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BOUNDED VERY WEAK SOLUTIONS FOR SOME NON-UNIFORMLY ELLIPTIC EQUATION WITH L^1 DATUM

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Abstract. In this paper we obtain the existence of bounded very weak solutions for the Dirichlet boundary value problem of a class of non-uniformly elliptic equations with L^1 integrability conditions by using the regularizing effect of the interaction between the coefficient of lower order term and the datum in the right-hand side.

1. Introduction

Suppose that Ω is a bounded domain of $\mathbf{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. In this paper we are concerned with the following non-uniformly elliptic Dirichlet boundary problem

$$\begin{cases} -\operatorname{div}\big(D_\xi\Phi(\nabla u)\big) + a(x)g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Phi\colon \mathbf{R}^N \mapsto \mathbf{R}_+$ is a C^1 nonnegative, strictly convex function, $D_\xi\Phi\colon \mathbf{R}^N \to \mathbf{R}^N$ represents the gradient of $\Phi(\xi)$ with respect to ξ and ∇u represents the gradient with respect to x. Without loss of generality we may assume that $\Phi(0) = 0$. Our main assumptions are that $\Phi(\xi)$ satisfies

(i) the super-linear condition

$$\lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|^q} = \infty, \tag{1.2}$$

where $q > 1$.

(ii) the symmetric condition: there exists a positive number $C > 0$ such that

$$\Phi(-\xi) \leq C\Phi(\xi), \quad \xi \in \mathbf{R}^N. \tag{1.3}$$

The continuous function $g(s)$ satisfies

$$\lim_{s\to-\infty} g(s) = -\infty, \quad \lim_{s\to\infty} g(s) = \infty, \tag{1.4}$$

and for all $s \in \mathbf{R}$,

$$|g(s)| \leq C_1|s|^\alpha + C_2, \tag{1.5}$$

where $\alpha = q - 1$, C_1, C_2 are positive constants.

Moreover, we assume that

$$a(x), f(x) \in L^1(\Omega), \tag{1.6}$$

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and there exists $Q \in (0, +\infty)$ such that,

$$|f(x)| \le Qa(x), \text{ a.e. } x \in \Omega. \tag{1.7}$$

There are several well-known examples of functions $\Phi(\xi)$ satisfying the assumptions (1.2) and (1.3). Some of them are listed here.

Example 1.1.

$$\Phi(\xi) = \frac{1}{p}|\xi|^p, \quad p > q.$$

In this case, equation (1.1) is the p-Laplacian equation.

Example 1.2.

$$\Phi(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_N}|\xi_N|^{p_N}, \quad p_i > q,\ i = 1, 2, \ldots, N,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. In this case, equation (1.1) is the anisotropic p-Laplacian equation.

Example 1.3.

$$\Phi(\xi) = e^{\frac{|\xi|^2}{2}} - 1.$$

The energy functional

$$E(u) = \int_\Omega \exp(|\nabla u|^2)\, dx \tag{1.8}$$

originates from the exponential harmonic mappings. It has been studied in [10, 14, 15], especially for the regularity theory.

The main purpose of this paper is to establish the existence of solutions for problem (1.1) under the integrability conditions (1.6) and (1.7). In general, a solution of an elliptic equation having a right-hand side in $L^1(\Omega)$ is not bounded and has no finite energy. The solutions may not belong to Sobolev space $W_0^{1,1}(\Omega)$. So in this case it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of entropy solutions was first proposed by Bénilan et al. in [4] for the nonlinear elliptic problems. It was then adapted to the study of some nonlinear elliptic and parabolic problems. We refer to [2, 5, 6, 16] for details. Recently, Arcoya and Boccardo in [3] studied the regularizing effect of the interaction between the coefficient of the zeroth order term and the datum in the following elliptic equations:

$$-\text{div}(M(x)\nabla u) + a(x)u = f(x),$$
$$-\text{div}(M(x)\nabla u) + a(x)g(u) = f(x),$$
$$-\text{div}(M(x, u)\nabla u) + a(x)u = B(x, u, \nabla u) + f(x),$$

