx &\le C\int_{B_1}|\nabla\eta|^p w^{p\tau}\,dx\\ &\quad + CA\left(\left(\int_{B_1}(\eta w^\tau)^{p^*}dx\right)^{\frac{p}{p^*}} + \left(\int_{B_1}(\eta w^\tau)^{p^*(s)}\,d\mu_s\right)^{\frac{p}{p^*(s)}}\right),\end{aligned} \tag{A.7}$$

where $C = C(N,p,s,\tau) > 0$ and

$$A = \left(\int_{B_1} a_1^{\frac{N}{p}}\,dx\right)^{\frac{p}{N}} + \left(\int_{B_1} a_2^{\frac{N-s}{p-s}}\,d\mu_s\right)^{\frac{p-s}{N-s}}.$$

By the Sobolev inequality and the Caffarelli–Kohn–Nirenberg inequality, we obtain that

$$\left(\int_{B_1}(\eta w^\tau)^{p^*}dx\right)^{\frac{p}{p^*}} + \left(\int_{B_1}(\eta w^\tau)^{p^*(s)}d\mu_s\right)^{\frac{p}{p^*(s)}} \le C(N,p,s)\int_{B_1}|\nabla(\eta w^\tau)|^p\,dx. \tag{A.8}$$

Combining (A.7) and (A.8) yields that

$$\begin{aligned}&\left(\int_{B_1}(\eta w^\tau)^{p^*}\,dx\right)^{\frac{p}{p^*}} + \left(\int_{B_1}(\eta w^\tau)^{p^*(s)}\,d\mu_s\right)^{\frac{p}{p^*(s)}}\\ &\le C\int_{B_1}|\nabla\eta|^p w^{p\tau}\,dx + CA\left(\left(\int_{B_1}(\eta w^\tau)^{p^*}\,dx\right)^{\frac{p}{p^*}} + \left(\int_{B_1}(\eta w^\tau)^{p^*(s)}\,d\mu_s\right)^{\frac{p}{p^*(s)}}\right).\end{aligned}$$

Thus we can choose

$$\delta = \delta(\tau) > 0 \tag{A.9}$$

small enough such that if $A < \delta = \delta(\tau)$, then $CA < 1/2$ and thus

$$\left(\int_{B_1} (\eta w^\tau)^{p^*}\,dx\right)^{\frac{p}{p^*}} + \left(\int_{B_1} (\eta w^\tau)^{p^*(s)}\,d\mu_s\right)^{\frac{p}{p^*(s)}} \le C\int_{B_1} |\nabla\eta|^p w^{p\tau}\,dx,$$

for $C = C(N,p,s,\tau) > 0$. In particular, if $A < \delta$, we have

$$\left(\int_{B_1} (\eta w^\tau)^{p^*}\,dx\right)^{\frac{p}{p^*}} \le C\int_{B_1} |\nabla\eta|^p w^{p\tau}\,dx \tag{A.10}$$

for $C = C(N,p,s,\tau) > 0$.

Let $0 < r < R \le 1$ and $\eta \in C_0^\infty(B_R)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_r and $|\nabla\eta| \le 2/(R-r)$. Substituting η into (A.10) gives us that

$$\left(\int_{B_r} w^{p\chi\tau}\,dx\right)^{\frac{1}{\chi}} \le \frac{C(\tau)}{(R-r)^p}\int_{B_R} w^{p\tau}\,dx, \tag{A.11}$$

where $\chi = p^*/p > 1$.

