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LIGHT SIDE OF COMPACTNESS IN LEBESGUE SPACES: SUDAKOV THEOREM

Przemysław Górka and Humberto Rafeiro

Warsaw University of Technology, Department of Mathematics and Information Sciences
Ul. Koszykowa 75, 00-662 Warsaw, Poland; p.gorka@mini.pw.edu.pl

Pontificia Universidad Javeriana, Facultad de Ciencias, Departamento de Matemáticas
Cra. 7 No. 43-82, Bogotá, Colombia; silva-h@javeriana.edu.co

Abstract. In this note we show that, in the case of bounded sets in metric spaces with some additional structure, the boundedness of a family of Lebesgue p-summable functions follow from a certain uniform limit norm condition. As a byproduct, the well known Riesz–Kolmogorov compactness theorem can be formulated only with one condition.

1. Introduction

The classical theorem of Kolmogorov [11], sometimes also called Riesz–Kolmogorov theorem, characterizes the compactness of sets of functions in Lebesgue spaces. In the original formulation of Kolmogorov the theorem is the following:

Theorem 1.1. (Kolmogorov) *Suppose $\mathcal{F}$ is a set of functions in $L^p([0,1])$ $(1 < p < \infty)$. In order that this set be relatively compact, it is necessary and sufficient that both of the following conditions be satisfied:*

(K1) *the set $\mathcal{F}$ is bounded in L^p;*
(K2) $\lim_{h\to 0} \|f_h - f\|_p = 0$ *uniformly with respect to $f \in \mathcal{F}$,*

where f_h denotes the well-known Steklov function, viz.

$$f_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt.$$

After that, Tamarkin [18] extended the result to the case where the underlying space can be unbounded, with an additional condition related to the behaviour at infinity. Tulajkov [19] showed that Tamarkin's result was true even when $p = 1$. Finally, Sudakov [16] showed that condition (K1) follows from condition (K2). All the previous results were proved in the framework of one dimensional Euclidean space.

The Riesz–Kolmogorov compactness theorem has also been extended to other function spaces, for example, it was extended by Takahashi [17] for Orlicz spaces satisfying the Δ_2-condition, by Goes and Welland [2] for continuously regular Köthe spaces, by Musielak [12] to Musielak–Orlicz spaces, by Rafeiro [14] to variable exponent Lebesgue spaces, by Rafeiro and Vargas [15] to grand Lebesgue spaces, by Górka and Rafeiro [8] to grand variable Lebesgue spaces, by Górka and Macios [6, 7] to Lebesgue spaces in metric measure spaces, just to name a few. Weil [20] showed the compactness theorem in $L^p(G)$, where G is a locally compact group. Pego [13] (see [4] and [5]) formulated Kolmogorov theorem for $p = 2$ in terms of the Fourier transform.

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For a more detailed account regarding the history of the Riesz–Kolmogorov theorem, see [9].

In this small note we want to show that, whenever we are working in the general framework of metric measure spaces, condition (K1) is superfluous since it is a consequence of condition (K2).

2. Preliminaries

We shall denote the average of locally integrable function f over the measurable set A in the following manner

$$(f)_A = \fint_A f\,d\mu = \frac{1}{\mu(A)}\int_A f\,d\mu.$$

Let (X,ρ,μ) be a metric measure space equipped with a metric ρ and a Borel regular measure μ. We assume throughout the paper that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. Additionally, we assume that the measure μ satisfies a doubling condition. This means that, there exists a constant $C_d > 0$ such that for every ball $B(x,r)$,

$$\mu\left(B(x,2r)\right) \le C_d \mu\left(B(x,r)\right).$$

Now, let us recall the notion of continuity of a measure with respect to a metric (see [3, 1]).

Definition 2.1. Let (X,ρ,μ) be a metric measure space. The measure μ is said to be continuous with respect to the metric ρ if for all $x \in X$ and $r > 0$ the following condition holds:

$$\lim_{y\to x} \mu(B(x,r)\Delta B(y,r)) = 0,$$

where $A\Delta B$ stands for the symmetric difference, i.e. $A\Delta B := A\setminus B \cup B\setminus A$.

For example, when (X,ρ,μ) is a geodesic space (cf. [10]) and the measure μ is doubling, then μ is continuous with respect to the metric ρ (see [1]).