and obtained some interesting and surprising results that the bounded solutions with finite energy exist for the corresponding Dirichlet problems of the above equations. Our work can be seen as a natural outgrowth of the results in [3] to the more general quasilinear problem (1.1). To this aim, we first employ a unifying method developed in [17] (see [7] for the parabolic case) to prove the existence of weak solutions for problem (1.1) under the integrability conditions that $f \in L^N(\Omega)$ and $a \in L^\infty(\Omega)$. It is worth pointing out that we do not assume polynomial or exponential growth for function Φ as in [1, 8, 14]. Based on this result and the regularizing effect of the interaction between the coefficient of lower order term and the datum, we obtain the existence of bounded very weak solutions for problem (1.1) under the L^1 integrability conditions (1.6) and (1.7) by using the approximation techniques.

The solutions of equation (1.1) are understood in the following sense.

Definition 1.4. A function $u \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ with $D_\xi\Phi(\nabla u) \in L^1(\Omega)$ is called a bounded very weak solution to problem (1.1) if for every $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega D_\xi\Phi(\nabla u) \cdot \nabla\varphi \, dx + \int_\Omega a g(u)\varphi \, dx = \int_\Omega f\varphi \, dx. \tag{1.9}$$

Remark 1.5. Notice that we only assume that $D_\xi\Phi(\nabla u) \in L^1(\Omega)$ instead of $D_\xi\Phi(\nabla u) \cdot \nabla u \in L^1(\Omega)$ in Definition 1.4. For this reason we call the solution "very weak".

Now we state our main result.

Theorem 1.6. *Assume that the structure conditions* (1.2)–(1.5) *and the integrability conditions* (1.6) *and* (1.7) *hold. Then there exists a bounded very weak solution* $u \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ *for problem* (1.1).

The rest of this paper is organized as follows. In Section 2, we first list some basic results that will be used later. Next we construct a sequence of the approximation solutions. Then we find the limit of a subsequence is the solution as required. In the following C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries and the proof of main result

2.1. Some properties about $\Phi(\xi)$. Let $\Phi(\xi)$ be a nonnegative convex function. We define the polar function of $\Phi(\xi)$ as

$$\Psi(\eta) = \sup_{\xi \in \mathbf{R}^N} \{\eta \cdot \xi - \Phi(\xi)\}, \tag{2.1}$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is easy to see that $\Psi(\eta)$ is a convex function. Observe that the super-linear condition (1.2) implies the 1-coercive condition (see [13], Chapter E)

$$\lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|} = \infty \tag{2.2}$$

holds. Suppose that $\Phi(\xi)$ is a nonnegative convex C^1 function with $\Phi(0) = 0$. Then, for all $\xi, \eta, \zeta \in \mathbf{R}^N$, we have the following inequalities:

$$\Phi(\xi) \le \xi \cdot D\Phi(\xi), \tag{2.3}$$

$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \ge 0, \tag{2.4}$$

$$\xi \cdot \eta \le \Phi(\xi) + \Psi(\eta), \tag{2.5}$$

$$\Psi(D\Phi(\zeta)) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta. \tag{2.6}$$

Moreover, if $\Phi(\xi)$ satisfies the super-linear condition (2.2), then its polar function $\Psi(\eta)$ also satisfies (2.2). We refer to [7, 11, 17] for the details.

2.2. The proof of main results. In this subsection we first give a reasonable definition of weak solutions and then prove the existence of weak solutions for problem (1.1). Let $q = 1 + \alpha > 1$ be the constant defined as in (1.2).

Definition 2.1. A function $u \in W_0^{1,q}(\Omega)$ with $D_\xi \Phi(\nabla u) \cdot \nabla u \in L^1(\Omega)$ and $a(x)g(u) \in L^1(\Omega)$ is called a weak solution to problem (1.1) if for every $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla \varphi \, dx + \int_\Omega a g(u) \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{2.7}$$

Theorem 2.2. *Assume that the structure conditions* (1.2)–(1.5) *hold. If* $f \in L^N(\Omega)$ *and* $a \in L^\infty(\Omega)$*, then there exists a weak solution* $u \in W_0^{1,q}(\Omega)$ *for problem* (1.1).