Now for any fixed $q > p^*$, there exists $k \in \mathbf{N}$ such that $p\chi^k \le q < p\chi^{k+1}$. Let $\tau_i = \chi^i$, $r_i = r + (R-r)/2^{i-1}$, $i = 1,\ldots,k$, and let

$$\delta = \min\{\delta(\tau_1),\cdots,\delta(\tau_k)\},$$

where $\delta(\tau_i)$ is defined by (A.9) with $\tau = \tau_i$, $i = 1,\cdots,k$. Then if $A < \delta$, we obtain by (A.11)

$$\left(\int_{B_{r_{i+1}}} w^{p\chi^{i+1}}\,dx\right)^{\frac{1}{p\chi^{i+1}}} \le \frac{C(\tau_i)}{(r_i - r_{i+1})^{1/\chi^i}}\left(\int_{B_{r_i}} w^{p\chi^i}\,dx\right)^{\frac{1}{p\chi^i}},$$

for $i = 1,\ldots,k$. Iterate the above inequality from $i = 1$ to $i = k$. We obtain

$$\left(\int_{B_{r_{k+1}}} w^{p\chi^{k+1}}\,dx\right)^{\frac{1}{p\chi^{k+1}}} \le \frac{C}{(R - r_{k+1})^\sigma}\left(\int_{B_R} w^{p^*}\,dx\right)^{\frac{1}{p^*}}$$

for some constants $C > 0$ and $\sigma > 0$. As a result, we have

$$\left(\int_{B_r} w^q\,dx\right)^{\frac{1}{q}} \le \frac{C}{(R-r)^\sigma}\left(\int_{B_R} w^{p^*}\,dx\right)^{\frac{1}{p^*}}, \tag{A.12}$$

since $q < p\chi^{k+1}$.

Fix $\gamma \in (0,p^*)$. There exists $\theta \in (0,1)$ such that

$$\frac{1}{p^*} = \frac{\theta}{\gamma} + \frac{1-\theta}{q}.$$

Thus by Hölder's inequality and Young's inequality, (A.12) implies that

$$\left(\int_{B_r} w^q\,dx\right)^{\frac{1}{q}} \le \frac{1}{2}\left(\int_{B_R} w^q\,dx\right)^{\frac{1}{q}} + \frac{C}{(R-r)^{\sigma/\theta}}\left(\int_{B_R} w^\gamma\,dx\right)^{\frac{1}{\gamma}}.$$

In particular, there holds

$$\left(\int_{B_r} w^q\,dx\right)^{\frac{1}{q}} \le \frac{1}{2}\left(\int_{B_R} w^q\,dx\right)^{\frac{1}{q}} + \frac{C}{(R-r)^{\sigma/\theta}}\left(\int_{B_1} w^\gamma\,dx\right)^{\frac{1}{\gamma}}.$$

Now Applying Lemma 4.3 of Han and Lin [22, Chapter 4] (see Lemma A.5 below) yields

$$\left(\int_{B_r} w^q\,dx\right)^{\frac{1}{q}} \le \frac{C}{(R-r)^{\sigma/\theta}}\left(\int_{B_1} w^\gamma\,dx\right)^{\frac{1}{\gamma}}$$

for some constants $C>0$. Choose $r=1/2$ and $R=1$. We complete the proof. □

We attach Lemma 4.3 of Han and Lin [22, Chapter 4] here for the readers' convenience.

Lemma A.5. *Let $f\ge 0$ be a bounded function in $[\tau_0,\tau_1]$ with $\tau_0\ge 0$. Suppose for $\tau_0\le t<s\le\tau_1$ we have*

$$f(t)\le \theta f(s)+\frac{A}{(s-t)^\alpha}+B$$

for some $\theta\in(0,1]$ and some nonnegative constants A,B,α. Then, there exists $c(\alpha,\theta)>0$ such that, for any $\tau_0\le t<s\le\tau_1$, there holds

$$f(t)\le c(\alpha,\theta)\left\{\frac{A}{(s-t)^\alpha}+B\right\}.$$

Appendix B. A decay estimate

We use $\mathbf{R}^N_*$ to denote either $\mathbf{R}^N$ or $\mathbf{R}^N_+$. Consider the following equation

$$\begin{cases}-\Delta_p u=\mu|u|^{p^*-2}u+\dfrac{|u|^{p^*(s)-2}u}{|x|^s} & \text{in } \mathbf{R}^N_*,\\ u\in\mathcal{D}^{1,p}_0(\mathbf{R}^N_*),\end{cases}\tag{B.1}$$