Now, we can recall the charecterization of relatively compact sets in $L^p(X,\rho,\mu)$ from [6].

Theorem 2.2. *Let (X,ρ,μ) be a metric measure space and $1 < p < \infty$. Suppose moreover, that there exists $\theta > 0$ such that $\mu(B(x,1)) \ge \theta$. Let $x_0 \in X$, then the subset $\mathcal{F}$ of $L^p(X,\mu)$ is relatively compact in $L^p(X,\mu)$ if and only if the following conditions are satisfied:*

$$\mathcal{F} \text{ is bounded,} \tag{2.1}$$

$$\lim_{R\to\infty} \int_{X\setminus B(x_0,R)} |f(x)|^p\,d\mu(x) = 0, \quad \text{uniformly for } f \in \mathcal{F}, \tag{2.2}$$

$$\lim_{r\to 0} \int_X |f(x) - (f)_{B(x,r)}|^p\,d\mu(x) = 0, \quad \text{uniformly for } f \in \mathcal{F}. \tag{2.3}$$

3. Main result

The main result of this paper is the following.

Theorem 3.1. *Assume that (X,ρ,μ) is a connected metric measure space with continuous measure satisfying the doubling condition. Suppose, moreover, that balls are relatively compact and there exists $\theta > 0$ such that $\mu(B(x,1)) \ge \theta$. Let $1 < p <$*

∞ and D be a bounded subset of X such that $X \setminus \overline{D} \neq \emptyset$. If the family $\mathcal{F}$ in $L^p(D,\mu)$ satisfies

$$\lim_{r\to 0} \int_X |f(x) - (f)_{B(x,r)}|^p \, d\mu(x) = 0 \quad \text{uniformly for } f \in \mathcal{F},$$

where we continue the function f by zero beyond D, then $\mathcal{F}$ is bounded.

In order to prove Theorem 3.1 we will need some auxiliary results. We start with the following lemma.

Lemma 3.2. *Let $h > 0$ and denote by $\mathbf{1}_D$ the characteristic function of the set D. Then the operator $U\colon L^p(X) \longrightarrow L^p(X)$ given by*

$$Uf(x) = \fint_{B(x,h)} \mathbf{1}_D f \, d\mu$$

is compact.

Proof. Let us take $V = B(0,1) \subset L^p(X)$. We shall show that the set $U(V)$ is relatively compact in $L^p(X)$. For this purpose, we shall use the characterization of relatively compact sets in $L^p(X)$ from Theorem 2.2. Since D is bounded, there exists a ball $B(x_0,r)$ such that $D \subset B(x_0,r)$. Thus, for $f \in V$ we have $\operatorname{supp}(Uf) \subset B(x_0, r+h) =: W_h$. Hence, by the Jensen inequality, we get

$$\begin{aligned}\|Uf\|^p_{L^p(X)} = \|Uf\|^p_{L^p(W_h)} &= \int_{W_h} |Uf|^p \, d\mu = \int_{W_h} \left| \fint_{B(x,h)} \mathbf{1}_D f \, d\mu \right|^p d\mu \\ &\leq \int_{W_h} \fint_{B(x,h)} |f|^p \, d\mu \, d\mu \leq \|f\|^p_{L^p(X)} \frac{\mu(W_h)}{\inf_{x\in X} \mu(B(x,h))}.\end{aligned}$$

Since μ is doubling and $\mu(B(x,1)) \geq \theta$, we have $\inf_{x\in X} \mu(B(x,h)) > 0$. Thus, we get that $U(V)$ is bounded. Moreover, since $\operatorname{supp}(Uf) \subset W_h$, we get that $\|Uf\|_{L^p(X\setminus W_h)} = 0$. Finally, it remains to show that the family $U(f)$ is uniformly L^p-equicontinuous. Let $h > r > 0$. We have

$$\begin{aligned}&\int_X \left| Uf(x) - \fint_{B(x,r)} Uf(y)\, d\mu(y) \right|^p d\mu(x) \\ &= \int_{W_{2h}} \left| \fint_{B(x,r)} \left(\fint_{B(x,h)} \mathbf{1}_D f(z)\, d\mu(z) - \fint_{B(y,h)} \mathbf{1}_D f(z)\, d\mu(z) \right) d\mu(y) \right|^p d\mu(x).\end{aligned}$$