Proof. We consider the variational problem

$$\min\{J(v) \mid v \in V\},$$

where $V = \{v \in W_0^{1,q}(\Omega) \mid \Phi(\nabla v) \in L^1(\Omega)\}$, and functional J is

$$J(v) = \int_\Omega \Phi(\nabla v) \, dx + \int_\Omega a G(v) \, dx - \int_\Omega f v \, dx$$

with $G(v) = \int_0^v g(s) \, ds$. It is straightforward to check that functional $J(v)$ is coercive, lower bounded and lower semi-continuous in V. Therefore, from the standard technique in Calculus of Variations (see for instance [9]), one can show $J(v)$ has a minimizer $u(x)$ in V. Then it is sufficient to prove that the minimizer $u(x)$ satisfies the Euler–Lagrange equation of functional J weakly.

Since $u \in V$ is a minimizer, we have $\lambda u \in V$, $\lambda \in (0,1)$, and

$$J(u) \le J(\lambda u),$$

which implies

$$\begin{aligned} &\int_\Omega \Phi(\nabla u) \, dx + \int_\Omega a G(u) \, dx - \int_\Omega f u \, dx \\ &\le \int_\Omega \Phi(\lambda \nabla u) \, dx + \int_\Omega a G(\lambda u) \, dx - \lambda \int_\Omega f u \, dx. \end{aligned}$$

Recalling (2.4), we know

$$\Phi(\nabla u) - \Phi(\lambda \nabla u) \ge (1-\lambda) D_\xi \Phi(\lambda \nabla u) \cdot \nabla u.$$

Then

$$\begin{aligned} (1-\lambda) \int_\Omega D_\xi \Phi(\lambda \nabla u) \cdot \nabla u \, dx &\le (1-\lambda) \int_\Omega f u \, dx + \int_\Omega a(G(\lambda u) - G(u)) \, dx \\ &\le (1-\lambda) \int_\Omega f u \, dx + C(1-\lambda) \|a\|_{L^\infty(\Omega)} \int_\Omega [|u|^{1+\alpha} + |u|] \, dx. \end{aligned}$$

Dividing the above inequality by $1-\lambda$, and passing to limits as $\lambda \to 1$, we have

$$\liminf_{\lambda \to 1} \int_\Omega D_\xi \Phi(\lambda \nabla u) \cdot \nabla u \, dx \le \int_\Omega f u \, dx + C \int_\Omega [|u|^{1+\alpha} + |u|] \, dx.$$

Since $D_\xi \Phi(\lambda \nabla u) \cdot \nabla u \ge 0$, by Fatou's Lemma we conclude that

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla u \, dx \le \int_\Omega f u \, dx + C \int_\Omega [|u|^{1+\alpha} + |u|] \, dx.$$

Due to (1.2) and (2.2), for every $\delta > 0$, there exist constants $C_\delta > 0$ such that

$$|\xi|^{1+\alpha} \le \delta \Phi(\xi) + C_\delta, \quad |\xi| \le \delta \Phi(\xi) + C_\delta. \tag{2.8}$$

By Hölder's and Sobolev's inequalities, (1.5) and (2.8), we have

$$\begin{aligned}\left|\int_\Omega fu\,dx\right| &\le \|f\|_{L^N(\Omega)}\|u\|_{L^{1^*}(\Omega)} \le C\|f\|_{L^N(\Omega)}\|\nabla u\|_{L^1(\Omega)}\\ &\le C\delta\int_\Omega \Phi(\nabla u)\,dx + C_\delta\end{aligned}\tag{2.9}$$

and

$$\int_\Omega |\nabla u|^{1+\alpha}\,dx + \int_\Omega |\nabla u|\,dx \le \delta\int_\Omega \Phi(\nabla u)\,dx + C_\delta.\tag{2.10}$$

By choosing δ sufficiently small we can deduce from (2.3) that

$$\frac{1}{2}\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx \le C.$$

It follows from (2.6) that $D_\xi\Phi(\nabla u)\cdot\nabla u \in L^1(\Omega)$ and $\Psi(D_\xi\Phi(\nabla u)) \in L^1(\Omega)$.