where $\mathcal{D}^{1,p}_0(\mathbf{R}^N_*)$ is the completion of $C_0^\infty(\mathbf{R}^N_*)$ in the seminorm $\|u\|_{\mathcal{D}^{1,p}_0(\mathbf{R}^N_*)}=\|\nabla u\|_{p,\mathbf{R}^N_*}$. In this section, we give an estimate for the decay of solutions to equation (B.1) at the infinity. We have the following result.

Proposition B.1. *Let u be a solution of* (B.1)*. Then there exists a constant $C>0$ such that*

$$|u(x)|\le\frac{C}{1+|x|^{\frac{N-p}{p-1}}},\quad \forall\,x\in\mathbf{R}^N_*.\tag{B.2}$$

To prove Proposition B.1, the following preliminary estimate is needed.

Lemma B.2. *Let u be a solution of* (B.1)*. Then there is a constant $C>0$ such that*

$$|u(x)|\le\frac{C}{1+|x|^{\frac{N-p}{p}+\sigma}},\quad \forall\,|x|\ge 1,$$

for some $\sigma>0$.

The same estimate of Lemma B.2 was obtained in [6, Lemma B.3] for solutions to equation

$$\begin{cases}-\Delta_p u=|u|^{p^*-2}u & \text{in } \mathbf{R}^N,\\ u\in W^{1,p}(\mathbf{R}^N),\end{cases}$$

and in [31, Proposition 2.1] for solutions to more general equations. The proof of Lemma B.2 is the same as that of Lemma B.3 in [6] and of Proposition 2.1 in [31]. So we omit the details. To prove Proposition B.1, we will use the following comparison principle which is a special case of [31, Theorem 1.5]. Denote $\mathcal{D}^{1,p}(\Omega)=\{u\in L^{p^*}(\Omega)\colon \nabla u\in L^p(\Omega)\}$.

Theorem B.3. *Let Ω be an exterior domain such that $\Omega^c = \mathbf{R}^N \setminus \Omega$ is bounded and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a subsolution of equation*

$$-\Delta_p u = f|u|^{p-2}u \quad \text{in } \Omega, \tag{B.3}$$

and $v \in \mathcal{D}^{1,p}(\Omega)$ a positive supersolution of

$$-\Delta_p v = g|v|^{p-2}v \quad \text{in } \Omega, \tag{B.4}$$

such that $\inf_{\partial\Omega} v > 0$, where $g \in L^{\frac{N}{p}}(\Omega)$ satisfies $f \le g$ in Ω. Moreover, assume that

$$\limsup_{R\to\infty} \frac{1}{R} \int_{B_{2R}(0)\setminus B_R(0)} |u|^p |\nabla \log v|^{p-1}\, dx = 0. \tag{B.5}$$

Then if $u \le v$ on $\partial\Omega$, we have that

$$u \le v \quad \text{in } \Omega.$$

Now we can prove Proposition B.1.

Proof of Proposition B.1. We will use Theorem B.3 to prove Proposition B.1. First we consider the case when $\mathbf{R}^N_* = \mathbf{R}^N$. Let u be a solution to equation (B.1) with $\mathbf{R}^N_* = \mathbf{R}^N$ and set

$$f(x) = \mu|u(x)|^{p^*-p} + \frac{|u(x)|^{p^*(s)-p}}{|x|^s}.$$

Then $f \ge 0$. By Lemma B.2, we have

$$f(x) \le C|x|^{-\alpha} \quad \text{for } |x| \ge 1, \tag{B.6}$$

where

$$\alpha = \min\left\{(p^*-p)\left(\frac{N-p}{p}+\sigma\right), s + (p^*(s)-p)\left(\frac{N-p}{p}+\sigma\right)\right\}.$$

Since $0 < s < p$ and $\sigma > 0$, we have $\alpha = p + (p^*(s)-p)\sigma > p$. Thus $f \in L^{\frac{N}{p}}(\mathbf{R}^N \setminus B_1(0))$.