On the other hand, from the proof of Lemma 4.3 in [1] and by the Hölder inequality we have

$$\begin{aligned}&\left| \fint_{B(x,h)} \mathbf{1}_D f(z)\, d\mu(z) - \fint_{B(y,h)} \mathbf{1}_D f(z)\, d\mu(z) \right| \\ &\leq \frac{1}{\mu(B(x,h))} \int_{B(x,h)\Delta B(y,h)} |\mathbf{1}_D f(z)|\, d\mu(z) + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))} \int_{B(y,h)} |\mathbf{1}_D f(z)|\, d\mu(z) \\ &\leq \|f\|_{L^p(X)} \left(\frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right).\end{aligned}$$

Hence, we obtain

$$\int_X \left| Uf(x) - \fint_{B(x,r)} Uf(y)\, d\mu(y) \right|^p d\mu(x) \leq \|f\|^p_{L^p(X)} \int_{W_{2h}} |I(x)|^p \, d\mu(x).$$

where

$$I(x) = \fint_{B(x,r)} \left(\frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y)$$

By virtue of Lebesgue differentiation theorem (see e.g., [10]) we have

$$\lim_{r\to 0} \fint_{B(x,r)} \left(\frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y) = 0.$$

Furthermore,

$$\begin{aligned} &\left| \fint_{B(x,r)} \left(\frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y) \right|^p \\ &\le \left(\frac{\mu(W_{4h})^{1-1/p}}{\inf_{x\in X} \mu(B(x,h))} + \frac{\mu(W_{4h})}{\inf_{x\in X} \mu(B(x,h))^{1+1/p}} \right)^p . \end{aligned}$$

Finally, the Lebesgue theorem finishes the proof. □

We will also need the following result.

Lemma 3.3. 1 *is not an eigenvalue of* U.

Proof. Let us take $f \in L^p(X)$ such that $Uf = f$. We shall show that $f = 0$. Since the measure μ is continuous, from the proof of the previous lemma we have $f \in C(X)$. Moreover, from the proof of the previous lemma we have that $\operatorname{supp} f \subset W_h = B(x_0, r+h)$. Next, let us take a ball B such that $W_h \subset B$ and $\bar{D} \subset B$. Suppose that $M = \sup_{x\in\bar{B}} f(x) > 0$ and let

$$C = \{x \in X : f(x) = M\}.$$

Next, let us take $x_0 \in \partial C$. Due to the fact that C is closed, we have that $B(x_0,h) \cap (X \setminus C)$ is an open nonempty set. Thus $\mu\left((X \setminus C) \cap B(x_0,h)\right) > 0$. This contradicts our assumption that $f(x_0) = Uf(x_0)$. □

We now prove the main result.

Proof of Theorem 3.1. For this purpose we use the Riesz–Schauder theory (see e.g., [21]). Since U is compact and 1 is not an eigenvalue of U, we get that $(U - I)^{-1}$ is bounded. On the other hand, we have $\|Uf - f\|_{L^p(X)} \le C$ for $f \in \mathcal{F}$ and some positive constant C. Thus,

$$\|f\|_{L^p(X)} \le C\|(U - I)^{-1}\|_{L^p(X)\to L^p(X)},$$

and we obtain the desired result. □

As a corollary from Theorem 2.2 and Theorem 3.1 we obtain the following characterization of relatively compact sets.

Theorem 3.4. *Assume that* (X, ρ, μ) *is a connected metric measure space with continuous measure satisfying the doubling condition. Suppose, moreover, that balls are relatively compact and there exists* $\theta > 0$ *such that* $\mu(B(x,1)) \ge \theta$. *Let* $1 < p < \infty$ *and* D *be a bounded subset of* X *such that* $X \setminus \overline{D} \ne \emptyset$. *Then, the family* $\mathcal{F}$ *in* $L^p(D,\mu)$ *is relatively compact in* $L^p(D,\mu)$ *if and only if*

$$\lim_{r\to 0} \int_X |f(x) - (f)_{B(x,r)}|^p \, d\mu(x) = 0 \quad \textit{uniformly for } f \in \mathcal{F},$$

where we continue the function f *by zero beyond* D.

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