For some fixed $\varphi(x)\in C_0^1(\Omega)$, we know that $J(u)\le J(\lambda u+(1-\lambda)\varphi)$, $\forall\lambda\in(0,1)$. Denote $\xi_\lambda = \lambda\nabla u + (1-\lambda)\nabla\varphi$. In light of (2.4), we find

$$\Phi(\nabla u) - \Phi(\xi_\lambda) \ge (1-\lambda)D_\xi\Phi(\xi_\lambda)\cdot(\nabla u - \nabla\varphi),$$

and deduce as above to have

$$\begin{aligned}&\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx\\ &\le \int_\Omega fu\,dx - \int_\Omega f\varphi\,dx + \frac{1}{1-\lambda}\int_\Omega a[G(\xi_\lambda)-G(u)]\,dx.\end{aligned}\tag{2.11}$$

Consider

$$h(\lambda) = \Phi(\xi_\lambda) = \Phi(\lambda\nabla u + (1-\lambda)\nabla\phi).$$

It is obvious that h is a convex function in $\mathbf{R}$. Then by the monotonicity of a convex function's derivative, we know

$$h'(0)\le h'(\lambda)\le h'(1),\quad \lambda\in(0,1),$$

which yields that

$$D_\xi\Phi(\nabla\phi)\cdot(\nabla u-\nabla\varphi) \le D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi) \le D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\phi).\tag{2.12}$$

Recalling (1.3) and (2.6), we have

$$\begin{aligned}|D_\xi\Phi(\nabla u)\cdot\nabla\varphi| &\le \Psi(D_\xi\Phi(\nabla u)) + \Phi(\nabla\varphi) + \Phi(-\nabla\varphi)\\ &\le \Psi(D_\xi\Phi(\nabla u)) + (C+1)\Phi(\nabla\varphi).\end{aligned}\tag{2.13}$$

As $\Psi(D_\xi\Phi(\nabla u))\in L^1(\Omega)$ and $\varphi\in C_0^1(\Omega)$, it is easy to know $D_\xi\Phi(\nabla\varphi)\cdot(\nabla u-\nabla\varphi)\in L^1(\Omega)$ and $D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\varphi)\in L^1(\Omega)$. By the Lebesgue dominated convergence theorem, we have

$$\int_\Omega \lim_{\lambda\to1} D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx = \lim_{\lambda\to1}\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx.$$

Since g is a continuous function, then

$$\begin{aligned}\lim_{\lambda\to1}\frac{1}{1-\lambda}\int_\Omega[G(\xi_\lambda)-G(u)]\,dx &= \int_\Omega\left[\lim_{\lambda\to1}\frac{1}{1-\lambda}\int_u^{\lambda u+(1-\lambda)\varphi} g(s)\,ds\right]dx\\ &= \int_\Omega g(u)(\varphi-u)\,dx.\end{aligned}$$

Furthermore, recalling (2.11) we have

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\varphi)\,dx \le \int_\Omega fu\,dx-\int_\Omega f\varphi\,dx+\int_\Omega ag(u)\varphi\,dx-\int_\Omega ag(u)u\,dx.$$

Denote

$$A_0=\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx-\int_\Omega fu\,dx+\int_\Omega ag(u)u\,dx.$$

Then we conclude that, for every $\varphi(x)\in C_0^1(\Omega)$,

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\varphi\,dx-\int_\Omega f\varphi\,dx+\int_\Omega ag(u)\varphi\,dx\ge A_0.$$

By a scaling argument, it follows that

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\varphi\,dx-\int_\Omega f\varphi\,dx+\int_\Omega ag(u)\varphi\,dx=0.$$

It means that $u(x)$ is a weak solution of problem (1.1). □

Now we are ready to prove the existence of bounded very weak solutions of problem (1.1). We would like to point out that our approach is much influenced by [3].