Let $\epsilon > 0$ and write $\gamma = (N-p)/(p-1)$. Let $v(x) = |x|^{-\gamma}(1+|x|^{-\epsilon})$ for $x \ne 0$. A simple calculation shows that

$$-\Delta_p v = g(x)v^{p-1} \quad \text{for } x \ne 0,$$

where

$$g(x) = \frac{(p-1)(\gamma+\epsilon)^{p-1}\epsilon}{(1+|x|^{-\epsilon})^{p-1}|x|^{p+(p-1)\epsilon}}.$$

It is easy to derive from the above formula that

$$C|x|^{-p-(p-1)\epsilon} \ge g(x) \ge C'|x|^{-p-(p-1)\epsilon} \quad \text{for } |x| \ge 1, \tag{B.7}$$

for $C, C' > 0$ depending on N, p and ϵ. As a result, we have $g \in L^{\frac{N}{p}}(\mathbf{R}^N \setminus B_1(0))$ since $\epsilon > 0$.

Choose $\epsilon > 0$ small such that $p + (p-1)\epsilon < \alpha$. Then by (B.6) and (B.7), we can find a large number $R_0 > 1$ such that

$$g(x) \ge f(x) \quad \text{for } |x| \ge R_0.$$

To verify (B.5), we note that $|\nabla \log v(x)| \le C|x|^{-1}$ for $|x|$ large enough. Hence by Lemma B.2, we have for $R \ge R_0$ large enough,

$$\begin{aligned}\frac{1}{R}\int_{B_{2R}(0)\backslash B_R(0)} |u|^p |\nabla \log v|^{p-1}\,dx &\le \frac{C}{R}\int_{B_{2R}(0)\backslash B_R(0)} |x|^{-\left(\frac{N-p}{p}+\sigma\right)p-(p-1)}\,dx \\ &\le CR^{-p-\left(\frac{N-p}{p}+\sigma\right)p+N} \\ &= CR^{-p\sigma},\end{aligned}$$

which implies that (B.5) holds. Therefore, we can apply Theorem B.3 with $\Omega = \mathbf{R}^N \backslash B_{R_0}(0)$ to conclude that

$$u(x) \le v(R_0)^{-1}\Big(\sup_{|y|=R_0} |u(y)|\Big) v(x) \quad \text{for } |x| \ge R_0,$$

where $v(R_0) = v(x)$ with $|x| = R_0$. We can prove the same estimate as above for $-u$ similarly. Therefore, we obtain that

$$|u(x)| \le C|x|^{-\frac{N-p}{p-1}} \quad \text{for } |x| \ge R_0. \tag{B.8}$$

So we obtain an estimate for the decay of u at infinity. To prove (B.2), it is enough to note that $u \in L^\infty_{\rm loc}(\mathbf{R}^N)$, which can be proved by Moser's iteration method [25]. In particular, we obtain that

$$|u(x)| \le C' \quad \text{for } |x| \le R_0 \tag{B.9}$$

for a constant $C' > 0$. Combining (B.8) and (B.9) yields (B.2). This finishes the proof of the case when $\mathbf{R}^N_* = \mathbf{R}^N$.

Next, consider the case when $\mathbf{R}^N_* = \mathbf{R}^N_+$. In this case, for any solution u, we use the odd extension

$$\tilde{u}(x) = \begin{cases} u(x', x_N) & \text{if } x_N \ge 0, \\ -u(x', -x_N) & \text{if } x_N < 0, \end{cases}$$

for $x = (x', x_N) \in \mathbf{R}^N$. It is direct to verify that $\tilde{u} \in \mathcal{D}^{1,p}(\mathbf{R}^N)$ and that $\tilde{u}$ is a solution of equation (B.1) with $\mathbf{R}^N_* = \mathbf{R}^N$. Thus our problem is reduced to the previous case. We then conclude easily that Proposition B.1 holds true for the second case. The proof of Proposition B.1 is complete. □

Appendix C. Estimates for solutions of p-Laplacian equations

In this section, we copy two results on p-Laplacian equation from [6] without proof. We assume that D is a bounded domain with $\Omega \subset\subset D$. The following result is Lemma 2.2 of [6].