Proof of Theorem 1.6. We first introduce the approximated problems. Let $\{f_n\},\{a_n\}$ defined by

$$f_n(x)=\frac{f(x)}{1+\frac{1}{n}|f(x)|},\quad a_n(x)=\frac{a(x)}{1+\frac{Q}{n}|a(x)|} \tag{2.14}$$

be two sequences of functions strongly convergent to f and a in $L^1(\Omega)$. By Theorem 2.2, we obtain the weak solution $u_n\in W_0^{1,q}(\Omega)$ of the approximation problem

$$-\mathrm{div}\big(D_\xi\Phi(\nabla u_n)\big)+a_ng(u_n)=f_n(x),$$

which satisfies

$$\int_\Omega D_\xi\Phi(\nabla u_n)\cdot\nabla\varphi\,dx+\int_\Omega a_ng(u_n)\varphi\,dx=\int_\Omega f_n\varphi\,dx,\quad \forall\varphi\in C_0^1(\Omega). \tag{2.15}$$

Since $\psi(s)=s\left(1+\frac{s}{n}\right)^{-1}$ is increasing, we know from (1.7) that

$$|f_n(x)|=\frac{|f(x)|}{1+\frac{1}{n}|f(x)|}\le\frac{Qa(x)}{1+\frac{Q}{n}a(x)}=Qa_n(x). \tag{2.16}$$

Recalling (1.4), we can choose $k_0>0$ such that

$$g(s)s\ge 0 \tag{2.17}$$

and

$$|g(s)|\ge Q \tag{2.18}$$

for every $s\in(k_0,+\infty)$. We define

$$G_{k_0}(s)=\begin{cases}0 & \text{if } |s|\le k_0,\\ s-k_0 & \text{if } s>k_0,\\ s+k_0 & \text{if } s<-k_0.\end{cases}$$

Choosing $G_{k_0}(u_n)$ as a test function in (2.15) yields

$$\begin{aligned}\int_\Omega D_\xi\Phi(\nabla u_n)\cdot\nabla G_{k_0}(u_n)\,dx+\int_\Omega a_n g(u_n)G_{k_0}(u_n)\,dx &\le \int_\Omega |f_n|\cdot|G_{k_0}(u_n)|\,dx\\ &\le Q\int_\Omega a_n|G_{k_0}(u_n)|\,dx,\end{aligned}$$

which further follows from (2.17) that

$$\int_\Omega D_\xi\Phi(\nabla G_{k_0}(u_n))\cdot\nabla G_{k_0}(u_n)\,dx+\int_\Omega a_n[|g(u_n)|-Q]|G_{k_0}(u_n)|\,dx\le 0.$$

Thus we conclude from (2.18) that $\|u_n\|_{L^\infty(\Omega)}\le k_0$ and the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

As a consequence, we take u_n as a test function in (2.15) to deduce

$$\begin{aligned}&\int_\Omega \Phi(\nabla u_n)\,dx-\max_{|s|\le k_0}|g(s)s|\int_\Omega a_n\,dx\\ &\le \int_\Omega D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dx+\int_\Omega a_n g(u_n)u_n\,dx\le k_0\int_\Omega |f_n|\,dx,\end{aligned}$$

that is

$$\int_\Omega \Phi(\nabla u_n)\,dx\le k_0\int_\Omega |f|\,dx+\max_{|s|\le k_0}|g(s)s|\int_\Omega a\,dx. \tag{2.19}$$

From (1.2) we may choose a subsequence of $\{u_n\}$ (denote it by the original sequence) and a function $u\in W_0^{1,q}(\Omega)$ such that

$$\begin{aligned}\nabla u_n &\rightharpoonup \nabla u \quad \text{weakly in } L^q(\Omega),\\ u_n &\to u \quad \text{strongly in } L^q(\Omega),\end{aligned}$$

and

$$\int_\Omega \Phi(\nabla u)\,dx\le \liminf_{n\to\infty}\int_\Omega \Phi(\nabla u_n)\,dx.$$

However, in order to obtain the existence of bounded very weak solutions, this is not enough to pass to a limit under the integral signs and more information is needed on the gradients. We shall prove that a subsequence of the sequence $\{\nabla u_n\}$ converges to ∇u almost everywhere in Ω .

We first claim that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let $\delta>0$, and denote

$$\begin{aligned}E_1 &:= \{x\in\Omega\colon |\nabla u_n|>h\}\cup\{|\nabla u_m|>h\},\\ E_2 &:= \{x\in\Omega\colon |u_n-u_m|>1\}\end{aligned}$$

and

$$E_3:=\{x\in\Omega\colon |\nabla u_n|\le h,\ |\nabla u_m|\le h,\ |u_n-u_m|\le 1,\ |\nabla u_n-\nabla u_m|>\delta\},$$

where h will be chosen later. It is obvious that

$$\{x\in\Omega\colon |\nabla u_n-\nabla u_m|>\delta\}\subset E_1\cup E_2\cup E_3.$$

In view of (2.19) and (2.8), there exists constant $C>0$ such that

$$\operatorname{meas}\{x\in\Omega\colon |\nabla u_n|\ge h\}\le \frac{\|\nabla u_n\|_{L^q(\Omega)}}{h^q}\le\frac{C}{h^q}.$$

Let $\varepsilon>0$. We may choose $h=h(\varepsilon)$ large enough such that

$$\operatorname{meas}(E_1)\le\varepsilon/3,\quad \text{for all } n,m\ge 0. \tag{2.20}$$

On the other hand, we know that $\{u_n\}$ converges to u strongly in $L^q(\Omega)$. Then there exists $N_1(\varepsilon) \in \mathbf{N}$ such that

$$\text{meas}(E_2) \le \varepsilon/3, \quad \text{for all } n, m \ge N_1(\varepsilon). \tag{2.21}$$

Moreover, since Φ is C^1 and strictly convex, then there exists a real valued function $m(h,\delta) > 0$ such that

$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \ge m(h,\delta) > 0, \tag{2.22}$$

for all $\xi, \zeta \in \mathbf{R}^N$ with $|\xi|, |\zeta| \le h$, $|\xi - \zeta| \ge \delta$. By taking $T_1(u_n - u_m)$ as a test function in (2.15), we obtain

$$\begin{aligned} m(h,\delta)\text{meas}(E_3) &\le \int_{E_3} [D_\xi\Phi(\nabla u_n) - D_\xi\Phi(\nabla u_m)] \cdot (\nabla u_n - \nabla u_m)\, dx \\ &= \int_{E_3} [f_n - f_m] T_1(u_n - u_m)\, dx \\ &\quad + \int_{E_3} [a_n g(u_n) - a_m g(u_m)] T_1(u_n - u_m)\, dx \\ &\le \|f_n - f_m\|_{L^1(\Omega)} + \|a_n g(u_n) - a_m g(u_m)\|_{L^1(\Omega)} := \alpha_{n,m}, \end{aligned}$$

which implies that

$$\text{meas}(E_3) \le \frac{\alpha_{n,m}}{m(h,\delta)} \le \varepsilon/3,$$

for all $n, m \ge N_2(\varepsilon, \delta)$. It follows from (2.20) and (2.21) that

$$\text{meas}\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \le \varepsilon, \quad \text{for all } n, m \ge \max\{N_1, N_2\},$$

that is $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we may choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega.$$

As ∇u_n converges ∇u weakly in $L^q(\Omega)$, we deduce that v coincides with the weak gradient of u. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{2.23}$$

In view of (2.19) and (2.6), we know that

$$\int_\Omega \Psi(D_\xi\Phi(\nabla u_n))\, dx \le C. \tag{2.24}$$

Applying Lemma 2.8 in [17] and (2.23), we conclude that (up to a subsequence)

$$D_\xi\Phi(\nabla u_n) \rightharpoonup D_\xi\Phi(\nabla u) \quad \text{weakly in } L^1(\Omega). \tag{2.25}$$

Finally, using the inequality

$$|a_n(x) g(u_n)| \le a(x) \max_{|s| \le k_0} |g(s)|,$$

we obtain the $L^1(\Omega)$ convergence of the sequence $\{a_n(x)g(u_n)\}$ to $a(x)g(u)$ by the Lebesgue dominated convergence theorem. Recalling (2.25) and the $L^1(\Omega)$ convergence of $f_n(x)$, we pass to the limits in (2.15) to conclude that u is a bounded very weak solution in the sense of Definition 1.4. □

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