Proposition C.1. *For any functions $f_1(x) \ge 0$ and $f_2(x) \ge 0$, let $w \ge 0$ be the solution of*

$$\begin{cases} -\Delta_p w = f_1 + f_2 & \textit{in } D, \\ w = 0 & \textit{on } \partial D. \end{cases}$$

Let $w_i,\ i = 1, 2$, be the solution of

$$\begin{cases} -\Delta_p w = f_i & \textit{in } D, \\ w = 0 & \textit{on } \partial D, \end{cases}$$

respectively. Then, there is a constant $C > 0$, *depending only on* $r = \frac{1}{3}dist(\Omega, \partial D)$, *such that*

$$w(x) \leq C \inf_{y \in B_r(x)} w(y) + Cw_1(x) + Cw_2(x), \quad \forall x \in \Omega.$$

Next result is Proposition C.1 of [6] which gives an estimate for solutions of p-Laplacian equation by Wolff potential.

Proposition C.2. *There is a constant* $\gamma \in (p-1, (p-1)N/(N-p+1))$, *such that for any solution* $u \in W^{1,p}(D) \cap L^\infty(D)$ *to equation*

$$-\Delta_p u = f \quad \text{in } D,$$

where $f \in L^1(D)$, $f \geq 0$, *there exists a constant* $C = C(N, p, \gamma) > 0$, *such that for any* $x \in D$ *and* $r \in (0, \mathrm{dist}(x, \partial D))$,

$$\left(\fint_{B_r(x)} u^\gamma \, dy \right)^{\frac{1}{\gamma}} \leq C + C \int_r^{\mathrm{dist}(x,\partial D)} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} f \, dy \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Appendix D. A global compactness result

Recall that by (2.1) we define, for any function u,

$$\rho_{x,\lambda}(u) = \lambda^{\frac{N-p}{p}} u(\lambda(\cdot - x))$$

for any $\lambda > 0$ and $x \in \mathbf{R}^N$. In this section, we give a global compactness result in the following proposition.

Proposition D.1. *Let* u_n, $n = 1, 2, \ldots$, *be a solution of equation* (1.7) *with* $\epsilon = \epsilon_n \to 0$, *satisfying* $\|u_n\| \leq C$ *for some constant* C *independent of* n. *Then* u_n *can be decomposed as*

$$u_n = u_0 + \sum_{j=1}^{k} \rho_{x_{n,j},\lambda_{n,j}}(U_j) + \sum_{j=k+1}^{m} \rho_{0,\lambda_{n,j}}(U_j) + \omega_n,$$

where u_0 *is a solution for* (1.1), $\omega_n \to 0$ *strongly in* $W_0^{1,p}(\Omega)$, $x_{n,j} \in \Omega$. *And as* $n \to \infty$, $\lambda_{n,j} \to \infty$ *for all* $1 \leq j \leq m$, $\lambda_{n,j}\mathrm{dist}(x_{n,j}, \partial\Omega) \to \infty$ *for all* $1 \leq j \leq k$. *For* $j = 1, 2, \cdots, k$, U_j *is a solution of*

$$\begin{cases} -\Delta_p u = b_j \mu |u|^{p^*-2} u & \text{in } \mathbf{R}^N, \\ u \in \mathcal{D}^{1,p}(\mathbf{R}^N), \end{cases}$$

for some $b_j \in (0, 1]$. *For* $j = k+1, k+2, \cdots, m$, U_j *is a solution of*

$$\begin{cases} -\Delta_p u = b_j \mu |u|^{p^*-2} u + b_j \frac{|u|^{p^*(s)-2} u}{|x|^s} & \text{in } \mathbf{R}^N_*, \\ u \in \mathcal{D}_0^{1,p}(\mathbf{R}^N_*), \end{cases}$$

for some $b_j \in (0, 1]$, *where* $\mathbf{R}^N_* = \mathbf{R}^N$ *when* $0 \in \Omega$, *and* $\mathbf{R}^N_* = \mathbf{R}^N_+$ *when* $0 \in \partial\Omega$. *Moreover, set* $x_{n,i} = 0$ *for* $i = k+1, \cdots, m$. *For* $i, j = 1, 2, \cdots, m$, *if* $i \neq j$, *then*

$$\frac{\lambda_{n,j}}{\lambda_{n,i}} + \frac{\lambda_{n,i}}{\lambda_{n,j}} + \lambda_{n,j}\lambda_{n,i}|x_{n,i} - x_{n,j}|^2 \to \infty$$

as $n \to \infty$.

Proof. The proof is standard, see e.g. [6, 7, 32]. We omit the details. □

References

[1] Ambrosetti, A., and P. H. Rabinowitz: Dual variational methods in critical point theory and applications. - J. Funct. Anal. 14, 1973, 349–381.

[2] Bahri, A., and J. Coron: On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. - Comm. Pure Appl. Math. 41:3, 1998, 253–294.

[3] Brézis, H.: Nonlinear elliptic equations involving the critical Sobolev exponent-survey and perspectives. - In: Directions in partial differential equations (Madison, WI, 1985), Publ. Math. Res. Center Univ. Wisconsin 54, Academic Press, Bonston, MA, 1987, 17–36.

[4] Brézis, H., and L. Nirenberg: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. - Comm. Pure Appl. Math. 36:4, 1983, 437–477.

[5] Caffarelli, L. A., R. Kohn, and L. Nirenberg: First order interpolation inequalities with weights. - Compos. Math. 53:3, 1984, 259–275.

[6] Cao, D., S. Peng, and S. Yan: Infinitely many solutions for p-Laplacian equation involving critical Sobolev growth. - J. Funct. Anal. 262:6, 2012, 2861–2902.

[7] Cao, D., and S. Yan: Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential. - Calc. Var. Partial Differential Equations 38:3-4, 2010, 471–501.

[8] Capozzi, A., D. Fortunato, and G. Palmieri: An existence result for nonlinear elliptic problems involving critical exponents. - Ann. Inst. H. Poinceré Anal. Non Linéaire 2:6, 1985, 463–470.

[9] Cerami, G. S. Solimini, and M. Struwe: Some existence results for superlinear elliptic boundary value problems involving critical exponents. - J. Funct. Anal. 69:3, 1986, 289–306.

[10] Chern, J. L., and C.-S. Lin: Minimizers of Caffarelli–Kohn–Nirenberg inequalities with the singularity on the boundary. - Arch. Ration. Mech. Anal. 197:2, 2010, 401–432.

[11] Cingolani, S., and G. Vannella: Multiple positive solutions for a critical quasilinear equation via Morse theory. - Ann. Inst. H. Poincaré Anal. Non Liné aire 26:2, 2009, 397–413.

[12] Cingolani, S., and G. Vannella: On the multiplicity of positive solutions for p-Laplace equations via Morse theory. - J. Differential Equations 247:11, 2009, 3011–3027.

[13] Coron, J. M.: Topologie et cas limite des injections de Sobolev (Topology and limit case of Sobolev embeddings). - C. R. Acad. Sci. Paris Ser. I Math. 299:7, 1984, 209–212 (in French).

[14] Degiovanni, M., and S. Lancelotti: Linking solutions for p-Laplace equations with nonlinearity at critical growth. - J. Funct. Anal. 256:11, 2009, 3643–3659.

[15] Devillanova, G., and S. Solimini: Concentration estimates and multiple solutions to elliptic problems at critical growth. - Adv. Differential Equations 7:10, 2002, 1257–1280.

[16] Egnell, H.: Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents. - Arch. Ration. Mech. Anal. 104:1, 1988, 57–77.

[17] Ghoussoub, N.: Duality and perturbation methods in critical point theory. - Cambridge Tracts in Math. 107, Cambridge Univ. Press, Cambridge, 1993.

[18] Ghoussoub, N., and X. S. Kang: Hardy–Sobolev critical elliptic equations with boundary singularities. - Ann. Inst. H. Poincaré Anal. Non Linéaire 21:6, 2004, 767–793.

[19] Ghoussoub, N., and F. Robert: The effect of curvature on the best constant in the Hardy-Sobolev inequalities. - Geom. Funct. Anal. 16:6, 2006, 1201–1245.

[20] Ghoussoub, N., and C. Yuan: Multiple solutions for quasilinear PDEs involving critical Sobolev and Hardy exponents. - Trans. Amer. Math. Soc. 352:12, 2000, 5703–5743.

[21] Guedda, M., and L. Véron: Quasilinear elliptic equations involving critical Sobolev exponents. - Nonlinear Anal. 13:8, 1989, 879–902.

[22] Han, Q., and F. H. Lin: Elliptic partial differential equations. - Courant Lect. Notes Math. 1, New York University, Courant Institute of Mathematical Sciences, New York, 1997.

[23] Hsia, C. H., C.-S. Lin, and H. Wadade: Revisiting an idea of Brézis and Nirenberg. - J. Funct. Anal. 259:7, 2010, 1816–1849.

[24] Li, Y. Y., and C.-S. Lin: A nonlinear elliptic PDE and two Sobolev–Hardy critical exponents. - Arch. Ration. Mech. Anal. 203:3, 2012, 943–968.

[25] Moser, J.: A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. - Comm. Pure Appl. Math. 13, 1960, 457–468.

[26] Papageorgiou, E. H., and N. S. Papageorgiou: A multiplicity theorem for problems with the p-Laplacian. - J. Funct. Anal. 244:1, 2007, 63–77.

[27] Rabinowitz, P. H.: Variational methods for nonlinear eigenvalue problems. - In: Eigenvalues of non-linear problems (Centro Internaz. Mat. Estivo (C.I.M.E.), III Ciclo, Varenna) 1974, 139–195.

[28] Rabinowitz, P. H.: Minimax methods in critical points theory with applications to differential equations. - CBMS Series 65, Providence, RI, 1986.

[29] Shen, Y., and X. Guo: On the existence of infinitely many critical points of the even functional $\int_\Omega F(x,u,Du)\,dx$ in $W_0^{1,p}$. - Acta Math. Sci. (English Ed.) 7:2, 1987, 187–195.

[30] Struwe, M.: A global compactness result for elliptic boundary value problems involving nonliearities. - Math. Z. 187:4, 1984, 511–517.

[31] Xiang, C.-L.: Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential. - J. Differential Equations 259:8, 2015, 3929–3954.

[32] Yan, S.: A global compactness result for quasilinear elliptic equations with critical Sobolev exponents. - Chinese J. Contemp. Math. 16:3, 1995, 227–234.

[33] Yan, S., and J. Yang: Infinitely many solutions for an elliptic problem involving critical Sobolev and Hardy–Sobolev exponents. - Calc. Var. Partial Differential Equations 48:3-4, 2013, 587–610.

[34] Zhu, X.: Nontrivial solution of quasilinear elliptic equations involving critical Sobolev exponent. - Sci. Sin. Ser. A 31:10, 1988, 1166–1181.

[35] Ziemer, W. P.: Weakly differentiable functions. - Grad. Texts in Math. 120, Springer-Verlag, New York, 1989.

Received 22 October 2015 • Revised received 4 April 2016 • Accepted 9 April 2016